NATIONAL BUREAU OF STANDARDS REPORT

7580

ON THE TRACE-CLASS OF OPERATORS I.

Seminar Lectures,

July 1962

by

Robert Schatten

U. S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

THE NATIONAL BUREAU OF STANDARDS

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1. Introduction.

Let $\mathcal{H}$ be a complex Hilbert space. If T is a completely continuous operator on T, then $(T^*T)^{\frac{1}{2}}$ is not only completely continuous but also non-negative (hence Hermitean). If $\lambda_1, \lambda_2, \ldots$ represent all the non-zero eigenvalues of $(T^*T)^{\frac{1}{2}}$--each eigenvalue repeated in the sequence the number of times equal to its multiplicity--we may form the sum $\Sigma_i \lambda_i$ which--to indicate its dependence on the operator T--will be also denoted by $\tau(T)$. By definition, the trace-class (τc) consists of all those operators T for which $\tau(T)$ is finite. It is not a simple argument to prove that (τc) forms a linear space and that $\tau(T)$ defines there a norm. Incidentally, the resulting normed linear space turns out to be complete, that is, forms a Banach space. It is also true that the operators of finite rank form a dense set in (τc).

We remark that for an operator T, the above definition of $\tau(T)$ involves notions which are meaningful only in linear spaces with an inner product. It is of interest--and in fact of significance--to observe that for operators T of finite rank, $\tau(T)$ may be also expressed via concepts meaningful in a perfectly general Banach space. This means, for operators T on $\mathcal{H}$ of finite rank, we have two versions for $\tau(T)$. While one immediately carries over to arbitrary Banach spaces, the other does not yield to a straightforward generalization. This observation permits then to carry over to perfectly general Banach spaces the concept of a trace-class of operators. To define the last, one simply proceeds as follows: One considers the linear space of all the operators T of finite rank on the given Banach space. There one defines $\tau(T)$

via the concepts meaningful in general Banach spaces. The customary metric completion of the so resulting normed linear space furnishes then the desired trace-class of operators.

It remains thus to sketch how for operators T of finite rank, $\tau(T)$ may be expressed via concepts meaningful in any Banach space. The argument follows: If $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are elements in $\mathfrak{H}$, then

$$Tg = \sum_{i=1}^{n} (g, g_i) f_i$$

represents an operator T of finite rank which we shall also denote symbolically by $\sum_{i=1}^{n} f_i \otimes \bar{g}_i$. The converse is also true, that is, every operator T of finite rank admits many such representations of the form $\sum_{j=1}^{m} \varphi_j \otimes \bar{\psi}_j$; the number of terms m will vary of course with the representation of T. It can be shown that

$$\tau(T) = \inf \sum_{j=1}^{m} \|\varphi_j\| \, \|\psi_j\|$$

where the above infimum extends over the set of all sums corresponding to all possible representations of the operator T of finite rank.

The details of all that was said above will be outlined in later sections.

2. Problems to be considered.

The last formula expresses $\tau(T)$ in a language meaningful for any Banach space. This suggests the desirability to investigate the above infimum for general normed linear spaces (not necessarily inner product spaces). The details follow:

Let $\mathfrak{F}$ and $\mathfrak{G}$ stand for any two normed linear spaces and $\mathfrak{F}^*$, $\mathfrak{G}^*$ represent their conjugate spaces, that is, the corresponding spaces of additive and bounded, that is, continuous, linear functional on $\mathfrak{F}$ and $\mathfrak{G}$ respectively:

Elements in $\mathfrak{F}$ will be denoted by $f_1, f_2, \ldots$;
in $\mathfrak{G}$ will be denoted by $g_1, g_2, \ldots$;
in $\mathfrak{F}^*$ will be denoted by $F_1, F_2, \ldots$;
in $\mathfrak{G}^*$ will be denoted by $G_1, G_2, \ldots$;.

Observe that for a fixed pair of elements $G_o \in \mathfrak{G}^*$ and $f_o \in \mathfrak{F}$, the expression

$$Tg = G_o(g)f_o$$

defines an operator of rank 1 from $\mathfrak{G}$ into $\mathfrak{F}$. More generally, if $G_1, \ldots, G_n$ are in $\mathfrak{G}^*$ and $f_1, \ldots, f_n$ are in $\mathfrak{F}$, then

$$Tg = \sum_{i=1}^{n} G_i(g)f_i$$

represents an operator from $\mathfrak{G}$ into $\mathfrak{F}$ of rank at most n. The last operator we shall also denote by $\Sigma_{i=1}^{n} f_i \otimes G_i$; one calls then $\Sigma_{i=1}^{n} f_i \otimes G_i$ a representation of the operator T. It is not difficult to see that conversely, every operator T from $\mathfrak{G}$ into $\mathfrak{F}$ of finite rank has many such representations. With each representation $\Sigma_{i=1}^{n} f_i \otimes G_i$ of a given operator T of finite rank, one associates the number $\Sigma_{i=1}^{n} ||f_i|| \, ||G_i||$ and then defines

$$\tau(T) = \inf \sum_{i=1}^{n} || f_i || \, || G_i ||$$

where the infimum extends over the set of all possible representations of T. We prove below that the so defined $\tau(T)$ is a norm on the linear space of all operators from $\mathfrak{G}$ into $\mathfrak{F}$ of finite rank. The metric completion of the above normed linear space is then defined as the trace-class of operators from $\mathfrak{G}$ into $\mathfrak{F}$.

Similarly, if $f_1, \ldots, f_n$ are in $\mathfrak{F}$ and $g_1, \ldots, g_n$ are in $\mathfrak{G}$, the expression $\Sigma_{i=1}^{n} f_i \otimes g_i$ represents an operator T of finite rank from $\mathfrak{G}^*$ into $\mathfrak{F}$ whose defining equation is given by $T(G) = \Sigma_{i=1}^{n} G(g_i) f_i$. Moreover, every operator T of finite rank from $\mathfrak{G}^*$ into $\mathfrak{F}$ has many such representations.

<u>Our problem</u> is to furnish some "enlightening" information concerning $\tau(T) = \inf \Sigma_{i=1}^{n} || f_i || \, || g_i ||$ where the infimum is extended over the set of sums corresponding to all possible representations $\Sigma_{i=1}^{n} f_i \otimes g_i$ of the given operator T of finite rank. In other words, we are interested in some characterizations of the above infimum. Perhaps in particular Banach spaces, it is possible to express $\tau(T)$ <u>"directly in terms of T"</u> just as it was done in the case of eompletely continuous operators T on a Hilbert space. In the last, $\tau(T) = \sum_i \lambda_i$,where the λ_i represent all the non-zero eigenvalues of $(T^* T)^{\frac{1}{2}}$; each eigenvalue appearing in the last sum the number of times equal to its multiplicity.

At this stage one should ask for the following question: Assume that $f_1, \ldots, f_n$ are in $\mathfrak{F}$ and $g_1, \ldots, g_n$ are in $\mathfrak{G}$. Assume also that $f'_1, \ldots, f'_m$ are in $\mathfrak{F}$ and $g'_1, \ldots, g'_m$ are in $\mathfrak{G}$. When do the expressions $\Sigma_{i=1}^{n} f_i \otimes g_i$ and $\Sigma_{j=1}^{m} f'_j \otimes g'_j$ represent the same operator of finite rank from $\mathfrak{G}^*$ into $\mathfrak{F}$. To answer this question observe (writing $f_1 \otimes g_1 +, \ldots, + f_n \otimes g_n$ instead of $\Sigma_{i=1}^{n} f_i \otimes g_i$) that

1. $$f_1 \otimes g_1 + f_2 \otimes g_2 +, \ldots, + f_n \otimes g_n =$$
$$= f_{1'} \otimes g_{1'} + f_{2'} \otimes g_{2'} +, \ldots, + f_{n'} \otimes g_{n'}$$

where 1', 2', ..., n' is any permutation of the 1, 2, ..., n.

2a. $$f_1 \otimes (g_1 + \bar{g}_1) + f_2 \otimes g_2 + \ldots + f_n \otimes g_n =$$
$$= f_1 \otimes g_1 + f_1 \otimes \bar{g}_1 + f_2 \otimes g_2 + \ldots + f_n \otimes g_n$$

2b. $$(f_1 + \bar{f}_1) \otimes g_1 + f_2 \otimes g_2 + \ldots + f_n \otimes g_n =$$
$$= f_1 \otimes g_1 + \bar{f}_1 \otimes g_1 + f_2 \otimes g_2 + \ldots + f_n \otimes g_n$$

3. $$(a_1 f_1) \otimes g_1 + (a_2 f_2) \otimes g_2 + \ldots + (a_n f_n) \otimes g_n =$$
$$= f_1 \otimes (a_1 g_1) + f_2 \otimes (a_2 g_2) + \ldots + f_n \otimes (a_n g_n)$$

where $a_1, a_2, \ldots, a_n$ are arbitrary scalars.

It is not difficult to see that two expressions $\Sigma_{i=1}^{n} f_i \otimes g_i$ and $\Sigma_{j=1}^{m} f'_j \otimes g'_j$ define the same operator T of finite rank if and only if one can be derived from the other by a finite number of successive applications of the above relations 1, 2a, 2b, 3.

Theorem. Let $\mathfrak{F}$ and $\mathfrak{G}$ represent two normed linear spaces and $\mathfrak{R}$ stand for the linear space of all operators from $\mathfrak{G}^*$ into $\mathfrak{F}$ of finite rank. Then, the above defined function $\tau(T)$ represents a norm on $\mathfrak{R}$. Also, $\tau(T)$ has the "cross-property" that is, $\tau(T)$ coincides with the bound $||T||$ for all operators T of rank ≤ 1. The last condition characterizes $\tau(T)$ completely in the following sense: $\tau(T)$ is the greatest norm on $\mathfrak{R}$ having the cross-property.

Proof: Let $T \in \mathfrak{R}$ and $G \in \mathfrak{G}^*$. Then for any representation $\Sigma_{i=1}^{n} f_i \otimes g_i$ of T we have

$$||T(G)|| = ||\sum_{i=1}^{n} G(g_i)f_i|| \leq ||G|| \sum_{i=1}^{n} ||f_i|| \, ||g_i||$$

Thus,

$$||T(G)|| \leq ||G|| \, \tau(T)$$

The last inequality implies

$$||T|| \leq \tau(T)$$

for all operators T in $\mathfrak{R}$.

(i). If $T = 0$, then obviously $\tau(T) = 0$.

If $T \neq 0$, then $0 < ||T|| \leq \tau(T)$, and thus $\tau(T) > 0$.

(ii.) It is also clear that for any scalar a we have ~~$\tau(T)$~~ $\tau(aT)$ $= |a| \tau(T)$.

(iii.) To prove that $\tau(T_1 + T_2) \leq \tau(T_1) + \tau(T_2)$ for any two operators T_1 and T_2 in $\mathfrak{R}$, we argue as follows:

Let $\epsilon > 0$ be given. Choose a representation $\Sigma_{i=1}^{n} f_i \otimes g_i$ of T_1, such that

$$\sum_{i=1}^{n} ||f_i|| \, ||g_i|| \leq \tau(T_1) + \frac{\epsilon}{2}$$

Similarly, we can find a representation $\Sigma_{j=1}^{m} f'_j \otimes g'_j$ of T_2 such that

$$\sum_{j=1}^{m} \| f'_j \| \, \| g'_j \| \leq \tau(T_2) + \frac{\varepsilon}{2}$$

But then $\Sigma_{i=1}^{n} f_i \otimes g_i + \Sigma_{j=1}^{m} f'_j \otimes g'_j$ is a representation for $T_1 + T_2$, and therefore

$$\tau(T_1 + T_2) \leq \sum_{i=1}^{n} \| f_i \| \, \| g_i \| +$$

$$+ \sum_{j=1}^{m} \| f'_j \| \, \| g'_j \| \leq \tau(T_1) + \tau(T_2) + \varepsilon .$$

(iv.) Assume now that T is of rank ≤ 1. Then T admits a representation in the form $T = f \otimes g$ and thus,

$$\tau(T) \leq \| f \| \, \| g \| = \| T \| .$$

We already know that $\| T \| \leq \tau(T)$ holds. Thus $\tau(T) = \| T \|$.

Finally, suppose that $\alpha(T)$ is a norm on $\mathcal{R}$ satisfying the condition $\alpha(f \otimes g) = \| f \| \, \| g \|$. Let $T \in \mathcal{R}$. For every representation $\Sigma_{i=1}^{n} f_i \otimes g_i$ of T we have

$$\alpha(T) = \alpha\left(\sum_{i=1}^{n} f_i \otimes g_i \right) \leq$$

$$\leq \sum_{i=1}^{n} \alpha(f_i \otimes g_i) = \sum_{i=1}^{n} \| f_i \| \, \| g_i \|$$

Thus, $\alpha(T) \leq \tau(T)$.

3. The trace-class of operators on a Hilbert space.

We start with some preliminaries.

The setting for our discussion is a fixed Hilbert space $\mathcal{H}$, that is, a complex inner product space for which the resulting metric space is complete. The inner product will be denoted by (f,g) and $\|f\| = (f,f)^{\frac{1}{2}}$ will stand for the norm that goes with it.

A <u>basis</u> is by definition a maximal orthonormal family of vectors $\{\varphi_i\}$. It follows from Zorn's Lemma (which is equivalent to the axiom of choice) that in any Hilbert space there is a basis. Incidentally, the cardinal number corresponding to any two bases is always the same and defines the dimension of the space.

Clearly, $\{\varphi_i\}$ is a basis if and only if,

$$\|f\|^2 = \sum_i |(f,\varphi_i)|^2$$

for all f in $\mathcal{H}$.

A continuous linear transformation A defined everywhere on $\mathcal{H}$ with values in $\mathcal{H}$ is termed an <u>operator</u>. Continuity is equivalent to boundedness, that is

$$\|A\| = \sup_{\|f\| \leq 1} \|Af\| < +\infty$$

The adjoint of A will be denoted by A^*; by definition A^* is the unique operator which satisfies the condition

$$(Af,g) = (f,A^*g)$$

for all pairs f,g.

<u>Theorem 1</u>: Let A be a given operator and $\{\varphi_i\}$, $\{\psi_j\}$ be any two bases. Then

$$\sum_i \| A\varphi_i \|^2$$

$$\sum_i \sum_j |(A\varphi_i, \psi_j)|^2$$

$$\sum_j \| A^* \psi_j \|^2$$

represent the same (finite or infinite) value; we denote the last by $|A|^2$.

Proof. For a fixed i, the "Pythagorean theorem" implies

$$\| A\varphi_i \|^2 = \sum_j |(A\varphi_i, \psi_j)|^2$$

and thus,

$$\sum_i \| A\varphi_i \|^2 = \sum_i \sum_j |(A\varphi_i, \psi_j)|^2$$

Replacing in the last A by A^*, $\{\varphi_i\}$ by $\{\psi_j\}$, and $\{\psi_j\}$ by $\{\varphi_i\}$ one gets

$$\sum_j \| A^* \psi_j \|^2 = \sum_j \sum_i |(A^* \psi_j, \varphi_i)|^2 =$$

$$= \sum_i \sum_j |(\psi_j, A\varphi_i)|^2 = \sum_i \sum_j |(A\varphi_i, \psi_j)|^2.$$

Thus, the values of the above three sums are the same. Observe finally that,

$$\sum_i \| A\varphi_i \|^2 = \sum_j \| A^* \psi_j \|^2 =$$

$$= \sum_j \| A^{**} \psi_j \|^2 = \sum_j \| A \psi_j \|^2$$

This concludes the proof.

<u>Definition</u>. For a given operator A, let $|A|^2$ stand for the common value of the three "sums" determined in Theorem 1.

<u>Theorem 2</u>. For any operator A, we have

$$||A|| \leq |A|$$

Proof. It is sufficient to prove that $||A\varphi|| \leq |A|$ for any vector φ such that $||\varphi|| = 1$. This is easy: Choose a basis $\{\varphi_i\}$ with φ as one of its elements. We then have

$$||A\varphi||^2 \leq \sum_i ||A\varphi_i||^2 = |A|^2.$$

<u>Theorem 3</u>. For any two operators A and B we have:

$$|A + B| \leq |A| + |B| \quad .$$

Proof. It is sufficient to consider the case in which both $|A| < +\infty$ and $|B| < +\infty$. Choose a fixed basis $\{\varphi_i\}$. Then

$$|A + B| = (\sum_i ||(A + B)\varphi_i||^2)^{\frac{1}{2}} \leq$$

$$(\sum_i (||A\varphi_i|| + ||B\varphi_i||)^2)^{\frac{1}{2}} \leq$$

$$(\sum_i ||A\varphi_i||^2)^{\frac{1}{2}} + (\sum_i ||B\varphi_i||^2)^{\frac{1}{2}} = |A| + |B|$$

<u>Theorem 4</u>. Let $\mathcal{B}$ stand for the set of all operators A for which $|A| < +\infty$. With the obvious definitions of addition and scalar multiplication, $\mathcal{B}$ is a complex linear space. There $|A|$ represents a norm.

Proof. This is an immediate consequence of the preceeding discussion.

Remark: Incidentally, the normed linear space $\mathcal{B}$ is also complete, hence a Banach space. Considering however, $\mathcal{B}$ only as a linear set of operators, and defining the bound $\|A\|$ of an operator as a new norm on $\mathcal{B}$, then the resulting normed linear space is not complete.

Remark: Incidentally, $|A|$ also satisfies the following conditions:

(i) $|A| = |A^*|$

(ii) For any operator B, we have

$$|AB| \leq |A| \; \|B\| \leq |A| \cdot |B|$$

As a consequence of (ii), $\mathcal{B}$ is also an algebra. In fact, $\mathcal{B}$ is an ideal in the algebra of all operators.

Definition: An operator A in $\mathcal{B}$ is commonly referred to as one which belongs to the E. Schmidt-class and $|A|$ is said to define its Hilbert-Schmidt norm.

Remark: That the Hilbert-Schmidt norm of an operator is always not smaller that its bound was stated in Theorem 2. Also, it can be readily verified that the Hilbert-Schmidt norm is a cross-norm; that is, $|A| = \|A\|$ wherever A is an operator of rank ≤ 1.

At this point, the following comment is in order: Let L_2 stand for the Hilbert space of all complex-valued Lebesgue measurable functions f(x) defined on the interval $0 \leq x \leq 1$ for which $|f(x)|^2$ is integrable; two functions being considered identical if and only if they differ on a set of measure zero. There, the linear operations are the usual ones in function spaces; the inner product is represented by

$$(f,g) = \int f(x)\overline{g(x)}dx$$

Similarly, let $\mathcal{L}_2$ represent the Hilbert space of complex-valued measurable functions $K(x,y)$ defined on $0 \leq x, y \leq 1$ for which $|K(x,y)|^2$ is integrable; the inner product being given by

$$(H,K) = \iint H(x,y)\overline{K(x,y)}dxdy.$$

One observes that if $K_1(x,y)$ and $K_2(x,y)$ are both in $\mathcal{L}_2$, then the function

$$H(x,y) = \int K_1(x,z)K_2(z,y)dz$$

is also in $\mathcal{L}_2$ and

$$\iint |H(x,y)|^2 dxdy \leq \iint |K_1(x,y)|^2 dxdy \cdot \iint |K_2(x,y)|^2 dxdy$$

Thus, if $H(x,y)$ is defined as the "product" of $K_1(x,y)$ and $K_2(x,y)$, the space $\mathcal{L}_2$ turns out to be an algebra.

Let $K(x,y)$ be a fixed element in $\mathcal{L}_2$. For $f(x)$ in L_2,

$$\int K(x,y)f(y)dy$$

is then defined for almost all x in $0 \leq x \leq 1$ and represents a function $g(x)$, again in L_2. It turns out that the equation

$$g(x) = \int K(x,y)f(y)dy$$

defines an operator K on L_2 which belongs to the Schmidt-class (of operators on L_2) and

$$|K| = \left(\iint |K(x,y)|^2 dxdy\right)^{\frac{1}{2}}.$$

Moreover, every operator on L_2 in the Schmidt-class is obtained in such a manner. This one-to-one correspondence between $\mathcal{L}_2$ and the Schmidt-class of operators on L_2 preserves addition, scalar-multiplication, products, and the norm. This means we have the following:

Theorem 5: The Schmidt-class of operators on L_2 and the Hilbert space $\mathfrak{L}_2$ are congruent not only as Banach spaces but also as Banach algebras.

Definition: An operator A is termed Hermitean if $A^* = A$, and non-negative in symbol $A \geq 0$ if $(Af,f) \geq 0$ for all f in $\mathfrak{H}$. Since $\mathfrak{H}$ is a space with complex scalars, every non-negative operator is necessarily Hermitean. It is also clear that for any operator A the products A^*A and AA^* are ≥ 0. It can be shown that for every operator $A \geq 0$, there is a unique operator $B \geq 0$ such that $A = B^2$. It is customary to write $[A]$ for the unique ≥ 0 operator such that $A^*A = [A]^2$. Clearly, $[A] = [A^*]$ if and only if A is normal, that is, $A^*A = AA^*$.

Theorem 6: Let A be a given operator and $\{\varphi_i\}$ a basis. Then

$$\sum_i([A]\varphi_i,\varphi_i)$$

is independent on the chosen basis.

Proof. Since $[A] \geq 0$, there is a unique operator $B \geq 0$ such that $[A] = B^2$. Now,

$$([A]\varphi_i,\varphi_i) = (B^2\varphi_i,\varphi_i) = (B\varphi_i,B\varphi_i) = ||B\varphi_i||^2$$

An application of Theorem 1 concludes the proof.

Definition: The operators A for which the sum in the Theorem 6 if finite, form the trace-class (τc).

<u>Theorem 7</u>: Let (τc) denote the class of all operators A for which

$$\sum_i ([A]\varphi_i, \varphi_i) < +\infty$$

for a fixed basis $\{\varphi_i\}$. With the obvious definition of addition and scalar multiplication (τc) is linear space. The last will be normed if the above sum represents the norm of an operator A. Moreover, the resulting normed linear space is complete, hence a Banach space: it contains the operators of finite rank as a dense subset. The operators in trace-class necessarily belong to the Schmidt-class. Moreover (τc) is a (two-sided) ideal in the algebra of all operators and a Banach algebra under its own norm.

An operator A is termed completely continuous if for every bounded sequence of vectors f_1, f_2, f_3, ... the transformed sequence Af_1, Af_2, Af_3, ... contains a subsequence convergent (in the strong sense) to some element of the space.

Theorem 8: Let $\mathcal{C}$ represent the set of all completely continuous operators on $\mathcal{H}$. With the conventional definition of sum, product, and scalar multiple for operators, $\mathcal{C}$ is an algebra in fact a two-sided ideal in the algebra of all operators on $\mathcal{H}$. It is readily seen that the algebra $\mathcal{C}$ will be normed if the bound of an operator stands for its norm. Also, the resulting normed algebra is complete, that is, is a Banach algebra.

We are about to exhibit a representation characteristic for the completely continuous operators. First, however, we state some preliminaries.

<u>The polar decomposition</u>: Let A be an operator. Then there exists an operator W isometric on the closure of the range of [A] and equal to 0 on its orthogonal complement, for which the following relations hold:

$$A = W[A]$$

$$[A] = W^* A \quad .$$

The above representation is unique in the following sense: If $A = W_1 B_1$ where $B_1 \geq 0$ and W_1 is isometric on the range of B_1 and equal to 0 on its orthogonal complement, then $B_1 = [A]$ and $W_1 = W$. In the case A is of finite rank we may assume that W is unitary (not unique however).

The last theorem implies that A is completely continuous if and only if [A] is such.

<u>Definition</u>: If φ and ψ are two elements of $\mathfrak{H}$ let

$$\varphi \otimes \overline{\psi}$$

represent the operator whose defining equation is given by

$$(\varphi \otimes \overline{\psi})f = (f, \psi)\varphi$$

for all f in $\mathfrak{H}$.

If $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are any 2n elements and $\mu_1, \ldots, \mu_n$ are any n scalars, the meaning of the symbol

$$\sum_{i=1}^{n} \mu_i f_i \otimes \overline{g_i}$$

is clear; it represents an operator of rank at most n.

One readily verifies the following simple relations:

(1) $(f_1 + f_2)\otimes\overline{g} = (f_1\otimes\overline{g}) + (f_2\otimes\overline{g})$

$f\otimes(\overline{g_1 + g_2}) = (f\otimes\overline{g}_1) + (f\otimes\overline{g}_2)$

(2) $(\alpha f)\otimes\overline{g} = \alpha(f\otimes\overline{g})$

$f\otimes(\overline{\alpha g}) = \overline{\alpha}(f\otimes\overline{g})$

for any scalar α.

(3) $(f_1\otimes\overline{g}_1)(f_2\otimes\overline{g}_2) = (f_2,g_1)(f_1\otimes\overline{g}_2)$

(4) $A(f\otimes\overline{g}) = (Af)\otimes\overline{g}$

for any operator A.

(5) $(f\otimes\overline{g})^* = g\otimes\overline{f}$.

Thus, if

$$A = \sum_{i=1}^{n} \mu_i f_i\otimes\overline{g}_i \text{ then } A^* = \sum_{i=1}^{n} \overline{u}_i g_i\otimes\overline{f}_i$$

In general, analogous infinite sums have no meaning. However, the following theorem will be useful for all our purposes:

<u>Theorem 9</u>:: Let $\{\omega_i\}$ and $\{\psi_i\}$ stand for any two orthonormal families of vectors and $\{\mu_i\}$ a <u>bounded</u> family of complex numbers indexed by the same set of subscripts. Then,

$$Tf = \sum_i \mu_i (f, \psi_i)\omega_i$$

is meaningful for every f in $\mathcal{H}$ and represents an operator T which we shall also denote by

$$\sum_i \mu_i \omega_i \otimes \overline{\psi}_i$$

The bound of T is given by

$$||T|| = \sup_i |u_i|$$

Proof: We have:

$$\sum_i |\mu_i (f, \psi_i)|^2 \leq$$

$$\sup_i |\mu_i| \sum_i |(f, \psi_i)|^2 \leq \|f\|^2 \sup |\mu_i|$$

Thus Tf is meaningful and

$$\|Tf\| \leq \|f\|^2 \sup |\mu_i|$$

This of course means that $\|T\| \leq \sup |\mu_i|$. On the other hand, we have $\|T\psi_i\| = |\mu_i|$. And therefore $\|T\| \geq \sup |\mu_i|$. Thus, $\|T\| = \sup |\mu_i|$.

The fundamental theorem of algebra implies that every operator on a finite dimensional complex inner product space, has eigenvalues. In the case the inner product space is infinite dimensional, it is always possible to construct (even completely continuous) operators which do not have a single eigenvalue. It may be added that thus far it is not known whether every operator T on a Hilbert space $\mathfrak{H}$ has at least a proper invariant subspace. We mean hereby, a subspace $\mathfrak{M}$ such that $0 \neq \mathfrak{M} \neq \mathfrak{H}$ and $T(\mathfrak{M}) \subseteq \mathfrak{M}$.

The story is however quite different when one deals with Hermitean operators. It is well known that every Hermitean operator A on an n-dimensional space $\mathfrak{H}_n$ always admits an orthonormal basis of eigenvectors. We mean hereby, that there is an orthonormal basis $\varphi_1, \ldots, \varphi_n$ in $\mathfrak{H}_n$ for which

$$A\varphi_i = \lambda_i \varphi_i \text{ for } 1 \leq i \leq n \quad .$$

Thus, A may be written in the form

$$A = \sum_{i=1}^{n} \lambda_i \varphi_i \otimes \overline{\varphi_i} \quad .$$

The corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are necessarily real.

Conversely, every sum of the above form with $\varphi_1, \ldots, \varphi_n$ orthonormal and $\lambda_1, \ldots, \lambda_n$ real, represents a Hermitean operator.

The infinite-dimensional extension of the above result which follows is also well known:

Theorem 10: Every Hermitean completely continuous operator A on a Hilbert space admits in that space an orthonormal basis of eigenvectors. The corresponding non-zero (necessarily real) eigenvalues are each of finite multiplicity and may be arranged either in a finite or denumerably infinite sequence $\lambda_1, \lambda_2, \ldots$ (each non-zero eigenvalue being repeated in the sequence the number of times equal to its multiplicity) such that $\lambda_i \to 0$. If $\varphi_1, \varphi_2, \ldots$ is a corresponding orthonormal sequence of eigenvectors (that is, $A\varphi_i = \lambda_i \varphi_i$ for $i = 1, 2, \ldots$), then

$$A = \sum_i \lambda_i \varphi_i \otimes \overline{\varphi_i} \quad .$$

Conversely, every sum of the above form, that is, with $\{\varphi_i\}$ orthonormal, λ_i real and $\lambda_i \to 0$, represents a completely continuous Hermitean operator.

The above yields a representation characterizing the completely continuous Hermitean operators. To obtain an analogous representation valid for all completely continuous operators, one makes use of the polar decomposition for operators.

Theorem 11: An operator A is completely continuous if and only if it admits a "polar representation"

$$A = \sum_i \lambda_i \varphi_i \otimes \overline{\psi_i}$$

where both $\{\varphi_i\}$ and $\{\psi_i\}$ are orthonormal sequences and the λ_i's are positive. The sum has either a finite or denumerably infinite number of terms. In the last case, we have also $\lambda_i \to 0$. The above representation is unique in the sense that the λ_i's are necessarily all the positive proper values (each represented in the sequence $\{\lambda_i\}$ the number of times equal to its multiplicity) of [A].

Proof: Since A is completely continuous, the same is true for [A]. Thus,

$$[A] = \sum_i \lambda_i \psi_i \otimes \overline{\psi_i} \quad .$$

Now, if A = W[A] is its polar decomposition, then W is isometric on the closed linear manifold determined by $\{\psi_i\}$. Thus, $\{W\psi_i\}$ is also an orthonormal family. Put $W\psi_i = \varphi_i$. Then

$$A = W[A] = W(\sum_i \lambda_i \psi_i \otimes \overline{\psi_i}) =$$

$$= \sum_i \lambda_i (W\psi_i) \otimes \overline{\psi_i} = \sum_i \lambda_i \varphi_i \otimes \overline{\psi_i} \quad .$$

Corollary: For an operator T of finite rank

$$\tau(T) = \sum_{j=1}^{m} \|f_j\| \, \|g_j\|$$

where the infimum is taken over the set of all numbers corresponding to all possible representations $\sum_{j=1}^{m} f_j \otimes \overline{g_j}$ of T.

Proof. Since $\tau(T)$ is a crossnorm, for any representation $\Sigma_{j=1}^{m} f_j \otimes \bar{g}_j$ of T we have $\tau(T) \leqq \Sigma_{j=1}^{m} \tau(f_j \otimes \bar{g}_j) = \Sigma_{j=1}^{m} \|f_j\| \; \|g_j\|$. However, if $\Sigma_{i=1}^{n} \lambda_i \varphi_i \otimes \overline{\psi_i}$ is a polar representation of T, then

$$T^*T = (\sum_{i=1}^{n} \lambda_i \psi_i \otimes \overline{\varphi_i})(\sum_{i=1}^{n} \lambda_i \varphi_i \otimes \overline{\psi_i}) =$$

$$= \sum_{i=1}^{n} \lambda_i^2 \psi_i \otimes \overline{\psi_i}$$

and $[T] = \Sigma_{i=1}^{n} \lambda_i \psi_i \otimes \overline{\psi_i}$. Now, $[T]\varphi = 0$ whenever φ is orthogonal to $\psi_1, \ldots, \psi_n$. Thus,

$$\tau(T) = \sum_{i=1}^{n} ([T]\psi_i, \psi_i) =$$

$$= \sum_{i=1}^{n} (\lambda_i \psi_i, \psi_i) = \sum_{i=1}^{n} \lambda_i =$$

$$= \sum_{i=1}^{n} \|\lambda_i \varphi_i\| \; \|\psi_i\| \quad .$$

<u>Theorem 12</u>: Every operator in the trace-class is necessarily in the Schmidt-class. Every operator in the Schmidt-class is completely continuous.

Let A be a completely continuous operator and $\sum_i \lambda_i \varphi_i \otimes \overline{\psi_i}$ its polar form. Then A is in the Schmidt-class if and only if $\sum_i \lambda_i^2 < +\infty$; we have $|A| = (\sum_i \lambda_i^2)^{\frac{1}{2}}$. The operator A is in the trace-class if and only if $\Sigma_i \lambda_i < +\infty$; we have $\tau(A) = \Sigma_i \lambda_i$.

U. S. DEPARTMENT OF COMMERCE
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