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A THEOREM ON OPTIMAL LOCATIONS

by

A.J. Goldman and P.R. Meyers

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to

U.S. Post Office Department

and

Office of High Speed Ground Transportation

Department of Commerce

NBS

U.S. DEPARTMENT OF COMMERCE

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Operations Research Section

Applied Mathematics Division

For

U.S. Post Office Department

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IMPORTANT NOTICE

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A THEOREM ON OPTIMAL LOCATION

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National Bureau of Standards

ABSTRACT

The problem studied concerns the location of a facility (e.g. a mail-sorting installation or a transport terminal) so as to minimize the sum of transport costs for the "source"-to-facility and facility-to-"sink" flows. Movements are assumed to occur along least costly paths in a network of roads or other conduits. The only assumption on cost structure is that marginal cost is a non-increasing function of trip length, this function possibly changing from trip segment to trip segment. The distributions of sources and sinks along the roads can be arbitrary, as can the network topology. Under further mild hypotheses, it is shown that the search for an optimal facility location can be confined to endpoints of roads and interchanges between roads. A detailed study is made of a radially symmetric circular-disk model in which only radial and circumferential movement is permitted (each with a constant unit cost), and the "density" of sources and sinks drops exponentially as a function of distance from the center of the disk.

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1. INTRODUCTION

Suppose given a collection of "customers" for the services of a central facility(mail sorting installation, transport terminal, firehouse, library, school, food distribution point, etc.). The common feature of the class of applications we have in mind is that performance of the service involves the movement of something between the customer location and the central facility --- e.g. of mail from patrons to sorting installation and from sorting installation to recipients, or of travellers from initial origins to transport terminal, or from terminal to final destinations.

Thus each customer location can act either as a source of flow to the facility, or a sink of flow from the facility, or both. We can weight each source location according to the amount of flow emanating from it, and each sink location according to how much flow it receives, thus giving rise to two mass distributions, denoted μ (for the sources) and μ^* (for the sinks).

The flow is required to move along a least costly among the "allowable paths"; the latter will be called roads for concreteness, though interpretations as pipelines, pneumatic tubes or other types of conduit are equally in order. We use the notation equally in order.

$c(P,L)$ = cost of transporting[2] one unit from source location P to facility location L,

$c^*(L,P^*)$ = cost of transporting one unit from facility location L to sink location P^*.

Then the total transport cost, including both incoming (to the facility) and outgoing flow, is

$$f(L) = \Sigma_P \, c(P,L) \, \mu(P) + \Sigma_{P^*} \, c^*(L,P^*) \, \mu^*(P^*) \, . \tag{1}$$

(1) Supported by the Post Office Department and by the Department of Commerce's High Speed Ground Transportation Project. No official endorsement is implied.

(2) Transport costs are considered exclusive of the costs at the central facility of unloading, processing (servicing) and loading. These costs are assumed independent of the location of the facility.

The "optimal location problem" for a single facility can be defined, for present purposes, as that of choosing the facility's location L so as to minimize f(L). It may be useful to "smear out" the representation of small customers to a continuous mass distribution, retaining some discrete mass-points or mass-curves as required to represent extra-large customers or concentrations of them. Thus (1) will be replaced by

$$f(L) = \int c(P,L)\, d\mu\,(P) + \int c^*(L,P^*)\, d\mu^*(P^*) \;. \qquad (2)$$

In recent (unpublished) work[(3)], C. Witzgall discussed a number of topics relating to this problem. As one of several concrete illustrative models, he considered the case of identical <u>uniform</u> distributions of sources and sinks over a circular disk. Only radial and circumferential movements were permitted, a situation suggestive of a city which has a radial web of major streets and is surrounded by a beltway. The two parameters of the model were both costs of transporting a unit over a unit distance, namely the costs for <u>radial</u> and for <u>peripheral</u> movements, which will be denoted c_r and c_p respectively. Witzgall showed that the condition for a peripheral location of the facility to be better than a central one is

$$c_p/c_r < 0.216 \;,\; \text{or}\; c_p/c_r < 2/3\pi = 0.212 \;, \qquad (3)$$

according as short-cuts through the center of town to a peripheral facility are or are not permitted.

No proof was given, however, that the optimum had to occur for one of these <u>extreme</u> cases (location at center or on circumference) rather than at an <u>intermediate</u> position along a radius. This certainly seemed plausible (and turned out to be true) for the simple disk model, but its main interest lay in the likelihood that it represented a special case of some <u>general principle</u>.

The present paper records our current understanding of this principle. As will be seen, it admits <u>arbitrary</u> mass distributions μ and μ^*, and road networks much more general than that of the disk model. The unit (per mile) transport cost along a road need not be constant; indeed, the <u>critical assumption</u> describing the principle's range of validity appears to be that transport cost along a road (per unit carried) is a <u>concave</u> function (perhaps differing from road to road) of the distance travelled. That is, long-haul costs no more per mile than does short-haul[(4)].

(3) C. Witzgall, "Optimal Location of a Central Facility; Mathematical Models and Concepts," 1964.

(4) This statement about average costs should really be replaced by the corresponding one about marginal costs.

In the following Section 2, we formulate the present version of the principle as a theorem, which is then proved in Section 3. Some extensions are described in Section 4. Section 5 contains an analysis of a more general "circular disk" model with radial symmetry; advantage is taken of the theorem, in that the search for an optimal location for the central facility is confined to the center and periphery of the disk.

2. FORMULATION OF THEOREM

We consider two systems Σ and Σ^* of roads; the first carries the source-to-facility flow, and the second the flow from facility to sinks. Some roads, or segments of roads, may be common to Σ and Σ^*. The set of points covered by the roads of Σ will for simplicity be identified with the set of possible locations of sources[5] (the distinction between $\mu > 0$ and $\mu = 0$ implicitly sorting out the "actual" locations among the possible ones), and this point-set is assumed compact[6]; similarly for Σ^* and the possible locations of sinks. It is assumed that each possible source location is connected within Σ to each possible location of the facility, and similarly for the facility-to-sink flow.

Each road is taken to be a continuous rectifiable curve in the usual mathematical sense; this implies an orientation[7], and a running "path length parameter" which increases from zero at the beginning of the road to some finite maximum value (the length of the road) at its end. Some roads may be closed curves, i.e. their beginning point and ending point coincide (as for a beltway).

As noted earlier, transport cost along any one road (per unit carried) is assumed a <u>concave</u> function of the road-length traversed, this function possibly differing from road to road. While this seems plausible when expressed as a comparison of long-haul per mile transport cost with that for short-haul (the former not exceeding the latter), it <u>may</u> introduce a bias into the choice between central or peripheral location for the facility. Flow from an outlying source to a central location may in its progress encounter successively more congested zones, and therefore succesively higher per mile costs; our concavity hypothesis in effect assumes that this effect is at least counter-balanced by the economies of long-haul operations. On the other hand, the reverse holds true for flow from central facility to outlying sink. Roughly speaking, the bias is likely to be for or against a central location according as μ or μ^* is more concentrated about the "center" of the region under study.

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(5) This evades any need to provide a more or less arbitrary description of how flow from off-Σ sources moves toward Σ. Such a description can be guided by reality for any specific application.

(6) The desire to encompass idealized models with infinitely many customers and roads and special points requires more technical mathematics than would be needed for discrete models. Readers unfamiliar with these concepts (e.g. compactness) can ignore them without serious loss.

(7) A bidirectional road can be represented as two oppositely oriented roads.

Certain special points on the roads of the two systems are distinguished. These are required to include the beginning and ending points of all roads which are not closed curves, and of all segments shared by any specific set of roads. (Thus the "focus" of a radial web of roads would be a special point.) They are also required to include all interchanges, i.e. points where two roads intersect and transition from one to the other is permitted. (For the the simple disk model described earlier, this means that all points on the beltway are special.) The set of special points is assumed compact.

From this last assumption we see that an allowed location for the central facility is either a special point, or else[8] lies on a road segment free of special points. In the first case the special point must be an interchange from a road of Σ to one of Σ^*, while in the second case the road segment must be common to a road of Σ and one of Σ^*. The set of allowed locations L is supposed compact; thus $f(L)$, if continuous, does actually attain a minimum value. The continuity of $f(L)$ is ensured by requiring $c(P,L)$ and $c^*(L,P^*)$ to be continuous.

As noted earlier, flow is assumed to move along a least costly of the allowed paths. More precisely, if $c(P,\pi,L)$ is the transport cost for unit flow along the allowed route π in Σ from P to L, then

$$c(P,L) = \inf_{\pi} c(P,\pi,L) \; ; \tag{4}$$

similarly for $c^*(L,P^*)$.

The remaining assumption is that all flow, whether to or from the facility, must be routed so as to pass through at least one special point. This suggests viewing the special points as seats of some recording or check-in operation, or perhaps of consolidation and break-bulk operations. The hypothesis is automatically satisfied when the facility is itself at a special point. It is automatically satisfied for flow from or to a customer location on a road not containing the central facility, since such flow must pass through at least one interchange. Thus the assumption is restrictive only if the facility's location L is on a road segment free of special points except for its endpoints, and even then it requires "unnatural" routing only for the (presumably small) fraction of flow originating or terminating on that particular segment.

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(8) The point is that an allowed location, if not itself a special point, cannot be a limit of special points.

THEOREM. <u>Under the above hypotheses</u>, <u>at least one special point is an optimal location</u>.

The important implication is that when seeking to determine an optimal location, one can simplify the task by restricting attention to special points. This may be useful not only in placing new facilities, but also in explaining historic patterns of location of "central places". In connection with this last remark, it should be noted that cities can be regarded (in part) as "service facilities" for the "customers" of their surrounding areas.

3. PROOF OF THEOREM

As observed above, the hypotheses of the theorem ensure the existence of at least one optimal location L_o. We assume that L_o is an <u>ordinary</u> (i.e., non-special) point, and prove the theorem by showing that this implies the existence of an optimal special point.

Because L_o is ordinary and the set of special points is compact, L_o must lie on some road segment σ_o whose interior contains only ordinary points, and which lies both on at least one road of Σ and on at least one road of Σ^*. Consider the continuation of σ_o through L_o in one of the two possible directions. A "branch-point" may be reached at which σ_o loses its identity because some road containing it diverges, or some new road portion merges with it; such a point is a special point. The beginning or end of a road may be encountered, or an interchange may be reached; again a special point must be present. If none of these occurs, then we must eventually return to L_o; in this case σ_o is a closed curve consisting of ordinary points only. Since this curve has no interchanges, and all customer locations must be connectable with L_o, it follows that all possible customer locations lie on σ_o, i.e. the points of all roads lie on σ_o. This is impossible because all flow must pass through a special point, whereas σ_o lacks special points. The conclusion is that the maximal continuation of σ_o through L_o in either direction ends in a special point.

From now on we reserve the symbol σ_o for the <u>maximal</u> segment of ordinary points containing L_o, including its (special) end-points E_o and E_1. (The possibility of a closed curve with $E_o = E_1$ as its sole special point offers no difficulty.) The points of σ_o will be denoted $L(\lambda)$, where λ is the arc-length parameter along σ_o measured from E_o, i.e. $L(0) = E_o$ and $L(\lambda_{max}) = E_1$, where

$$\lambda_{max} = \text{length of } \sigma_o .$$

It will be shown that either E_o or E_1 is optimal. This will be done by proving that the function

$$g(\lambda) = f(L(\lambda)) , \tag{5}$$

defined for $0 \leq \lambda \leq \lambda_{max}$, attains its minimum at an endpoint of this interval. Since L_o lies on σ_o, we have

$$\min_{L} f(L) \leq \min_{\lambda} g(\lambda) = \min_{L \in \sigma_o} f(L) \leq f(L_o) = \min_{L} f(L) ,$$

from which the optimality of E_o or E_1 will follow.

Let Q be the set of all special points on all roads of Σ which contain σ_o, and assume $0 < \lambda < \lambda_{max}$. Consider any possible location P. If the flow from P to $L(\lambda)$ would move wholly along a single road of Σ, then the last special point it would encounter is either E_o or E_1, which both lie in Q. If not, the flow enters the <u>last</u> road it traverses at some interchange which is a point of Q. So eq(4) becomes

$$c(P,L(\lambda)) = \inf_{q \in Q} \{c(P,q) + c(q,L(\lambda))\} . \tag{6}$$

We will use the abbreviations

$$c_q(\lambda) = c(q,L(\lambda)) ,$$
$$\bar{c}_q(\lambda,P) = c(P,q) + c_q(\lambda) .$$

The appearance of $c_q(\lambda)$ in (6) refers to movement along a <u>single</u> road, and so by hypothesis this function is <u>concave</u> in λ. That is, for each triple $(\lambda^-, \lambda^+, t)$ with

$$0 < \lambda^- < \lambda^+ < \lambda_{max} , \qquad 0 < t < 1 , \tag{7}$$

the equality

$$c_q(t\lambda^- + (1-t)\lambda^+) \geq tc_q(\lambda^-) + (1-t)c_q(\lambda^+) \tag{8}$$

is valid. Since $c(P,q)$ is independent of λ, it follows that for each $q \epsilon Q$ and each P, $\bar{c}_q(\lambda,P)$ is concave in λ.

This will be used to prove[9] that for each P,

$$c(P,L(\lambda)) = \inf \{\bar{c}_q(\lambda,P): q \epsilon Q\} \tag{9}$$

is concave in λ. Let (λ^-,λ^+,t) be as in (7). For any $\delta > 0$, there is a $q \epsilon Q$ such that

$$c(P,L(t\lambda^- + (1-t)\lambda^+)) + \delta > \bar{c}_q(t\lambda^- + (1-t)\lambda^+,P) . \tag{10}$$

From (8) and (9), we have

$$\bar{c}_q(t\lambda^- + (1-t)\lambda^+,P) \geq tc(P,L(\lambda^-)) + (1-t)\, c(P,L(\lambda^+)) . \tag{11}$$

Combining (10) and (11), and recalling that δ was arbitrary, we have

$$c(P,L(t\lambda^- + (1-t)\lambda^+)) \geq tc(P,L(\lambda^-)) + (1-t)\, c(P,L(\lambda^+)) , \tag{12}$$

and so (by definition) $c(P,L(\lambda))$ is concave in λ.

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(9) For completeness, the proofs of known properties of concave functions are incorporated.

Inequality (12) can be rewritten as

$$c(P,L(t\lambda^- + (1-t)\lambda^+)) - tc(P,L(\lambda^-)) - (1-t)\ c(P,L(\lambda^+)) \geq 0\ . \qquad (13)$$

A non-negative integrand yields a non-negative integral with respect to any mass distribution μ ; applying this to (13), we obtain

$$\int c(P,L(t\lambda^- + (1-t)\lambda^+)d\mu(P) \geq t\int c(P,L(\lambda^-))d\mu(P) + (1-t)\int c(P,L(\lambda^+))d\mu(P)\ .$$

This is simply the assertion that the function

$$g_1(\lambda) = \int c(P,L(\lambda))d\mu(P)$$

is concave in λ . By eqs(2) and (5), $g(\lambda)$ is the sum of $g_1(\lambda)$ and a similar term for the facility-to-sinks flow. The latter can be proved concave by similar arguments, and so $g(\lambda)$ is itself concave. Thus, for (λ^-,λ^+,t) as in (7),

$$\begin{aligned} g(t\lambda^-+(1-t)\lambda^+) &\geq tg(\lambda^-) + (1-t)g(\lambda^+) \\ &\geq t\ \min(g(\lambda^-),\ g(\lambda^+)) + (1-t)\ \min(g(\lambda^-),g(\lambda^+)) \\ &= \min\ (g(\lambda^-),\ g(\lambda^+))\ . \end{aligned}$$

That is

$$g(\lambda) \geq \min\ (g(\lambda^-),\ g(\lambda^+)) \qquad \text{for}\quad \lambda^- \leq \lambda \leq \lambda^+\ .$$

By continuity this remains true for $\lambda^- = 0$ and $\lambda^+ = \lambda_{max}$ so that, as desired,

$$\min\ \{g(\lambda)\ :\ 0 \leq \lambda \leq \lambda_{max}\} = \min\ \{g(0),\ g(\lambda_{max})\}\ ,$$

from which it follows as shown above that E_0 or E_1 is an optimal location.

4. SOME EXTENSIONS

One extension of the theorem arises if its hypotheses are weakened to require concavity of transport costs along only <u>some</u> roads and road segments. The preceding proof can easily be adapted to establish the existence of at least one optimal location which is not an ordinary point on any of those roads and segments for which concavity is postulated.

The description of a second extension requires another definition. A function $h(\lambda)$, defined (say) on an interval $0 \leq \lambda \leq \lambda_{max}$, is called <u>strictly</u> <u>concave</u> in λ if for all $(\lambda^-, \lambda^+, t)$ obeying (7), the <u>strict</u> inequality

$$h(t\lambda^- + (1-t)\lambda^+) > th(\lambda^-) + (1-t)h(\lambda^+)$$

holds.

If $g(\lambda)$ is strictly concave in λ, the last part of the preceding proof can be modified to show that the minimum of $g(\lambda)$ occurs <u>only</u> for $\lambda = 0$ or $\lambda = \lambda_{max}$. It would follow that <u>only</u> special points can be optimal.

One might hope that the strict concavity of $g(\lambda)$ could be inferred if the hypothesis of concavity of transport costs along any road were strengthened to <u>strict</u> concavity. This will not always work. The functions $\bar{c}_q(\lambda, P)$ will each be strictly concave, but the infimum of an infinite collection of strictly concave functions need not be strictly concave (it can for example be linear), so that "strictness" can be lost in the transition to the function $c(P, L(\lambda))$ defined in (9).

It is however true that the minimum of a <u>finite</u> collection of strictly concave functions is strictly concave. If $\mathcal{U}$ is discrete, then the passage from $c(P, L(\lambda))$ to $g_1(\lambda)$ also preserves strict concavity. So for <u>discrete</u> models, <u>only</u> special points can be optimal if transport costs are strictly concave.

5. APPLICATION TO A CIRCULAR DISC MODEL

In this section we illustrate the previous material by examining in some detail a generalization of the circular disc model mentioned in Section 1. The mass distributions μ (for sources) and μ^* (for sinks) are assumed to coincide, as are the road systems Σ and Σ^* and the associated transport costs. Thus, for any location of the facility, the cost of the sources-to-facility flow will be exactly half that of the total (two-way) flow.

Polar coordinates (r,θ), with origin at the center of the disc, will be employed. The disc itself is characterized by $0 \leq r \leq 1$, i.e. its radius is (for simplicity) chosen as the unit of length. The mass density is assumed to fall off <u>exponentially</u> from the disc-center in a radially symmetric way [10], i.e.

$$d\mu(r,\vartheta) = K \exp(-ar)\, r\, dr\, d\vartheta$$

where K is a normalization constant, $r\, dr\, d\theta$ is the element of area in polar coordinates, and $a > 0$ is a parameter representing the degree of central concentration of the distribution. The limiting case $a=0$ corresponds to the <u>uniform</u> distribution studied by Witzgall (<u>op cit</u>), and our study generalizes his precisely in permitting $a > 0$.

The system of available roads is taken to consist of the periphery of the disk and (initially) all of its radii, regarded as bidirectional.

(10) This assumption is a fairly common one. See for example H.K. Weiss, The Distribution of Urban Population and an Application to a Servicing Problem, Operations Research 9 (1961), pp. 860-874.

The intersection of each radial road with the periphery is taken as a transfer point and also as an end-point of that road, the other end-point being the center of the disk. Unit costs of peripheral and radial travel are assumed to be constants (c_p and c_r , respectively); thus the transport cost along any road is a linear (hence, concave) function of distance travelled. It follows that our general principle indeed applies, to show that the optimal location for the facility is either at the center of the disc or on the periphery. By symmetry, we need consider only one among the peripheral locations, and we choose the one given by $\theta=0$.

First suppose the facility is located at the center of the disc. Then the least costly path from any point to the facility consists of a single radial segment, and so the total cost (for the two-way flow) is

$$\begin{aligned} f_c &= 2 \int_0^1 \int_{-\pi}^{\pi} c_r r d\mu(r,\theta) \\ &= 4\pi K c_r \int_0^1 r^2 \exp(-ar) dr \\ &= 4\pi K c_r a^{-3}[2-(2+2a+a^2)\exp(-a)] . \end{aligned}$$

Second, suppose the facility is located on the periphery, at the point given by $\theta=0$. There are really two cases to consider, depending on whether or not short-cuts through the central district are permitted, i.e. whether or not the center of the disc is assumed to be an interchange point between distinct radial roads. We assume for the moment that such short-cuts are forbidden, so that the least costly path from any point consists of a radial trip to the periphery, followed by a trip to the facility along a minor arc of the periphery. The corresponding total cost is

$$\begin{aligned} f_p &= 2 \int_0^1 \int_{-\pi}^{\pi} [c_r(1-r) + c_p\, |\theta|]\, d\mu(r,\theta) \\ &= 4\, Kc_r \int_0^1 (1-r)r \exp(-ar)dr \\ &\quad + 4Kc_p \int_0^1 r \exp(-ar) \int_0^{\pi} \theta\, d\theta\, dr \\ &= 4\pi\, Kc_r a^{-3}[(a-2) + (a+2) \exp(-a)] \\ &\quad + 2\pi^2\, Kc_p a^{-2}[1-(1+a) \exp(-a)]\ . \end{aligned}$$

Our objective is to determine the "critical value" $\tau^* = \tau^*(a)$, of the ratio

$$\tau = c_p/c_r\ , \tag{14}$$

which divides the cases for which the peripheral location of the facility is optimal from those for which the central location is optimal, i.e.

$$f_p < f_c \quad \text{if} \quad \tau < \tau^*(a)\ ,$$

$$f_p > f_c \quad \text{if} \quad \tau > \tau^*(a)\ .$$

Thus τ^* can be found by equating f_p to f_c , and solving for τ . The result (for the case of forbidden short-cuts) is

$$\tau^*(a) = (2/\pi)[(4-a)\exp(a) - (4+3a+a^2)]/a[\exp(a) - (1+a)]\ , \tag{15}$$

which for small _a_ has the form

$$\tau^*(a) = (2/\pi)[1+o(a)]/[3+a+o(a)] \ .$$

Thus $\tau^*(0) = 2/3\pi = 0.212^+$, in conformance with Witzgall's result for the uniform distribution.

Increasing values of _a_ correspond to greater central concentration, and therefore to a greater tendency for the central location to be optimal. We should therefore expect $\tau^*(a)$ to be a _decreasing_ function. This can indeed be verified analytically (details omitted), and is also indicated by the second column of Table 1. It is especially interesting to note that $\tau^*(a)$ turns _negative_ at roughly $a=2.7$, representing a degree of concentration at (and beyond) which _subsidization_ of circumferential travel would be required to overcome the cost advantage of the central location.

Next we assume that short-cuts through the center are _permitted_. It is easily seen that the least-cost path from any point (r,θ) to the facility is either the same one assumed in the previous discussion, or the one involving a radial trip to the center followed by a radial trip along the ray $\theta=0$ to the periphery. The former is less costly than the latter if and only if

$$c_r(1-r) + c_p\,|\theta| < c_r\,(1+r) \ ,$$

i.e. if and only if

TABLE 1: Values(*) of $T^*(a)$

	$T^*(a)$	
a	A. Short cuts through center forbidden	B. Short cuts through center permitted
0.05	.208	.214
0.1	.205	.210
0.2	.198	.202
0.3	.191	.194
0.4	.183	.186
0.5	.176	.179
0.6	.168	.171
0.7	.161	.163
0.8	.153	.155
0.9	.145	.147
1.0	.137	.139
1.2	.121	.123
1.4	.105	.106
1.6	.089	.090
1.8	.073	.073
2.0	.056	.057
2.3	.032	.032
2.6	.007	.007
2.7	-.001	.000

(*) Recall that $T^*(a)$ is the value of c_p/c_r for which the transport costs associated with peripheral and central facility locations are equal.

where

$$F(x,a) = ax + x(4+3a+a^2)\exp(-a) - x^2(1+a)\exp(-a)$$
$$- [6-6\exp(-x) - 2x\exp(-x)] \ . \tag{18}$$

Before considering this equation in detail, we will dispose of the remaining case $\tau > 2/\pi$, corresponding to $x > a$. In this case the short-cut <u>fails</u> to be best if and only if

$$\tau|\theta|/2 \leq r \leq 1 \ ,$$

which can only occur for $|\theta| \leq 2/\tau$. Thus the total cost is

$$f_p = 2 \int_0^{2/\tau} \int_0^{\tau\theta/2} c_r(1+r)d\mu(r,\theta)$$

$$+2 \int_0^{2/\tau} \int_{\tau\theta/2}^1 [c_r(1-r)+c_p\theta]d\mu(r,\theta)$$

$$+2 \int_{2/\tau}^{\pi} \int_0^1 c_r(1+r)d\mu(r,\theta)$$

$$= 2 Kc_r \{\int_0^{\pi} \int_0^1 r \exp(-ar)dr \ d\theta$$

$$+ \int_0^{2/\tau} \int_0^{\tau\theta/2} r^2 \exp(-ar)dr \ d\theta$$

$$+ \int_0^{2/\tau} \int_{\tau\theta/2}^1 (\tau\theta r-r^2)\exp(-ar)dr \ d\theta$$

$$+ \int_{2/\tau}^{\pi} \int_0^1 r^2 \exp(-ar)dr \ d\theta\} \ .$$

We can also write

$$f_c = 2\,Kc_r\,\{\int_0^{2/\tau}\int_0^{\tau\theta/2} r^2\exp(-ar)dr\,d\theta$$

$$+\int_0^{2/\tau}\int_{\tau\theta/2}^{1} r^2\exp(-ar)dr\,d\theta$$

$$+\int_{2/\tau}^{\pi}\int_0^{1} r^2\exp(-ar)dr\,d\theta\}\ ,$$

and so

$$(f_p-f_c)/2Kc_r = \int_0^{\pi}\int_0^{1} r\exp(-ar)dr\,d\theta$$

$$+\int_0^{2/\tau}\int_{\tau\theta/2}^{1}(\tau\theta r-2r^2)\exp(-ar)dr\,d\theta$$

$$-\int_0^{1}\int_0^{\pi} r\exp(-ar)d\theta\,dr$$

$$+\int_0^{1}\int_0^{2r/\tau}(\tau\theta r-2r^2)\exp(-ar)d\theta\,dr$$

$$=\int_0^{1}\{\pi+\int_0^{2r/\tau}(\tau\theta-2r)d\theta\}\,r\exp(-ar)dr$$

$$=\int_0^{1}\{\pi-2r^2/\tau\}\,r\exp(-ar)dr\ ,$$

which has a <u>positive</u> integrand. It follows that for $\tau > 2/\pi$, a central location is always preferable.

Thus it suffices to consider (17) and (18), for $x \leq a$. Here (17) has an extraneous root x=0 for all $\underline{a}$, arising from a multiplication by τ in simplifying to reach (16). The remaining root $x^*(a)$ was determined numerically, and then

$$\tau^*(a) = 2x^*(a)/\pi a$$

was calculated[11]. It appears as the third column in Table 1. We see that the availability of the short-cut has no great effect on $\tau^*(a)$; it is again a decreasing function which turns negative at approximately a=2.9.

So far our illustrations have dealt exclusively with a continuous mass distribution. To help assess the consequences of such an idealization, we conclude by considering a more discrete version of the circular model. Here the roads consist of N radial "spokes" uniformly spread around the disk, lying along the rays

$$\vartheta = k(2\pi/N) , \qquad k = 0,1,\ldots,N-1 ,$$

and also the circumference itself. Along each radius, the mass distribution (as before) decays exponentially as a function of r. For simplicity we treat only the case in which short-cuts through the center are forbidden.

By our general theorem, the optimal location for the facility will lie either at the center of the disk, or at the tip of one of the

(11) The assistance of Miss S. Rose (NBS Technical Analysis Division) in the algebraic and numerical calculations is gratefully acknowledged.

N spokes (which we take to be the one at $\theta=0$.) For a central facility, the total cost of the flow to and from it is

$$f_c = 2NKc_r \int_0^1 r^2 \exp(-ar)dr ,$$

which is $N/2\pi$ times the corresponding expression for the continuous model.

For a peripheral facility, the cost if N is <u>even</u> is

$$f_p = 2NKc_r \int_0^1 (1-r)r \exp(-ar)dr$$

$$+ 2Kc_p(2\pi/N)(N/2+2 \sum_{k=0}^{(N-2)/2} k) \int_0^1 r \exp(-ar)dr$$

$$= 2NKc_r \int_0^1 (r-r^2) \exp(-ar)dr$$

$$+ \pi NKc_p \int_0^1 r \exp(-ar)dr ,$$

which is again $N/2\pi$ times the corresponding expression for the continuous model. This difference from the continuous model is of course simply due to failure to "rescale" by changing K ; the conclusion (for <u>even</u> N) is that $\tau^*(a)$ is exactly the same as for the continuous model.

When N is <u>odd</u>, the cost for a peripheral facility becomes

$$f_p = 2NKc_r \int_0^1 (1-r) r \exp(-ar)dr$$

$$+ 2Kc_p(2\pi/N)2 \sum_{k=0}^{(N-1)/2} k \int_0^1 r \exp(-ar)dr$$

$$= 2NKc_r \int_0^1 (r-r^2)\exp(-ar)dr$$

$$+ \pi(N-N^{-1})Kc_p \int_0^1 r \exp(-ar)dr \ .$$

Thus

$$(f_p-f_c)/2NKc_r = \int_0^1 (r-2r^2)\exp(-ar)dr$$

$$+ (\pi\tau/2)(1-N^{-2}) \int_0^1 r \exp(-ar)dr \ .$$

For the continuous case, on the other hand, we find from previous formulas that

$$(f_p-f_c)/4\pi Kc_r = \int_0^1 (r-2r^2)\exp(-ar)dr$$

$$+ (\pi\tau/2) \int_0^1 r \exp(-ar)dr \ .$$

It follows that $\tau^*(a)$, for the present model, is $(1-N^{-2})^{-1}$ times its value for the continuous model.

These results give at least some reassurance, that using a continuous idealization in such problems leads neither to qualitative nor to unacceptable quantitative errors.