NATIONAL BUREAU OF STANDARDS REPORT

7513

ANALYSIS OF OVERFLOW RATE

IN

A SORTING SYSTEM

by

B.K. Bender

A.J. Goldman

Technical Report

to

U.S. Post Office Department

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U. S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

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B.K. Bender and A.J. Goldman

Operations Research Section

Applied Mathematics Division

For

Data Processing Systems Division

Post Office Mechanization Project

To

U.S. Post Office Department

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ABSTRACT

A method is developed for determining the proportion of parcels which "overflow" in an automatic sorter, in terms of the capacities of the buffers employed, the number of operator lanes, and other system parameters. The treatment extends previous ones by permitting consideration of arbitrary buffer sizes, and by yielding time-varying as well as steady state solutions. Numerical results for the steady state case are given, and display a drastic increase in overflow rate when the number of operator lanes exceeds a critical value.

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1. INTRODUCTION

This report treats a question arising in connection with the parcel post sorter being designed for the Post Office Department by Food Machinery and Chemical Corporation (FMC). A queueing approach to the problem, assuming random (Poisson) arrival and service times, was discussed in Volume 9 of FMC's 3/31/61 Gateway Project Report. Under the assumptions of uniform arrival and service rates, plus a hypothesis of equal distribution of parcels among cross belts, the "steady-state no-buffer" case was solved by John Weaver of FMC in the 9/26/60 Technical Memorandum No. STM-8, "Analysis of the Jamming Phenomenon in the Matrix Sorting Machine".

Using the same "uniformity" assumptions as in the Technical Memorandum cited, we shall describe a solution method valid for

(a) arbitrary buffer sizes,

(b) the time-varying case as well as the steady-state case, and

(c) arbitrary distribution of parcels among cross belts.

The generality of the solution is perhaps greater than seems necessary now, but may prove useful in future work. It seems appropriate to point out explicitly that the methods employed are purely analytic (i.e., no simulation is required) and also mainly elementary; only simple algebra and probability theory, together with mathematical induction, are used.

A brief description of the problem is given in Section 2, while Section 3 contains a detailed account of the "mathematical model" to be employed. The presentation of the general solution method is begun in Section 4, but the discussion is then narrowed to a special "steady state" situation believed to be of particular interest. The resulting formulas, which represent a generalization to arbitrary buffer size of those given in FMC's memo, are used to derive numerical results for realistic values of the of the relevant parameters. These results show that the overflow rate increases drastically when the number of operator lanes reaches a certain level (determined by the parameters), so that additional lanes would accomplish relatively little sorting. The reader's attention is especially directed to this material. Numerical results for a wider range of cases are being calculated, and will be reported separately.

In Section 5 the description of the general solution method is resumed and completed. The method is illustrated by three applications. The first sketches the details for treating a small buffer, with room for only one parcel. The second refers to a steady state situation in which cross belts move faster than operator lanes. The third, given in Section 6, extends FMC's "no buffer" analysis to the case in which the distribution by destinations of parcels varies with time during the sorting system's operation.

2. STATEMENT OF THE PROBLEM

Conceptually, each parcel arriving at the sorter is "coded" by an operator and placed on a moving belt called an operator lane. This belt passes over a series of cross belts, also moving. At the end of each cross belt is a set of bins for accumulating parcels to a particular set of destinations. When the parcel moving along the operator lane passes over the proper cross belt, it is transferred to that cross belt and eventually to the proper destination bin.

In practice, however, parcels are coded not by one but by any of several operators, and are placed on any of several operator lanes, all passing over the same set of cross belts. The complication arises when a parcel intended for a particular cross belt reaches that belt and finds the passing section of it to be already occupied by another parcel (from an "earlier" operator lane). The new parcel cannot occupy that same section (assuming this were physically possible), since the code cannot be retained for two different destinations; one parcel would end in the wrong bin.

One possibility is to have each parcel, for which space on the appropriate cross belt is not available when needed, simply continue to the end of the operator lane and then be rehandled or recycled. This is the "no buffer" case mentioned above. An alternative is to place buffers at the intersection points of operator lanes and cross belts, so that a parcel could wait in the appropriate buffer until space for it became available on the passing cross belt. Since definite costs are associated with both the addition of buffers and the recycling of parcels, it was desired to ascertain by how much installation of buffers of various capacities would reduce the fraction of parcels requiring recycling, and how this fraction depends on the number of operator lanes.

A somewhat different technique may occur to the reader. One would place bins for an especially frequent destination at the ends of several cross belts; if a parcel for such a destination was unable to transfer to the first appropriate cross belt, it would have "another chance" further along the operator lane. This would involve certain equipment complications; what is done instead is to install bins for major destinations directly "on" the operator lanes, between the operators' coding stations and the system under analysis here.

3. THE MATHEMATICAL MODEL

Our mathematical model of the situation is a discrete one. A belt section is in effect identified with some appropriate point on that section; the parcel (if any) carried by the section is thought of as concentrated at the corresponding point, and the section is considered to pass another belt at the moment the corresponding point passes it. This idealization simplifies the description of the analysis, but does not appear to restrict its validity. It also permits us to think of time as moving in discrete steps.

Some more specific assumptions will now be listed. The second of them could apparently be considerably weakened (e.g., k_{ij} instead of k_j) without causing much extra conceptual difficulty, though the details of the derivations would become substantially more complicated. The notation (i,j) will be used for the intersection of the i-th operator lane and the j-th cross belt.

(1) Uniform Motion. Sections of the i-th operator lane arrive at (i,j) at a constant rate, and the same holds for the j-th cross belt. Thus we can define

t_{ij}=time between arrivals of successive sections of operator lane at (i,j),

$t_{ij}^{(c)}$=time between arrivals of successive sections of cross belt at (i,j).

(2) Synchronization. This assumption has three parts. First, for each (i,j) with $i > 1$ there is a number $c_{ij} \geq 1$ such that

$c_{ij}t_{ij}^{(c)}$=time between the arrival of a section of cross belt at $(i-1,j)$ and the same section's arrival at (i,j).

For constant belt speed, this says that $(i-1,j)$ and (i,j) are c_{ij} "section lengths" apart along the j-th cross belt. Second, there is an integer k_j, representing the ratio of cross belt speed to operator lane speed, such that

$$t_{ij}=k_j t_{ij}^{(c)}.$$

Third, at some initial time $t=t_{ij}^*$, an operator lane section and a cross belt section arrive simultaneously at (i,j). From this and the second assumption it follows that <u>every</u> arrival of an operator lane section at (i,j) is simultaneous with the arrival of a cross belt lane at (i,j), but not conversely if $k_j > 1$.

One consequence of assumption (2) is that all times of interest in connection with (i,j) will be of the form

$$t=t_{ij}^* + Nt_{ij}^{(c)}, \qquad N=0,1,2,\ldots;$$

a cross belt section will pass (i,j) at each of these times, but an operator lane section will pass if and only if N is a multiple of k_j. To simplify notation, we adopt the abbreviation

$$t_{ij}^{(N)}= t_{ij}^* + Nt_{ij}^{(c)}.$$

(3) <u>Independence</u>. Suppose we set

$p_{ij}(t)$=probability that at time t a parcel destined for the j-th cross belt arrives at (i,j) along the i-th operator lane,

$q_{ij}(t)= 1-p_{ij}(t)$.

We assume that the event defining $p_{ij}(t)$ is independent of both (a) the contents of the buffer at (i,j) at time t, and (b) the state (occupied or unoccupied) of the section of the j-th cross belt (if any) arriving at (i,j) at time t. In effect, we assume that the destination of a particular parcel entering the sorter is independent of that of any other parcel, and also of the choice of an operator lane for the parcel. This topic is discussed in the FMC Technical Memorandum cited earlier ("Step V"). Note that $p_{ij}(t_{ij}^{(N)})=0$ unless N is divisible by k_j.

The mathematical model will now be described in more detail, by specifying what may happen at any intersection (i,j) at any of the (discrete) times t concerning us. These times are those at which sections of cross belt arrive at (i,j), i.e. $t_{ij}^{(N)} = t_{ij}^{*} + N\, t_{ij}^{(c)}$ for N=0,1,2,etc.

Suppose first that N is not a multiple of k_j, so that no section of the i-th operator lane also arrives at time t. The arriving section of the j-th cross belt may either be full (i.e., already occupied by a particle) or empty; the latter is always the case if i=1 (i.e., for the first operator lane). If the buffer at (i,j) is empty, or the passing section of the cross belt is full, then no change ensues. But if the buffer is not empty and the passing section is empty then one of the parcels in the buffer (it doesn't matter which, for our purposes) is transferred to the passing section. In this connection it is convenient to define

$E_{ij}(t)$ = probability that the section of cross belt passing (i,j) at time t is empty.

Next suppose that N is a multiple of k_j. If the arriving section of the operator lane carries no parcel for the j-th cross belt, and the passing section of the cross belt is full, then no change ensues. The same is true if no parcel for the cross belt arrives, the passing section of the cross belt is empty, and the buffer at (i,j) is empty. If the buffer is non-empty and a space on the cross belt is available then one of the parcels in the buffer is transferred to the cross belt; the new parcel (if one arrives) goes into the buffer. If a new parcel arrives to find an empty buffer and an empty passing section of cross belt, it is transferred directly to the cross belt. To describe the remaining cases, it is convenient to define

$b(i,j)$ = capacity of the buffer at (i,j),

$B_{ijk}(t)$ = probability that buffer at (i,j) contains exactly k parcels at time t, so that $B_{ijk}(t) = 0$ for $k > b(i,j)$. Assume that a new parcel for the j-th cross belt arrives, but that the passing section of this belt is occupied. If the buffer at (i,j) contains fewer than $b(i,j)$ parcels, the new parcel simply goes into the buffer. But if the buffer is full (i.e., contains $b(i,j)$ parcels), then the new parcel counts as overflow and disappears so far as further analysis is concerned. We set

$V_{ij}(t)$ = probability that an overflow at (i,j) occurs at time t.

Finally, to simplify notation we shall generally abbreviate

$$p_{ij}(t_{ij}^{(N)}), q_{ij}(t_{ij}^{(N)}), E_{ij}(t_{ij}^{(N)}), B_{ijk}(t_{ij}^{(N)}), V_{ij}(t^{(N)})$$

to

$$p_{ij}^{(N)}, q_{ij}^{(N)}, E_{ij}^{(N)}, B_{ijk}^{(N)}, V_{ij}^{(N)} .$$

4. SOLUTION OF THE PRINCIPAL CASE

Since the performance of the sorter, in the present context, is measured by the rate of overflow, determination of the functions $V_{ij}^{(N)}=V_{ij}(t_{ij}^{(N)})$ is our main objective. The preceding description of the mathematical model, in particular the Independence Assumption, enables us at once to write the equation

(1) $$V_{ij}^{(N)}=p_{ij}^{(N)}B_{ij,b(i,j)}^{(N)}[1-E_{ij}^{(N)}] .$$

Regarding the functions $p_{ij}^{(N)}$ as "known", we see that eq(1) permits the calculation of the functions $V_{ij}^{(N)}$ if we can somehow determine the functions $E_{ij}^{(N)}$ and $B_{ijk}^{(N)}$, especially for $k=b(i,j)$. A method for determining these two families of functions will now be described.

The method involves a fixed value of j, i.e. a single cross-belt, and proceeds by induction on the operator lane index i. For i=1 (referring to the operator lane met "first" by every cross belt) we clearly have

(2) $$B_{1jo}^{(N)} = 1, \qquad B_{1jk}^{(N)} = 0 \qquad \text{for } k > o,$$

(3) $$E_{1j}^{(N)}= 1.$$

Of course no buffer would be needed at (1,j), i.e. b(1,j)=o.

For the "induction step", suppose that the functions $E_{i-1,j}^{(N)}$ and $B_{i-1,jk}^{(N)}$ have been determined; we must show how to find the functions $E_{ij}^{(N)}$ and $B_{ijk}^{(N)}$. Since the section of the j-th cross belt which passes (i,j) at time $t_{ij}^{(N)}$ is the same one that passed (i-1,j) at time $t_{ij}^{(N)}-c_{ij}t_{ij}^{(c)}$, we have

(4) $$E_{ij}^{(N)}=E_{i-1,j}(t_{ij}^{(N)}-c_{ij}t_{ij}^{(c)})q_{ij}(t_{ij}^{(N)}-c_{ij}t_{ij}^{(c)})B_{i-1,jo}(t_{ij}^{(N)}-c_{ij}t_{ij}^{(c)}).$$

This shows how to determine the function $E_{ij}^{(N)}$ from the known functions

$$q_{ij}^{(N)} = 1-p_{ij}^{(N)},\ E_{i-1,j}^{(N)},\ \text{and } B_{i-1,jo}^{(N)}.$$

(We note in passing that (2), (3) and (4) admit the somewhat more explicit solution

$$(5)\quad E_{ij}^{(N)} = \prod_{r=2}^{i} q_{ij}\left(t_{ij}^{(N)} - \sum_{s=r}^{i} c_{sj} t_{sj}^{(c)}\right) \prod_{r=2}^{i-1} B_{rjo}\left(t_{ij}^{(N)} - \sum_{s=r+1}^{i} c_{sj} t_{sj}^{(c)}\right)$$

for $i > 1$, where the second product is taken as unity for $i=2$; this follows by induction on i.) The remaining task is to show how the functions $B_{ijk}^{(N)}$ can be determined from the functions $E_{ij}^{(N)}$ and $p_{ij}^{(N)}$.

For this purpose, we examine what possible numbers of parcels in buffer (i,j) at time $t_{ij}^{(N)}$, and what intervening events, could lead to the buffer containing exactly k parcels at time $t_{ij}^{(N+1)}$. Using the detailed description of the model given in the last section, we obtain the following equations:

$$(6)\quad B_{ijo}^{(N)} = B_{ijo}^{(N)}\left[q_{ij}^{(N)} + E_{ij}^{(N)} p_{ij}^{(N)}\right] + B_{ij1}^{(N)} E_{ij}^{(N)} q_{ij}^{(N)} \qquad \text{for } k=0,$$

$$(7)\quad B_{ijk}^{(N+1)} = B_{ij,k-1}^{(N)} p_{ij}^{(N)}\left[1-E_{ij}^{(N)}\right] + B_{ijk}^{(N)}\left[q_{ij}^{(N)}\left(1-E_{ij}^{(N)}\right) + p_{ij}^{(N)} E_{ij}^{(N)}\right]$$
$$+ B_{ij,k+1}^{(N)} q_{ij}^{(N)} E_{ij}^{(N)} \qquad \text{for } 0 < k < b(i,j),$$

$$(8)\quad B_{ij,b(i,j)}^{(N+1)} = B_{ij,b(i,j)-1}^{(N)} p_{ij}^{(N)}\left[1-E_{ij}^{(N)}\right] + B_{ij,b(i,j)}^{(N)}\left[\left(1-E_{ij}^{(N)}\right)\right.$$
$$\left. + p_{ij}^{(N)} E_{ij}^{(N)}\right] \qquad \text{for } k=b(i,j).$$

In addition, the obvious identity

$$(9)\quad \sum_{k=o}^{b(i,j)} B_{ijk}^{(N)} = 1$$

expresses the fact that the number of parcels in the buffer at (i,j) is always some integer between 0 and the buffer's capacity, inclusive.

So far the analysis has been carried out on a rather high level of generality. The remainder of this general analysis will be deferred to Section 5, and attention will now be focussed on a more special situation of particular interest, which we might call the principal case. The first hypothesis delimiting this case is

(10) $k_j = 1,$

i.e. equal speeds for operator lanes and the j-th cross belt. The second, involving a kind of "homogeneity", is that

(10a) $c_{ij} = c_j = \text{integer},\ t_{ij}^{(c)} = t_j^{(c)},\ t^*_{ij} = t^*_j,$

all independent of i. The third assumption is that, to a satisfactory approximation, the steady state input condition

(11) $p_{ij}(t^{(N)}) = p_{ij}^{(N)} = p_{ij}$, independent of N, for all i

holds; this might occur if the fluctuations in $p_{ij}^{(N)}$ are small enough to be ignored over the long run. The same is then true of $q_{ij} = 1 - p_{ij}$. Under these hypotheses, it is known (a few more details are given at the end of Section 5) that the j-th cross belt will gradually settle down so that the functions $E_{ij}^{(N)}$ and $B_{ijk}^{(N)}$, for all operator lanes "i", obey conditions

(12) $E_{ij}^{(N)} = E_{ij},\ B_{ijk}^{(N)} = B_{ijk}$, independent of N

analogous to (11).

The determination of the functions $B_{ijk}^{(N)}$ (which are now constants B_{ijk}) can be carried out especially simply for the principal case. Since $B_{ijk}^{(N+1)} = B_{ijk}^{(N)}$ and $E_{ij}^{(N+1)} = E_{ij}^{(N)}$, eqs (6) and (7) can be rearranged to yield

(13) $B_{ij1}\, q_{ij}\, E_{ij} = B_{ijo}\, p_{ij}\, [1-E_{ij}]$,

(14) $B_{ij,k+1}\, q_{ij}\, E_{ij} = B_{ijk}\, [q_{ij}\, E_{ij} + p_{ij}\, (1-E_{ij})] - B_{ij,k-1}\, p_{ij}\, [1-E_{ij}]$

for $o < k < b\ (i,j)$.

From this we can prove, by induction on k, that

(15) $B_{ij,k+1}\ q_{ij}\ E_{ij} = B_{ijk}\ p_{ij}\ [1-E_{ij}]$ for $o \leq k < b(i,j)$.

For k=o, eq (15) is precisely (13). If eq (15) holds with k replaced by k-1, then eq (14) yields

$$B_{ij,k+1}\ q_{ij}\ E_{ij} = B_{ijk}\ [q_{ij}\ E_{ij} + p_{ij}\ (1-E_{ij})] - B_{ijk}\ q_{ij}\ E_{ij}$$

$$= B_{ijk}\ p_{ij}\ [1-E_{ij}],$$

so that (15) also holds for k. This completes the induction proof.

If we now define an auxiliary quantity

(16) $\nu_{ij} = p_{ij}\ [1-E_{ij}]/\ q_{ij}\ E_{ij}$,

then it readily follows from eq (15) that

(17) $B_{ijk} = (\nu_{ij})^k\ B_{ijo}$ for $o \leq k \leq b\ (i,j)$.

Substitution from (17) into (9) yields

$$1 = \sum_{k=o}^{b(i,j)} (\nu_{ij})^k\ B_{ijo} = B_{ijo} \sum_{k=o}^{b(i,j)} (\nu_{ij})^k,$$

so that

(18) $B_{ijo} = 1/ \sum_{k=o}^{b(i,j)} (\nu_{ij})^k = (1-\nu_{ij})/(1-(\nu_{ij})^{b(i,j)+1})$,

where the first formula must be used if $\nu_{ij} = 1$, and is preferable for numerical work if ν_{ij} is near unity. As a check, the combination (17) and (18) can be shown to be consistent with eq (18).

We can now summarize the solution procedure for the principal case. Note first that by the Independence Assumption, the constant p_{ij} is independent of i and can be abbreviated $p_j = \alpha f_j$, where

(19) α = probability that a random section of operator lane carries a parcel to be sorted,

TABLE 1. Values(*) of V_{ij}/p_j

	p_j=0.1				p_j=0.2			
b=	0	1	2	3	0	1	2	3
i= 1	.000	.000	.000	.000	.000	.000	.000	.000
2	.100	.001	.000	.000	.200	.012	.001	.000
3	.190	.005	.000	.000	.360	.055	.009	.002
4	.271	.014	.001	.000	.488	.154	.054	.020
5	.344	.027	.002	.000	.590	.330	.240	.191
6	.410	.049	.006	.001	.672	.589	.703	.788
7	.469	.082	.014	.002	.738	.868	.995	___
8	.522	.132	.034	.009	.790	.996	___	___
9	.570	.208	.091	.041	.832	___	___	___
10	.613	.326	.192	.204	.866	___	___	___
11	.652	.502	.710	.735	.893	___	___	___
12	.687	.735	.985	___	.914	___	___	___

(*) Omitted values are 1.00.

Buffer capacity b(i,j)=b for i > 1.

TABLE 2. Values(*) of $1-E_{1j}$

	$p_j=0.1$				$p_j=0.2$			
b=	0	1	2	3	0	1	2	3
i= 1	.000	.000	.000	.000	.000	.000	.000	.000
2	.100	.100	.100	.100	.200	.200	.200	.200
3	.190	.200	.200	.200	.360	.398	.400	.400
4	.271	.299	.300	.300	.488	.587	.597	.600
5	.344	.399	.400	.400	.590	.756	.786	.796
6	.410	.496	.500	.500	.672	.890	.938	.958
7	.469	.591	.599	.600	.738	.971	.997	___
8	.522	.683	.698	.699	.790	.998	___	___
9	.570	.770	.794	.799	.832	___	___	___
10	.613	.849	.885	.895	.866	___	___	___
11	.652	.916	.969	.973	.893	___	___	___
12	.687	.966	.998	___	.914	___	___	___

(*) Omitted values are 1.00.

Buffer capacity $b(i,j)=b$ for $i > 1$.

(20) f_j = probability that a random parcel is destined for the j-th cross belt.

In practice, α might be interpreted as the ratio (if ≤ 1) of operator coding speed in parcels / minute to operator lane speed in sections / minute, and f_j as the relative frequency of parcels to the destinations of the j-th cross belt. The initial conditions (for i=1) are

(21) $B_{1jo} = 1, \quad b(1,j) = o,$

(22) $E_{1j} = 1.$

For the recursion step, suppose $E_{i-1,j}$ and $\nu_{i-1,j}$ are known. First calculate E_{ij} by the formula

(23) $E_{ij} = E_{i-1,j}\, q_j\, (1-\nu_{i-1,j})/(1-(\nu_{i-1,j})^{b(i-1,j)+1}),$

obtainable from (4) and (18). Then compute ν_{ij} by (16). The overflow rate at (i,j) is found from the formula

(24) $V_{ij} = p_j\, [1-E_{ij}]\, (\nu_{ij})^{b(ij)}(1-\nu_{ij})/(1-(\nu_{ij})^{b(i,j)+1})$

obtainable from (1), (17) and (18).

This procedure was used to obtain the numerical results presented in Tables 1 and 2. The tabulated quantities are

V_{ij}/p_j = expected fraction of parcels on the i-th operator lane, destined for the j-th cross belt, which overflow at (i,j),

$1 - E_{ij}$ = expected fraction of the cross belt sections already occupied when they reach (i,j).

Although the table refers to situations in which the buffer capacity has the same value b(i,j)=b for all i > 1, the method is not limited to such cases. Additional tables are in preparation.

In discussing these numerical results, we note first that it was obvious from the start that adding a buffer (or increasing buffer capacity) at an "early" intersection corresponding to a small value of i, while reducing the overflow rate at that intersection, would also tend to make cross belt sections leave that intersection "loaded" more often than before and thus to increase the overflow rate at intersections further "downstream". It is not too surprising, then, to find for example that increasing b from 1 to 2 diminishes the overflow rate for the initial lanes, but increases the rate for subsequent lanes; the point is that our analysis gives precise information as to which lane marks the beginning of this "inversion" ($i=11$ for $p_j=0.1$, $i=6$ for $p_j=0.2$). Also, it was not obvious without specific investigation and computation that for fixed b and p_j the values of V_{ij}/p_j, instead of increasing gradually with i, would increase extremely sharply for a certain "critical value" of i. If $p_j=0.1$ and $b=2$ or 3 for instance, then operator lanes past the tenth will contribute relatively little sorting according to Table 1---Table 2 shows that cross belt sections will be occupied at least 97% of the time when they reach such lanes---so that really compelling reasons due to other factors would be required to justify a system with more than ten lanes under these circumstances. Similarly, if $p_j=0.2$ and $b=2$ or 3 then the addition of operator lanes past the fifth seems questionable. These examples should indicate the usefulness of data like those in Tables 1 and 2.

5. SOLUTION OF THE GENERAL CASE; APPLICATIONS

We return now to the general analysis interrupted in Section 4. Recall that the remaining task was to utilize eqs (6) through (9) to determine the functions $B_{ijk}^{(N)}$ in terms of $E_{ij}^{(N)}$ and $p_{ij}^{(N)}$. For this purpose we shall prove, by induction on k, that

$$(25)\quad B_{ij,k+1}^{(N)}\; q_{ij}^{(N)}\; E_{ij}^{(N)} = \sum_{s=o}^{k} [B_{ijs}^{(N+1)} - B_{ijs}^{(N)}] + B_{ijk}^{(N)}\; p_{ij}^{(N)} [1-E_{ij}^{(N)}]$$

$$\text{for } o \leq k < b(i,j).$$

Note that under the conditions (12) of the principal case, the first sum on the right vanishes term by term and the previous eq (15) is regained; this shows why the principal case admitted so especially elegant a solution.

To prove eq (25), we first observe that for k=o it is just a rearrangement of eq (6). Next, for $o < k < b(i,j)$ we can rewrite eq (7) as

$$B_{ij,k+1}^{(N)}\; q_{ij}^{(N)}\; E_{ij}^{(N)} = [B_{ijk}^{(N+1)} - B_{ijk}^{(N)}] + B_{ijk}^{(N)}\; p_{ij}^{(N)} [1-E_{ij}^{(N)}]$$

$$+ \{B_{ijk}^{(N)}\; q_{ij}^{(N)}\; E_{ij}^{(N)} - B_{ij,k-1}^{(N)}\; p_{ij}^{(N)}\; [1-E_{ij}^{(N)}]\}\;;$$

if (25) holds with k replaced by k-1, then the term {---} in the last equation is equal to

$$\sum_{s=o}^{k-1} [B_{ijs}^{(N+1)} - B_{ijs}^{(N)}]\;,$$

so that eq (25) holds for k as well as k-1. This completes the induction proof.

The general procedure for finding the functions $B_{ijk}^{(N)}$ can be described as follows. Eq (25) with k=o, yields $B_{ij1}^{(N)}$ in terms of

$B_{ijo}^{(N)}$ and $B_{ijo}^{(N+1)}$. Therefore it also yields $B_{ij1}^{(N+1)}$ in terms of $B_{ijo}^{(N+1)}$ and $B_{ijo}^{(N+2)}$. Next, eq (25) with k=1 yields $B_{ij2}^{(N)}$ in terms of $B_{ij1}^{(N)}$, $B_{ij1}^{(N+1)}$, $B_{ijo}^{(N+1)}$ and $B_{ijo}^{(N)}$; by the last two sentences, it therefore yields an expression for $B_{ij2}^{(N)}$ in terms of $B_{ijo}^{(N)}$, $B_{ijo}^{(N+1)}$ and $B_{ijo}^{(N+2)}$. We now continue in the same way; for $o \leq k \leq b(i,j)$, $B_{ijk}^{(N)}$ is determined as an expression in $B_{ijo}^{(N)}, B_{ijo}^{(N+1)}, \ldots, B_{ijo}^{(N+k)}$. After this is done, we substitute the resulting expressions into eq (9) and obtain a difference equation of order b(i,j) for the function $B_{ijo}(t)$. Once this equation (with appropriate initial conditions) has been solved, the functions $B_{ijk}(t)$ are determined by the expressions already found for them in terms of $B_{ijo}(t)$.

The simplest non-trivial application arises for b(i,j)=1, i.e. a buffer with room for only one parcel. Here eq (25) with k=o yields

$$(26) \quad B_{ij1}^{(N)} q_{ij}^{(N)} E_{ij}^{(N)} = [B_{ijo}^{(N+1)} - B_{ijo}^{(N)}] + B_{ijo}^{(N)} p_{ij}^{(N)} [1-E_{ij}^{(N)}]$$

$$= B_{ijo}^{(N+1)} - B_{ijo}^{(N)} [q_{ij}^{(N)} + p_{ij}^{(N)} E_{ij}^{(N)}] ;$$

if this is solved for $B_{ij1}^{(N)}$ and the result substituted into the appropriate version

$$B_{ijo}^{(N)} + B_{ij1}^{(N)} = 1$$

of eq (9), we obtain the first-order difference equation

$$(27) \quad B_{ijo}^{(N+1)} = q_{ij}^{(N)} E_{ij}^{(N)} + B_{ijo}^{(N)} [p_{ij}^{(N)} E_{ij}^{(N)} + q_{ij}^{(N)} (1-E_{ij}^{(N)})]$$

for the function $B_{ijo}(t)$. If the time $t_{ij}^{(o)} = t^*_{ij}$ was sufficiently early in the system's operation, we may take $B_{ijo}^{(o)} = 1$ as initial condition. The solution of the difference equation is then found to be

$$(28) \quad B_{ijo}^{(N+1)} = q_{ij}^{(N)} E_{ij}^{(N)} + \prod_{s=o}^{N} [p_{ij}^{(s)} E_{ij}^{(s)} + q_{ij}^{(s)} (1-E_{ij}^{(s)})]$$

$$+ \sum_{s=o}^{N-1} q_{ij}^{(s)} E_{ij}^{(s)} \prod_{r=s+1}^{N} [p_{ij}^{(r)} E_{ij}^{(r)} + q_{ij}^{(r)} (1-E_{ij}^{(r)})]$$

where the last sum is omitted for $N=o$; this formula can be verified by induction on N.

For any particular "reasonable" value of $b(i,j) > 1$, the difference equation analogous to (27) and the explicit solution analogous to (28) can be worked out. It is important to realize, however, that for many purposes the explicit solution is unnecessary; particular values of it can be calculated from the initial conditions and the difference equation, while its theoretical properties can be proved by induction with the aid of the difference equation.

As a second application, let us see how much the analysis of the principal case is complicated if (10) must be changed to

$$(29) \qquad k_j = 2 ,$$

i.e. if the cross belt is moving twice as fast as the operation lanes. The appropriate modification of the "steady state input" condition (11) for this case is

$$(30) \qquad p_{ij}^{(2m)} = p_{ij} , \quad \text{independent of } m,$$

since we automatically have $p_{ij}^{(2m+1)} = o$ when (29) holds. Conditions (10a) are retained, and the Independence Assumption permits us to abbreviate p_{ij} to p_j. Under these hypotheses, it is known (see the end of this section) that the j-th cross belt will gradually settle down so that the analogs

$$(31) \quad E_{ij}^{(N+2)} = E_{ij}^{(N)}, \; B_{ijk}^{(N+2)} = B_{ijk}^{(N)}$$

of (12) are satisfied. (Eq (12) would be obtained if "N+2" in (31) were replaced by "N+1".) Thus the functions $E_{ij}(t)$ and $B_{ijk}(t)$ each take on at most two different values, which we abbreviate

$$E_{ij} = E_{ij}^{(2m)}, \quad E_{ij}^{+} = E_{ij}^{(2m+1)},$$

$$B_{ijk} = B_{ijk}^{(2m)}, \quad B_{ijk}^{+} = B_{ijk}^{(2m+1)}.$$

Eq (25) splits into two equations:

(32) $$B_{ij,k+1}\, q_j\, E_{ij} = \sum_{s=o}^{k} [B_{ijs}^{+} - B_{ijs}] + B_{ijk}\, p_j\, [1-E_{ij}] ,$$

(33) $$B_{ij,k+1}^{+}\, E_{ij}^{+} = \sum_{s=o}^{k} [B_{ijs} - B_{ijs}^{+}] \qquad \text{for } o \leq k < b(i,j) .$$

We now define the auxiliary quantity

(34) $$\rho_{ij} = 1 - (1/E_{ij}^{+}) < o ,$$

and prove that

(35) $$\sum_{s=o}^{k} [B_{ijs}^{+} - B_{ijs}] = (\rho_{ij})^{k}\, B_{o}^{+} - \sum_{s=o}^{k} (\rho_{ij})^{k-s}\, B_{ijs} \quad \text{for } o \leq k \leq b(i,j).$$

For k=o, eq (35) is an identity. By eqs (33) and (34) ,

$$\sum_{s=o}^{k+1} [B_{ijs}^{+} - B_{ijs}] = \sum_{s=o}^{k} [B_{ijs}^{+} - B_{ijs}] + B_{ij,k+1}^{+} - B_{ij,k+1}$$

$$= \sum_{s=o}^{k} [B_{ijs}^{+} - B_{ijs}] - (1/E_{ij}^{+}) \sum_{s=o}^{k} [B_{ijs}^{+} - B_{ijs}] - B_{ij,k+1}$$

$$= \rho_{ij} \sum_{s=o}^{k} [B_{ijs}^{+} - B_{ijs}] - B_{ij,k+1} ;$$

if eq (35) is true for k, substitution in the last equation shows it to be true for k+1 as well. This completes the induction proof.

The case b(i,j)=1 has been treated earlier (eqs (27) and (28)), while the situation b(i,j)=o will be discussed in Section 6. We therefore assume b(i,j)=b > 1, and proceed to show that

$$(36)\quad B_{ij,k+2}\, q_j\, E_{ij} = B_{ij,k+1} \{(\rho_{ij}\, q_j - p_{ij})\, E_{ij} - q_j\}$$

$$- B_{ijk}\, \rho_{ij}\, p_j\, (1-E_{ij}) \quad \text{for } o \le k \le b-2\ ;$$

this is an analog of the eq (15) obtained for the principal case. To prove (36), replace k by k+1 in (32) and (35) to obtain

$$B_{ij,k+2}\, q_j\, E_{ij} = (\rho_{ij})^{k+1}\, B^{+}_{ijo} - \sum_{s=o}^{k+1} (\rho_{ij})^{k+1-s}\, B_{ijs}$$

$$+ B_{ij,k+1}\, p_j\, [1-E_{ij}]$$

$$= \rho_{ij} \{(\rho_{ij})^{k}\, B^{+}_{ijo} - \sum_{s=o}^{k} (\rho_{ij})^{k-s}\, B_{ijs}\} - B_{ij,k+1}$$

$$+ B_{ij,k+1}\, p_j\, [1-E_{ij}]$$

$$= \rho_{ij} \sum_{s=o}^{k} [B^{+}_{ijs} - B_{ijs}] - B_{ij,k+1}\, [q_j + p_j\, E_{ij}]$$

$$= \rho_{ij} \{B_{ij,k+1}\, q_j\, E_{ij} - B_{ijk}\, p_j\, [1-E_{ij}]\} - B_{ij,k+1}\, [q_j + p_j\, E_{ij}],$$

yielding (36). With the abbreviations

$$(37)\quad \lambda_{ij} = - \rho_{ij}\, (1-E_{ij})/q_j E_{ij} = -\rho_{ij}\, \nu_{ij} \ge o$$

$$(38)\quad \mu_{ij} = - \{q_j + (p_{ij} - \rho_{ij}\, q_j)\, E_{ij}\}/q_j\, E_{ij}\ ,$$

we rewrite eq (36) as

$$(39)\quad B_{ij,k+2} = \lambda_{ij}\, B_{ijk} + \mu_{ij}\, B_{ij,k+1}\ .$$

This second-order linear difference equation with constant coefficients can be solved by standard methods; the result is

$$(40)\quad B_{ijk} = \{\gamma^k (B_{ij1} - \delta B_{ijo}) - \delta^k (B_{ij1} - \gamma B_{ijo})\}/(\gamma - \delta)$$

where γ and δ are the roots of the equation $x^2 = \lambda_{ij} + \mu_{ij} x$. By using (40) to substitute in the version $\sum_{k=o}^{b} B_{ijk} = 1$ of eq (9), we obtain the following relation between B_{ijo} and B_{ij1}:

$$(41)\quad B_{ij1} \{(1-\delta)(1-\gamma^{b+1}) - (1-\gamma)(1-\delta^{b+1})\}$$

$$- B_{ijo} \{\delta(1-\delta)(1-\gamma^{b+1}) - \gamma (1-\gamma)(1-\delta^{b+1})\} = (1-\gamma)(1-\delta)(\gamma-\delta).$$

Another relation between B_{ijo} and B_{ij1} will be derived by equating two expressions for B^+_{ijo}. To find the first expression, we set k=o in (32) to obtain

$$(42)\quad B^+_{ijo} = B_{ij1} q_j E_{ij} + B_{ijo} [q_j + p_j E_{ij}] .$$

To find the second expression, we set k=b in eq (35) and use the facts

$$\sum_{s=o}^{b} B_{ijs} = \sum_{s=o}^{b} B^+_{ijs} = 1$$

to obtain

$$o = (\rho_{ij})^b B^+_{ijo} - \sum_{s=o}^{b} (\rho_{ij})^{b-s} B_{ijs}$$

or equivalently

$$(43)\quad B^+_{ijo} = \sum_{s=o}^{b} (\rho_{ij})^{-s} B_{ijs} ;$$

use of eq (40) to substitute in the last equation yields

$$(44)\quad B^+_{ijo} = \{B_{ij1} [(\rho_{ij} - \delta)((\rho_{ij})^{b+1}-\gamma^{b+1}) - (\rho_{ij} -\gamma)((\rho_{ij})^{b+1}-\delta^{b+1})]$$

$$- B_{ijo} [\delta(\rho_{ij}-\delta)((\rho_{ij})^{b+1}-\gamma (\rho_{ij}-\gamma)((\rho_{ij})^{b+1}-\delta^{b+1})]\}$$

$$\div (\rho_{ij} -\gamma)(\rho_{ij} -\delta)(\rho_{ij})^b (\gamma -\delta).$$

The relation obtained by equating the right-hand sides of eqs (42) and (44), when solved simultaneously with (41), determine B_{ij1} and B_{ijo}. All B_{ijk} are then determined by (40), B^{+}_{ijo} is given by (42), and the remaining B^{+}_{ijk} can be found recursively from eq (33). This completes the solution.

We conclude this section by sketching how (12) follows from (10) and (10a) and (11), and how (31) follows from (29) and (10a) and (30). These are the special cases $k_j=1,2$ of the assertion that if (10a) holds, and if the inputs to the j-th cross belt are "steady state" in the sense that

$$(45)\quad p_{ij}^{(N+k_j)} = p_{ij}^{(N)} \qquad \text{for all } N,$$

then the cross belt will gradually settle down so that the analogs

$$(46)\quad E_{ij}^{(N+k_j)} = E_{ij}^{(N)},\ B_{ijk}^{(N+k_j)} = B_{ijk}^{(N)}$$

of (45) are satisfied.

Number the "section positions" along the cross belt consecutively; the first position is at (1,j), the (c_j+1)-st at (2,j), the $(2c_j+1)$-st at (3,j), etc. Define the <u>state</u> of the system at time $t=t_j+mk_jt_j^{(c)}$ to consist of the number of parcels in each buffer, and the status (occupied or empty) of every section position, at time t and at each of the preceding k_j-1 moments. It can then be shown that the probability of transition from any particular state at time t to any particular state at time $t+k_jt_j^{(c)}$ depends only on the states, not on t. Thus the belt constitutes what is called a <u>Markov chain</u>, and the assertion follows by appeal to the theory of such chains. In most practical situations the "settling down" is rapid enough that, unless especially interested in transient effects, one can confine attention to the problem as characterized by the equations analogous to (46).

6. THE NO-BUFFER CASE

If $b(i,j) = o$, then the functions $B_{ijk}(t)$ are known without further work as

$$B_{ijo}(t) = 1, \; B_{ijk}(t) = o \quad \text{for } k > o.$$

Therefore it is relatively easy to study a cross belt with no buffers at <u>any</u> of its intersections, or merely with no buffers at a set of intersections at the beginning of the belt; in the second instance the investigation is confined to the "buffer-free" zone.

For this situation, eq (5) simplifies to

$$(47) \quad E_{ij}^{(N)} = \prod_{r=2}^{i} q_{ij}(t_{ij}^{(N)} - \sum_{s=r}^{i} c_{sj} \, t_{sj}^{(c)}) \qquad (i > 1),$$

and so eq (1) yields

$$(48) \quad V_{ij}^{(N)} = p_{ij}^{(N)}[1 - \prod_{r=2}^{i} q_{ij} \, (t_{ij}^{(N)} - \sum_{s=r}^{i} c_{sj} \, t_{sj}^{(c)})] \qquad (i > 1).$$

Suppose we specialize further by imposing the "homogeneity" conditions (10a). Then the Independence Assumption shows $p_{ij}(t)$ and $q_{ij}(t)$ to be independent of i, and eq (48) becomes

$$V_{ij}^{(N)} = p_j^{(N)}[1 - \prod_{r=2}^{i} q_j \, (t_j^{(N)} - (i - r+1) \, c_j t_j^{(c)})] \qquad (i > 1).$$

Since $p_{ij}^{(N)} = o$ if N is not divisible by k_j, a non-zero probability of overflow is possible only in the case

$$(49) \quad V_{ij}(t_j^* + mk_j t_j^{(c)}) = p_j(t_j^* + mk_j t_j^{(c)})[1 - \prod_{r=2}^{i} q_j(t_j^* + \{mk_j - (i-r+1)c_j\} t_j^{(c)})]$$

of the last equation. In (49), the only q_j-values not necessarily equal to unity are those for which $(i-r+1)c_j$ is divisible by k_j. If for example k_j and c_j are relatively prime (have no common factor except unity), then the only non-unity values for q_j in (49) are those for which i-r+1 is a multiple of k_j, i.e.

$$r = i+1-k_j,\ i+1-2k_j,\ i+1-3k_j,\ \text{etc.} \qquad (r \geq 2).$$

The number of such values is $< (i-1)/k_j >$, where $< x >$ denotes the largest integer not exceeding x.

Assume now, in addition to (10a), that k_j and c_j are relatively prime and that the steady state condition (45) holds; this last condition can be written

$$p_j(t^*_j + mk_j t_j^{(c)}) = p_j \ , \quad \text{independent of } m,$$

and (46) permits us to abbreviate

$$V_{ij}(t^*_j + mk_j t_j^{(c)}) = V_{ij}, \quad \text{independent of } m.$$

By the preceding remarks, the overflow probability (49) takes the form

$$(50) \quad V_{ij} = p_j \left(1-q_j^{< (i-1)/k_j >}\right) \qquad (i > 1).$$

This generalizes a result obtained in the FMC technical memo cited (see "Step III", pp. 7-8 of the memo), showing a cross belt with $k_j > 1$, under the stated hypotheses, to be equivalent to several shorter belts each with $k_j=1$. Our symbols k_j and c_j correspond respectively to I and J in FMC's notation.

If we specialize still further to the case $k_j=1$ and let

N= number of operator lanes

(this should not be confused with our earlier use of the symbol N), then it follows from (50) that the expected rate of overflows along the j-th cross belt is

$$\sum_{i=2}^{N} V_{ij} = \sum_{i=2}^{N} p_j \ (1-q_j^{i-1}) = p_j \ (N-1-\sum_{i=2}^{N} q_j^{i-1}) \ ,$$

or finally

$$(51) \quad \sum_{i=2}^{N} V_{ij} = p_j N - (1-q_j^N) \ .$$

If there are R cross belts, all bufferless and with $k_j=1$, and if all c_j and $t_j^{(c)}$ are equal, then the expected overflow rate for the <u>entire system</u> is

$$(52) \quad \sum_{j=1}^{R} \sum_{i=2}^{N} V_{ij} = N \sum_{j=1}^{R} p_j - \sum_{j=1}^{R} (1-q_j^N).$$

Setting $p_j = \alpha f_j$ (see eqs (19) and (20)), we find that the total overflow rate becomes

$$(53) \quad Q = \alpha N \sum_{j=1}^{R} f_j - \sum_{j=1}^{R} (1-(1-\alpha f_j)^N).$$

This is a generalization of the corresponding formula on p. 9 of the FMC memo, which can be obtained by setting all $f_j=1/R$ in our eq (53), i.e. by assuming equal distribution of parcels among cross belts.

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