NATIONAL BUREAU OF STANDARDS REPORT

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MARKOV CHAIN ANALYSIS OF A SORTING MACHINE

by

Joel Levy

Technical Report

to

U.S. Post Office Department

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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Operations Research Section

Applied Mathematics Division

For

Data Processing Systems Division

Post Office Mechanization Project

IMPORTANT NOTICE

U. S. DEPARTMENT OF COMMERCE

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ABSTRACT

Markov Chain Analysis of a Sorting Machine

A method is developed for describing an automatic sorter which is being used simultaneously for sorting operations by several clerks. Some information concerning the behaviour of differently designed machines under varying conditions can be obtained from this description.

CONTENTS

1. BACKGROUND

For the motivation of the model to be discussed and its general formulation reference is made to "Analysis of Overflow Rate in a Sorting System" by B.K. Bender and A.J. Goldman (National Bureau of Standards Report # 7513, May 1, 1962). In particular see sections one thru four of that report. We translate the Bender-Goldman report into the notation of Markov matrices and its terminology. This is done so that the methods of Markov Chains can be applied.

2. LOCAL PICTURE OF MACHINE

The discussion will be confined to events at the intersection of a given operator belt and a given destination belt when the feed in on the destination belt and operator belt are sequentially independent, i.e. whether or not a package arrives at a given moment is independent of whether or not one arrived at previous moments. (We use "destination belt" for "cross belt" in the report cited above). If an operator-destination intersection at one of the intersection points from which the destination belt is coming has a buffer with positive capacity, the transitions of the local state of the machine are not governed by a Markov process. A more complete description of the machine than is given here would be necessary in order that the change of state may be probabilistically determined by the present state of the machine. A treatment of the more general case is in preparation.

3. SEQUENCE OF STEPS IN EVENT

The following sequence of steps in the machine operation is postulated:

a) the operator belt acts (it either puts out a package or it

does not put a package off at the intersection). The package, if one is put out, enters the destination tray, if it is empty, or enters the buffer, if the buffer is not filled to capacity and the destination tray is occupied, or overflows (this is recorded in step d),

b) new destination tray (empty or occupied) arrives,

c) buffer discharges, i.e. if buffer is occupied and tray arrives empty then one package moves from the buffer to the destination tray; if one of these conditions does not hold then the number of packages in the buffer is unchanged,

d) new state of the system (defined in the following section) is recorded and back to step a.

4. LABELLING OF STATES

The possible states and the labelling system that will be used to identify them are the following for $b > 0$:

$S_1 \equiv$ destination belt empty (this implies buffer empty)

$S_2 \equiv$ destination belt occupied, buffer empty

$S_3 \equiv$ buffer has one package

* * * * * * * * *

$S_{b+2} \equiv$ buffer has b packages (filled to capacity)

$S_{b+3} \equiv$ buffer has b-1 packages and overflow just occurred

$S_{b+4} \equiv$ buffer has b packages and overflow just occurred

For b=0, S_1 and S_2 are as above and the remaining states are:

$S_3 \equiv$ destination belt empty and overflow has just occurred

$S_4 \equiv$ destination belt occupied and overflow has just occurred.

5. LOCAL TRANSFORMATIONS

Four transformations on the local state of the machine are assumed. We denote them as follows:

$$T_0 = (0,0)$$

$$T_1 = (0,1)$$

$$T_2 = (1,0)$$

$$T_3 = (1,1)$$

In the ordered pairs above a zero on the right indicates that the operator belt did not put out a package, and a one on the right indicates the operator belt did put out a package. A zero on the left indicates the absence of a package on the newly arrived destination tray. A one on the left indicates the presence of a package on the newly arrived destination tray.

The mnemonic justification of the above notation is that the symbols indicating given operations are written from right to left in the order of application specified in section 3. Then the ordered pairs indicating the composition of the two operations are labelled with the integer whose binary representation they are.

6. OPERATION TRANSITION MATRIX (b=0)

The effect of the above transformations on the machine states is conveniently described by a matrix, i.e. a rectangular array of the symbols labelling the states of the machine; no algebraic structure is intended.

The matrix for b=0 is as follows:

	S_1	S_2	S_3	S_4
T_o	S_1	S_1	S_1	S_1
T_1	S_1	S_3	S_1	S_3
T_2	S_2	S_2	S_2	S_2
T_3	S_2	S_4	S_2	S_4

The matrix is to be read as follows: Given the machine is in state S_i and tranformation operation T_j is applied, the new state of the machine is the entry in column i, row j.

7. MARKOV MATRIX (b=0)

Assuming that transition operator T_j will be used with probability p_j, the Markov matrix appropiate to the process governing the machine states can be derived from the operation transition matrix. The entries of each row of the Markov matrix are gotten from the corresponding column of the operation transition matrix as follows:

1) operation transition matrix column i, row j, entry S_k, yields
2) Markov transition matrix row i, column k, entry p_j.

If a given column in the operation transition matrix has more than one entry S_k, the different p_j's are summed to form the appropriate Markov matrix entry. In other words, the probability of transition from S_i to S_k (i.e. the entry in row i and column k of the Markov transition matrix) is the sum of those p_j's for which T_j operating on S_i yields S_k (i.e. for which S_k appears in row j and column i of the operation transition matrix), and is zero if there are no such p_j's.

The Markov transition matrix for b=0 is:

	S_1	S_2	S_3	S_4
S_1	p_o+p_1	p_2+p_3	0	0
S_2	p_o	p_2	p_1	p_3
S_3	p_o+p_1	p_2+p_3	0	0
S_4	p_o	p_2	p_1	p_3

8. OPERATION TRANSITION MATRIX (b > 0)

For b > 0, the operation transition matrix has 4 rows and b+4 columns:

1) The first column is: $[S_1, S_1, S_2, S_2,]^T$,

2) The jth column, for j=2,...,b+1, consists of:

   $[S_{j-1}, S_j, S_j, S_{j+1}]^T$.

3) Column (b+3) is the same as column (b+1).

4) Columns (b+2) and (b+4) are:

   $[S_{b+1}, S_{b+3}, S_{b+2}, S_{b+4}]^T$.

	S_1	S_2					S_{b+1}	S_{b+2}	S_{b+3}	S_{b+4}
T_o	S_1	S_1	*				S_b	S_{b+1}	S_b	S_{b+1}
T_1	S_1	S_2		*			S_{b+1}	S_{b+3}	S_{b+1}	S_{b+3}
T_2	S_2	S_2			*		S_{b+1}	S_{b+2}	S_{b+1}	S_{b+2}
T_3	S_2	S_3				*	S_{b+2}	S_{b+4}	S_{b+2}	S_{b+4}

It is readily verified from the definitions of the states and of the operation transition matrix that when the number of local states in the machine is b+4 the column headed S_{b+3} (S_{b+4}) is identical with the column headed S_{b+1} (S_{b+2}) .

9. MARKOV MATRIX ($b > 0$)

The Markov matrix is constructed from the operation transition matrix as explained above in section 7, where the process was carried out for the case $b=0$. For $b > 3$ the corner entries will be as follows:

	S_1	S_2	S_b	S_{b+1}	S_{b+2}	S_{b+3}	S_{b+4}
S_1	p_o+p_1	p_2+p_3	0	0	0	0	0
S_2	p_o	p_1+p_2	0	0	0	0	0
S_3	0	p_o	0	0	0	0	0
*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*
S_{b+1}	0	0	p_o	p_1+p_2	p_3	0	0
S_{b+2}	0	0	0	p_o	p_2	p_1	p_3
S_{b+3}	0	0	p_o	p_1+p_2	p_3	0	0
S_{b+4}	0	0	0	p_o	p_2	p_1	p_3

10. FUNCTIONS OF b and $p_j (j=0,1,2,3)$

For a range of values of b, and $p_j (j=0,1,2,3)$ the following quantities are of interest:

1) the expected, i.e. mean, proportion of time that the machine will be in each of the states,

2) the expected proportion of time there will be an overflow,

3) the expected proportion of time the destination tray will leave the operator position intersection empty,

4) from the frequency function of (1):

   a) distribution function of the number of empty buffer positions

      (distribution function of the number of occupied buffer positions)

   b) frequency function of the number of empty buffer positions

      (frequency function of the number of occupied buffer positions)

   c) expected number of empty buffer positions.

11. DEFICIENCY MEASURES OF MACHINE

Of the quantities above, 2) the expected proportion of time there will be an overflow, 3) the expected proportion of time the destination tray will leave the operator position empty, and 4c) the expected number of empty buffer positions for the steady-state probability of the given Markov chain, are each a measure of deficiency of the machine for the given parameter values.

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