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THE GENERATION AND RECOGNITION OF FLICKER NOISE

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U. S. DEPARTMENT OF COMMERCE
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THE GENERATION AND RECOGNITION OF FLICKER NOISE

James A. Barnes

Abstract

Flicker noise is defined as noise whose power spectral density varies inversely proportional to the Fourier frequency. Flicker noise appears to be present in every electronic device, but comprehensive explanations as to its source and level are lacking. It is shown that one can approximate flicker noise by analog methods which are amenable to the use of analog computers. It is also shown that certain diffusion processes and certain Poisson processes can generate flicker noise. A method of generating flicker noise using a digital computer is presented.

With a little practice one can learn to recognize noise processes with various power law spectra--some examples are shown. A review of techniques for inferring the power spectral density from time domain measurements which have been developed recently by several authors is presented.

Key Words: Noise, Flicker Noise, Diffusion, Poisson Process, Spectral Density, $\frac{1}{f}$ noise, Effective Stationarity.

Introduction

Flicker noise is the name given to a random noise whose power spectral density varies as $\frac{1}{|\omega|}$ over a very large spectral region. Equivalently in the time domain, a time series whose standard deviation is independent of the sampling time is a flicker noise. While it is possible to present compelling arguments why this behavior cannot persist to zero Fourier frequency, experiments to determine a low frequency break in behavior have failed for most (if not all) devices. Measurements have indicated a continuation of this behavior at one cycle per year in some devices [Atkinson, et al., 1963].

Part of the difficulty in analysis of this type of noise may be ascribed to its very mild divergence characteristic. That is, if one assumes that an oscillator's frequency is modulated by a flicker noise, and one can measure the frequency with an uncertainty of, say, $\pm A$ in five milliseconds, then one can show that the oscillator will be within about $\pm 5A$ of that frequency at an age of the Universe later! This is a logarithmic divergence in the time domain, and it is so mild as to give experimenters a false sense of security and to cause them to consider the process to be stationary. It is interesting to note that the false assumption of stationarity can, in specific situations, yield correct answers (see Appendix).

The present paper places emphasis on noise processes which have a power spectral density varying as $|\omega|^{-1}$. However, it is of value to consider this noise in its relation to noises whose power spectral densities are proportional to $|\omega|^{-\alpha}$ for constant values of α.

Implicit in most of the discussions in this paper is the assumption that one may meaningfully consider power spectral densities of the form

$$S_\phi(\omega) = h\,|\omega|^{-\alpha}.$$

This is not a trivial assumption because for $\alpha \geq 1$ one can show that $\phi(t)$ is not covariant stationary [Barnes and Allan, 1966]. Thus the assumption that $S_\phi(\omega) = h|\omega|^{-\alpha}$ raises doubt that $S_\phi(\omega)$ can even be defined for $\phi(t)$. There are given in this paper two plausibility arguments to support the assumption that

$$S_\phi(\omega) = h|\omega|^{-\alpha}, \ \alpha \geq 1$$

is a meaningful relation.

First and foremost, experimentally one encounters noise processes which appear, in fact, to be described by such power spectra and very often $\alpha \geq 1$. Here one may well wish to compare the actual experimental methods with theory.

Secondly, it is entirely possible that these power spectra break into a well behaved, convergent spectral law as $\omega \rightarrow 0$. Such cutoffs are at extremely low frequencies for some devices since, in general, these cutoff frequencies have not been observed. One may thus assume a lower cutoff frequency, ω_ℓ, and construct the theory accordingly. Since ω_ℓ has, to date, not been an observable in most cases, one considers only those quantities which are insensitive to ω_ℓ for very small ω_ℓ. This could be called a cutoff independent formulation. This method of formulating the problems of power spectra with the specific form

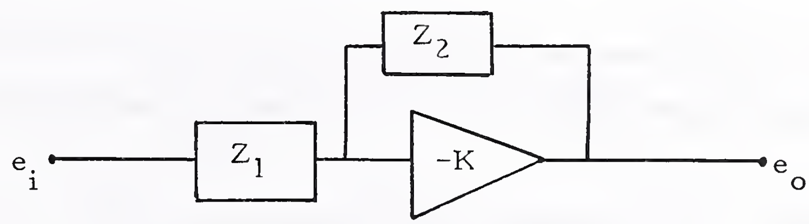
$$S_\phi(\omega) \approx h\left|\omega_\ell + j\omega\right|^{-\alpha}$$

is considered in some detail in the Appendix. One can show [Cutler and Searle, 1966] that many quantities of interest are not critically dependent on the form of $S_\phi(\omega)$ for ω smaller than the reciprocal observation time--indeed, this is precisely why ω_ℓ is not normally an observable.

1. The Generation of Flicker Noise

1.1 Analog methods

An operational amplifier is a device whose transfer function, $g(\omega)$, is approximated by a large, real, negative number, $-K$, over a very large spectral region $(0 \le \omega \le \omega_h)$ which is assumed to include the frequency domain of interest. If one connects such an operational amplifier with the two impedances, Z_1 and Z_2, as shown,



the overall transfer function, $G(\omega)$, is given by

$$G(\omega) = \frac{e_o(\omega)}{e_i(\omega)} = \frac{-K}{1 + (1+K)\frac{Z_1}{Z_2}}.$$

Thus, if $K \gg \left|\frac{Z_2}{Z_1}\right|$,

$$G(\omega) \approx -\frac{Z_2}{Z_1}.$$

Consider the case where the first impedance is a resistor, $Z_1 = R$, and the second a "fractional capacitor" [Carlson and Halijak, 1964], $Z_2 = \left(\frac{1}{j\omega c}\right)^{\frac{\alpha}{2}}$, where Z_2 is defined as the impedance of a $\frac{\alpha}{2}$ - th "fraction" of a capacitor. Actual networks which can realize impedances of this form are considered below. The transfer function of this device then becomes

$$G(\omega) \approx -\frac{1}{R}\left(\frac{1}{j\omega c}\right)^{\frac{\alpha}{2}}$$

$$\approx -\left(\frac{1}{j\omega\tau}\right)^{\frac{\alpha}{2}}$$

where $RC^{\frac{\alpha}{2}} = \tau^{\frac{\alpha}{2}}$.

For convenience and simplicity it is desirable to follow this device by an inverting amplifier to give an overall transfer function of

$$G'(\omega) \approx \left(\frac{1}{j\omega\tau}\right)^{\frac{\alpha}{2}} \text{ for } K >> \left|\frac{1}{j\omega\tau}\right|^{\frac{\alpha}{2}}. \tag{1-1}$$

If the input signal, e_i, has a power spectral density, $S_i(\omega)$, and, similarly, the output signal, e_o, has a power spectral density, $S_o(\omega)$, then

$$S_o(\omega) = |G'(\omega)|^2 S_i(\omega). \tag{1-2}$$

For a white input noise of power spectral density h and $\tau = 1$ second,

$$S_o(\omega) \approx h|\omega|^{-\alpha} \tag{1-3}$$

for $|\omega|^{\alpha}K^2 >> 1$. Thus, one may generate $|\omega|^{-\alpha}$ noise from a white noise source, an operational amplifier, and a "fractional capacitor."

It is of interest to determine the operation of this device in the time domain, also. The impulse response function of a filter is the Fourier transform of the transfer function. For a filter whose transfer function is $\left(\frac{1}{j\omega}\right)^{\frac{\alpha}{2}}$, one can show that the impulse response function is [Campbell and Foster, 1948 (Pair 516)]

$$\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \tau^{\frac{\alpha}{2}-1} .$$

Thus, if $e_i(t)$ is the input signal in the time domain, the output, $e_o(t)$, is the convolution of $e_i(t)$ with the impulse response function. That is,

$$e_o(t) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^t (t-t_o)^{\frac{\alpha}{2}-1} e_i(t_o)dt_o, \tag{1-4}$$

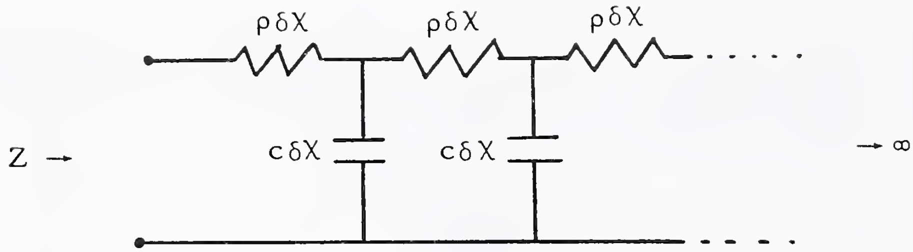
or $e_o(t)$ is the $\frac{\alpha}{2}$ - fold integral [Courant, 1957] of $e_i(t)$. Note that for $\frac{\alpha}{2} = 1$, one has a "whole" capacitor and the output is

$$e_o(t) \approx \int_0^t e_i(t_o)dt_o. \tag{1-5}$$

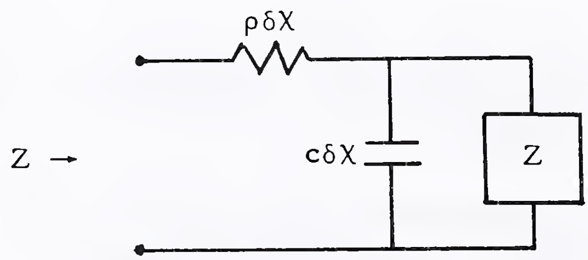
This is the proper result for an integrator as used in analog computers. Typically, integrators used in analog computers realize a $K \geq 10^6$. Thus one, theoretically, has an integrator working down to ~ one cycle per 12 days. There are, of course, other effects which limit its operation long before this low frequency is actually realized.

1.2 Realization of the impedance $(j\omega)^{-\frac{\alpha}{2}}$

O. Heaviside [1950] noted that the impedance of a semi-infinite lossy transmission line has the form $\sqrt{\frac{A}{j\omega}}$. One may show that this is reasonable by considering the lumped-constant transmission line shown.



Since the line is infinite in extent and has impedance Z, nothing is changed by adding one more section:



The solution of this network is

$$Z^2 = \frac{\rho}{j\omega c} + Z\rho\delta X. \tag{1-6}$$

In the limit as $\delta X \to 0^+$ (a continuous line), (1-6) reduces to

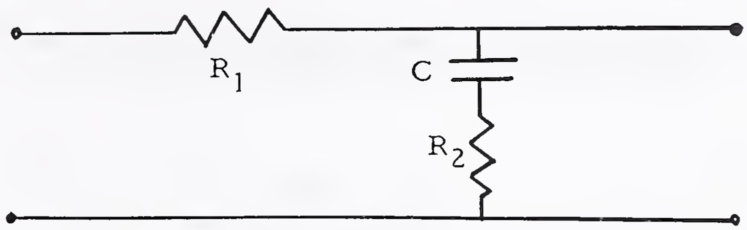
$$Z^2 = \frac{\rho}{j\omega c}$$

or

$$Z = \sqrt{\frac{\rho}{j\omega c}}$$

where ρ and c are the resistance and capacitance per unit length of the line.

Additional methods of constructing impedances of the form $(j\omega)^{-\alpha}$ may be found in the literature [Carlson and Halijak, 1961; Lerner, 1963; and Roy and Shenoi, 1966]. An intuitive and relatively simple method of constructing such impedances was devised by the author. Consider the network:



The transfer function of this network is

$$g(\omega) = \frac{1 + j\omega\tau_2}{1 + j\omega(\tau_1 + \tau_2)} \tag{1-7}$$

where $\tau_1 = R_1C$ and $\tau_2 = R_2C$. For small frequencies $\left(\omega(\tau_1+\tau_2) << 1\right)$, $g(\omega) \approx 1$; for large frequencies $\left(\omega\tau_2 >> 1\right)$, $g(\omega) \approx \frac{\tau_1}{\tau_1+\tau_2}$.

If one has a sequence of such filters such that for the i-th filter

$$\tau_1^{(i)} = (\beta)^i \tau_1^{(o)}$$

and

$$\tau_2^{(i)} = (\beta)^i \tau_2^{(o)} \text{ for } i = 1, 2, \cdots N, \text{ and } \beta < 1,$$

and these filters are cascaded with appropriate isolation amplifiers between successive filters, the over-all transfer function becomes

$$G(\omega) = \prod_{i=1}^{N} \left[\frac{1 + j\omega\tau_1^{(o)}(\beta)^i}{1 + j\omega(\beta)^i\left(\tau_1^{(o)} + \tau_2^{(o)}\right)}\right]. \tag{1-8}$$

At frequencies near where $\omega\tau^{(o)}(\beta)^N \approx 1$, the transfer function has approximately the value

$$G(\omega) \approx \left[\frac{\tau_1^{(o)}}{\tau_1^{(o)} + \tau_2^{(o)}}\right]^N$$

At frequencies near $\omega\tau_1^{(o)} \approx 1$ the transfer function has approximately the value unity. Thus, over an interval of frequency from

$\omega_1 = \frac{1}{\tau_1^{(o)}}$ to $\omega_2 = \frac{1}{\beta^N \tau_1^{(o)}}$ the (voltage) transfer function is reduced by the factor

$$\left(\frac{\tau_1^{(o)}}{\tau_1^{(o)} + \tau_2^{(o)}}\right)^N .$$

If this filter is to approximate a filter whose transfer function (in magnitude) varies according to the relation

$$|G'(\omega)| \approx |\omega|^{-\frac{\alpha}{2}},$$

then

$$\frac{|G'(\omega_2)|}{|G(\omega_1)|} = \frac{|\omega_1|^{\frac{\alpha}{2}}}{|\omega_2|^{\frac{\alpha}{2}}} \approx \beta^{\frac{N\alpha}{2}} \approx \left(\frac{\tau_1^{(o)}}{\tau_1^{(o)} + \tau_2^{(o)}}\right)^N .$$

$$\therefore \; \beta \approx \left(\frac{\tau_1^{(o)}}{\tau_1^{(o)} + \tau_2^{(o)}}\right)^{\frac{2}{\alpha}} . \qquad (1-9)$$

For flicker noise, $\alpha = 1$ and (1-9) becomes

$$\beta \approx \left(\frac{\tau_1^{(o)}}{\tau_1^{(o)} + \tau_2^{(o)}}\right)^2 .$$

Setting N = 4, and $\tau_1^{(o)} + \tau_2^{(o)} = 3\tau_1^{(o)}$, one has $\beta = \frac{1}{9}$ and (1-8) becomes

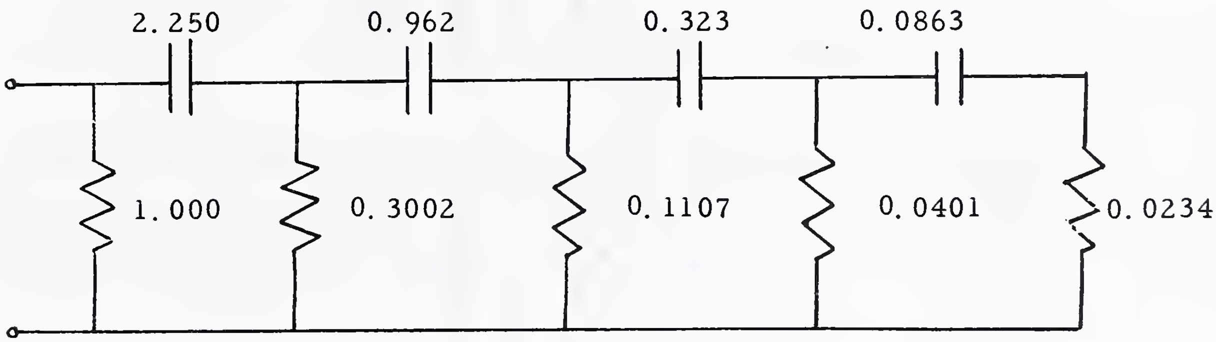
$$G(\omega) = \left(\frac{S+1}{3S+1}\right)\left(\frac{S+9}{3S+9}\right)\left(\frac{S+81}{3S+81}\right)\left(\frac{S+729}{3S+729}\right), \qquad (1\text{-}10)$$

where $S = j\omega\tau_1^{(o)}$.

Equation (1-10) is the transfer function of an active filter (with isolation amplifiers between sections) which should approximate $\left(\frac{A}{j\omega}\right)^{\frac{1}{2}}$ over a relative frequency range of $\sim 9^4$ or nearly 4 decades of frequency. If one were to consider (1-10) as a function which approximates $\left(\frac{A}{j\omega}\right)^{\frac{1}{2}}$, one could consider an impedance, $Z(\omega)$, which has exactly this same functional form. That is,

$$Z(\omega) = \left(\frac{S+1}{3S+1}\right)\left(\frac{S+9}{3S+9}\right)\left(\frac{S+81}{3S+81}\right)\left(\frac{S+729}{3S+729}\right). \qquad (1\text{-}11)$$

One can decompose (1-11) into a realizable impedance:



By scaling the resistor and capacitor values, one obtains a circuit whose impedance approximates $\sqrt{\frac{A}{j\omega}}$ for $A = 1816 \ \frac{(K\Omega)^2}{\text{sec.}}$, within ±0.5 dB from about 1 Hz to 4 kHz. By using such an impedance with an operational amplifier as in Section 1.1 (impedance Z_2), one obtains an amplifier whose (magnitude) gain differs from $\left|\frac{A}{j\omega}\right|^{\frac{1}{2}}$ as shown in Figure 1.

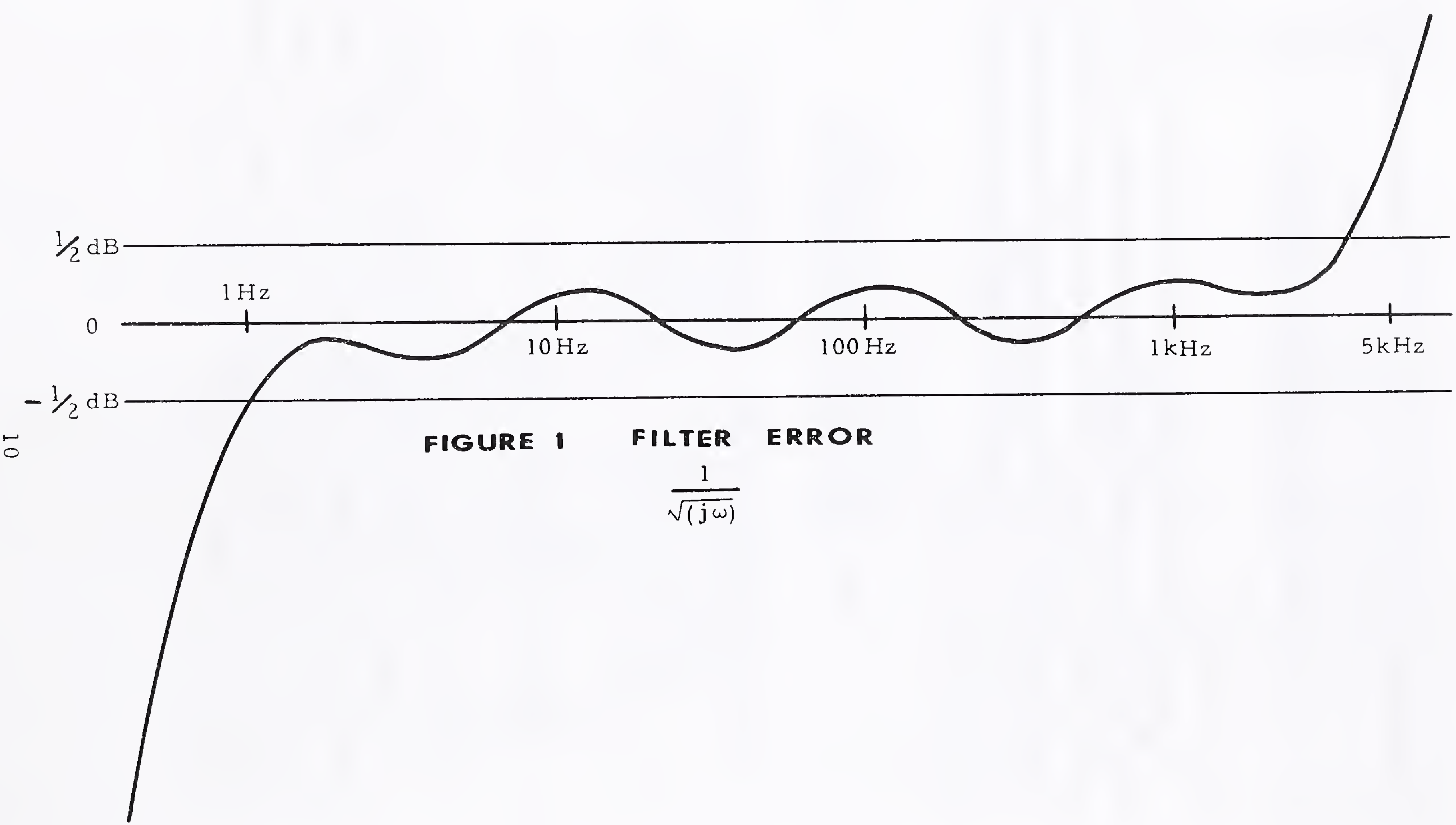
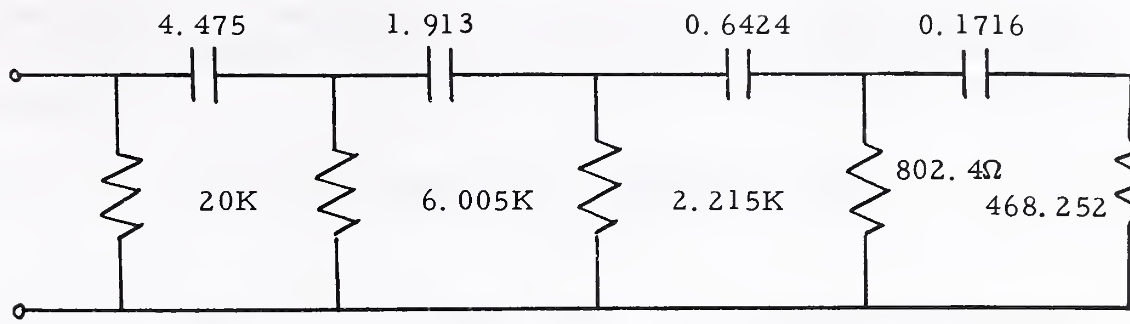


FIGURE 1 FILTER ERROR

$$\frac{1}{\sqrt{(j\omega)}}$$

The circuit diagram of the impedance used for Figure 1 is shown below.



Also of interest, if one intercharges Z_1 and Z_2 using the previous filter, one obtains a filter with a transfer function proportional $(j\omega)^{+\frac{1}{2}}$. Thus, if one applies a flicker noise signal to the input, the output is white noise.

1.3 Flicker noise in Diffusion Processes

Heaviside's semi-infinite lossy transmission line may be considered the electrical analog of a diffusion process. Consider a semi-infinite rod which obeys the diffusion equation

$$\frac{\partial^2 \psi(X,t)}{\partial X^2} - a^2 \frac{\partial \psi(X,t)}{\partial t} = -4\pi\rho(X,t) \qquad (1\text{-}12)$$

where $\psi(X,t)$ might be temperature or an impurity concentration, and $\rho(X,t)$ is the source function, e.g., a heat source. The Green's function [Morse and Feshbach, 1953] for this problem is

$$g(X-X_o, t-t_o) = \frac{2\pi}{a}\left(\frac{1}{\sqrt{\pi}\,\sqrt{t-t_o}}\right) e^{-\frac{a^2(X-X_o)^2}{4(t-t_o)}} \; U(t-t_o)$$

where $U(\tau)$ is the unit step function defined by the relation

$$U(\tau) = \begin{cases} 1, & \tau \geq 0 \\ 0, & \tau < 0. \end{cases}$$

Thus, if the source for this system is $e_i(t)$ located at $\chi = 0$, the end of the rod, the solution of (1-12) may be written

$$\psi(\chi, t) = \left(\frac{2\pi}{a}\right)\frac{1}{\sqrt{\pi}}\int_0^t \frac{e_i(t_o)}{\sqrt{t - t_o}} \exp\left[-\frac{a^2\chi^2}{4(t - t_o)}\right] dt_o . \tag{1-13}$$

At the end of the rod, $\chi = 0$, (1-13) is formally identical to (1-4) for $\alpha = 1$ (except for the factor $\frac{2\pi}{a}$), and thus, if $e_i(t)$ is a white noise process, $\psi(0, t)$ is a flicker noise process.

An obvious implication is that impurity diffusion in a semiconductor should cause a flicker noise process at the surface--a critical region for semiconductor devices. It is interesting to note that flicker noise is often referred to as "semiconductor noise." The apparent lack of a strong temperature dependence of actual flicker processes, however, does not seem to support the present model of impurity diffusion as the cause of flicker noise. A different diffusion model has been considered by J. M. Richardson [1950] and shown to give flicker noise and yet not be significantly temperature sensitive. However, some assumptions in the Richardson model do not seem entirely physical.

1.4 Quantum Mechanical Model

The treatment here, due primarily to Dr. D. Halford [1967], is based on discrete events in a Poisson process and hence is easily amenable to quantum statistics.

Consider a real, "localizable" function $h(\chi)$. For convenience, let

$$\left.\begin{aligned} \left|\int_{-\infty}^{\infty} h(\chi)d\chi\right| &< \infty \\ \int_{-\infty}^{\infty} h^2(\chi)d\chi &= 1 \end{aligned}\right\} . \tag{1-14}$$

It is, thus, true that $h(\chi)$ has a Fourier transform, $g(y)$, defined by the relations

$$h(\chi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)e^{+i\chi y}dy \tag{1-15}$$

and

$$g(y) = \int_{-\infty}^{\infty} h(\chi)e^{-i\chi y}d\chi. \tag{1-16}$$

One requires $h(\chi)$ to be a "reasonable" function in the sense of being nearly realizable in some physical device. Therefore, one further assumes $h(\chi)$ is finite for all values of χ.

It is true that, for $y = 0$, (1-16) may be written in the form

$$|g(0)| = \left|\int_{-\infty}^{\infty} h(\chi)d\chi\right| < \infty \tag{1-17}$$

by virtue of (1-14). Thus, in the region of small y, if one were to approximate the energy spectrum,

$$U(y) \equiv |g(y)|^2, \tag{1-18}$$

by a power law, $A|y|^{\gamma_o}$, one would have

$$U(y) \approx A|y|^{\gamma_o} < \infty$$

for "small" y, and, hence,

$$\gamma_o \geq 0. \tag{1-19}$$

In the region of "large" y, the total energy must also be finite; i. e.,

$$\int_{y_1}^{\infty} |g(y)|^2 dy < \infty, \tag{1-20}$$

where y_1 is a "large" number. This implies that if U(y) asymptotically approaches the function $B|y|^{\gamma_\infty}$ as $y \to \infty$,

$$\gamma_\infty < -1.$$

It can be shown that the conditions on $h(\chi)$ are sufficient to guarantee that

$$\gamma_\infty \leq -2. \tag{1-21}$$

While it is possible to let $h(\chi)$ be a very general function, it is sufficient for the present treatment to further restrict the acceptable class of functions, $h(\chi)$. In particular, define $\gamma(y)$ such that

$$U(y) = A|y|^{\gamma(y)} \text{ for all } y.$$

Only those functions, $h(\chi)$, will be considered for which $\gamma(y)$ is a monotonic function of y from γ_o to γ_∞ as y goes from 0 to ∞. This avoids some problems in defining effective times of the functions generated by $h(\chi)$.

One may now consider a Poisson process composed of a superposition of functions obtained from $h(\chi)$ by scaling amplitude and time and also by translating time; i. e.,

$$m_i(t) = a_i h\left(\frac{t - t_i}{\tau_i}\right). \tag{1-22}$$

Since this is assumed to be a Poisson process, the distribution of occurrence times, t_i, is random, uncorrelated and uniformly distributed over the time interval of interest.

Let the τ_i's and a_i's be distributed according to some probability distribution density function, $\rho(\tau, a)$. That is, the probability of observing a "life-time" between τ and $\tau + d\tau$ and an amplitude between a and a + da is

$$\rho(\tau, a) d\tau\, da.$$

The Fourier transform of $m_i(t)$ may easily be shown to be

$$\eta_i(\omega) = \int_{-\infty}^{\infty} m_i(t) e^{-i\omega t} dt \tag{1-23}$$

$$= a_i \tau_i e^{-i\omega t_i} g(\omega\tau_i),$$

where g(y) is defined in (1-15) and (1-16). The energy spectrum associated with $m_i(t)$ is then

$$|\eta_i(\omega)|^2 = a_i^2 \tau_i^2 |g(\omega\tau_i)|^2.$$

If R is the average <u>rate</u> of occurrence of events, $m_i(t)$, then the average rate of energy expenditure (i. e., power) at frequency ω due to events of life-times between τ and $\tau + d\tau$ and of amplitudes between a and a + da is given by

$$d^2 S(\omega) = R a^2 \tau^2 \rho(\tau, a) |g(\omega\tau)|^2 da\, d\tau .$$

Thus the total power spectrum due to all events is

$$S(\omega) = R \int_0^{\infty}\int_{-\infty}^{\infty} a^2 \tau^2 \rho(\tau, a) |g(\omega\tau)|^2 da\, d\tau . \tag{1-24}$$

Consider the function $A^2(\tau)P(\tau)$ defined by the relation

$$A^2(\tau)P(\tau) = \int_{-\infty}^{\infty} a^2 \rho(\tau, a)da,$$

and $P(\tau)$ is the probability density for τ without regard to amplitude and $A^2(\tau)$ is the mean square amplitude for the actual events of lifetime τ. Equation (1-24) can now be written in the form

$$S(\omega) = R\int_{-\infty}^{\infty} \tau^2 A^2(\tau)P(\tau)\,|g(\omega\tau)|^2\,d\tau. \tag{1-25}$$

If it should happen that

$$\tau^2 A^2(\tau)P(\tau) = B\tau^{\delta} \tag{1-26}$$

over some range of τ (say $\tau_o < \tau < \tau_\infty$) and be zero (or practically so) outside this range, then (1-25) becomes

$$S(\omega) = \frac{RB}{|\omega|^{1+\delta}} \int_{\omega\tau_o}^{\omega\tau_\infty} |g(y)|^2 y^{\delta} dy. \tag{1-27}$$

If it is true that $y_o = \omega\tau_o$ and $y_\infty = \omega\tau_\infty$ reasonably include the major portion of the area under the function $|g(y)|^2 y^{\delta}$ for positive y, then (1-27) may be approximated by

$$S(\omega) \approx \frac{RB}{|\omega|^{1+\delta}} \int_{0}^{\infty} |g(y)|^2 y^{\delta} dy$$

and for $\delta = 0$ one obtains

$$S(\omega) \approx \frac{RB\pi}{|\omega|} \tag{1-28}$$

by virtue of (1-14), (1-15), and (1-16) and the symmetry of $|g(y)|^2$. Within the restrictions noted above, (1-28) is valid and hence the process generates a flicker noise. These restrictions on ω may be expressed in the following forms for $\delta = 0$:

$$1 - \frac{1}{\pi}\int_{|\omega|\tau_0}^{|\omega|\tau_\infty} |g(y)|^2 dy << 1 \qquad (1\text{-}29)$$

or

$$\frac{1}{\pi}\int_{|\omega|\tau_\infty}^{\infty} |g(y)|^2 dy << 1,$$

$$(1\text{-}30)$$

$$\frac{1}{\pi}\int_{0}^{|\omega|\tau_0} |g(y)|^2 dy << 1.$$

Obviously these conditions cannot be satisfied at $\omega = 0$.

More extensive and complete solutions of the integral equation

$$h|\omega|^{-\alpha} = R\int_{-\infty}^{\infty} \tau^2 A^2(\tau) P(\tau) |g(\omega\tau)|^2 d\tau$$

have not been considered in the literature to this author's knowledge.

1.5 Comments on Current Papers

For $\delta = 0$, (1-26) may be written in the form

$$A^2(\tau)P(\tau) = \frac{B}{\tau^2}. \qquad (1\text{-}31)$$

That is, a necessary condition for the generation of flicker noise by the process just described is that (1-31) is valid over a sufficient region. Unfortunately, one can find in the literature [van der Ziel, 1950; Mueller, 1965; and Jordan and Jordan, 1965] this condition reduced nearest integer. Starting at location t_i, the integer 1 is added to every location up to and including a location congruent to $(t_i + \tau_i)$ mod T. By using modular arithmetic in the storage process one insures that the entire data are equivalent to a segment of an infinite periodic process. An alternative is to limit $t_i + \tau_i \leq T$ and use only the latter portion of the data.

Repeating the above process a total of 4096 times with T = 1024, yielded the results shown in figure 2. Also included in figure 2 are the (finite) integrals of the flicker noise samples. Using methods described in Section 2, six independent samples of flicker noise generated by this scheme were analyzed. The data were consistent with a power spectral density of the form

$$S(\omega) = h|\omega|^{-\alpha}$$

where α was determined to be $0.98 \pm .02$. The random numbers used for these tests were computer generated by a pseudorandom process [Andrew, 1966; and Chambers, 1967].

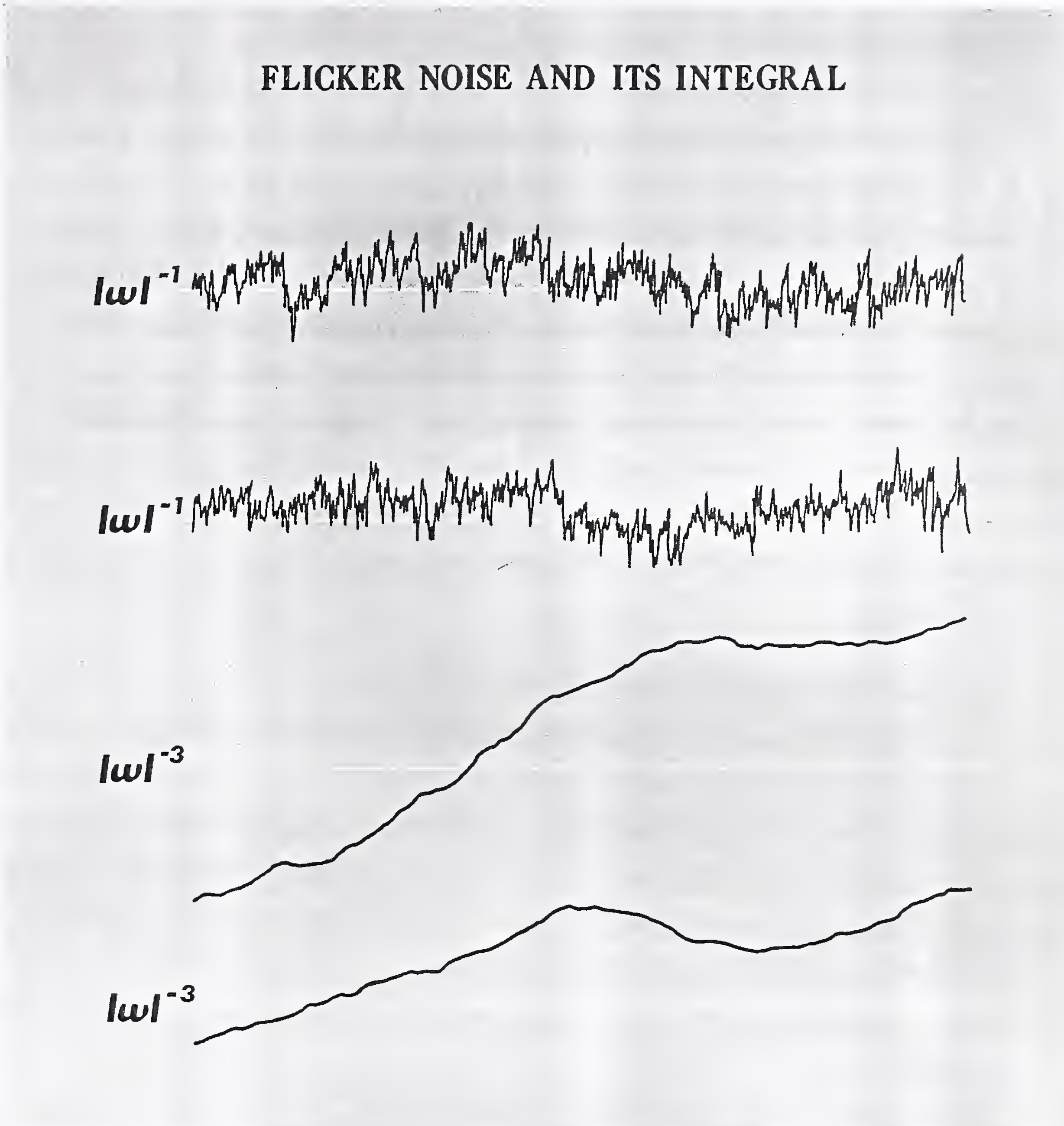

FLICKER NOISE AND ITS INTEGRAL
$|\omega|^{-1}$
$|\omega|^{-1}$
$|\omega|^{-3}$
$|\omega|^{-3}$

Figure 2

2. Noise Recognition

2.1 General Remarks

The most comprehensive treatment of recognition of noises with the power law types of power spectral densities is to be found in the Proceedings of the IEEE, Vol. 54, No. 2, February 1966. Of particular note are the papers by Cutler and Searle [1966], Vessot, et al. [1966], Allan [1966], and Barnes [1966]. These papers deal with the inference of power spectral densities from certain time domain analyses. While one can approach the classification problem directly in the frequency domain either by analog techniques or other analysis [Blackman and Tukey, 1958], one is normally interested in predicting performance both in the frequency domain and the time domain. The time (frequency) domain implications of the frequency (time) domain behavior are of great value.

2.2 Visual Inspection

One of the most surprising aspects of noise characterized by a power law type of power spectral density is the fact that, with a little practice, a visual inspection of the data is often adequate to classify the spectral type. Of course, one often has other information about the noise source and hence often one has reason to suspect one type of noise over another before the data are acquired.

It is apparent that, if one has a noise whose spectral density is of the form

$$S(\omega) = h|\omega|^{-\alpha},$$

then for $\alpha \leq 1$ the process is high frequency divergent. One certainly does not, in reality, observe such a process since the measuring apparatus has a finite high frequency limit. What one does observe, however, is that for processes where $\alpha < 2$, the visual appearance can be drastically changed by the high frequency cutoff.

While it is true that for $\alpha \geq 1$ the process is low frequency divergent, Cutler [1966] points out that a finite data sample analysis is equivalent to high-pass filtration. Thus, the low frequency divergent character is obscured to some extent. Indeed, the operation of "zeroing" amplifiers, oscilliscopes, and various measuring devices, is, to some extent, high-pass filtration.

Thus for valid noise comparisons, one should compare those noise processes with similar products of data length with high frequency cutoff. In figure 3 a number of noise samples which were computer generated are shown. The cutoff (cycle) frequency data length product is about 500 for figure 3.

2.3 Standard Deviation Measurements

A familiar statistic of a noise process is the standard deviation of different values of the noise, $X(t)$. Normally when considering a continuous noise process one averages the noise value over some time interval, τ, to obtain a finite set of, say, N measurements. As pointed out above, for some noises the system bandwidth, ω_h, is a significant parameter. If there exists "dead time", τ_d, between successive averages of χ, then this dead time may also be of significance. Thus one may compute a quantity $s^2(N, \tau, \tau_d, \omega_h)$ defined by typical equations for the sample standard deviation:

$$s^2(N, \tau, \tau_d, \omega_h) = \frac{1}{N-1} \sum_{n=1}^{N} \left(\bar{X}_n - \frac{1}{N} \sum_{i=1}^{N} \bar{X}_i \right)^2, \tag{2-1}$$

where $\bar{X}_n$ is the average of $X(t)$ over the n-th time interval of duration τ.

Normally one is interested in a sample standard deviation as an estimate of the "true" standard deviation which would be obtained in the limit as N approaches infinity. For many of the types of noises being

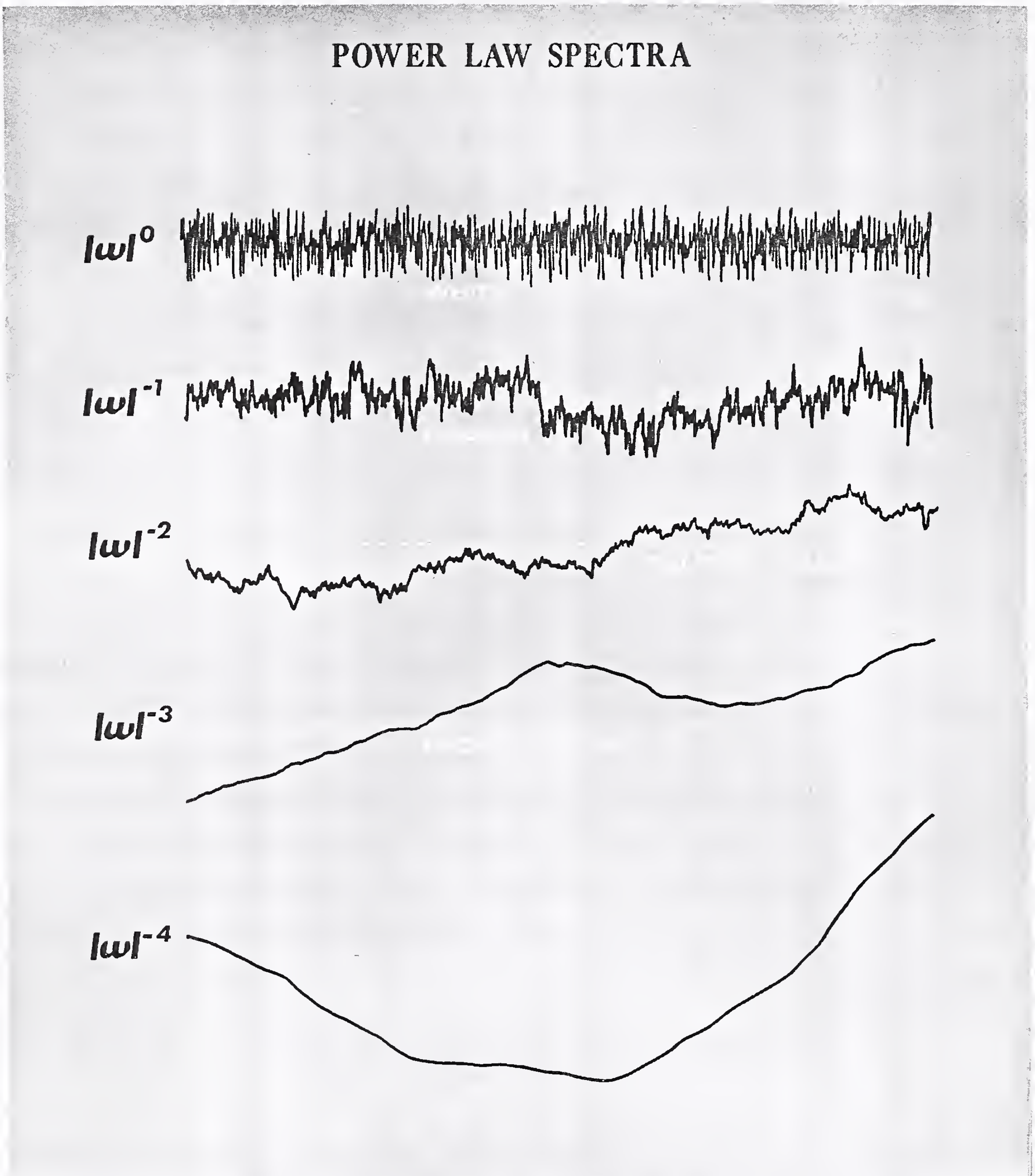


Figure 3

considered here, however, this limit does not exist (i. e. s diverges as $N \to \infty$). For the work here, the "true" standard deviation $\sigma\left(N, \tau, \tau_d, \omega_h\right)$ is defined as the limit

$$\sigma^2\left(N, \tau, \tau_d, \omega_h\right) = \lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^{M} s_i^2 \left(N, \tau, \tau_d, \omega_h\right)$$

where $s_i\left(N, \tau, \tau_d, \omega_h\right)$ is the sample standard deviation of the i-th independent set of measurements characterized by the parameters N, τ, τ_d, and ω_h. This approach extends the utility of the concept of the standard deviation to noises whose power spectral densities may be described as

$$S_\chi(\omega) = h|\omega|^{-\alpha}$$

where $\alpha < 3$. For noise processes where $\alpha \geq 3$ other techniques must be used. Since one can never actually pass to the limit $M \to \infty$, one measures again a sample standard deviation.

It has been shown by Vessot, et al. [1966] and Allan [1966] that if

$$S_\chi(\omega) = h|\omega|^{-\alpha}$$

then

$$\sigma^2\left(N, \tau, \tau_d, \omega_h\right) = k\tau^{\mu}$$

where $\tau\omega_h >> 1$, k is a function of N, τ_d, ω_h, and μ and α are related as shown in figure 4.

It is thus possible to plot $\ln \sigma^2\left(N, \tau, \tau_d, \omega_h\right)$ vs. $\ln \tau$ and infer α from the slope of this plot and figure 4.

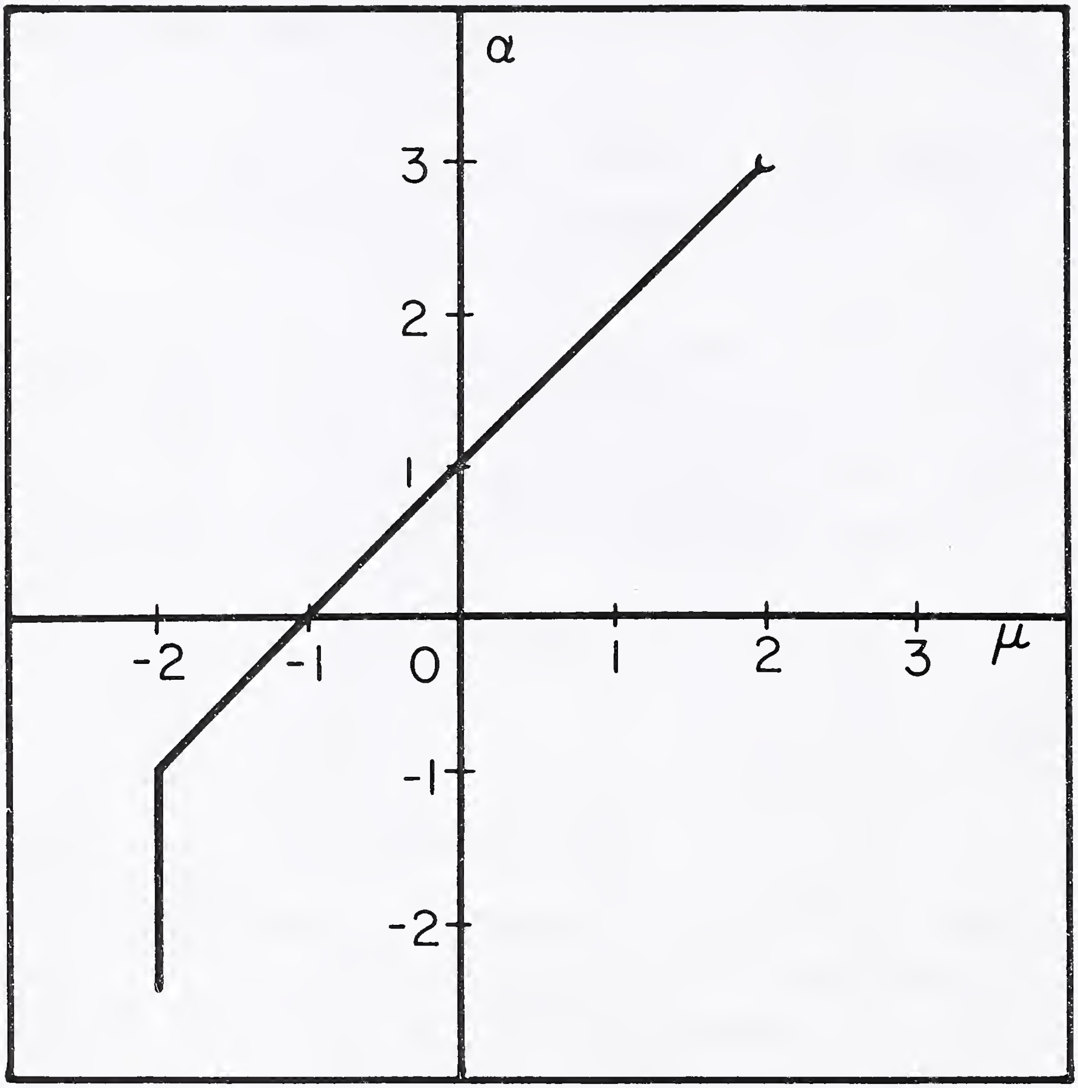


Figure 4 μ - α Mapping

For $\alpha > -1$, Allan [1966] shows that

$$\sigma^2\left(N, \tau, \tau_d, \omega_h\right) = A|\tau|^{\mu}\left\{1 + \sum_{n=1}^{N-1} \frac{(N-n)(nr)^{\mu+2}}{N(N-1)}\left[2 - \left(1 + \frac{1}{nr}\right)^{\mu+2} - \left(1 - \frac{1}{nr}\right)^{\mu+2}\right]\right\}$$

where $r \equiv \dfrac{\tau + \tau_d}{\tau}$, $\tau\omega_h >> 1$, and A depends only on ω_h, the noise level, and noise spectral type.

2.4 The Chi Test

Following Allan [1966], one may define the quantity, χ, by the relation

$$\chi(N, \alpha) = \frac{\sigma^2\left(N, \tau, 0, \omega_h\right)}{\sigma^2\left(2, \tau, 0, \omega_h\right)}. \tag{2-2}$$

This should not be confused with the statistician's chi-squared test. . Provided $\tau\omega_h >> 1$, $\chi(N,\alpha)$ is a function only of N and the spectral type of noise being considered. Table 1 lists values of $\chi(N, \alpha)$ for typical values of N and α.

2.5 Finite Differences

The first finite difference of the quantity $\chi(t)$ may be defined as

$$\Delta\chi_i = \chi(t_i + \tau) - \chi(t_i).$$

Similarly, the second, third, and fourth differences are

$$\Delta^2\chi_i = \chi(t_i + 2\tau) - 2\chi(t_i + \tau) + \chi(t_i)$$

$$\Delta^3\chi_i = \chi(t_i + 3\tau) - 3\chi(t_i + 2\tau) + 3\chi(t_i + \tau) - \chi(t_i)$$

and

$$\Delta^4\chi_i = \chi(t_i + 4\tau) - 4\chi(t_i + 3\tau) + 6\chi(t_i + 2\tau) - 4(t_i + \tau) + \chi(t_i).$$

N

μ	4	8	16	32
-2.0	.833	.750	.708	.687
-1.9	.845	.765	.724	.704
-1.8	.858	.782	.743	.722
-1.7	.871	.801	.763	.743
-1.6	.886	.822	.786	.767
-1.5	.902	.844	.812	.793
-1.4	.919	.870	.841	.824
-1.3	.937	.897	.873	.859
-1.2	.956	.928	.910	.899
-1.1	.977	.962	.952	.946
-1.0	1.000	1.000	1.000	1.000
- .9	1.023	1.041	1.054	1.062
- .8	1.049	1.088	1.116	1.136
- .7	1.077	1.139	1.188	1.223
- .6	1.106	1.197	1.270	1.327
- .5	1.138	1.261	1.365	1.450
- .4	1.171	1.332	1.475	1.598
- .3	1.208	1.412	1.604	1.777
- .2	1.247	1.501	1.753	1.993
- .1	1.288	1.602	1.928	2.257
.0	1.333	1.714	2.133	2.580
.1	1.381	1.840	2.374	2.978
.2	1.432	1.981	2.658	3.470
.3	1.487	2.141	2.993	4.082
.4	1.546	2.320	3.390	4.846
.5	1.609	2.522	3.862	5.802
.6	1.677	2.750	4.424	7.005
.7	1.749	3.007	5.093	8.523
.8	1.827	3.298	5.893	10.446
.9	1.910	3.627	6.851	12.888
1.0	1.999	4.000	8.000	16.000
1.1	2.095	4.421	9.379	19.974
1.2	2.198	4.900	11.040	25.062
1.3	2.308	5.442	13.041	31.593
1.4	2.426	6.059	15.457	39.992
1.5	2.552	6.759	18.376	50.815
1.6	2.687	7.554	21.908	64.788
1.7	2.832	8.460	26.187	82.855
1.8	2.988	9.490	31.377	106.253
1.9	3.154	10.663	37.677	136.597
2.0	3.333	11.999	45.333	176.000

TABLE OF X VALUES

TABLE 1

N

64	128	256	512	1024
.677	.671	.669	.667	.667
.693	.688	.685	.684	.683
.712	.706	.704	.702	.702
.733	.727	.725	.723	.723
.756	.751	.748	.747	.746
.784	.779	.776	.774	.774
.815	.810	.807	.806	.805
.851	.847	.844	.843	.842
.893	.889	.887	.886	.886
.942	.940	.938	.938	.937
1.000	1.000	1.000	1.000	1.000
1.068	1.072	1.074	1.075	1.076
1.150	1.159	1.165	1.168	1.171
1.249	1.266	1.278	1.286	1.291
1.369	1.400	1.422	1.437	1.447
1.517	1.568	1.606	1.634	1.655
1.700	1.782	1.847	1.898	1.937
1.928	2.058	2.167	2.257	2.332
2.215	2.417	2.598	2.758	2.899
2.580	2.892	3.190	3.472	3.736
3.047	3.527	4.015	4.508	5.004
3.649	4.384	5.183	6.045	6.973
4.431	5.554	6.857	8.362	10.097
5.454	7.166	9.290	11.916	15.156
6.800	9.407	12.866	17.444	23.496
8.583	12.547	18.177	26.157	37.456
10.957	16.982	26.141	40.046	61.139
14.135	23.285	38.181	62.405	101.779
18.407	32.301	56.521	98.717	172.209
24.177	45.265	84.639	158.134	295.300
32.000	64.000	128.000	256.000	511.999
42.643	91.192	195.211	418.126	895.897
57.172	130.818	299.864	688.060	1579.703
77.064	188.766	463.475	1139.479	2803.483
104.373	273.778	720.174	1897.334	5002.701
141.955	398.853	1124.206	3174.015	8969.196
193.791	583.344	1761.937	5331.243	16146.011
265.432	856.107	2771.095	8986.198	29168.198
364.630	1260.222	4371.641	15193.658	52856.122
502.214	1860.075	6915.323	25758.904	96042.085
693.333	2752.000	10965.336	43776.034	174933.41

TABLE OF χ VALUES
Table 1 (continued)

$$\chi = \frac{\langle \sigma^2 (N, \tau) \rangle}{\langle \sigma^2 (2, \tau) \rangle}; \quad \langle \sigma^2 (N, \tau) \rangle = a(\mu) \frac{N(N^{\mu} - 1)}{N - 1} \quad |\tau|^{\mu}$$

and $S(\omega) = h|\omega|^{-\alpha}$; $\mu = \alpha - 1$, $-2 < \mu < 2$

For Fourier frequencies, ω, such that

$$\omega\tau >> 1,$$

the power spectral density of n-th finite difference, $S_{(\Delta^n \chi)}(\omega)$, of the function χ is related to the power spectral density of χ by the approximate relation

$$S_{(\Delta^n \chi)}(\omega) \approx S_\chi(\omega) \cdot (\omega\tau)^{2n}.$$

Thus, if $S_\chi(\omega)$ is low frequency divergent in a power-law sense, there always exists some finite n for which the n-th finite difference has a convergent power spectral density. This fact is exploited by Barnes [1966] for the recognition and classification of flicker noise. Provided n is sufficiently large, the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left(\Delta^n \chi_i\right)^2 = \left\langle \left(\Delta^n \chi_i\right)^2 \right\rangle \tag{2-3}$$

always exists. As before, if the measuring system has a high frequency cutoff, ω_h, then one normally works in the range where

$$\omega_h \tau >> 1.$$

The mean square of the n-th finite difference is, in general, a function of the delay time, τ, and the high frequency cutoff, ω_h.

It is true that if N (the number of independent differences used) is not large or the order of differencing, n, is not adequately large, the expected value of

$$\frac{1}{N} \sum_{i=1}^{N} \left(\Delta^n \chi_i\right)^2$$

for fixed N may differ from $\left\langle \left(\Delta^n \chi_i\right)^2 \right\rangle$ as defined in (2-3). One may, in fact, use Allan's [1966] theory to treat these cases (see also Section 2. 3).

2.6 Ratio of Variances of Finite Differences

In analogy to the χ-function of Allan [1966] (see Section 2.4), one may compute the ratio of the variance (or mean square if appropriate) of the (n+1)-th finite difference to that of the n-th finite difference. This ratio is a function only of the order of differencing used and the noise spectral type. By extending the work of Barnes [1966], figure 5 can be obtained for the ratio of variances of finite differences. As shown in figure 4, the point at $\alpha = 1$ is actually degenerate for all α such that $\alpha \leq 1$. One might consider that for this situation μ is the more fundamental quantity and $\mu \geq -2$ as indicated in figure 4 regardless of α.

2.7 Bandwidth Dependence

Vessot, et al. [1966] consider the case where one may, roughly, equate

$$\tau \approx \frac{1}{\omega_h}$$

and consider $\sigma^2\left(\infty, \frac{1}{\omega_h}, 0, \omega_h\right)$. Obviously this will exist only for noises where $\alpha < 1$. This has been referred to as the "marching bandwidth" method. As one may expect there does exist noise spectral types where the variation of σ with ω_h is one of the better means of diagnosis.

2.8 Summary of Relations

The use of precision oscillators in frequency standards, clocks, spectrum analyzers, and Doppler radar ranging has resulted in the development of several statistical quantities of significance. By virtue of the variable use of oscillators it is of value to relate the various quantities and consider their sensitivity as a diagnostic tool.

Consider $\phi(t)$ to be the fluctuations in phase of some oscillator relative to an ideal definition. The instantaneous frequency, $\dot{\phi}(t) = \frac{d}{dt}\phi(t)$, is assumed to have a power spectral density, $S_{\dot{\phi}}(\omega)$, which may be

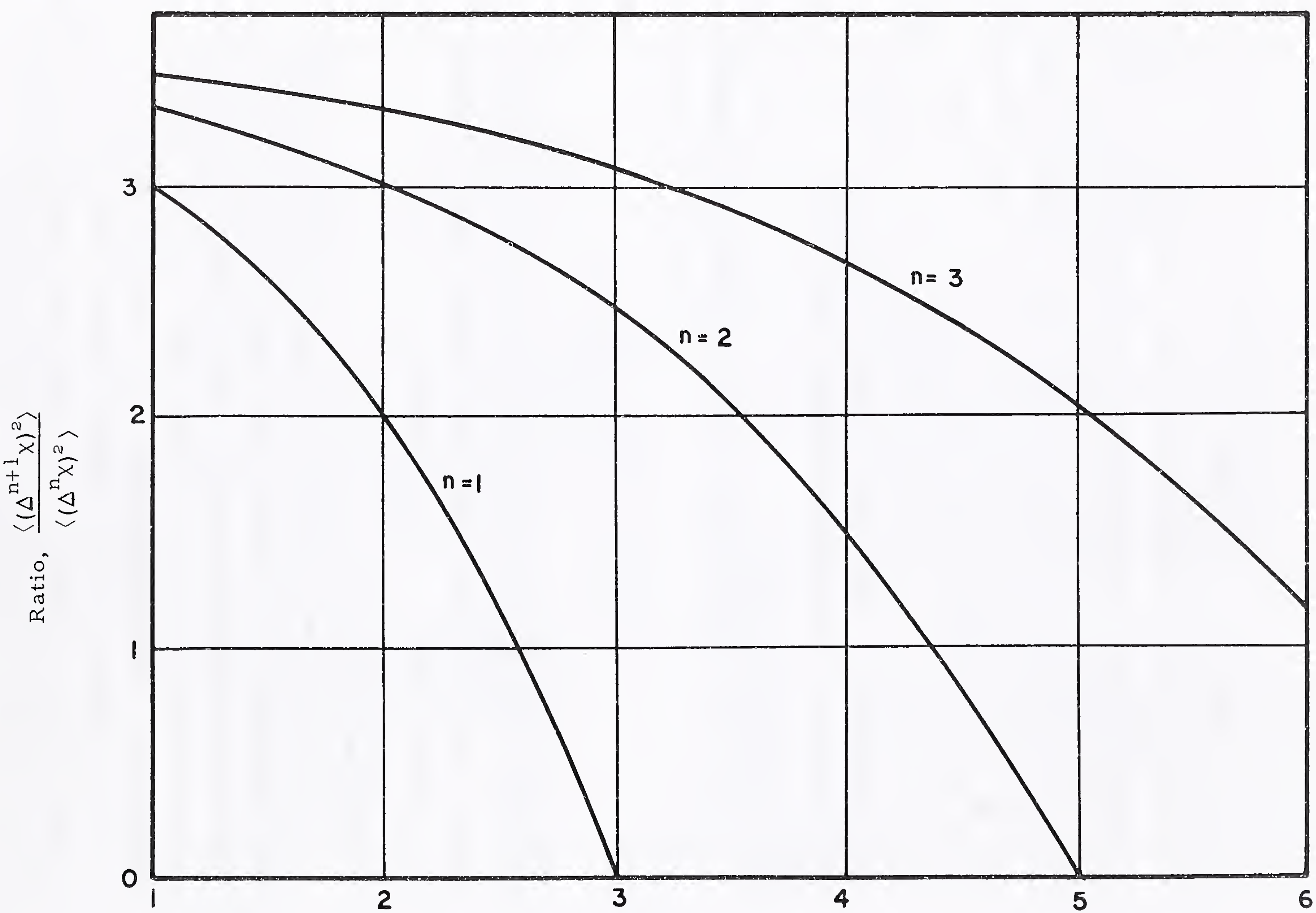


Figure 5

Exponent, α, for $S(\omega) = h|\omega|^{-\alpha}$

expanded in a Laurent series where only four terms are of any significance in its use. That is,

$$S_{\dot{\phi}}(\omega) = \begin{cases} A|\omega|^{-1} + B + C|\omega| + D\omega^2, & |\omega| < \omega_h \\ 0, & \text{otherwise.} \end{cases} \tag{2-4}$$

This specifically assumes that the lowest order noise process affecting the frequency is a flicker noise. In actual practice if the oscillator is subjected to various external perturbations one would expect a term proportional to $|\omega|^{-2}$ to be of significance. The present model, thus, assumes "ideal" ambient conditions.

It is of value to consider four different statistics of the process: First, the standard deviation of the average frequency, $\sigma^2\left(N, \tau, \tau_d, \omega_h\right)$, as defined in Section 2. 3 is important. Second, when the dead time, τ_d, is zero, the equations simplify significantly and it is of value to consider this special case. The third quantity of interest is the mean square second difference of the phase, $D^2(\tau, \omega_h) = \langle (\Delta^2\phi)^2 \rangle$.

When used in Doppler radar ranging, one compares the phase elapsed during some interval, τ, with that elapsed during a delayed interval of the same duration. Thus, the fourth quantity may be defined as

$$\psi^2(\tau, T, \omega_h) = \langle [\phi(t+T+\tau) - \phi(t+T) - \phi(t+\tau) + \phi(t)]^2 \rangle$$

and will be referred to as the Doppler range error. Table 2 lists the definition of the four quantities and relates them to the autocovariance functions and the power spectral densities for general (effective stationary) noise processes.

Table 3 shows general relations between the four statistical quantities for any noise process. For the specific model, (2-4), the terms arising from each term of the Laurent expansion are given for each of the four statistical quantities.

Inspection of Table 4 reveals various general rules for diagnostics of power-law spectra:

(1) Changing N in the variance (χ-test of Section 2.4) is useful in recognizing white or flicker frequency modulation (A and B terms) but not for distinguishing between white and flicker phase modulation (C and D terms).

(2) Changing τ for any of the quantities is similar to changing N ((1) above) in the variance.

(3) Changing ω_h is the most sensitive method of distinguishing white phase noise (D-terms) from the others (A, B, and C-terms).

(4) Increasing τ_d from zero in the variance has the general result of losing sensitivity for distinguishing among any of the terms.

Table 2

List of Definitions

QUANTITY	DEFINITION	AUTOCOVARIANCE NOTATION	SPECTRAL NOTATION
$\langle\sigma^2(N,\tau,\tau_d,\omega_h)\rangle$ (Variance)	$\frac{1}{N-1}\left\langle\sum_{n=0}^{N-1}\left[\frac{\phi\left(n\tau_d+(n+1)t\right)-\phi\left(n(\tau+\tau_d)\right)}{\tau}\right]^2 - \frac{1}{N}\left[\sum_{n=0}^{N-1}\frac{\phi\left(n\tau_d+(n+1)\tau\right)-\phi\left(n(\tau+\tau_d)\right)}{\tau}\right]^2\right\rangle$	$\frac{2}{\tau^2}\left\{R_\phi(0)-R_\phi(\tau)-\frac{1}{N(N-1)}\sum_{n=1}^{N-1}(N-n)\left[2R_\phi\left(n(\tau+\tau_d)\right) - R_\phi\left(n(\tau+\tau_d)+\tau\right)-R_\phi\left(n(\tau+\tau_d)-\tau\right)\right]\right\}$	$\frac{4}{\tau^2}\int_0^\infty S_\phi(\omega)\left\{1-\cos(\omega\tau)-\frac{1}{N(N-1)}\sum_{n=1}^{N-1}(N-n)\cdot\left\{2\cos n\omega(\tau_d+\tau) - \left[\cos\omega\left(n(\tau+\tau_d)+\tau\right)+\cos\omega\left(n(\tau+\tau_d)-\tau\right)\right]\right\}\right\}d\omega$
$\langle\sigma^2(N,\tau,0,\omega_h)\rangle$ (Variance - no dead time)	$\frac{1}{N-1}\left\langle\sum_{n=1}^{N}\left[\frac{\Delta\phi_n}{\tau}-\frac{\phi_{N+1}-\phi_1}{N\tau}\right]^2\right\rangle$	$\frac{2}{(N-1)\tau^2}\left[\left(N-\frac{1}{N}\right)R_\phi(0)-NR_\phi(\tau)+\frac{1}{N}R_\phi(N\tau)\right]$	$\frac{4}{N(N-1)\tau^2}\int_0^\infty S_\phi(\omega)\left[N^2(1-\cos\omega\tau)-1+\cos N\omega\tau\right]d\omega$
$D^2(\tau,\omega_h)$ (second difference)	$\left\langle\left[\phi(t+2\tau)-2\phi(t+\tau)+\phi(t)\right]^2\right\rangle$	$6R_\phi(0)-8R_\phi(\tau)+2R_\phi(2\tau)$	$4\int_0^\infty S_\phi(\omega)\left[3-4\cos(\omega\tau)+\cos(2\omega\tau)\right]d\omega$
$\Psi^2(\tau,T,\omega_h)$ (Doppler range error)	$\left\langle\left[\phi(t+T+\tau)-\phi(t+T)-\phi(t+\tau)+\phi(t)\right]^2\right\rangle$ $\left(\Psi^2(\tau,T,\omega_h)=\Psi^2(T,\tau,\omega_h)\right)$	$4R_\phi(0)-4R_\phi(\tau)-4R_\phi(T) + 2R_\phi(T-\tau)+2R_\phi(T+\tau)$	$4\int_0^\infty S_\phi(\omega)\left[2-2\cos\omega\tau-2\cos\omega T+\cos\omega(T-\tau) + \cos\omega(T+\tau)\right]d\omega$

Table 3

General Relations

	General Relations between Quantities $\langle \sigma^2(N, \tau, \tau_d, \omega_h) \rangle$ Variance	Doppler Range Error $\Psi^2(\tau, T, \omega_h)$	Second Difference $D^2(\tau, \omega_h)$
$D^2(\tau, \omega_h)$	$2\tau^2 \langle \sigma^2(2, \tau, 0, \omega_h) \rangle$	$\Psi^2(\tau, \tau, \omega_h)$	Same
$\Psi^2(\tau, T, \omega_h)$	$2\tau^2 \langle \sigma^2(2, \tau, \tau + \tau_d, \omega_h) \rangle$	Same	$D^2(\tau, \omega_h)$ for $\tau_d = 0$ only
$\langle \sigma^2(N, \tau, \tau_d, \omega_h) \rangle$	Same	$\frac{1}{2\tau^2} \Psi^2(\tau, \tau + \tau_d, \omega_h)$ for N = 2 only	$\frac{1}{\tau^2 2^n (2^n - 1)} \sum_{i=1}^{n} 4^{(i-1)} D^2(2^{(n-i)}\tau, \omega_h)$ for $N = 2^n$ and $\tau_d = 0$ only

Table 4

LAURENT EXPANSION OF SPECTRAL DENSITY

QUANTITY	$S_\phi(\omega) = \frac{A}{\lvert\omega\rvert}$	$+ B$	$+ C\lvert\omega\rvert$	$+ D\omega^2$
Variance $\langle\sigma^2(N, \tau, \tau_d, \omega_h)\rangle$	$A \cdot \frac{8 \ln 2}{N(N-1)} \sum_{n=1}^{N-1} (N-n) \Big[-2(n\rho)^2 \ln(n\rho)$ $+ (n\rho+1)^2 \ln(n\rho+1) + (n\rho-1^2)\ln(n\rho-1)\Big]$ $\rho = \frac{t+\tau_d}{\tau}$	$B \cdot \frac{2\pi}{\tau};\ \tau_d \le 0$ $B \cdot \frac{2\pi T}{\tau^2};\ \tau_d \ge 0$	$C \cdot \frac{4}{\tau^2}\Big[2 + \ln(\tau\omega_h)$ $+ \frac{1}{N(N-1)} \sum_{n=1}^{N-1} (N-n) \ln \left\lvert \frac{n^2}{n^2-\rho^2} \right\rvert\Big]$ $\rho > 1$	$D \cdot \frac{N+\delta(\tau_d)}{N} \quad \frac{2\omega_h}{\pi\tau^2}$ $\delta = \begin{cases} 1, & \tau_d = 0 \\ 0, & \tau_d > 0 \end{cases}$
Variance $\langle\sigma^2(N, \tau, 0, \omega_h)\rangle$ (no dead time)	$A \cdot \frac{2N \ln(N)}{N-1}$	$B \cdot \frac{2\pi}{\tau}$	$C \cdot \frac{4(N+1)}{N\tau^2}\Big[2 + \ln(\tau\omega_h)$ $- \frac{1}{N^2-1} \ln(N)\Big]$	$D \cdot \frac{N+1}{N} \cdot \frac{2\omega_h}{\pi\tau^2}$
Second Difference $D^2(\tau, \omega_h)$	$A \cdot 8 \ln(2)\tau^2$	$B \cdot 4\pi\tau$	$C \cdot 4\Big[\big(6 - \ln(2)\big) + 3\ln(\tau\omega_h)\Big]$	$D \cdot \frac{6\omega_h}{\pi}$
Doppler range error $\Psi^2(\tau, T, \omega_h)$	$- A \cdot 2\tau^2 \Big[2\rho^2 \ln(\rho)$ $- (\rho+1)^2 \ln(\rho+1) - (\rho-1)^2 \ln(\rho-1)\Big]$ $\sim A \cdot 4\tau^2\big(2 + \ln(\rho)\big)$ for $\rho >> 1$ $\sim A \cdot 4T^2\big(2 + \ln(1/\rho)\big)$ for $\rho << 1$	$B \cdot 4\pi\tau;\ T \ge \tau$ $B \cdot 4\pi T;\ \tau \ge T$	$C \cdot 8\left[2 + \ln\left(\left\lvert\tau\omega_h \sqrt{\frac{\rho^2}{\rho^2-1}}\right\rvert\right)\right]$ $\rho = \frac{T}{\tau} \neq 1$	$D \cdot \big[2 + \delta(T-\tau)\big] \frac{2\omega_h}{\pi}$ $\delta = \begin{cases} 1, & T = \tau \\ 0, & T \neq \tau \end{cases}$

A. 1 The Random Walk Problem

Consider a filter whose transfer function may be expressed as

$$g(\omega) = \frac{1}{\left(\omega_\ell + j\omega\right)} \tag{A-1}$$

where ω_ℓ is the low frequency cutoff. It is easy to realize this exact transfer function with a resistor and a capacitor.

As in Section 1. 1, assume the input, e_i, is a white noise whose spectral density is

$$S_i(\omega) = h.$$

The output, e_o, has a spectral density which is, then,

$$S_o(\omega) = |g(\omega)|^2 S_i(\omega), \tag{A-2}$$

or

$$S_o(\omega) = \frac{h}{|\omega_\ell + j\omega|^2} \tag{A-3}$$

From work with analog computers, one knows that as ω_ℓ becomes smaller, the filter becomes more nearly a perfect integrator. Thus one is tempted to consider the limit function of the sequence $\left\{\frac{h}{|\omega_\ell + j\omega|^2}\right\}$ as $\omega_\ell \to 0$ to be the true power spectral density of a (continuous) random walk. This limit function is just $h|\omega|^{-2}$.

For non-zero ω_ℓ, one may obtain the autocovariance function of the output defined by the relation

$$R_o(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e_o(t)e_o(t+\tau)dt. \tag{A-4}$$

It is often easier to obtain $R_o(\tau)$ as the inverse Fourier transform of $S_o(\omega)$ (Wiener-Khinchin theorem). For $\omega_\ell \neq 0$ no problem is encountered because $e_o(t)$ is a stationary time function. The result is [Campbell and Foster, 1948] (pair 444)

$$R_o(\tau) = \frac{\pi h}{\omega_\ell} \exp\left(-\omega_\ell |\tau|\right). \tag{A-5}$$

It is apparent now, however, that the sequence $\{R_o(\tau)\}$ does not have a limit function as $\omega_\ell \to 0$. While one can consider the limit of the spectral density sequence, the limit autocovariance function does not exist, and, thus, in the limit, the Wiener-Khinchin theorem does not apply. With the autocovariance not defined, one has difficulty working in the time domain.

There is, however, an interesting approach to this problem. One knows that a random walk has "stationary increments" [Parzen, 1962]. Assume $\omega_\ell \neq 0$ temporarily. One may consider the quantity

$$\Delta e_o(t) = e_o(t+\tau) - e_o(t),$$

and, thus,

$$\left\langle \left(\Delta e_o(t)\right)^2 \right\rangle = 2\left[R_o(0) - R_o(\tau)\right] \tag{A-6}$$

where the angular brackets denote time average, and use has been made of the definition of the autocovariance function, (A-4). Substitution of (A-5) into (A-6) yields

$$\left\langle \left(\Delta e_o(t)\right)^2 \right\rangle = \frac{2\pi h}{\omega_\ell}\left[1 - \exp\left(-\omega_\ell |\tau|\right)\right]. \tag{A-7}$$

One may, now, consider the limit of (A-7) as $\omega_\ell \to 0$. This limit is

$$\lim_{\omega_\ell \to 0} \left\langle \left(\Delta e_o(t)\right)^2 \right\rangle = 2\pi h |\tau|. \tag{A-8}$$

The usual factor of 2π is present because $S_o(\omega)$ is defined to be a density relative to angular Fourier frequencies. Equation (A-8) does, in fact, give the true mean square value of the increment in a random walk or Brownian Motion [Parzen, 1962; and Cutler and Searle, 1966].

Some insight may be gained by considering the Taylor series expansion of (A-5); i. e.,

$$R_o(\tau) = \pi h \left\{ \frac{1}{\omega_\ell} - |\tau| + \frac{1}{\omega_\ell} \sum_{n=2}^{\infty} \frac{\left(-\omega_\ell |\tau|\right)^n}{n!} \right\}. \tag{A-9}$$

It is apparent that the summation on the right of (A-9) vanishes in the limit as $\omega_\ell \to 0$. The term $-|\tau|$ is independent of ω_ℓ as $\omega_\ell \to 0$. It is apparent that if functionals of $R_o(\tau)$ are considered whose form is

$$F\left(R_o(\tau)\right) = \sum_{n=1}^{N} \beta_n R_o(\tau_n),$$

then the limit as $\omega_\ell \to 0$ will always be given by

$$\lim_{\omega_\ell \to 0} F(R_o(\tau)) = -\sum_{n=1}^{N} \beta_n |\tau_n| \pi h$$

provided only that $\sum_{n=1}^{N} \beta_n = 0$. With this constraint satisfied, one may consider the "effective, stationary (E-S) autocovariance function" of $S(\omega) = h|\omega|^{-2}$ to be $R_{ES}(\tau) = -\pi h |\tau|$.

One may, therefore, treat a random walk as a stationary process and use the E-S autocovariance functions in the time domain to solve the resulting time domain problems. One must, however, be certain that the constraint, $\sum \beta_n = 0$, is satisfied. A significant and interesting point is that

$$R_{ES}(\tau) = -\pi|\tau|$$

is the (generalized) Fourier transform of the generalized function $|\omega|^{-2}$ [Lighthill, 1962]. It is obvious that E-S autocovariance functions do not necessarily satisfy the usual inequality

$$R(0) \geq |R(\tau)|.$$

A. 2 Extension of Effective Stationarity

Consider the power spectral density function

$$S(\omega) = h\left|\omega_{\ell} + j\omega\right|^{-\alpha}. \tag{A-10}$$

The autocovariance function, $R(\tau)$, corresponding to (A-10) may be found using the Wiener-Khinchin theorem and pair 569 of Campbell and Foster [1948]. That is

$$R(\tau) = 2\pi h \frac{|\tau|^{\frac{\alpha-1}{2}} K_{\frac{\alpha-1}{2}}(\omega_{\ell}|\tau|)}{\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)(2\omega_{\ell})^{\frac{\alpha-1}{2}}}, \tag{A-11}$$

where $K_{\nu}(X)$ is the Bessel function of the second kind for imaginary argument. Two significant relations are

$$K_{\nu}(X) = \frac{\pi}{2 \sin \nu\pi}\left[I_{-\nu}(X) - I_{\nu}(X)\right] \tag{A-12}$$

and

$$I_{\nu}(X) = \sum_{k=0}^{\infty} \frac{X^{\nu+2k}}{\left(2^{\nu+2k}\right)(k!)\Gamma(\nu+k+1)}. \tag{A-13}$$

From (A-13) the term corresponding to $k = 0$ for $\nu = \frac{\alpha-1}{2}$ and $X = \omega_{\ell}|\tau|$ is

$$\frac{(\omega_{\ell}|\tau|)^{\frac{\alpha-1}{2}}}{\left(2^{\frac{\alpha-1}{2}}\right)\Gamma\left(\frac{\alpha+1}{2}\right)}.$$

Substitution in (A-12) and then (A-11) gives a term in $R(\tau)$ which is

$$R_{ES}(\tau) = \frac{-\pi\sqrt{\pi}\,h\left|\frac{\tau}{2}\right|^{\alpha-1}}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right)\sin\left[\frac{\pi(\alpha-1)}{2}\right]}. \tag{A-14}$$

If $\frac{\alpha - 1}{2}$ is an integer, then the term in (A-13) for $k = \frac{\alpha - 1}{2}$ and $\nu = -\frac{\alpha - 1}{2}$ also contributes to the ω_ℓ-independent term but with inverted sign. However, with $\frac{\alpha - 1}{2}$ being an integer, the denominator of (A-14) also vanishes. Always the special cases of integer values of $\frac{\alpha - 1}{2}$ (flicker noise and its integrals) must be treated separately.

For $\alpha = 2$ in (A-14) one obtains, in fact, $R_{ES}(\tau) = -\pi h |\tau|$ in agreement with Section A. 1. For α being an even integer, (A-14) reduces to

$$R_{ES} = (-1)^{\frac{\alpha}{2}} \pi h \frac{|\tau|^{\alpha - 1}}{(\alpha - 1)!} \qquad \text{(A-15)}$$

which is in exact agreement with the Fourier transform of the appropriate generalized functions [Lighthill, 1962]. Making use of the duplication formula [Whittaker and Watson, 1952] for gamma functions,

$$2^{2z - 1}\Gamma(z)\Gamma(z + \tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2z),$$

(A-14) may be written in the form

$$R_{ES}(\tau) = \pi h |\tau|^{\alpha - 1} \frac{1}{\Gamma(\alpha)\cos\left(\frac{\pi \alpha}{2}\right)} \qquad \text{(A-16)}$$

which is also in exact agreement with generalized functions. Equation (A-16) does not apply for α being an odd integer, however.

The special cases of $\frac{\alpha - 1}{2}$ being an integer may be evaluated as the limit of the sum of the two important terms in the expansion of (A-11) as $\frac{\alpha - 1}{2}$ approaches the integer n. The first term is just that given by (A-14).

A. 3 Constraints on Effective Stationarity

Consider a function, $\phi(t)$, and a second function, $\psi(t)$, obtained from $\phi(t)$ by the relation

$$\psi(t) = \sum_{i=1}^{N} b_n \phi(t + C_n \tau) \tag{A-18}$$

where the set $\{b_n, C_n\}$ and τ are constants. If one is interested in calculating the mean square of $\psi(t)$, this can always be expressed in the form

$$\langle (\psi(t))^2 \rangle = \sum_{i=1}^{M} \beta_i R_\phi(\gamma_i \tau) \tag{A-19}$$

where the set $\{\beta_i, \gamma_i\}$ are obtained from the $\{b_n, C_n\}$ of (A-18) and it is assumed that $\phi(t)$ is effectively stationary.

It has already been shown that, for

$$S_\phi(\omega) = h|\omega|^{-2},$$

the constraints on $\{\beta_i, \gamma_i\}$ are given simply by the relation

$$\sum_{i=1}^{M} \beta_i = 0.$$

It is a simple matter to show that the condition

$$\sum_{n=1}^{N} b_n = 0$$

for the coefficients $\{b_n, C_n\}$ is also sufficient to guarantee correct results.

As shown by Barnes [1966], for $\alpha \geq 3$, the constraints on $\{b_n, C_n\}$ become

$$\sum_{n=1}^{N} b_n = 0$$

and

$$\sum_{n<\ell}^{N} b_n b_\ell (C_\ell - C_n)^2 = 0.$$

Intuitively one would expect these constraints to break down at $\alpha \geq 5$ since a second difference does satisfy the constraints but fails to converge for $\alpha \geq 5$ (see [Barnes and Allan, 1967]). These conditions become

$$\sum_{i=1}^{M} \beta_i = 0$$

and

$$\sum_{i=1}^{M} \beta_i \gamma_i^2 = 0$$

for the $\{\beta_i, \gamma_i\}$. By an extension, one would expect that the constraints

$$\sum_{i=1}^{M} \beta_i = 0$$

$$\sum_{i=1}^{M} \beta_i \gamma_i^2 = 0$$

$$\sum_{i=1}^{M} \beta_i \gamma_i^4 = 0$$

should guarantee effective stationarity to the region $\alpha < 7$. One may quickly verify that the third finite difference satisfies all of these conditions. Similarly one could add the condition

$$\sum_{n<\ell}^{N} b_n b_\ell (C_\ell - C_n)^4 = 0$$

for the $\{b_n, C_n\}$. These results are compiled in Table 6.

TABLE 6

COEFFICIENT CONSTRAINTS FOR EFFECTIVE STATIONARITY

Range for α $S_\phi(\omega) = h\|\omega\|^{-\alpha}$	$\{b_n C_n\}$ Constraints $\psi(t) = \sum_{n=1}^{N} b_n \phi(t + C_n \tau)$	$\{\beta_i \gamma_i\}$ Constraints $\langle \psi^2(t) \rangle = \sum_{i=1}^{M} \beta_i R_\phi(\gamma_i \tau)$
$\alpha < 1$	None (ϕ is stationary)	None (ϕ is stationary)
$\alpha < 3$	$\sum_{n=1}^{N} b_n = 0$	$\sum_{i=1}^{M} \beta_i = 0$
$\alpha < 5$	$\sum_{n=1}^{N} b_n = 0$ $\sum_{n<\ell}^{N} b_n b_\ell (C_\ell - C_n)^2 = 0$	$\sum_{i=1}^{M} \beta_i = 0$ $\sum_{i=1}^{M} \beta_i \gamma_i^2 = 0$
$\alpha < 7$	$\sum_{n=1}^{N} b_n = 0$ $\sum_{n<\ell}^{N} b_n b_\ell (C_\ell - C_n)^2 = 0$ $\sum_{n<\ell}^{N} b_n b_\ell (C_\ell - C_n)^4 = 0$	$\sum_{i=1}^{M} \beta_i = 0$ $\sum_{i=1}^{M} \beta_i \gamma_i^2 = 0$ $\sum_{i=1}^{M} \beta_i \gamma_i^4 = 0$

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