INCIDENCE SPACES AND ALGEBRAS

by

E. C. Dade and K. Goldberg
THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to government agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. Research projects are also performed for other government agencies when the work relates to and supplements the basic program of the Bureau or when the Bureau's unique competence is required. The scope of activities is suggested by the listing of divisions and sections on the inside of the back cover.

Publications

The results of the Bureau's research are published either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three periodicals available from the Government Printing Office: The Journal of Research, published in four separate sections, presents complete scientific and technical papers; the Technical News Bulletin presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of non-periodical publications: Monographs, Applied Mathematics Series, Handbooks, Miscellaneous Publications, and Technical Notes.

A complete listing of the Bureau's publications can be found in National Bureau of Standards Circular 460, Publications of the National Bureau of Standards, 1901 to June 1947 ($1.25), and the Supplement to National Bureau of Standards Circular 460, July 1947 to June 1957 ($1.50), and Miscellaneous Publication 240, July 1957 to June 1960 (Includes Titles of Papers Published in Outside Journals 1950 to 1959) ($2.25); available from the Superintendent of Documents, Government Printing Office, Washington 25, D. C.
INCIDENCE SPACES AND ALGEBRAS*

by

E. C. Dade** and K. Goldberg

*This work was supported in part by the Office of Naval Research.

**California Institute of Technology

IMPORTANT NOTICE

Approved for public release by the director of the National Institute of Standards and Technology (NIST) on October 9, 2015

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS
Incidence Spaces and Algebras

E. C. Dade and K. Goldberg

An incidence matrix is a matrix whose entries are either 0 or 1. We use a fixed notation for the incidence matrices $I$, the identity matrix, and $J$, the matrix with 1 in every position. The fact that $I$, $J-I$ are disjoint incidence matrices which form a basis for a linear algebra, lies behind much of the manipulation of the celebrated "incidence equation"

$$AA^T = kI + \lambda(J-I)$$

in which $A$ is an incidence matrix of order $v$. For given integers $v,k,\lambda$ the existence of $A$ is equivalent to the existence of a $v,k,\lambda$ design, special cases of which are finite projective planes and Hadamard matrices.

Starting in 1955 several mathematicians, including R. C. Bose, D. Mesner and the authors, began considering more general algebras with a basis of incidence matrices whose sum was $J$. At first each of these matrices was symmetric. Bose, and later Mesner, considered the properties of a design with incidence matrix $A$ such that

$$AA^T = k_1A_1 + k_2A_2 + \ldots + k_dA_d$$

$$J = A_1 + A_2 + \ldots + A_d$$

*This work was supported in part by the Office of Naval Research.*
where the $A_i$ are incidence matrices which form the basis of an algebra. We came upon a similar generalization through the graph problem mentioned at the end of this note. In each case $A_i = I$.

Our purpose is to summarize the results we obtained in generalizing these concepts by not demanding $A_i^T = A_i$ or $A_i = I$, or even that the basis be an algebra basis.
1. **Incidence Spaces.** A set $\mathcal{B} = \{A_1, A_2, \ldots, A_d\}$ of $m \times n$ incidence matrices is called an **incidence basis** if

$$A_1 + A_2 + \cdots + A_d = J_{m,n}^r$$

where $J_{m,n}$ is the $m \times n$ matrix with 1 in every position. This implies that the $A_p$ are mutually disjoint (i.e., if $A_p$ has 1 in the $i,j$ position then $A_q \neq A_p$ has 0 in the $i,j$ position), and that for each $i$ and $j$ there is a unique $p$ such that the $i,j$ element of $A_p$ is 1. Thus the $A_p$ are linearly independent.

The linear span

$$\mathcal{X} = \{A_1, A_2, \ldots, A_d\}$$

of the incidence basis $\mathcal{B}$ (over some unspecified field $\mathcal{K}$ of characteristic 0) is called an $m \times n$ **incidence space**. The only incidence matrices in $\mathcal{X}$ are the $2^d$ sums of distinct elements of $\mathcal{B}$. Therefore, $\mathcal{B}$ is the only incidence basis which is a vector basis for $\mathcal{X}$, so that we may speak of the incidence basis of an incidence space.

If $m = n$, so that the $A_p$ are square, and if $\mathcal{X}$ is closed under matrix multiplication, it is called an **incidence algebra**. In this case there exist non-negative integers $a_{pq}^{(r)}$, $p, q, r = 1, 2, \ldots, d$ such that

$$A_p A_q = \sum_{r=1}^{d} a_{pq}^{(r)} A_r \quad p,q=1,2,\ldots,d.$$ 

These integers are called the **structure constants** of $\mathcal{X}$.
Let $\mathcal{O} = [A_1, A_2, \ldots, A_d]$ be an $m \times n$ incidence space closed under right multiplication by $J_{m,n}$. Then for each $p = 1, 2, \ldots, d$ the non-zero rows of $A_p$ have the same row sum $a_p$, and hence the same number of non-zero elements. Let $U_p$ be the set of indices of the non-zero rows of $A_p$. Then $U_p$ and $U_q$ are either disjoint or equal. We let $u_p$ denote the order of $U_p$. If $\mathcal{O}$ is closed under left multiplication by $J_{n,m}$ we similarly define the column sets $V_p$ of order $v_p$.

An $n \times n$ incidence algebra is closed under both right and left multiplication by $J = J_{n,n}$, hence it has both row and column sets. Any basis element is concentrated in one row set and one column set, and the number of non-zero elements in any two rows (columns) are equal. For a fixed $U_p$ and $V_q$ let $M_{p,q}^{pq}$ denote the sum of those $A_r$ with $U_r = U_p$ and $V_r = V_q$. Then the set of $M_{p,q}^{pq}$ is an incidence basis whose linear span is a direct summand of $\mathcal{O}$. $\mathcal{O}$ is simple if and only if it equals this summand and has a unit element.

An incidence space is called *stochastic* if each basis element has constant row sums and constant column sums. In that case it has just one row set and just one column set.

An $n \times n$ incidence space is called *semi-stochastic* if each row set $U_p$ is also a column set $V_p$ (then each column set is a row set). In this case let $C_{p,q}^{pq}$ be the subspace spanned by those $A_r$ with row set $U_p$ and column set $V_q$. Then $C_{p,q}^{pq}$ is isomorphic to a
stochastic $u_p \times u_q$ incidence space, called an associated incidence space, and

$$C_{pq} \cdot C_{rs} = \begin{cases} C_{ps} & \text{if } q = r \\ \{0\} & \text{if } q \neq r \end{cases}$$

for all $p, q, r, s$. Conversely, given any set of stochastic $u_p \times u_q$ incidence spaces $C_{pq}$ satisfying the above multiplicative condition, we can construct a semi-stochastic incidence space having these as its associated incidence spaces.

If an incidence space contains the identity matrix $I$ and is closed under left and right multiplication by $J$, then it is semi-stochastic.
2. Semi-Symmetric Incidence Algebras. The incidence algebra \( \mathcal{A} = [A_1, A_2, \ldots, A_d] \) is called semi-symmetric if it is closed under matrix transposition; i.e., if for each \( p = 1, 2, \ldots, d \), there is a \( p' \) such that \( A_p^T = A_{p'} \). A semi-symmetric incidence algebra is clearly semi-stochastic and, algebraically, semi-simple.

After a suitable simultaneous renumbering of rows and columns, which does not alter the incidence or algebraic properties of the algebra, certain of the basis elements have the form

\[
A_1 = \begin{bmatrix} \cdots & J_{a_1} & 0 & \cdots \\
J_{a_1} & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & J_{a_1} \\
\cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix} \cdots & 0 & \cdots & \cdots \\
0 & \cdots & J_{a_2} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

\[
A_8 = \begin{bmatrix} 0 & \cdots & 0 & J_{a_8} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

where the empty spaces are zeros. The row and column sets of \( \mathcal{A} \) are the row sets \( U_1, U_2, \ldots, U_8 \) of \( A_1, A_2, \ldots, A_8 \) respectively.

If \( A_p \) has row set \( U_i \) and column set \( U_j \), then \( A_p \) has various submatrices equal to \( J_{a_i, a_j} \) arranged as entries in an \( n_i \times n_j \) incidence matrix, e.g.,
again with zeros in the blanks. If we replace the $J'$ by 1 (and the $a_i', a_j$ by 0) we arrive at a new $(n_1+n_2+...+n_9) \times (n_1+n_2+...+n_9)$ incidence algebra, also semi-symmetric, which contains the identity $I$. These two algebras are clearly isomorphic. Thus the study of semi-symmetric incidence algebras may be reduced to the study of those containing $I$.

The unit element of $\mathcal{A}$ is clearly $a_1^{-1}A_1 + a_2^{-1}A_2 + ... + a_9^{-1}A_9$. One may also prove that if an arbitrary incidence algebra $\mathcal{A} = [A_1, A_2, ..., A_d]$ contains a unit element of the form $x_1A_1 + x_2A_2 + ... + x_dA_d$ with each $x_i$ non-negative, then $\mathcal{A}$ is equivalent, in the above sense, to an incidence algebra containing $I$.

An incidence algebra is called **symmetric** if each of the semi-symmetric but symmetric basis matrices are symmetric, and **anti-symmetric** if, the only symmetric basis matrices are those with ones on the main diagonal. A symmetric incidence algebra is commutative and stochastic.

A stochastic, semi-symmetric incidence algebra is symmetric if and only if all the basis matrices have only real characteristic roots.
In an arbitrary incidence algebra the subset of those matrices whose transposes are also in the algebra is a semi-symmetric incidence algebra. The subset of symmetric matrices is an incidence space, and an algebra if and only if it is commutative.
In this section we assume that \( \mathcal{K} \) is algebraically closed.

3. Characters of Commutative Incidence Algebras. Let \( \mathcal{O} \) be a commutative \( n \times n \) incidence algebra.

It is stochastic because each \( \Lambda_p \) commutes with \( J = J_{n,n} \).

For any \( \lambda_1, \ldots, \lambda_d \) in \( \mathcal{K} \) let \( M(\lambda_1, \ldots, \lambda_d) \) denote the maximal space of \( n \times 1 \) vectors over \( \mathcal{K} \) such that for each \( p = 1, 2, \ldots, d \)

\[
(\Lambda_p - \lambda I)^{\mu_p} \cdot M(\lambda_1, \ldots, \lambda_d) = \{0\}
\]

with \( \mu_p \) some positive integer. Since \( \mathcal{O} \) is commutative, \( \mathcal{O} \cdot M(\lambda_1, \ldots, \lambda_d) \) is contained in \( M(\lambda_1, \ldots, \lambda_d) \).

There are only a finite number of \( M(\lambda_1, \ldots, \lambda_d) \) different from \( \{0\} \). Index those with not all \( \lambda_1 = 0 \) by \( \eta_1, \ldots, \eta_t \), with

\[
M_{\eta_k} = M(\lambda_1^{(k)}, \ldots, \lambda_d^{(k)}) \quad k = 1, 2, \ldots, t
\]

and let \( M_0 = M(0, \ldots, 0) \). Note that \( \eta_0 = \{0\} \) if and only if \( \mathcal{O} \) contains \( I \). The set \( M \) of all \( n \times 1 \) vectors over \( \mathcal{K} \) is the direct sum of \( M_0, M_1, \ldots, M_t \). The maximal subspace of \( M \) annihilated by a power of \( \Lambda_p - \lambda I \) is the direct sum of those \( M_{\eta_k} \) with \( \lambda_1^{(k)} = \lambda \).

The non-zero characters of \( \mathcal{O} \) are the \( t \) homomorphisms of \( \mathcal{O} \) onto \( \mathcal{K} \):

\[
\chi_{\eta_k} : \sum_{p=1}^d x_p \Lambda_p \rightarrow \sum_{p=1}^d x_p \lambda_p^{(k)} \quad k = 1, 2, \ldots, t.
\]
If we denote the dimension of $\mathcal{M}_k$ by $v_k$, then $v_k$ is called the multiplicity of $\chi_k$. We have $v_o + v_1 + \ldots + v_t = n$, with $v_o$ non-negative and the other $v_k$ positive integers. If $\sigma$ is any automorphism of $\mathcal{K}$, then $\chi_k^\sigma$ maps $\Lambda_p$ into $(\chi_k(\Lambda_p))^\sigma$ and thus is a character of $\sigma$. Denote it by $\chi_{\sigma(k)}$. Then $v_{\sigma(k)} = v_k$.

Thus, if $\lambda_p^{(1)}$ and $\lambda_p^{(1)}$ are roots of the same irreducible polynomial over a ground field of $\mathcal{K}$ then $v_i = v_j$.

Since $\sigma$ is stochastic, for each $p = 1, 2, \ldots, d, a_p$ is a characteristic root of $\Lambda_p$ (the one of largest absolute value) and $e = (1 \ 1 \ \ldots \ 1)^T$ is a characteristic vector. Since the characteristic root $n = a_1 + \ldots + a_d$ of $J$ is not multiple, one of the $\mathcal{M}_k$, say $\mathcal{M}_1$, is just the scalar multiples of $e$. That is, $v_1 = 1$ and $\chi_1(\Lambda_p) = \lambda_p^{(1)} = a_p$ for all $p$.

Let

$$\Lambda = (\lambda_i^{(1)})$$

denote the $d \times t$ matrix of characteristic roots of the $\Lambda_p$ as defined above. Since $J$ has one characteristic root equal to $n$ and the others equal to 0, the column sums of $\Lambda$ are, in order, $n, 0, \ldots, 0$.

Let

$$D_v = \text{diag}(v_1, \ldots, v_t)$$.
Then the trace of $A_p$ is the $i$-th row sum of $\Lambda D_v$. Let $t_{pq}$ denote the trace of $A^p A^q$, and let

$$ T = (t_{ij}) $$

Then

$$ T = \Lambda D_v \Lambda^T. \tag{3.1} $$

Let $e_1, \ldots, e_d$ denote the primitive idempotents of $\mathcal{O}$. Then $\chi_i(e_j) = \delta_{ij}$. Let $\mu_{ip}$ be defined by $e_i = \sum_{p=1}^d \mu_{ip} A^p$, and let $M$ denote the $t \times d$ matrix

$$ X = (\mu_{ij}). $$

Then

$$ M \Lambda = I_t, \tag{3.2} $$

the identity matrix of order $t$. Using this in (3.1) we get

$$ T^T D_v^{-1} = \Lambda \tag{3.3} $$

From these equations we have

$$ \text{rank } T = \text{rank } \Lambda = t $$

$$ \text{nullity } T = d-t = \text{dimension of radical of } \mathcal{O} $$

Thus $T$ and $\Lambda$ are non-singular if and only if $d = t$ or, equivalently, $\mathcal{O}$ is semi-simple. In that case (3.1) yields

$$ (\det \Lambda)^2 = (\det T) / \prod_{i=1}^t v_i \tag{3.4} $$

Suppose $\mathcal{O}$ is semi-symmetric, and let $A^T_p = A_p$, as before. Then

$$ \chi_i(A^p) = \chi_i(A^p) $$

- 11 -
\[ \lambda^{(1)}_{p'} = \overline{\lambda^{(1)}_p} \]

where the bar denotes complex conjugation. In this case \( \mathcal{A} \) is semi-simple and (3.4) becomes

\[ (\det \Lambda)^2 = (-1)^{s/2} n^d \prod_{p=1}^d s_p / \prod_{i=1}^d v_i \]

where \( s \) is the number of \( p \neq p' \). In this case \( \prod v_i \) must divide \( n^d \prod a_p \). We also have \( \mu_{ip} = v_i \lambda^{(1)}_{p'} / n a_p \) and the two equalities

\[ \sum_{i=1}^d v_i \lambda^{(1)}_p \lambda^{(1)}_{q'} = \delta_{pq} n a_p \quad p, q=1, 2, \ldots, d \]

\[ \sum_{i=1}^d v_i \lambda^{(1)}_p \lambda^{(1)}_{q} \lambda^{(1)}_{r'} = n a_{pq} (r) \quad p, q, r=1, 2, \ldots, d. \]
4. **Group Generation.** Let $G$ be a group of $n \times n$ permutation matrices. Let $\mathcal{A}(G)$ be the vector space of all matrices in $\mathcal{X}_n$ commuting with every element of $G$. Then $\mathcal{A}(G)$ is an incidence algebra, called a *group-generated* incidence algebra. We let $\mathcal{P}(G)$ denote the algebra spanned by $G$ over $\mathcal{X}$.

Let $P$ be the permutation group on $\{1, 2, \ldots, n\}$ corresponding to $G$. Then $(x_{ij})$ is an element of $\mathcal{A}(G)$ if and only if

$$x_{\pi(i)\pi(j)} = x_{ij} \quad i, j = 1, 2, \ldots, n$$

for all $\pi$ in $P$. We shall say that $G$ is transitive if $P$ is transitive, and generally speak of the permutations in $G$ as if they were the corresponding permutations in $P$. The elements of the incidence basis of $\mathcal{A}(G)$ are the incidence matrices of the equivalence sets of the equivalence: $(i, j) \sim (i', j')$ if and only if there is an element of $P$ taking $i$ into $i'$ and $j$ into $j'$. That is, the $i,j$ element of the $p$-th incidence matrix is 1 if and only if $(i, j)$ is in the $p$-th equivalence set.

A fundamental fact about $\mathcal{A}(G)$ is that it and $\mathcal{P}(G)$ are centralizers of each other and of no larger subsets in $\mathcal{X}_n$. The center of each is their intersection, and is spanned by the matrices formed by summing the elements in each conjugacy class of $G$.

If $\mathcal{A}$ is an arbitrary incidence algebra we let $G(\mathcal{A})$ denote the set of permutation matrices which commute with every element of $\mathcal{A}$. Then $\mathcal{A}(G(\mathcal{A}))$ contains $\mathcal{A}$, and $\mathcal{A}(G')$ contains $\mathcal{A}$ if and only if $G(\mathcal{A})$ contains $G'$. Thus $\mathcal{A}$ is group-generated if and only if it is generated by $G(\mathcal{A})$. Also if $G(\mathcal{A})$ is a proper subgroup of $G'$ then $\mathcal{A}(G')$ is a proper subalgebra of $\mathcal{A}(G(\mathcal{A}))$. 

- 13 -
Some of the properties of $\mathcal{O}(G) = \{A_1, A_2, \ldots, A_d\}$ are

1) $\mathcal{O}(G)$ is semi-symmetric (and thus semi-simple) and contains $I$.
2) $\mathcal{O}(G)$ is stochastic if and only if $G$ is transitive.
3) $\mathcal{O}(G)$ is commutative if and only if it is contained in $\mathcal{P}(G)$.
4) $\mathcal{O}(G)$ is anti-symmetric if and only if $G$ is of odd order.
5) $\mathcal{O}(G)$ is symmetric if and only if each transposition occurs as a disjoint cycle in $G$.

6) The row and column sets of $\mathcal{H}(G)$ are the transitivity sets of $G$.
7) If $G_i$ is the subgroup of $G$ leaving $i$ fixed then $a_p = [G_i : G_i \cap G_j]$ for any $i, j$ such that the $i, j$ element of $a_p$ is 1.
8) If $G$ is transitive then $d$ is the number of transitivity sets of $G_i$.
9) The order of the row set $U_p$ is $[G : G_i]$ for any $i$ in $U_p$.
10) $\mathcal{O}(G) = \mathcal{P}(G)$ if and only if $G$ is transitive and abelian.
11) If $G$ contains a full-cycle permutation the elements of $\mathcal{O}(G)$ are polynomials in that matrix.
12) If $Q$ is a permutation matrix $\mathcal{O}(QGQ^{-1}) = Q \mathcal{O}(G)Q^{-1}$.
13) If $\mathcal{P}(G)$ is written as a sum of irreducible representations $\Gamma_i$ of dimension $k_i$: $\mathcal{P}(G) = n_1 \Gamma_1 + n_2 \Gamma_2 + \ldots + n_r \Gamma_r$, then $n = \sum_{i=1}^r n_i k_i$, $d = \sum_{i=1}^r n_i^2$, and $\mathcal{O}(G)$ is isomorphic to $\mathcal{H}_{n_1} + \mathcal{H}_{n_2} + \ldots + \mathcal{H}_{n_r}$.
14) $\mathcal{O}(G) = [1, J-I]$ if and only if $G$ is doubly transitive.
As an illustration of the applications of these properties consider the following. If G is doubly transitive then \( \mathcal{O}(G) = [I,J-J] \). The centralizer of this \( \mathcal{O}(G) \) is the set of matrices which commute with \( J \), the set of all matrices with constant row and column sum. The centralizer of \( \mathcal{O}(G) \) is also \( \mathcal{Q}(G) \). Therefore any matrix with constant row and column sums is a linear combination of matrices taken from a doubly transitive group of permutation matrices.

We use a similar method to construct an incidence space \( S(G,H) \) from a pair of groups of permutation matrices \( G,H \) whose corresponding permutation groups \( P,Q \) are representations of degrees \( m \) and \( n \) respectively of some finite group. To each element \( M \) of \( G \) and \( \pi \) of \( P \) we denote by \( M^\pi \) and \( \pi^M \) the corresponding elements of \( H \) and \( Q \) respectively.

The \( S(G,H) \) has a similar set of three equivalent definitions as does \( \mathcal{O}(G) \) above: \( S(G,H) \) is the set of all \( m \times n \) matrices \( X = (x_{ij}) \) such that

\[
X_{\pi^i_M}(i,j) = x_{i\pi^j_M}(j) \quad i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n
\]

for all \( \pi \) in \( P \); (ii) \( PX = X\pi^P \) for all \( P \) in \( G \); and (iii) \( S(G,H) \) has as a basis the set of incidence matrices of the equivalence sets of the equivalence: \( (i,j) \sim (i',j') \) if and only if there is an element \( \pi \) of \( P \) such that \( \pi \) takes \( i \) into \( i' \) and \( \pi^M \) takes \( j \) into \( j' \).

Among the properties of \( S(G,H) \) are: it is closed under left multiplication by \( J_m \) and under right multiplication by \( J_n \); the
transposes form the incidence space $S(H,G)$; if $G$ is transitive then the dimension of $S(G,H)$ is equal to the number of transitivity sets of $G^T$ and the row sums of the incidence basis are the orders of these transitivity sets; and if $U,V,W$ are any elements of $S(G,H)$ then $UV^TW$ is an element of $S(G,H)$. It follows that the set
\[ \{UV^T : U,V \in S(G,H) \} \]
which is a subset of $A(G)$, is closed under multiplication.

An example of the results of this method is the following.

Let $K$ be a group with a subgroup $M$ of double index 2 (i.e. $K = N + M \times M$) and index $v$. Let $N$ be any other subgroup of $K$ of index $v$. Then a $v,k,\lambda$ design exists with $k$ equal to the index of $MN$ in $M$.

What amounts to this method is described by J. S. Frame in [4]. His primary interest was in the derivable group properties, so that his results and ours do not intersect.
5. Dimension 3 with identity. Suppose $\mathcal{U}$ has dimension 3 and contains 1. Then $\mathcal{U}$ is stochastic and commutative, and we may write

$$\mathcal{U} = [I, A, J-I-A].$$

Let $n$ be the order of $A$, $a$ its constant row and column sum, and $c_1, c_2, c_3$ non-negative integers such that

$$A^2 = c_1 I + c_2 A + c_3 (J-I-A).$$

The other structure constants of $\mathcal{U}$, each of which is a non-negative integer, may be determined from

$$A(J-I-A) = (a-c_1) I + (a-c_2 - 1) A + (a-c_3)(J-I-A)$$

$$(J-I-A)^2 = (n-2a-1+c_1) I + (n-2a+c_2) A + (n-2a-2+c_3)(J-I-A)$$

By interchanging $A$ and $J-I-A$, if necessary, we can assume that

$$a \leq \frac{n-1}{2}$$

Then the $c_i$ are non-negative integers satisfying

$$a \geq c_1, c_2 + 1, c_3 \geq 0$$

and $c_3 \geq 1$ if $a = (n-1)/2$.

$A$ is normal if and only if $\mathcal{U}$ is either symmetric or anti-symmetric. $\mathcal{U}$ is symmetric if and only if $c_1 = a$, and anti-symmetric if and only if $c_1 = 0$. 

- 16 -
\( \mathcal{A} \) is always semi-simple, and therefore has three characters \( \chi_1, \chi_2, \chi_3 \). Set \( a = \chi_1(\lambda) \) and let \( \lambda_2 = \chi_2(\lambda) \) and \( \lambda_3 = \chi_3(\lambda) \). The characteristic roots \( \lambda_2, \lambda_3 \) have multiplicities \( v_2, v_3 \) and satisfy the equation

\[
x^2 + (c_3 - c_2)x + c_3 - c_1 = 0
\]

We also have

\[
v_2 + v_3 = n-1
\]

\[
v_2\lambda_2 + v_3\lambda_3 = -a
\]

and

\[
(\lambda_2 - \lambda_3)^2v_2v_3 = n(c_1(n-1) - a^2)
\]

If \( \lambda_2, \lambda_3 \) are irrational then \( a = v_2 = v_3 = (n-1)/2 \) and either

1. \( \mathcal{A} \) is symmetric, \( n \equiv 1 \) (mod 4), \( A^2 + A = ((n-1)/4)(J+I) \), \( \lambda_2, \lambda_3 = (-1+\sqrt{n^2})/2 \)
2. \( \mathcal{A} \) is anti-symmetric; \( n \equiv 3 \) (mod 4), \( A^2 + A = ((n+1)/4)(J-I) \), \( \lambda_2, \lambda_3 = (-1-\sqrt{n^2})/2 \). This latter is the only way for \( \mathcal{A} \) to be anti-symmetric. It leads to the equation

\[
AA^T = \frac{n+1}{4} I + \frac{n-3}{4} J
\]

which means that \( A \) is a v,k,\( \lambda \) design from which one can construct a Hadamard matrix of order \( n+1 \). From our method of the previous section (see properties 4 and 8), a sufficient condition for \( A \) to exist is that there be a transitive permutation group on \( n \) letters, of odd order, with the subgroups leaving one letter fixed having exactly three transitivity sets. Unfortunately, when \( n \) is not a prime power such a group would be primitive, and therefore insoluble of odd order, contrary to the classical conjecture.
Now assume that $\lambda_2$ and $\lambda_3$ are rational. Then one is negative and the other non-negative. Let

$$s = \lambda_2 \geq 0 \text{ and } t = -\lambda_3 \geq 1.$$  

We can show that if $\theta$ is semi-symmetric it is symmetric, that $a$ is never a prime, that $a^2 \geq a^2 + at$ with equality if and only if $A$ is singular, that $A^T \neq A$ implies $a^2 + a \geq n$, that $s \neq a$ implies $a \leq a + 2 - 2(a+1)^{\frac{1}{2}}$ and $t \neq a$ implies $t \leq (a+1)/2$.  

- 18 -
6. A graph theory problem. Given $2t$ points, consider the set $\mathcal{L}$ of all (linear, undirected) graphs on these points consisting of $t$ disjoint lines. That is, in each graph each point is the neighbor of exactly one other point. There are

$$(6.1) \quad n = (2t-1)(2t-3)...3.1$$

such graphs:

$$\mathcal{L} = \{L_1, L_2, ..., L_n\}.$$

The union of two elements of $\mathcal{L}$ is a collection of disjoint cycles each having an even number of points. A cycle with $2k$ points is said to be of length $k$. Thus the lengths of the cycles form a partition of $t$.

Given a partition $\pi = (j_1^1, j_2^2, ..., j_t^t)$ of $t$:

$$t = j_1 + 2j_2 + ... + tj_t,$$

let $\mathcal{H}_\pi$ denote the set of all graphs on the $2t$ fixed points have $j_k$ cycles of length $k$ for $k = 1, 2, ..., t$. Let $\mathcal{R}$ be the set of partitions of $t$:

$$\mathcal{R} = \{\pi_1, \pi_2, ..., \pi_d\} \quad d = p(t),$$

where $p(t)$ denotes the number of partitions of $t$.

Given fixed elements $L$ of $\mathcal{L}$ and $\pi = (j_1^1, j_2^2, ..., j_t^t)$ of $\mathcal{R}$, it is easy to show that there are
elements of $L$ whose union with $L$ lies in $H_p$.

The following related question is more difficult to answer. Given fixed elements $L_i$ and $L_j$ of $L$, and $x_p$ and $x_q$ of $R$, for how many elements $L_k$ of $L$ do we have

\[(6.2) \quad L_i \cup L_k \in H_p \quad \text{and} \quad L_k \cup L_j \in H_q\]

It can be shown that the answer $a_{pq}^{(r)}$ only depends upon $x_p$, $x_q$, and $x_r$ where

\[(6.3) \quad L_i \cup L_j \in H_x\]

To solve this problem, we shall construct an incidence algebra whose structure constants are the above mentioned $a_{pq}^{(r)}$.

Let $S_{2t}$ be the set of permutations of the $2t$ fixed points. Let $H_1$ be the subgroup of $S_{2t}$ leaving $L_1$ fixed. Let $H_1, H_2, \ldots, H_n$ be the left cosets of $H_1$ in $S_{2t}$, such that any element of $H_i$ takes $L_1$ into $L_i$. This is the same $n$ as in (6.1).

For any element $\sigma$ of $S_{2t}$ we have

\[(6.4) \quad H_i = H_{\gamma_\sigma(i)} \quad i = 1, 2, \ldots, n\]

where $\gamma_\sigma$ is a permutation of the integers $1, 2, \ldots, n$. Let
\[ P = \{ \gamma_\sigma : \sigma \in S_{2t} \} \]

be the group of all permutations \( \gamma_\sigma \). Let \( G \) be the group of permutation matrices of order \( n \) corresponding to \( P \).

The incidence algebra we are look-for is \( \mathcal{O}(G) \).

The proof of this depends on two facts: (1) any element of \( S_{2t} \) permutes the elements of \( \mathbb{H}_\pi \), for any \( \pi \) in \( P \); and (2) if \( L_i \cup L_j \) and \( L_i' \cup L_j' \) are both in \( \mathbb{H}_\pi \), then there is an element of \( S_{2t} \) taking \( L_i \) into \( L_i' \) and \( L_j \) into \( L_j' \).

Let us restate (1) and (2) in terms of ordered pairs of the cosets \( H_1, H_2, \ldots, H_n \) and the equivalence:

\[ (H_i, H_j) \sim (H_i', H_j') \]

if and only if there is an element of \( P \) taking \( i \) into \( i' \) and \( j \) into \( j' \). Then (1) and (2) imply that the equivalence sets \( \overline{\mathbb{H}}_\pi', \overline{\mathbb{H}}_\pi_2', \ldots, \overline{\mathbb{H}}_\pi_d \) of this equivalence correspond to the sets \( \mathbb{H}_\pi', \mathbb{H}_\pi_2', \ldots, \mathbb{H}_\pi_d \) in such a way that \( (H_i, H_j) \) is in \( \overline{\mathbb{H}}_\pi \) if and only if \( L_i \cup L_j \) is in \( \mathbb{H}_\pi \).

Let \( A_p = (a_{ij}) \) be the incidence matrix of \( \overline{\mathbb{H}}_\pi \). That is,

\[ a_{ij} = \begin{cases} 1 & \text{if } (H_i, H_j) \in \overline{\mathbb{H}}_\pi \\ 0 & \text{otherwise} \end{cases} \]

From our discussion in section 4, we know that \( A_p \) is one of the elements of the incidence basis of \( \mathcal{O}(G) \). If \( \{ a_{pq}^{(r)} \} \) is the set of structure constants of this basis we have
\[ A_p A_q = \sum_{r=1}^{d} a_{pq}^{(r)} A_r. \]

Comparing the \( i, j \) elements we have

\[ \sum_{k=1}^{n} a_{pik} a_{qkj} = a_{pq}^{(r)} \]

where

\[ a_{rij} = 1. \]

Since \( a_{pik} a_{qkj} = 1 \) if and only if \( a_{pik} = a_{qkj} = 1 \), the constant \( a_{pq}^{(r)} \) is just the number of \( H_k \) such that

\[ (H_i, H_k) \in \overline{\mathcal{F}}_{p} \quad \text{and} \quad (H_k, H_j) \in \overline{\mathcal{F}}_{q} \]

where

\[ (H_i, H_j) \in \overline{\mathcal{F}}_r. \]

Comparing these conditions with (6.2) and (6.3), we see that the structure constant \( a_{pq}^{(r)} \) is indeed the answer to our problem.

That this is a reasonable solution follows from that fact that we can actually construct the incidence basis of \( \mathcal{O}(G) \) given the group \( P \). In turn this group only depends upon the group \( \Pi_1 \) which leaves the graph \( L_1 \) fixed.

This same method applies to any set of graphs \( \{L_1\} \) which are permuted among themselves by permutations of their points, so long as the sets \( \{\mathcal{H}_1\} \) into which the unions of \( L_1 \) are partitioned satisfy conditions corresponding to \( (F_1) \) and \( (F_2) \).
Under any conditions the incidence algebra $A(G)$ is symmetric (and therefore commutative and stochastic) because $L_i \cup L_j = L_j \cup L_i$, and thus $a_{pij} = a_{pj i}$ for all $p, i, j$. Furthermore, the characteristic roots of its incidence basis matrices are all rational integers. This follows from property 3 in section 4, the fact that $P$ is a representation of a symmetric group, and the fact that the characters of any representation of a symmetric group are rational (see [5], vol. II, pages 190-193).
References

1) R. C. Bose, "Versuche in unvollstandiger Blocken'',
Gastuarlesung Universitat Frankfort/M., Naturwissenschaftliche
Fakultat, 1955.

2) R. C. Bose and D. M. Mesner, "On linear associative algebras
corresponding to association schemes of partially balanced

3) D. M. Mesner, "An investigation of certain combinatorial
properties of partially balanced incomplete block experimental
designs and association schemes, with a detailed study of
designs of Latin squares and related types", unpublished

4) J. S. Frame, "Double coset matrices and group characters",

5) B. L. Van der Waerden, Modern Algebra, Frederick Ungar,
New York, 2nd ed., 1940.

6) E. C. Dade and K. Goldberg, "The construction of Hadamard matrices",

California Institute of Technology
and the
National Bureau of Standards
THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards at its major laboratories in Washington, D.C., and Boulder, Colorado, is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section carries out specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant publications, appears on the inside of the front cover.

WASHINGTON, D.C.


Office of Weights and Measures.

BOULDER, COLO.


CENTRAL RADIO PROPAGATION LABORATORY


RADIO STANDARDS LABORATORY

