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ANALYSIS OF A MARKET-SPLIT MODEL

by

A.J. Goldman, J.M. McLynn

P.R. Meyers, R.H. Watkins

Technical Report

to

Office of High Speed Ground Transportation

Department of Commerce

U.S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

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Applied Mathematics Division

and

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Davidson, Talbird and McLynn

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IMPORTANT NOTICE

U.S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

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ABSTRACT

A mathematical analysis is given for a class of models describing how the market might divide its patronage among p competing products $(p > 1)$. The distinctive feature is that the elasticities of the split fractions, with respect to changes in the parameters characterizing the products, are assumed equal to linear or separable functions of the split fractions themselves. (The permitted functional forms are actually somewhat more general.) The self-consistent models of this type are determined (for $p > 2$ they are linear), and their solutions derived.

ANALYSIS OF A MARKET SPLIT MODEL[1]

by

A.J. Goldman, J.M. McLynn[2]

P.R. Meyers, R.H. Watkins[2]

1. INTRODUCTION

This paper is concerned with a class of mathematical models for how the "market" (i.e., the consuming public) might divide itself among several ($p > 1$) competing products. For $j=1,2,\ldots,p$, we set

$$w_j = \text{fraction of market which selects the j-th product,} \tag{1}$$

so that

$$w_j \geq 0 \quad (j=1,2,\ldots,p) , \tag{2}$$

$$\sum_{j=1}^{p} w_j = 1 . \tag{3}$$

The choice-influencing attributes of the j-th product are described by the numerical values of certain parameters $x_{1j}, x_{2j}, \ldots, x_{n(j),j}$. Since the market share of the j-th product depends on the relative attractiveness of the other products, the <u>split fraction</u> w_j is a function not only of the x_{ij}'s, but also of the x_{ik}'s for $k \neq j$, i.e. of <u>all</u> the x's.

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(2) Davidson, Talbird and McLynn, Bethesda, Md.

We assume that the $\underline{x}$'s have been redefined (if necessary) so that

$$x_{ij} > 0 , \tag{4}$$

and so that increasing x_{ij} makes the j-th product <u>less</u> attractive (or at least no more attractive), thus tending to decrease w_j and increase the other w_k's. Formally,

$$\partial w_j/\partial x_{ij} \leq 0 , \tag{5}$$

$$\partial w_k/\partial x_{ij} \geq 0 \qquad \text{for } k \neq j . \tag{6}$$

Let $\vec{x}$ denote the ensemble of $\underline{x}$'s, while $\vec{w}=(w_1,\ldots,w_p)$. An additional assumption about $\vec{w}(\vec{x})$ is that, for each j , there exists at least one $\vec{x}$ for which

$$w_j(\vec{x}) = 1 , \quad w_k(\vec{x}) = 0 \qquad \text{for } k \neq j . \tag{7}$$

The intended interpretation is that none of the products has a guaranteed minimum market share, nor is any of them artificially precluded from gaining the entire market if its superiority would lead to this result. We shall also need the stronger hypothesis, that for any w with $0 \leq w \leq 1$ and any (j,k) with $j \neq k$ there exists at least one $\vec{x}$ such that

$$w_j(\vec{x})=w , \quad w_k(\vec{x})=1-w , \quad w_q(\vec{x})=0 \quad \text{for } q \neq j,k . \tag{8}$$

(It suffices to assume that $\vec{x}$'s can be chosen so as to bring $\vec{w}$ arbitrarily close to satisfying (7) or (8).)

It is traditional in economics to consider the <u>elasticities</u>

$$E_{ijk} = (\partial w_k/\partial x_{ij})/(w_k/x_{ij}) ; \tag{9}$$

E_{ijk} is the rate of relative change in w_k , (dw_k/w_k) , per unit relative change in x_{ij}, (dx_{ij}/x_{ij}) , for either $k=j$ (self-elasticity of j-th product) or $k \neq j$ (cross-elasticity). We introduce the variables

$$y_{ij} = \log x_{ij} \ , \tag{10}$$

in order to simplify (9) to

$$E_{ijk} = w_k^{-1}(\partial w_k/\partial y_{ij}) \ . \tag{11}$$

Note that (5) and (6) imply

$$\partial w_j/\partial y_{ij} \leq 0 \ , \tag{12}$$

$$\partial w_k/\partial y_{ij} \geq 0 \qquad \text{for} \quad k \neq j \ . \tag{13}$$

The elasticities are (initially unknown) functions of the $\sum_{j=1}^{p} n(j)$ components of $\vec{x}$, but it would clearly be much more convenient if they could be determined by observing only the p split fractions (the components of $\vec{w}$) . This suggests examining models of the form

$$E_{ijk} = F_{ijk}(\vec{w}) \ ,$$

which by (11) is equivalent to the system

$$\partial w_k/\partial y_{ij} = w_k \ F_{ijk}(\vec{w}) \tag{14}$$

of partial differential equations.

It is natural to begin with the simple case in which each F_{ijk} is linear, so that (14) becomes

$$\partial w_k/\partial y_{ij} = \sum_{m=1}^{p} b_{ijkm} w_k w_m \tag{15}$$

where the b's are constants. We shall deal with a generalization

$$\partial w_k/\partial y_{ij} = \sum_{m=1}^{p} b_{ijkm} f_k(w_k) g_m(w_m) \ , \tag{16}$$

where the functions f_k and g_k $(k=1,2,...,p)$, defined on $0 \le w \le 1$, satisfy

$$f_k(0) = g_k(0) = 0 \tag{17}$$

and have continuous derivatives f'_k, g'_k such that

$$f'_k > 0 \ , \ g'_k > 0 \ . \tag{18}$$

These last two conditions are of course satisfied for the particular choices

$$f_k(w) = g_k(w) = w$$

which specialize (16) to (15). It will be clear from the proofs to come that (18) need only be required to hold on a "sufficiently large" subset of $0 \le w \le 1$.

There are three reasons for passing from the linear model (15) to the (possibly) nonlinear (16). One is simply intellectual curiousity as to how the generalization will affect the analysis. Second is the possibility that some special insight into the competitive situation will strongly suggest that linearity is implausible. Third, if it should prove impossible to obtain a satisfactory "fit" to empirical data using the linear model, then perhaps more parameters (which can be adjusted to improve the fit) can be smuggled in *via* the f_k's and g_k's. Our *conclusions*, however, will show that the second and third of these hopes are *in vain* (for $p > 2$) .

In the application motivating this work, the "market" in question is to consist of a single "cell" in some stratification of the population of travellers between a particular origin and a particular destination. The "products" are the services offered by the various transport alternatives; the latter might be taken as the traditional transportation "modes" (air, rail, bus, private auto) plus whatever novelties social and technological change may produce, or might reflect a finer classification (e.g. particular auto routes, particular airlines, first-class vs. coach service). The components of $\vec{x}$ might be measures of trip time, trip cost, variability from published schedules, trip fatigue, frequency and severity of accidents, etc. Validation and subsequent *use* (for prediction) of such a model would of course require operationally meaningful specifications of the transport alternatives (more generally, the products) and of the $\underline{x}$'s, and also appropriate "calibration" based on empirical data. In the present paper, however, we are solely concerned with the mathematical consequences of the model's assumptions.

It will be shown in what follows that among the models described by (16) and the other assumptions listed above, the only ones which are consistent (have a solution $\vec{w}(\vec{x})$) are a subclass of the linear models given by (15). For this subclass, the general solution will be derived in closed form. The principal results appear in displays (42) and (45)-(48).

These conclusions only hold for $p > 2$, however. For p=2, the class of consistent models is shown in Section 4 to contain many nonlinear ones, and for these the general solution is derived in semi-explicit form.

Call two of the p products connected if, roughly speaking, the parameters of the first influence the elasticities of the second. The analyses of Sections 3 and 4 are carried out under a "connectivity hypothesis" which requires that each pair of products is connected. In Section 5 the results of dropping this hypothesis are investigated. For p=2, there is no essential change in the nature of the consistent models or their solutions. For $p > 2$, it is shown that the hypothesis must hold; disconnectedness would lead to a contradiction of our previous assumption (7). This suggests that it may be desirable to analyze the effects of replacing (7) and (8) by weaker assumptions about what market splits are theoretically "attainable".

2. PRELIMINARIES

We observe first that the g's can be assumed normalized so that

$$g_j(1) = 1 . \tag{19}$$

For, (17) and (18) imply that all $g_m(1) > 0$, so that in (16) we could replace each g_m by $g_m/g_m(1)$ and each b_{ijkm} by $g_m(1)b_{ijkm}$.

From (3) we have

$$\sum_{k=1}^{p} \partial w_k/\partial y_{ij} = \partial(\sum_k w_k)/\partial y_{ij} = 0 .$$

Substituting (16) into this, we obtain

$$\sum_k \sum_m b_{ijkm} f_k(w_k) g_m(w_m) = 0 . \tag{20}$$

By (7), $\vec{x}$ can be chosen so that $w_k(\vec{x})= 1$ and therefore $w_m(\vec{x})= 0$ for $m \neq k$. Applying this and (17) to (20) gives

$$b_{ijkk} f_k(1)=0$$

and since (17) and (18) imply $f_k(1) > 0$, it follows that

$$b_{ijkk}= 0 . \tag{21}$$

Next, for any w with $0 \leq w \leq 1$ and any (k,m) with $k \neq m$, we can by (8) choose $\vec{x}$ so that

$$w_k(\vec{x})=w,\ w_m(\vec{x})=1-w,\ w_q(\vec{x})=0 \quad \text{for} \quad q \neq k,m .$$

It follows from (20) that

$$b_{ijkm} f_k(w) g_m(1-w) + b_{ijmk} f_m(1-w) g_k(w)= 0 . \tag{22}$$

By (21) this also holds for k=m.

Suppose now that j,k,m are distinct. By (13) and (16),

$$\sum_m b_{ijkm} f_k(w_k) g_m(w_m) \geq 0 \ .$$

Choosing $\vec{x}$ as in (8), we see from this and (17) that

$$b_{ijkm} \geq 0 \quad (j \neq k,m) \ .$$

Thus both b's in (22) are non-negative, while for $0 < w < 1$ the f-values and g-values in (22) are positive. So (22) implies

$$b_{ijkm} = 0 \quad (j \neq k,m) \ . \tag{23}$$

We conclude this section by remarking that

$$\partial(\partial w_k/\partial y_{ij})/\partial y_{IJ} = \partial(\partial w_k/\partial y_{IJ})/\partial y_{ij} \ . \tag{24}$$

As is well-known, to prove this it suffices to show that the two second partial derivatives exist and are continuous. For the derivatives in (16) to exist, $\vec{w}(\vec{x})$ must be continuous. Since the f's and g's are continuous, it follows from (16) that all the first partial derivatives of $\vec{w}(\vec{x})$ are continuous. We can evaluate the left-hand side of (24) by applying the chain rule to (16):

$$\partial(\partial w_k/\partial y_{ij})/\partial y_{IJ} = \sum_m b_{ijkm} \{f_k'(w_k) g_m(w_m) \partial w_k/\partial y_{IJ} + f_k(w_k) g_m'(w_m) \partial w_m/\partial y_{IJ}\} \ . \tag{25}$$

Since the f's and g's and their derivatives are continuous, and the first-order partial derivatives were just proved continuous, it follows that the left-hand side of (24) is continuous; similarly for the right-hand side.

3. MAIN ANALYSIS

Throughout this section, we shall impose two additional restrictions. The first is that at least three products are involved, i.e. $p > 2$. The second might be called a <u>connectivity condition</u>; it asserts that for each pair of distinct products P_j and P_k $(k \neq j)$, there is an index $\underline{i}$ such that

$$b_{ijkj} \neq 0 \ . \tag{26}$$

The situations in which these restrictions are not satisfied will be discussed later.

Suppose (j,k,J) distinct. Then use of (23) simplifies (25) to

$$\partial(\partial w_k/\partial y_{ij})/\partial y_{IJ} = b_{ijkj}\{f_k'(w_k)g_j(w_j)\partial w_k/\partial y_{IJ} + f_k(w_k)g_j'(w_j)\partial w_j/\partial y_{IJ}\} \ .$$

By (16) and (23)

$$\partial w_k/\partial y_{IJ} = b_{IJkJ}f_k(w_k)g_J(w_J) \ ,$$

$$\partial w_j/\partial y_{IJ} = b_{IJjJ}f_j(w_j)g_J(w_J) \ ,$$

so that the expression for the left-hand side of (24) becomes

$$\partial(\partial w_k/\partial y_{ij})/\partial y_{IJ} = b_{ijkj}f_k(w_k)g_J(w_J) \ \{b_{IJkJ}f_k'(w_k)g_j(w_j) + b_{IJjJ}f_j(w_j)g_j'(w_j)\} \tag{26a}$$

The expression for the right-hand side can be obtained by interchanging (i,j) and (I,J):

$$\partial(\partial w_k/\partial y_{IJ})/\partial y_{ij} = b_{IJkJ} f_k(w_k) g_j(w_j)\{b_{ijkj} f_k'(w_k) g_J(w_J) + b_{ijJj} f_J(w_J) g_J'(w_J)\} .$$

Equating the two, we find that for $w_k > 0$ (and hence, by continuity, for $w_k = 0$ as well)

$$b_{ijkj} b_{IJjJ} g_J(w_J) f_j(w_j) g_j'(w_j) = b_{IJkJ} b_{ijJj} g_j(w_j) f_J(w_J) g_J'(w_J) .$$

Thus for $w_j > 0$ and $w_J > 0$,

$$b_{ijkj} b_{IJjJ} f_j(w_j) g_j'(w_j)/g_j(w_j) = b_{IJkJ} b_{ijJj} f_J(w_J) g_J'(w_J)/g_J(w_J) . \quad (27)$$

The left-hand side is a function of w_j , the right-hand one a function of w_J . Hence[3] each is <u>constant</u>.

We wish to deduce from this the existence of constants d_j (necessarily positive) such that for $w_j > 0$

$$f_j(w_j) g_j'(w_j)/g_j(w_j) = d_j , \quad (28)$$

i.e. (by continuity for $w_j = 0$ also)

$$f_j = d_j g_j/g_j' . \quad (29)$$

(3) Note the implicit use of the hypothesis $p > 2$.

This would not be justified if $b_{ijkj}b_{IJjJ} = 0$ in (27) could not be avoided. The connectivity condition ensures that i and I can be chosen so as to permit passing from (27) to (28). Thus the f's are uniquely determined by the g's (in a consistent model).

Now we return to (22), set $m=j$ and assume (j,k) distinct, and apply (29) to obtain

$$(b_{ijkj}d_k g_k(w)g_j(1-w)/g_k'(w)) + (b_{ijjk}d_j g_j(1-w)g_k(w)/g_j'(1-w)) = 0 \ .$$

Thus, for $0 \leq w < 1$, and hence by continuity for $w=1$ as well,

$$b_{ijkj}d_k g_j'(1-w) + b_{ijjk}d_j g_k'(w) = 0 \ . \tag{30}$$

Indefinite integration with lower limit zero gives

$$b_{ijjk}d_j g_k(w) - b_{ijkj}d_k g_j(1-w) = -b_{ijkj}d_k \ . \tag{31}$$

Setting $w=1$, we have

$$b_{ijjk}d_j = -b_{ijkj}d_k \ . \tag{32}$$

Substitution into (30) and (31), and use of the connectivity condition, yields

$$g_j'(1-w) - g_k'(w) = 0 \ ,$$

$$g_j(1-w) + g_k(w) = 1 \ , \tag{33}$$

for $k \neq j$. Since $p > 2$, there follows from (33) the existence of a function $g(w)$ such that

$$g_j(w) = g(w) \qquad \text{for } 1 \leq j \leq p \ . \tag{34}$$

That is, the $\underline{g}$'s coincide. In some of the analysis below, however, we shall keep subscripts on the $\underline{g}$'s to make the derivations easier to follow.

So far (24) has been exploited with (j,k,J) distinct. Now we apply it with $j=k\neq J$. The left-hand side is, by (25),

$$\sum_m b_{ijjm}\,\{f_j'(w_j)g_m(w_m)b_{IJjJ}f_j(w_j)g_J(w_J)$$

$$+ f_j(w_j)g_m'(w_m)\sum_n b_{IJmn}f_m(w_m)g_n(w_n)\}\ .$$

The right-hand side, by (23), is

$$\partial(b_{IJjJ}f_j(w_j)g_J(w_J))/\partial y_{ij}$$

$$= b_{IJjJ}\{f_j'(w_j)g_J(w_J)\partial w_j/\partial y_{ij} + f_j(w_j)g_J'(w_J)\partial w_J/\partial y_{ij}\}$$

$$= b_{IJjJ}f_j(w_j)\{f_j'(w_j)g_J(w_J)\sum_m b_{ijjm}g_m(w_m)$$

$$+ g_J'(w_J)b_{ijJj}f_J(w_J)g_j(w_j)\}\ .$$

Equating the two yields

$$\sum_m \sum_n b_{ijjm}\,b_{IJmn}g_m'(w_m)f_m(w_m)g_n(w_n) = b_{IJjJ}b_{ijJj}g_J'(w_J)f_J(w_J)g_j(w_j)\ .$$

Since, by (28),

$$g_m'(w_m)f_m(w_m) = d_m g_m(w_m)\ ,$$

$$g_J'(w_J)f_J(w_J) = d_J g_J(w_J)\ ,$$

the last equation becomes

$$\sum_m \sum_n b_{ijjm}b_{IJmn}d_m g_m(w_m)g_n(w_n) = b_{IJjJ}b_{ijJj}d_J g_J(w_J)g_j(w_j)\ .$$

Application of (23) reduces this to

$$\sum_m \{b_{ijjJ} b_{IJJm} d_J + b_{ijjm} b_{IJmJ} d_m\} g_m(w_m) = b_{IJjJ} b_{ijJj} d_J g_j(w_j) \ . \tag{35}$$

By (32), however,

$$b_{IJmJ} d_m = - b_{IJJm} d_J \ ,$$

and so (33) becomes

$$\sum_m (b_{ijjJ} - b_{ijjm}) b_{IJJm} g_m(w_m) = b_{IJjJ} b_{ijJj} g_j(w_j) \ . \tag{36}$$

Choose k distinct from j and J, and choose $\vec{x}$ so that $w_k(\vec{x}) = 1$. Then it follows from (36) that

$$(b_{ijjJ} - b_{ijjk})\, b_{IJJk} = 0 \qquad (j, J, k \text{ distinct}). \tag{37}$$

This and the connectivity hypothesis imply the existence of constants b_{ij} such that

$$b_{ijjk} = b_{ij} \qquad \text{for all } k \neq j \ . \tag{38}$$

Next we employ (20), which by (21) and (23) can be written

$$\sum_{m \neq j} b_{ijjm} f_j(w_j) g_m(w_m) + \sum_{k \neq j} b_{ijkj} f_k(w_k) g_j(w_j) = 0 \ .$$

A neater form is

$$\sum_{k \neq j} \{b_{ijjk} f_j(w_j) g_k(w_k) + b_{ijkj} f_k(w_k) g_j(w_j)\} = 0 \ .$$

Application of (29) and (38) yields

$$g_j(w_j) \sum_{k \neq j} \{b_{ij} d_j g_k(w_k)/g_j'(w_j) + b_{ijkj} d_k g_k(w_k)/g_k'(w_k)\} = 0 ,$$

which with the aid of (32) becomes

$$b_{ij} d_j g_j(w_j) \sum_{k \neq j} g_k(w_k)\{1/g_j'(w_j) - 1/g_k'(w_k)\} = 0 .$$

Now $d_j > 0$, and the connectivity hypothesis permits choosing i so that $b_{ij} \neq 0$. Hence

$$\sum_{k \neq j} g_k(w_k)\{1/g'_j(w_j) - 1/g_k'(w_k)\} = 0 . \tag{39}$$

Choose k and m so that (j,k,m) are distinct, and observe that $\vec{x}$ can be chosen as in (8). Thus (39) implies that

$$g_k(w)\{1/g_j'(0) - 1/g_k'(w)\} + g_m(1-w)\{1/g_j'(0) - 1/g_m'(1-w)\} = 0 .$$

By (33) and (34), this can be rewritten as

$$g(w)\{1/g'(0) - 1/g'(w)\} + (1-g(w))\{1/g'(0) - 1/g'(w)\} = 0 ,$$

implying that $g'(w) = g'(0)$. That is, g' is constant, and so $g(w)$ is linear. Since $g(0) = 0$ and $g(1) = 1$, we have

$$g(w) = w . \tag{40}$$

By (29),

$$f_j(w) = d_j w \ . \qquad (41)$$

We now return to the original model equations (16). For $k \neq j$, these become

$$\partial w_k/\partial y_{ij} = b_{ijkj} f_k(w_k) g_j(w_j)$$

$$= b_{ijkj} d_k w_k w_j = - b_{ij} d_j w_k w_j \ ,$$

while for $k = j$ we obtain

$$\partial w_j/\partial y_{ij} = \sum_m b_{ijjm} f_j(w_j) g_m(w_m)$$

$$= \sum_{m \neq j} b_{ij} d_j w_j w_m$$

$$= b_{ij} d_j w_j \sum_{m \neq j} w_m = b_{ij} d_j w_j (1 - w_j) \ .$$

Both forms can be combined in

$$\partial w_k/\partial y_{ij} = b_{ij} d_j w_k (\delta_{jk} - w_j) \ . \qquad (42)$$

From (42) we have

$$(1/w_k) dw_k = (1/w_k) \sum_{i,j} (\partial w_k/\partial y_{ij}) dy_{ij}$$

$$= \sum_{i,j} b_{ij} d_j (\delta_{jk} - w_j) dy_{ij} \ .$$

Therefore

$$dw_k/w_k - dw_1/w_1 = \sum_{i,j} b_{ij}d_j(\delta_{jk} - \delta_{j1})dy_{ij}$$

$$= \sum_i b_{ik}d_k dy_{ik} - \sum_i b_{i1}d_1 dy_{i1} \quad (43)$$

There is therefore a constant c_k such that

$$\log(w_k/w_1) = \sum_i b_{ik}d_k y_{ik} - \sum_i b_{i1}d_1 y_{i1} + c_k ,$$

and hence such that

$$w_k = C_k w_1 W_k/W_1 , \quad (44)$$

where

$$C_k = \exp(c_k) > 0 \qquad (C_1=1) ,$$

$$W_k = \exp(\sum_i b_{ik}d_k y_{ik}) .$$

From the definition (10) of y_{ij} , we have

$$W_k = (\prod_i x_{ik}^{b_{ik}})^{d_k} . \quad (45)$$

By summing (44) over $1 \leq k \leq p$ and applying (3), we obtain

$$1 = (w_1/W_1) \sum_k C_k W_k ,$$

$$w_k = C_k W_k / \sum_j C_j W_j . \quad (46)$$

We have shown that <u>if</u> the model (16) is to be consistent then it must have the special form (42), in which case its solutions $\vec{w}(\vec{x})$ must have the form given by (45) and (46). The parameters of these solutions are[4]

$$d_k > 0 \ , \ C_k > 0 \ , \ b_{ik} \leq 0 \qquad . \tag{47}$$

where the last inequality comes from (12). The connectivity hypothesis takes the form

$$\min_i b_{ik} < 0 \quad \text{for all } k \ ; \tag{48}$$

if it were violated for some $\underline{k}$, then from (45) and (46) we see that $\vec{w}(\vec{x})$ would not depend on the $\underline{x}$'s of the k-th product.

Although (42) admits the singular solution $w_k(\vec{x}) \equiv 0$ corresponding to $C_k=0$, this is ruled out by requirements (7) and (8).

Conversely, consider any sets of $\underline{b}$'s, $\underline{C}$'s and $\underline{d}$'s satisfying (47) and (48), and <u>define</u> $\vec{w}(\vec{x})$ by (45) and (46). It is readily verified that (1),(2) and (42) are satisfied; hence (12) and (13) are satisfied. For each $\underline{j}$ there is an $\underline{i}$ with $b_{ij} < 0$; (7) can be satisfied as closely as desired by letting the corresponding $x_{ij} \to 0$ and keeping all other $\underline{x}$'s fixed. Similarly, (8) can be satisfied as closely as desired. So (45) through (48) do give precisely the class of consistent models and their general solutions.

(4) Note from (45) that d_k appears only in the products $b_{ik}d_k$, so that parameter-fitting to empirical data would deal with these products.

4. THE TWO-PRODUCT CASE

In this section we continue to impose the connectivity condition (26), but now assume p=2. For this case the situation will be shown to be quite different from that with $p > 2$, in that there is an abundance of consistent non-linear models.

It is convenient to introduce the functions

$$h_1(w) = f_1(w)g_2(1-w) \ , \tag{49}$$

$$h_2(w) = f_2(w)g_1(1-w) \ , \tag{50}$$

so that for j=1,2

$$h_j(0) = h_j(1)=0 \ , \ h_j(w) > 0 \quad \text{for } 0 < w < 1 \ . \tag{51}$$

With the aid of (21), the model (16) is found to take the form

$$\partial w_1/\partial y_{i1} = b_{i112}h_1(w_1) \ , \tag{52}$$

$$\partial w_1/\partial y_{i2} = b_{i212}h_1(w_1) \ , \tag{53}$$

$$\partial w_2/\partial y_{i1} = b_{i121}h_2(w_2) \ , \tag{54}$$

$$\partial w_2/\partial y_{i2} = b_{i221}h_2(w_2) \ , \tag{55}$$

while (22) yields

$$b_{ij12}h_1(w) + b_{ij21}h_2(1-w) = 0 \ . \tag{56}$$

From (56) it follows that

$$b_{ij12}h_1(w) = - b_{ij21}h_2(1-w) ,$$

$$b_{IJ21}h_2(1-w) = - b_{IJ12}h_1(w) .$$

Multiplying these two equations together leads to

$$b_{ij12}b_{IJ21} = b_{ij21}b_{IJ12} . \tag{57}$$

Take $J = 1$; then connectivity ensures the existence of an index I with $b_{I121} \neq 0$. With

$$\lambda = b_{I112}/b_{I121} ,$$

it follows from (57) that

$$b_{ij12} = \lambda \, b_{ij21} . \tag{58}$$

Conditions (12), (13), (52) and (54) show that $\lambda \leq 0$, and the connectivity condition (choose j=2 in (58)) permits sharpening this to

$$\lambda < 0 . \tag{59}$$

It will now be shown that model consistency places no further conditions on the <u>b</u>'s, and no further conditions on the <u>f</u>'s and <u>g</u>'s except (51) and the relation

$$\lambda \, h_1(w) + h_2(1-w) = 0 \tag{60}$$

which follows from (56), (58) and the connectivity condition.

For j=1,2, and $0 < w < 1$, let

$$H_j(w) = \text{an indefinite integral of } 1/h_j(w) \ . \tag{61}$$

Then $H'_j(w) = 1/h_j(w) > 0$, and so $u = H_j(w)$ has an inverse function $w = \bar{H}_j(u)$ defined for all real $\underline{u}$ and taking values between 0 and 1 . $\bar{H}_j(-\infty) = 0$ and $\bar{H}_j(\infty) = 1$. These inverse functions will be used in expressing the explicit solution of the model. Note that (60) yields

$$H_1(w) - \lambda H_2(1-w) = 0 \tag{62}$$

for a proper choice --- which we assume made --- of the integrals[5] H_j .

With the use of (58), the model (52)-(55) becomes

$$\partial w_1/\partial y_{i1} = \lambda \, b_{i121}h_1(w_1) \ ,$$

$$\partial w_1/\partial y_{i2} = \lambda \, b_{i221}h_1(w_1) \ ,$$

$$\partial w_2/\partial y_{i1} = b_{i121}h_2(w_2) \ ,$$

$$\partial w_2/\partial y_{i2} = b_{i221}h_2(w_2) \ .$$

The simplifying substitutions

$$z_{ij} = b_{ij21}y_{ij} \tag{63}$$

(5) The choice is that $H_j(w_{oj}) = 0$ $(j = 1,2)$ where $0 < w_{oj} < 1$ and $w_{01} + w_{02} = 1$.

transform this into

$$\partial w_1/\partial z_{i1} = \partial w_1/\partial z_{i2} = \lambda\ h_1(w_1) \ , \tag{64}$$

$$\partial w_2/\partial z_{i1} = \partial w_2/\partial z_{i2} = h_2(w_2) \ . \tag{65}$$

If all independent variables except a particular one z_{11} are held fixed, then (64) yields the separable ordinary differential equation

$$dw_1/dz_{11} = \lambda\ h_1(w_1) \ ,$$

whose general solution is

$$H_1(w_1) = \lambda\ z_{11} + K_1(z|z_{11}) \ , \tag{66}$$

where the arguments of the (so far arbitrary) function K_1 consist of all $\underline{z}$'s <u>except</u> z_{11} . We invert this solution as

$$w_1 = \bar{H}_1(\lambda\ z_{11} + K_1(z|z_{11})) \ . \tag{67}$$

If $n(1) > 1$, we go on to z_{21} . It follows from (66) that $\partial K_1/\partial z_{21}$ exists. By (67) ,

$$\partial w_1/\partial z_{21} = (\partial K_1/\partial z_{21})\ \bar{H}_1'\ (\lambda\ z_{11} + K_1) \ . \tag{68}$$

Since

$$\bar{H}_1'(u) = 1/H_1'(\bar{H}_1(u)) = h_1(\bar{H}_1(u)) \ ,$$

it follows from (67) and (68) that

$$\partial w_1/\partial z_{21} = (\partial K_1/\partial z_{21}) h_1(w_1) \ .$$

Comparing this with the assertion of (64) that

$$\partial w_1/\partial z_{21} = \lambda\, h_1(w_1) \ ,$$

we see that

$$\partial K_1/\partial z_{21} = \lambda \ ,$$

so that there is a decomposition

$$K_1(z|z_{11}) = \lambda\, z_{21} + K_2(z|z_{11}, z_{21}) \ .$$

Thus (67) becomes

$$w_1 = \bar{H}_1(\lambda(z_{11} + z_{21}) + K_2(Z|z_{11}, z_{21})) \ . \tag{69}$$

Repetition of this argument leads finally to

$$w_1 = \bar{H}_1(\lambda \sum_{i=1}^{n(1)} z_{i1} + \lambda \sum_{i=1}^{n(2)} z_{i2} + c_1) \tag{70}$$

where c_1 is a constant. Working similarly with (65) leads to

$$w_2 = \bar{H}_2(\sum_{i=1}^{n(1)} z_{i1} + \sum_{i=1}^{n(2)} z_{i2} + c_2) \tag{71}$$

where c_2 is a constant.

Conversely, for any choices of c_1 and c_2 , (70) and (71) are easily verified to yield positive-valued solutions of (64) and (65). It follows from (62) that

$$\bar{H}_1(\lambda u) + \bar{H}_2(u) = 1 \ ,$$

and so choosing

$$c_1 = \lambda c_2 \tag{72}$$

is necessary and sufficient for the requirement $w_1 + w_2 = 1$ to be satisfied. The conditions (12) and (13) follow automatically from (59) and the known signs of the b's in the display above (63). Satisfaction of (7) and (8) is easily verified.

We continue this section by illustrating the solution with the linear case

$$f_j(w) = g_j(w) = w \quad (j=1,2) ,$$

leading to

$$h_j(w) = w(1-w) .$$

Choosing w_{oj} with $0 < w_{oj} < 1$, we have

$$H_j(w) = \int_{w_{oj}}^{w} [t(1-t)]^{-1} dt$$

$$= \log [w/(1-w)] - \log [w_{oj}/(1-w_{oj})] .$$

Taking $w_{01} + w_{02} = 1$ guarantees (62). Abbreviate

$$k_j = \log [w_{oj}/(1-w_{oj})] .$$

Then from

$$u = H_j = \log [w/(1-w)] - k_j$$

we obtain

$$w = \bar{H}_j(u) = w_{oj} \exp(u)/[1-w_{oj} + w_{oj} \exp(u)] .$$

Thus (70) and (71) yield

$$w_1 = w_{o1}\exp(u_1)/[w_{o2}+w_{o1}\exp(u_1)] ,$$

$$w_2 = w_{o2}\exp(u_2)/[w_{o1}+w_{o2}\exp(u_2)]$$

where

$$u_2 = \sum_{i=1}^{n(1)} z_{i1} + \sum_{i=1}^{n(2)} z_{i2} + c_2 ,$$

$$u_1 = \lambda u_2 .$$

From (63) and (10) we find that

$$\exp(u_2) = C_2P$$

where $C_2 = \exp(c_2) > 0$,

$$P = \prod_{i=1}^{n(1)} (x_{i1}^{b_{i121}}) \prod_{i=1}^{n(2)} (x_{i2}^{b_{i221}}) .$$

Hence

$$\exp(u_1) = C_1P^\lambda$$

where $C_1 = C_2^\lambda$ and, by (58),

$$P^\lambda = \prod_{i=1}^{n(1)} (x_{i1}^{b_{i112}}) \prod_{i=1}^{n(2)} (x_{i2}^{b_{i212}}) .$$

Taking $w_{o1}=w_{o2}=1/2$ for simplicity, we have

$$w_1 = C_2^\lambda P^\lambda/[1+C_2^\lambda P^\lambda] ,$$

$$w_2 = C_2P/[1+C_2P] .$$

To relate this to the material of Section 3, it is simplest to compare model equations rather than solution forms. If (42) of Section 3 is specialized to p=2 (for which it was <u>not</u> proved necessary), the result is

$$\partial w_1/\partial y_{i1} = b_{i1}d_1w_1(1-w_1) ,$$

$$\partial w_1/\partial y_{i2} = - b_{i2}d_2w_1(1-w_1) ,$$

$$\partial w_2/\partial y_{i1} = - b_{i1}d_1w_2(1-w_2) ,$$

$$\partial w_2/\partial y_{i1} = b_{i2}d_2w_2(1-w_2) .$$

Comparing these with the corresponding specializations of the model equations before (63), we have

$$\lambda b_{i121} = b_{i1}d_1 ,$$

$$\lambda b_{i221} = - b_{i2}d_2 ,$$

$$b_{i121} = - b_{i1}d_1 ,$$

$$b_{i221} = b_{i2}d_2 .$$

Thus the solution form of Section 3, if applied to p=2, yields the special category $\lambda=(-1)$ of the subclass of "linear" models among the models considered in the present section.

One specific example of a <u>non</u>linear model for p=2 which is consistent (since it satisfies the conditions given above) is characterized by

$$f_1(w) = f_2(w) = g_1(w) = g_2(w) = w^2 ,$$

and $\lambda = (-1)$.

5. THE DISCONNECTED CASE

Recall that the connectivity condition (26) required, for each pair of distinct products P_j and P_k $(j \neq k)$, the existence of an index $\underline{i}$ such that

$$b_{ijkj} \neq 0 \ .$$

In this concluding section, we investigate the situations in which this condition is violated. It is useful to observe in advance that

$$b_{ijkj} = 0 \quad \text{implies} \quad b_{ijjk} = 0 \ , \tag{72}$$

$$b_{ijjk} = 0 \quad \text{implies} \quad b_{ijkj} = 0 \ . \tag{73}$$

These results follow from (22) with $m=j$ if $j \neq k$, or from (21) otherwise.

As before, the case $p=2$ will be considered separately. The model is given by the previous equations (52)-(55). Let us assume that the connectivity condition is violated, in that

$$b_{i212} = 0$$

for all $\underline{i}$. Then by (72) it follows that $b_{i122} = 0$ for all i, so that from (52) and (54) we see that the values of the first products's parameters have no effect on the market split. (This makes the connectivity condition an increasingly reasonable hypothesis, for $p=2$.)

We cannot also have

$$b_{i121} = 0$$

for all $\underline{i}$, for this would make w_2 constant, in contradiction of (8). From (22), we have

$$b_{i112}h_1(w) = - b_{i121}h_2(1-w) \ ,$$

$$b_{I121}h_2(1-w) = - b_{I112}h_1(w) \ .$$

With I chosen such that $b_{I121} \neq 0$, we can proceed just as in Section 4 except that the y_{i1} and hence the z_{i1} do not appear. The result is

$$w_1 = \bar{H}_1(\lambda \sum_{i=1}^{n(2)} z_{i2} + c_1) \ ,$$

$$w_2 = \bar{H}_2(\sum_{i=1}^{n(2)} z_{i2} + c_2) \ ,$$

where

$$\lambda = b_{I112}/b_{I121} < 0 .$$

We turn now to the case $p > 2$. It can be assumed that for each pair (i,j), with $1 \leq j \leq p$ and $1 \leq i \leq n(j)$, there is a $\underline{k}$ such that

$$\partial w_k/\partial x_{ij} \neq 0 ,$$

for otherwise the parameter x_{ij} would have no influence on the market split and so could (and should) have been omitted from the model. We can choose k so that $k \neq j$, for if $\partial w_k/\partial x_{ij}$ vanished for all $k \neq j$, then so also would

$$\partial w_j/\partial x_{ij} = \partial(1 - \sum_{k \neq j} w_k)/\partial x_{ij} .$$

By (16), (21) and (23), then, it follows that for each (i,j) there is a $k \neq j$ for which $b_{ijkj} \neq 0$.

Consider two products P_j and P_J, $j \neq J$. We will call P_j <u>weakly</u> (strongly) <u>disconnected from</u> P_J if $b_{ijJj} = 0$ holds for some $\underline{i}$ (for <u>all</u> $\underline{i}$), $1 \leq i \leq n(j)$. Clearly strong disconnectedness implies weak disconnectedness. It will now be shown that, conversely, weak disconnectedness implies strong disconnectedness, so that we can speak simply of "disconnectedness" and its opposite, "connectedness". It will also be shown that disconnectedness is a <u>symmetric</u> relation, i.e if P_j is disconnected from P_J then P_J is disconnected from P_j.

For the proof, assume $b_{ijJj} = 0$ for some $\underline{i}$. There is a $\underline{k}$, with $k \neq j$, such that $b_{ijkj} \neq 0$; hence $k \neq J$. Since (j,k,J) are distinct we can apply (27)---whose derivation did not use the connectivity condition---to infer $b_{IJjJ} = 0$ for <u>all</u> I with $1 \leq I \leq n(J)$, i.e. P_J strongly disconnected from P_j. Applying the same argument with j and J interchanged, we have P_j strongly disconnected from P_J.

Since disconnectedness is symmetric, the same is true of connectedness. We now show that connectedness is also a <u>transitive</u> relation, i.e. if P_j is connected to P_k, and P_k to P_J, then P_j is connected to P_J (j,k,J distinct).

For this purpose we cite the equation, derived without use of the connectivity condition, which appears several lines above (38):

$$\sum_{mn}\sum b_{ijjm}b_{IJmn}g_m'(w_m)f_m(w_m)g_n(w_n) = b_{IJjJ}b_{ijJj}g_J'(w_J)f_J(w_J)g_j(w_j)$$

for distinct (j,J). By use of (21) and (23), this can be written as

$$\sum_{m\neq J} b_{ijjm}b_{IJmJ}g_m'(w_m)f_m(w_m)g_J(w_J)$$

$$+ \sum_n b_{ijjJ}b_{IJJn}g_J'(w_J)f_J(w_J)g_n(w_n) = b_{IJjJ}b_{ijJj}g_J'(w_J)f_J(w_J)g_j(w_j) ,$$

and terms collected to obtain

$$\sum_{m\neq J,j} \{b_{ijjm}b_{IJmJ}g_m'(w_m)f_m(w_m)g_J(w_J) + b_{ijjJ}b_{IJJm}g_J'(w_J)f_J(w_J)g_m(w_m)\}$$

$$+ b_{ijjJ}b_{IJJj}g_J'(w_J)f_J(w_J)g_j(w_j) = b_{IJjJ}b_{ijJj}g_J'(w_J)f_J(w_J)g_j(w_j) .$$

Now choose $\vec{x}$ so that $\vec{w}(\vec{x})$ has $w_q = 0$ for $q \neq j,k,J$. Then the last equation becomes

$$b_{ijjk}b_{IJkJ}g_k'(w_k)f_k(w_k)g_J(w_J) + b_{ijjJ}b_{IJJk}g_J'(w_J)f_J(w_J)g_k(w_k)$$

$$+ b_{ijjJ}b_{IJJj}g_J'(w_J)f_J(w_J)g_j(w_j) = b_{IJjJ}b_{ijJj}g_J'(w_J)f_J(w_J)g_j(w_j) .$$

If P_j were <u>dis</u>connected from P_J, the right-hand side and (using (72)) the last two summands on the left would vanish, leaving

$$b_{ijjk} b_{IJkJ} g'_k(w_k) f_k(w_k) g_J(w_J) = 0 \tag{74}$$

for $w_j + w_k + w_J = 1$. This is impossible since the connectedness of P_j with P_k and P_k with P_J guarantees that the b's in (74) are non-zero. So P_j and P_J are connected.

From the fact that connectedness is both symmetric and transitive, it follows that unless the connectivity condition holds (i.e. unless any two products are connected), the set of p products decomposes into two or more subsets such that

(a) any two products in the same subset are connected, but

(b) no two products in different subsets are connected.

Suppose for example that products P_1 and P_2 lie in different subsets S_1 and S_2. Then by (16) and (23) $\partial w_1/\partial y_{ij} = 0$ unless P_j is in S_1, i.e. w_1 depends only on the parameters of the products in S_1, and similarly for w_2 and S_2. By (7) we can choose the parameters of the products in S_1 so that $\vec{w}(\vec{x})$ has $w_1 = 1$. This is impossible because it requires $w_2 = 0$, whereas the parameters of products in S_1 cannot influence S_2. It follows from this contradiction that, for $p > 2$, the connectivity condition must hold.

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