NATIONAL BUREAU OF STANDARDS REPORT

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A TRANSPORT IMPROVEMENT PROBLEM

TRANSFORMABLE

TO A BEST-PATH PROBLEM

by

A.J. Goldman

Technical Report

to

Office of High Speed Ground Transportation

Department of Commerce

NBS

U.S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

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Operations Research Section

Applied Mathematics Division

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ABSTRACT

Suppose given, for each arc (i,j) of a transport network, both an initial traversal disutility $d(i,j)$ and a function $f(i,j,r)$ describing the reduction in $d(i,j)$ resulting from applying $\underline{r}$ units of resources to "improve" the arc. For each origin-destination pair, there arises the problem of how a limited resource budget should be allocated among the arcs so as to optimize a best path from origin to destination in the improved network. It is shown here that this problem is transformable to a best-path problem in a enlarged network, if the functions $f(i,j,r)$ are super-additive in $\underline{r}$.

A TRANSPORT IMPROVEMENT PROBLEM

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1. INTRODUCTION

Let $[N;A]$ be a network. That is, N is a finite non-empty set of objects called nodes, and A is a set of ordered pairs (i,j) of distinct nodes; each such pair is called an arc of the network. A path in the network is a sequence of nodes,

$$P:\ i_o, i_1, \ldots, i_{m-1}, i_m \tag{1}$$

such that $(i_{t-1}, i_t) \epsilon A$ for $1 \leq t \leq m$; more specifically, P is a path from i_o to i_m . The (i_{t-1}, i_t) are called the arcs of P . P will be called simple if it has no repeated arc; repetition of nodes is not excluded.

We assume given a function d which assigns a non-negative number $d(i,j)$ to each arc $(i,j) \epsilon A$. Here d is interpreted as a disutility function; i.e., $d(i,j)$ is the cost or time or risk, etc., involved in

(1) Supported by the Office of High Speed Ground Transportation, Department of Commerce. No official endorsement implied.

traversing arc (i,j)---or whatever mathematical combination of these "elementary factors" is found appropriate for the application at hand. The function $\underline{d}$ is extended "additively" to the paths of the network, so that its value for the path (1) is the sum

$$d(P) = \sum_{t=1}^{m} d(i_{t-1}, i_t) .$$

A particular node $\nu \in N$, such that for each $i \in N$ the network contains at least one path from ν to $\underline{i}$, is distinguished and called the origin. For each $i \in N$, there is the problem B_i of finding a path from ν to $\underline{i}$ which is "d-best" in the sense of having minimum total disutility (i.e., d-value). There is also the problem of efficiently solving the whole ensemble of problems

$$\{B_i : i \in N\} .$$

These "best-path" problems arise frequently in the analysis and simulation of transport or communication systems (and in some production planning models as well); a number of algorithms for their solution have been developed and implemented as digital computer programs[2].

The transport improvement problem of interest here has the following additional structure: For each arc $(i,j) \in A$, there is assumed given a

(2) For brevity, we refer only to S.N.N. Pandit's "Some Observations on the Routing Problem", Operations Research 10 (1962), pp.726-727---and by inference to its bibliography. A more computer-oriented survey by C. Witzgall is in preparation.

positive-valued function f_{ij}, defined for non-negative integer arguments, such that the application of $r(i,j)$ units of "resources" to improve (i,j) reduces the disutility of traversing this arc from $d(i,j)$ to

$$\max\{0, d(i,j) - f_{ij}(r(i,j))\} .$$

Each function f_{ij} is supposed to be non-decreasing, with $f_{ij}(0) = 0$, and to be <u>super-additive</u>, i.e.

$$f_{ij}(x+y) \geq f_{ij}(x) + f_{ij}(y)$$

for $x \geq 0$, $y \geq 0$. (Note that this includes the case of <u>linear</u> f_{ij}). There is also given an integer $R \geq 0$ representing the <u>total budget level</u>, i.e. the amount of resources available for transport.

A <u>feasible allocation</u> of resources among the arcs is thus an indexed system

$$\vec{r} = \{r(i,j):(i,j)\in A\}$$

of non-negative integers, satisfying the budget constraint

$$\Sigma\{r(i,j):(i,j)\in A\} = R . \tag{2}$$

To each such $\vec{r}$, we associate the "improved" disutility function

$$d_r(i,j) = \max\{0, d(i,j) - f_{ij}(r(i,j))\} \tag{3}$$

on $[N;A]$ resulting from improving the various arcs in accordance with $\vec{r}$.

For each node $k \in N$, we can now pose the following <u>problem</u> $P(k,R)$: To find, among all feasible allocations $\vec{r}$, one which minimizes the d_r-value of d_r-best paths from the origin ν to k. Furthermore there is the composite problem, denoted $P(R)$, of solving the whole ensemble of problems

$$\{P(k,R):k \in N\} \ . \tag{4}$$

To avoid possible misunderstanding, we remark explicitly that this problem $P(R)$ is <u>not</u> that of allocating limited resources so as to maximize some measure of <u>system</u>-wide improvement. Rather, it is the simultaneous solution of <u>many</u> problems $P(k,R)$, each seeking a resource allocation which is optimal from the viewpoint of a <u>single</u> origin-destination pair $[\nu,k]$, on the assumption that travellers in an $\vec{r}$-improved network will choose a d_r-best path from origin to destination.

Our purpose in this note is to show that the "optimal improvement" problem posed above for $[N;A]$ can be rather simply transformed into a best-path problem for an enlarged network $[N^*;A^*]$ with an associated disutility function d^*. The transformation will be presented in the next section, with some final remarks left for the concluding Section 3.

Wollner[3] has considered a special case of the optimal improvement problem, in which the resource allocation to each arc had to be "extreme" in the sense of being either 0, or else large enough to reduce the arc disutility to zero. This restriction is <u>not</u> imposed here.

(3) R. Wollner, "Removing Arcs from a Network", Operations Research <u>12</u> (1964), pp.934-940.

2. THE TRANSFORMATION

We begin by describing the enlarged network $[N^*;A^*]$, and its disutility function d^*. The node-set N^* is given by

$$N^* = \{< i,u > : i \in N \ , \ 0 \leq u \leq R \} \ ,$$

i.e. it contains R+1 "replicas"

$$< i,0 > \ , < i,1 > \ , \ \ldots, < i,R >$$

of each node $i \in N$. The arc-set A^* is given by

$$A^* = \{(< i,u > \ , < j,v >) : (i,j) \in A \ , \ 0 \leq u \leq v \leq R\} \ .$$

Finally, d^* is defined on A^* by

$$d^*(< i,u > \ , < j,v >) = \max \{0, d(i,j) - f_{ij}(v-u)\} \ , \qquad (5)$$

and extended additively to the paths of $[N^*;A^*]$.

Note that in $[N^*;A^*]$ there is at least one path P^* from $< \nu,0 >$ to any other node $< i,u >$. For, we can begin with a path

$$P: i_o, i_1, \ldots, i_{m-1}, i_m$$

in $[N;A]$ from $\nu = i_o$ to $i = i_m$, and form P^* as

$$P^* : < i_o,0 > \ , < i_1,0 > \ , \ldots, < i_{m-1},0 > \ , < i_m,u > \ .$$

One other concept must be introduced before the problem transformation can be presented. Consider any $k \in N$. For the optimal improvement problem $P(k,R)$, we can clearly confine attention to feasible allocations $\vec{r}$ such that there exists at least one path P_r , from ν to k in $[N;A]$, for which $r(i,j) = 0$ for all arcs (i,j) <u>not</u> in P_r . Such an allocation $\vec{r}$ will be called k-<u>special</u>, as will the pair $(\vec{r},P_r)$.

We next define a function which <u>associates</u>, to each path

$$P^*: < \nu,0 > = < i_o,0 > , < i_1,R_1 > ,\ldots, < i_m,R_m > = < k,R >$$

from $< \nu , 0 >$ to $< k,R >$ in $[N^*;A]$, a k-special pair $(\vec{r},P_r)$. It is specified by

$$P_r:\nu = i_o,i_1,\ldots,i_{m-1},i_m=k ,$$

$$r(i,j) = \Sigma \{R_t-R_{t-1} :(i_{t-1},i_t) = (i,j)\} .$$

If P_r happens to be a <u>simple</u> path, then comparison of (3) and (5) shows that

$$d_r(P_r) = d^*(P^*) \qquad (P_r \text{ simple}) . \tag{6}$$

In the general case, we assert that

$$d_r(P_r) \leq d^*(P^*) . \tag{7}$$

To prove this, consider any arc (i,j) of P_r, and let

$$T = \{t:(i_{t-1},i_t) = (i,j)\} ,$$

$$x_t = R_t - R_{t-1} \geq 0 .$$

Then (7) will follow, in view of (3) and (5), if we can show that

$$\max \{0, d(i,j) - f_{ij}(\sum_T x_t)\} \leq \sum_T \max \{0, d(i,j) - f_{ij}(x_t)\} . \qquad (8)$$

This will certainly be true if

$$d(i,j) \leq f_{ij}(\sum_T x_t)$$

so we assume the contrary. Then (8) becomes

$$d(i,j) - f_{ij}(\sum_T x_t) \leq \sum_T \{d(i,j) - f_{ij}(x_t)\}$$

or equivalently

$$\sum_T f_{ij}(x_t) - f_{ij}(\sum_T x_t) \leq (|T|-1)\, d(i,j) \qquad (9)$$

where $|T|$ is the number of elements of T . But (9) is true because f_{ij} is super-additive, T is non-empty, and $d(i,j) \geq 0$.

The desired problem transformation is stated in the following theorem, and justified by the theorem's proof.

THEOREM. Let P^* be a d^*-best path in $[N^*;A^*]$ from $< \nu,0 >$ to $< k,R >$, and let $(\vec{r},P_r)$ be the associated k-special pair. Then $\vec{r}$ is optimal for the problem $P(k,R)$, and P_r is a d_r-best path in $[N;A]$ from ν to k .

PROOF. It suffices to rule out the existence of a feasible allocation

$$\vec{s} = \{s(i,j):(i,j)\epsilon A\} ,$$

and a path

$$Q:\nu = i_o, i_1, \ldots, i_{m-1}, i_m = k$$

in [N;A] , such that

$$d_s(Q) < d_r(P_r) . \tag{10}$$

If such a path Q exists, we can assume it to be simple. Then a path

$$Q^*: < \nu,0 > = < i_o,0 >, < i_1,R_1 > ,\ldots, < i_{m-1},R_{m-1} > , < i_m,R_m > = < k,R >$$

in [N*;A*] is defined by

$$R_t = s(i_o,i_1) + s(i_1,i_2) + \ldots + s(i_{t-1},i_t) \leq R$$

for $1 \leq t \leq m-1$. There is associated to Q^* a k-special pair $(\vec{q},Q)$ such that, by (6),

$$d^*(Q^*) = d_q(Q) , \tag{11}$$

and such that

$$q(i_{t-1},i_t) = s(i_{t-1},i_t) \qquad 1 \leq t \leq m-1 ,$$

$$\begin{aligned} q(i_{m-1},i_m) &= R - R_{m-1} \\ &= R - \Sigma \{s(i_{t-1},i_t) : 1 \leq t \leq m-1\} \\ &\geq R - \Sigma \{s(i,j) : (i,j) \in A - \{(i_{m-1},i_m)\}\} \\ &= s(i_{m-1},i_m) . \end{aligned}$$

Thus $\vec{q}$ allocates at least as much to each arc of Q as $\vec{s}$ does, so

$$d_q(Q) \leq d_s(Q) . \tag{12}$$

Combining (11), (12), (10) and (7), we have $d^*(Q^*) < d^*(P^*)$, contradicting the choice of P^* . This completes the proof.

3. DISCUSSION

We have shown that the optimal improvement problem for $[N;A]$, when the functions f_{ij} are super-additive, can be transformed into a best-path problem for the enlarged network $[N^*;A^*]$. The main import of this result, of course, is that best-path algorithms can be applied to solve the optimal improvement problem. Such an inference must however be taken with a grain of sophistication. In particular, "applied" should be changed to "adapted", and since we have <u>not</u> specified any "good" adaptation, the subject is not closed --- though hopefully a substantial step has been taken.

The point here is one familiar to readers who have been charged with implementing an algorithm described in a "theoretical" paper: The computational feasibility and efficiency of an algorithm (and hence, in a strict sense, the algorithm itself) are often not determined without quite formal specification of how data are to be stored and manipulated. In the present instance, $[N^*;A^*]$ has a "special structure" due to the way it is formed from $[N;A]$. Use of programming ingenuity to exploit this feature, rather than <u>literal</u> application of a standard best-path computer program to $[N^*;A^*]$, will be essential for the efficient solution of large problems. This is obvious from the facts that the "enlargement" multiplies the number of nodes by $R+1$ and the number of arcs by $(R+1)(R+2)/2$. If a best-path algorithm of the "labelling" variety is adapted, as seems most likely, then for example labels on the individual nodes $< i,u > \epsilon N^*$ would probably be replaced by a <u>set</u> of "augmented" labels, each with a "u-component", for each $i \epsilon N$.

Changing the topic, we observe that most methods for solving the best-path problem in $[N^*;A^*]$, between $< \nu ,0 >$ and each of the nodes

$$\{ < i,R > : i \in N \} ,$$

will in fact simultaneously solve it between $< \nu , 0 >$ and <u>all</u> nodes in N^* . Since any path from $< \nu , 0 >$ to $< i,u >$ in $[N^*;A^*]$ will contain only nodes $< j,v >$ with $v \leq u$, this process will yield solutions of the optimal improvement problem for all budget levels up to R , not just for R alone. This is fortunate since our formulation, in particular the use of "=R" rather than " $\leq$ R" in (2), does not explicitly indicate cases in which the optimum can be attained without using all of the available resources.

Another comment concerns the restrictiveness of requiring the functions f_{ij} to be super-additive. We note that this class of functions includes all polynomials with non-negative coefficients, and also "threshold" type functions like

$$f_{ij}(x) = \max \{0,c(x-x^o)\}$$

where <u>c</u> and x^o are positive constants.

A final speculative remark: Suppose for simplicity that each problem $P(k,R)$ has a unique solution $\vec{r}(k)$, and let $z_{ik}(R)$ be the $d_{r(k)}$-value of a $d_{r(k)}$-best path from the origin to node <u>i</u> . Then the numbers $z_{ik}(R)$ and $z_{ki}(R)$ might perhaps form the basis for a useful index of "community of interest between <u>i</u> and <u>k</u> at resource level R" .

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