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INTERMEDIARY EQUATORIAL ORBITS OF AN ARTIFICIAL SATELLITE

by

John P. Vinti

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INTERMEDIARY EQUATORIAL ORBITS OF AN ARTIFICIAL SATELLITE[1]

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John P. Vinti
Mathematical Physics Section

[1]This work was supported by the U.S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

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A previous paper derived the solution for the drag-free motion of an artificial satellite in the gravitational field of an oblate planet. The corresponding potential, expressed in oblate spheroidal coordinates, leads to separability and represents the even zonal harmonics exactly through the second, for any oblate planet, and approximately through the fourth, in the case of the earth.

The previous paper contained a restriction on the orbital inclination I, viz., $I_c < I < 180° - I_c$, where I_c might be as large as $1° 54'$ for an orbit sufficiently close to the earth. The present paper removes this restriction and shows that many of the formulae for the periodic terms may be simplified, when the orbit is equatorial or almost so. The results agree with those obtained by a direct two-dimensional solution, when the orbit is purely equatorial.

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1. This work was supported by the U.S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

1. Introduction

This paper is a sequel to a recent paper[2], concerning an accurate intermediary orbit for satellite astronomy, and will accordingly follow the notation thereof. It there followed that if

$$\lambda \equiv b_1/b_2 < 1, \tag{1}$$

all the ρ-integrals are expressible in terms of rapidly converging series involving products of Legendre polynomials with arguments λ and $(1-e^2)^{-\frac{1}{2}}$. Condition (1) is equivalent to a restriction on the orbital inclination I, viz.,

$$I_c < I < 180° - I_c, \tag{2}$$

where, to the first order in k,

$$\tan^2 I_c = k \equiv (r_e/p)^2 J_2 \tag{3}$$

For the earth $J_2 = 0.00108$, so that for orbits so close that $p \approx r_e$,

$$I_c = 1° \; 54' \tag{4}$$

I imposed the condition $\lambda < 1$ in order that

$$(1 + A\rho^{-1} + B\rho^{-2})^{-\frac{1}{2}} \equiv (1-2\lambda h + h^2)^{-\frac{1}{2}} \tag{5}$$

should be a generating function for the Legendre polynomials $P_n(\lambda)$. It now appears that such a restriction is unnecessary. Thus

$$(1 - 2\lambda h + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(\lambda) \tag{6}$$

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2. J.P. Vinti, J. Research National Bureau of Standards, 65B,169-201,(1961), hereafter referred to as(A). Any reference in the present paper to an equation with a decimal number, such as(5.30), denotes an equation in (A).

even when $\lambda \geqq 1$, provided only that[3]

$$h < \lambda - (\lambda^2 - 1)^{\frac{1}{2}} \tag{7}$$

We need show only that(7) is always satisfied, in order that all the results of (A) hold for all orbital inclinations. The only changes will be a few simplifications, especially for the cases $I = 0°$ or $180°$ of purely equatorial orbits.

2. The ρ- Integrals

To show that (7) is always satisfied when $\lambda \geqq 1$, note that

$$\frac{b_1}{\rho} = \frac{b_1}{b_2}\frac{b_2}{\rho} = \lambda h \tag{8}$$

By (4.12), (4.13), and (5.6) it follows that b_1 and b_2 are both real and non-negative for $c^2 < ap$, a relation that always holds for satellite orbits. Thus (7) is satisfied if and only if

$$b_1/\rho < g(\lambda), \tag{9}$$

where

$$g(\lambda) \equiv \lambda^2 - \lambda(\lambda^2 - 1)^{\frac{1}{2}} \tag{10}$$

We shall next show that

$$1/2 < g(\lambda) \leqq 1 \qquad (\lambda \geqq 1) \tag{11}$$

To show that $g(\lambda) \leqq 1$, note that for $\lambda \geqq 1$, we have $1 > 1-\lambda^{-2} \geqq 0$, so that $(1-\lambda^{-2})^{\frac{1}{2}} \geqq 1 - \lambda^{-2}$, $1-(1-\lambda^{-2})^{\frac{1}{2}} \leqq \lambda^{-2}$, and finally $\lambda^2-\lambda(\lambda^2-1)^{\frac{1}{2}} \leqq 1$. Thus $g(\lambda) \leqq 1$.

To show that $g(\lambda) > 1/2$, note that for $\lambda \geqq 1$

$$0 \leqq 1 - \lambda^{-2} < (1-\tfrac{1}{2}\lambda^{-2})^2, \tag{12}$$

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3. E.W. Hobson, "The Theory of Spherical and Ellipsoidal Harmonics", page 15, Cambridge University Press, Cambridge 1931.

so that

$$(1-\lambda^{-2})^{\frac{1}{2}} < 1 - \tfrac{1}{2}\lambda^{-2} \tag{13}$$

or

$$1-(1-\lambda^{-2})^{\frac{1}{2}} > \tfrac{1}{2}\lambda^{-2} \tag{14}$$

Then

$$g(\lambda) \equiv \lambda^2-\lambda(\lambda^2-1)^{\frac{1}{2}} > \tfrac{1}{2}, \tag{15}$$

as was to be shown.

From (9) and (11) it follows that when $\lambda \geqq 1$ the condition $b_1/\rho < 1$ is necessary for the validity of (7) and that the condition $b_1/\rho < 1/2$ is sufficient for its validity.

To show that $b_1/\rho < 1/2$ for all bound[4] orbits, note that $\rho \geqq \rho_1$, so that

$$b_1/\rho \leqq b_1/\rho_1 \tag{16}$$

From the relation $c^2 = kp^2$ and the relation $\rho_1 = a(1-e)$, $e \leqq 1$, for a bound orbit, we then find from (3.25), (4.12), and (5.6) that

$$\frac{b_1}{\rho_1} = \frac{k(1+e)(1-\eta_0^2)[1-k\eta_0^2(1-e^2)]}{[1-k(1-e^2)][1-k\eta_0^2(1-e^2)]+4k\eta_0^2} \tag{17}$$

Here the numerator $\leqq k(1+e)$ and the denominator $\geqq (1-k)^2$, so that

$$\frac{b_1}{\rho_1} \leqq \frac{k(1+e)}{(1-k)^2} \leqq \frac{2k}{(1-k)^2}, \tag{18}$$

a function monotonic in k for $0 < k < 1$.

For the earth $k < 0.00108$, so that

$$b_1/\rho_1 < 0.00216 < \tfrac{1}{2} \tag{19}$$

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4. Past participle of the verb "to bind", taken from the terminology of atomic theory.

From (16) and (19) it follows that for the earth

$$b_1/\rho < 1/2 \tag{20}$$

This is the condition sufficient for the validity of (7) and thus of the expansion (6), for all $\lambda \geqq 1$.

All the developments of (A) then hold for the ρ-integrals, in particular (5.30) through (5.33) for the integral R_1, (5.35) through (5.40) for R_2, and (5.60) through (5.65) for R_3, where the D_m's are again given by (5.50) and (5.53).

To show the rapid convergence of the various series that there occur, we note first that since

$$P_n(\lambda) = \pi^{-1} \int_0^{\pi} [\lambda + (\lambda^2-1)^{\frac{1}{2}} \cos x]^n dx \tag{21}$$

and since $\lambda \equiv b_1/b_2 \geqq 1$, we have

$$(b_2/p)^n \, P_n(\lambda) = \pi^{-1} \int_0^{\pi} [b_1 p^{-1} + p^{-1}(b_1^2 - b_2^2)^{\frac{1}{2}} \cos x]^n dx \tag{22}$$

Then

$$\left| \left(\frac{b_2}{p}\right)^n P_n(\lambda) \right| \leqq \left[\frac{b_1}{p} + \frac{b_1}{p} (1 - \lambda^{-2})^{\frac{1}{2}} \right]^n \leqq \left(\frac{2b_1}{p} \right)^n \tag{23}$$

From (5.14) and (5.34) the series S_1 and S_2 that occur in the expressions for the ρ-integrals R_1 and R_2 are

$$S_j \equiv \sum_{n=n_j}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda) \int_0^{v} (1 + e \cos x)^{n-n_j} dx \, , \qquad (j=1,2) \tag{24}$$

where $n_1 = 2$ and $n_2 = 0$. Thus

$$|S_j| \leqq v \sum_{n=n_j}^{\infty} \left(\frac{2b_1}{p}\right)^n (1+e)^{n-n_j} \tag{25}$$

$$\leqq v \sum_{n=0}^{\infty} \left(\frac{2b_1}{p}\right)^{n+n_j} (1+e)^n \leqq v\left(\frac{2b_1}{p}\right)^{n_j} \sum_{n=0}^{\infty} \left[\frac{2b_1}{p}(1+e)\right]^n \tag{26}$$

$$\leqq \frac{v\left(\frac{2b_1}{p}\right)^{n_j}}{1 - \frac{2b_1}{p}(1+e)} \qquad (j = 1,2) \tag{27}$$

But $p = a(1-e^2) = \rho_1(1+e)$, so that by (18)

$$b_1 p^{-1} \leqq k(1-k)^{-2} \tag{28}$$

and

$$2b_1(1+e)\, p^{-1} \leqq 2k(1+e)(1-k)^{-2} \leqq 4k(1-k)^{-2}, \tag{29}$$

where $4k(1-k)^{-2} < 0.0043$ for the earth. It follows that the series for R_1 and R_2, and thus the series for the secular coefficients A_1 and A_2, converge absolutely and more rapidly than a geometric series of common ratio 0.0043.

By (5.49), (5.50), and (5.53) the series S_3 that occurs in the expression for the ρ-integral R_3 is

$$S_3 = S_e + S_o, \tag{30}$$

$$S_e \equiv \sum_{n=0}^{\infty} D_{2n} \int_0^v (1 + e\cos x)^{2n+2} dx \tag{31}$$

$$S_o \equiv \sum_{n=0}^{\infty} D_{2n+1} \int_0^v (1 + e\cos x)^{2n+3} dx \tag{32}$$

$$D_{2n} \equiv \sum_{j=0}^{n} (-1)^{n-j} (c/p)^{2n-2j} (b_2/p)^{2j} P_{2j}(\lambda) \tag{33}$$

$$D_{2n+1} \equiv \sum_{j=0}^{n} (-1)^{n-j} (c/p)^{2n-2j} (b_2/p)^{2j+1} P_{2j+1}(\lambda) \tag{34}$$

From (33) and (23) and the relation $c^2 = kp^2$, it follows that

$$|D_{2n}| \leqq k^n \sum_{j=0}^{n} (2b_1/c)^{2j} \leqq k^n \sum_{j=0}^{\infty} (2b_1/c)^{2j}$$

$$\leqq \frac{k^n}{1 - (2b_1/c)^2} \tag{35}$$

and from (34) and (23) that

$$|D_{2n+1}| \leqq k^n (2b_1/p) \sum_{j=0}^{n} (2b_1/c)^{2j} \leqq k^n (2b_1/p) \sum_{j=0}^{\infty} (2b_1/c)^{2j}$$

$$\leqq \frac{k^n (2b_1/p)}{1 - (2b_1/c)^2} \tag{36}$$

Thus, by (31) and (35)

$$|S_e| \leqq \frac{(1+e)^2 v}{1 - (2b_1/c)^2} \sum_{n=0}^{\infty} [k(1+e)^2]^n$$

$$\leqq \frac{(1 + e)^2 v}{[1-(2b_1/c)^2][1-k(1+e)^2]} \tag{37}$$

and by (32) and (36) that

$$|S_o| \leqq \frac{(1+e)^3 v(2b_1/p)}{[1-(2b_1/c)^2]} \sum_{n=0}^{\infty} [k(1+e)^2]^n$$

$$\leqq \frac{(1+e)^3 v(2b_1/p)}{[1-(2b_1/c)^2][1-k(1+e)^2]} \tag{38}$$

Then, by (30), (37), and (38), we have

$$|S_3| \leqq \frac{(1+e)^2 v[1+(1+e)(2b_1/p)]}{[1-(2b_1/c)^2][1-k(1+e)^2]} \tag{39}$$

But $p = \rho_1(1+e)$, so that by (18)

$$2b_1/p \leqq 2k(1-k)^{-2} \tag{40}$$

and

$$2b_1/c \equiv (2b_1/p)\, k^{-\frac{1}{2}} \leqq 2k^{\frac{1}{2}}(1-k)^{-2} \tag{41}$$

Thus

$$|S_3| \leqq \frac{(1+e)^2 v[1+2k(1+e)(1-k)^{-2}]}{[1-4k(1-k)^{-4}][1-k(1+e)^2]} \tag{42}$$

The series for S_3 thus converges absolutely and more rapidly than the power series in k of the function on the right side of (42), where $k < 0.0011$ for the earth. On replacing v by π, we can then say the same thing about the secular coefficient A_3.

3. Simplification of the ρ-Coefficients when $\lambda \geqq 1$

From $\lambda \equiv b_1/b_2$ it follows that

$$b_2 = b_1 \lambda^{-1} \leqq b_1 \qquad (\lambda \geqq 1) \tag{43}$$

Since $b_1=O(k)$, it then follows that b_2 is also of order k when $\lambda \geqq 1$. This fact enables us to simplify the coefficients A_{1n} and A_{2n}, which are needed only to $O(k^2)$, and the coefficients A_{3n}, which are needed only to O(k).

Thus (5.32) and (5.33) lead to

$$A_{11} = O(k^3) \qquad A_{12} = O(k^4), \tag{44}$$

(5.39) and (5.40) lead to

$$A_{23} = O(k^3) \qquad A_{24} = O(k^4), \tag{45}$$

and (5.37) and (5.38) lead to

$$A_{21} = (1-e^2)^{\frac{1}{2}} p^{-1} e[b_1 p^{-1} + (3-\lambda^{-2}) k^2 \cos^4 I] \tag{46}$$

$$A_{22} = (1-e^2)^{\frac{1}{2}} p^{-1} \frac{e^2}{8} (3-\lambda^{-2}) k^2 \cos^4 I \tag{47}$$

Here we have used

$$b_1 p^{-1} = k \cos^2 I + O(k^2) \tag{48}$$

in the terms involving $b_1^2 p^{-2}$.

Similarly, for the coefficients A_{3n}, which are needed only to O(k), we obtain from (5.62) through (5.65)

$$A_{31} = (1-e^2)^{\frac{1}{2}} p^{-3} e[2+ (3+ \frac{3}{4} e^2) k \cos^2 I-(4+ 3e^2)k] \tag{49}$$

$$A_{32} = (1-e^2)^{\frac{1}{2}} p^{-3} e^2[\frac{1}{4} + \frac{3}{4} k \cos^2 I - (\frac{e^2}{4}+ \frac{3}{2})k] \tag{50}$$

$$A_{33} = (1-e^2)^{\frac{1}{2}} p^{-3} e^3[\frac{k}{12} \cos^2 I - \frac{k}{3}) \tag{51}$$

$$A_{34} = - (1-e^2)^{\frac{1}{2}} p^{-3} e^4 k/32 \tag{52}$$

For $\lambda \geqq 1$ we need also to rewrite (5.31), (5.36), (5.50), and (5.53) for the secular coefficients A_1, A_2, D_{2n}, and D_{2n+1}, which all contain terms of the form $(b_2/p)^m P_m(\lambda)$. As sin I approaches zero, b_2 also approaches zero and λ becomes infinite, so that such a term takes the indeterminate form zero times infinity. To remove this indeterminacy, note that

$$\frac{b_2}{p} \equiv \frac{b_2}{b_1}\frac{b_1}{p} \equiv \frac{b_1}{p}\lambda^{-1}, \tag{53}$$

so that

$$\left(\frac{b_2}{p}\right)^m P_m(\lambda) = \left(\frac{b_1}{p}\right)^m \lambda^{-m} P_m(\lambda) \tag{54}$$

$$= \left(\frac{b_1}{p}\right)^m R_m(\lambda^{-1}) \tag{55}$$

Here $R_m(x)$ is a function that has already appeared in (A), viz.,

$$R_m(x) \equiv x^m P_m(x^{-1}) \tag{56}$$

a polynomial of degree $[m/2]$ in x^2.

To determine $(b_2/p)^m P_m(\lambda)$ for sin I = 0, first write

$$P_m(\lambda) = \sum_{j=0}^{[m/2]} \frac{(-1)^j (2m-2j)!\, \lambda^{m-2j}}{2^m\, j!(m-j)!(m-2j)!}, \tag{57}$$

so that

$$R_m(\lambda^{-1}) \equiv \lambda^{-m} P_m(\lambda) = 1 \qquad (m=0,1)$$

$$= \frac{(2m)!}{2^m (m!)^2} + \sum_{j=1}^{[m/2]} \frac{(-1)^j (2m-2j)!\,\lambda^{-2j}}{2^m\, j!(m-j)!(m-2j)!} \qquad (m=2,3,4,\ldots) \tag{58}$$

Thus

$$R_m(0) = \frac{(2m)!}{2^m (m!)^2}, \qquad (m = 0,1,2,3,\ldots) \tag{59}$$

so that by (55) and (59)

$$(b_2/p)^m P_m(\lambda) = \frac{(2m)!}{(m!)^2} \left(\frac{b_{10}}{2p}\right)^m, \qquad (\sin I = 0) \tag{60}$$

where b_{10} is the value of b_1 for $\eta_0 \equiv \sin I = 0$.

From (5.31), (5.36), (55), and (60) it then follows that

$$A_1 = (1-e^2)^{\frac{1}{2}} p \sum_{n=2}^{\infty} (b_1/p)^n R_n(\lambda^{-1}) R_{n-2}(\sqrt{1-e^2}) \qquad (\lambda \geqq 1) \tag{61}$$

$$= (1-e^2)^{\frac{1}{2}} p \sum_{n=2}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{b_{10}}{2p}\right)^n R_{n-2} (\sqrt{1-e^2}) \qquad (\sin I = 0) \tag{62}$$

$$A_2 = (1-e^2)^{\frac{1}{2}} p^{-1} \sum_{n=0}^{\infty} (b_1/p)^n R_n(\lambda^{-1}) R_n (\sqrt{1-e^2}) \qquad (\lambda \geqq 1) \tag{63}$$

$$= (1-e^2)^{\frac{1}{2}} p^{-1} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{b_{10}}{2p}\right)^n R_n(\sqrt{1-e^2}) \qquad (\sin I = 0) \tag{64}$$

Note that $A_1 = O(k^2)$ and $A_2 = O(k^0)$.

Then, from (5.61)

$$A_3 = (1-e^2)^{\frac{1}{2}} p^{-3} \sum_{m=0}^{\infty} D_m R_{m+2} (\sqrt{1-e^2}), \tag{65}$$

where, by (5.50), (5.53), (55), (60), and the relation $c^2 = kp^2$, we find

$$D_{2n} = \sum_{j=0}^{n} (-1)^{n-j} (c/p)^{2n-2j} (b_2/p)^{2j} P_{2j}(\lambda)$$

$$= \sum_{j=0}^{n} (-1)^{n-j}\, k^{n-j} (b_1/p)^{2j}\, R_{2j}(\lambda^{-1}) \qquad (\lambda \geqq 1) \qquad (66)$$

$$= \sum_{j=0}^{n} (-1)^{n-j}\, k^{n-j}\, \frac{(4j)!}{[(2j)!]^2} \left(\frac{b_{10}}{2p}\right)^{2j} \qquad (\sin I = 0) \qquad (67)$$

$$D_{2n+1} = \sum_{j=0}^{n} (-1)^{n-j}\, (c/p)^{2n-2j} (b_2/p)^{2j+1}\, P_{2j+1}(\lambda)$$

$$= \sum_{j=0}^{n} (-1)^{n-j}\, k^{n-j} (b_1/p)^{2j+1}\, R_{2j+1}(\lambda^{-1}) \qquad (\lambda \geqq 1) \qquad (68)$$

$$= \sum_{j=0}^{n-1} (-1)^{n-j}\, k^{n-j}\, \frac{(4j+2)!}{[(2j+1)!]^2} \left(\frac{b_{10}}{2p}\right)^{2j+1} \qquad (\sin I = 0) \qquad (69)$$

4. Simplification of Other Coefficients when $\lambda \geqq 1$

To simplify the η-coefficients when $\lambda \geqq 1$, we must first note that from page (176) of (A) we have

$$b_1 \approx k\, p \cos^2 I \qquad\qquad b_2 \approx k^{\frac{1}{2}} p \sin I \,, \qquad (70)$$

so that

$$\lambda \equiv b_1/b_2 \approx k^{\frac{1}{2}} \cos^2 I \csc I \qquad (71)$$

or

$$\eta_0 \equiv \sin I \simeq k^{\frac{1}{2}} \cdot \lambda^{-1} \cos^2 I \tag{72}$$

Thus

$$\eta_0 = O(k^{\frac{1}{2}}) \qquad (\lambda \geqq 1) \tag{73}$$

Also, since $\eta_2^{-1} = O(k^{\frac{1}{2}})$, by (3.42), it follows that

$$q \equiv \eta_0/\eta_2 = O(k) \qquad (\lambda \geqq 1) \tag{74}$$

There also arises an indeterminacy in the η-coefficients, from the quantity $(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0$, which takes the form infinity times zero as $\eta_0 \equiv \sin I$ approaches zero. To remove this indeterminacy, use (4.15a), viz.,

$$\alpha_3 = \alpha_2\left(1 - \frac{c^2\eta_0^2}{a_0\, p_0}\right)^{\frac{1}{2}} \cos I, \tag{75}$$

to find

$$(\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}}\eta_0 = \alpha_2^{-1}\left(1 + \frac{c^2}{a_0\, p_0}\cos^2 I\right)^{-\frac{1}{2}} \tag{76}$$

Then, from (8.27) and (7.23)

$$\psi_s = 2\pi \nu_2 [t + \beta_1 + \beta_2 \alpha_2^{-1}(a + b_1 + A_1) A_2^{-1}] \tag{77}$$

$$2\pi \nu_2 = \alpha_2\left(1 + \frac{c^2}{a_0\, p_0}\cos^2 I\right)^{\frac{1}{2}} A_2 B_2^{-1}(a + b_1 + A_1 + c^2\eta_0^2 A_2 B_1 B_2^{-1})^{-1} \tag{78}$$

From (8.33)

$$\psi_0 = (-2\alpha_1)^{-\frac{1}{2}} \alpha_2\left(1 + \frac{c^2}{a_0\, p_0}\cos^2 I\right)^{\frac{1}{2}} A_2 B_2^{-1} v_0 \tag{79}$$

From (8.37)

$$M_1 = (a + b_1)^{-1}[-(A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}) v_0$$

$$+ \frac{c^2}{4} (-2\alpha_1)^{\frac{1}{2}} \alpha_2^{-1}(1 + \frac{c^2}{a_0 p_0} \cos^2 I)^{-\frac{1}{2}} \eta_0^2 \sin(2\psi_s + 2\psi_0)], \tag{80}$$

of order k^2. Then, by (8.39)

$$E_1 = [1 - e'\cos(M_s + E_0)]^{-1} M_1, \tag{81}$$

since $M_1^2 = O(k^4)$. Then E_1 and v_1 are both of order k^2. Also, by (8.40),

$$\psi_1 = (-2\alpha_1)^{-\frac{1}{2}} \alpha_2 (1 + \frac{c^2}{a_0 p_0} \cos^2 I)^{\frac{1}{2}} B_2^{-1} [A_2 v_1 + A_{21}\sin(M_s + v_0) + A_{22}\sin(2M_s + 2v_0)]$$

$$+ \frac{q^2}{8} B_2^{-1} \sin(2\psi_s + 2\psi_0) \tag{82}$$

By (8.45), M_2 and thus E_2 and v_2 are of order k^3 and, by (8.48), ψ_2 is also of order k^3. Thus all the second-order periodic terms of (A) become of the third order and thus negligible, when $\lambda \geqq 1$.

By (8.50) the right ascension ϕ becomes

$$\phi = \beta_3 + \alpha_3\alpha_2^{-1}(1 + \frac{c^2}{a_0 p_0} \cos^2 I)^{-\frac{1}{2}}[(1 - \eta_0^2)^{-\frac{1}{2}}(1 - \eta_2^{-2})^{-\frac{1}{2}} \chi + B_3\psi]$$

$$- c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}(A_3 v + \sum_{n=1}^{4} A_{3n}\sin nv) \tag{83}$$

Here we have dropped the periodic term $(3/32)\, \eta_0^2 \eta_2^{-4} \sin 2\psi$ of (8.50), since it is of order k^3 for $\lambda \geqq 1$.

5. Summary for $\lambda \geqq 1$

For an almost equatorial orbit, corresponding to $\lambda \geqq 1$, the right ascension ϕ is given by (83). The spheroidal coordinates ρ and η are given by

$$\rho = a(1 - e \cos E) = (1 + e \cos v)^{-1} p$$

$$\eta = \eta_0 \sin \psi$$

Here the expressions

$$E = M_s + E_0 + E_1 \qquad v = M_s + v_0 + v_1 \qquad \psi = \psi_s + \psi_0 + \psi_1$$

are sufficiently accurate to give the secular terms exactly and the periodic terms correctly through order k^2, provided that M_s is calculated by (8.24), ψ_s by (77), E_0 by (8.31), E_1 by (80) and (81), v_0 and v_1 by the anomaly relations (8.1), ψ_0 by (79), and ψ_1 by (82).

6. The Case of a Purely Equatorial Orbit, I = 0° or 180°

For I = 0° or 180° we have $\eta_0 = 0$, $\cos^2 I = 1$, $\chi = \psi$ by (6.51), $\gamma_m = 0$ by (6.66),

$$B_3 = 1-(1 - \eta_2^{-2})^{-\frac{1}{2}} \tag{84}$$

by (6.65), and $|\alpha_3| = \alpha_2$, so that

$$\alpha_3/\alpha_2 = \operatorname{sgn} \alpha_3 \tag{85}$$

Then, by (83),

$$\phi = \beta_3 + (\operatorname{sgn} \alpha_3)(1 + \frac{c^2}{a_0 p_0})^{-\frac{1}{2}} \psi - c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}(A_3 v + \sum_{n=1}^{4} A_{3n} \sin nv) \tag{86}$$

Also, by (8.24) with $\eta_0 = 0$,

$$M_s = (-2\alpha_1)^{\frac{1}{2}}(a + b_1 + A_1)^{-1}(t + \beta_1) \tag{87}$$

and by (77) and (78), with $\eta_0 = 0$, $\cos^2 I = 1$, and $B_2 = 1$,

$$\psi_s = \beta_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}} + \alpha_2\,A_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}} (a + b_1 + A_1)^{-1}(t + \beta_1) \tag{88}$$

Thus, by (87) and (88),

$$\psi_s = \beta_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}} + \alpha_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}}(-2\alpha_1)^{-\frac{1}{2}}\,A_2\,M_s \tag{89}$$

Then, by (79) and (82), with $\eta_0 = 0$,

$$\psi_0 = \alpha_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}}(-2\alpha_1)^{-\frac{1}{2}}\,A_2\,v_0 \tag{90}$$

$$\psi_1 = \alpha_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}}(-2\alpha_1)^{-\frac{1}{2}}\left[A_2\,v_1 + \sum_{n=1}^{2} A_{2n}\sin(n\,M_s + n\,v_0)\right] \tag{91}$$

Addition of (89) through (91) then gives

$$\psi = \beta_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}} + \alpha_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}}(-2\alpha_1)^{-\frac{1}{2}}\left[A_2\,v + \sum_{n=1}^{2} A_{2n}\sin(n\,M_s + n\,v_0)\right] \tag{92}$$

Since $v = M_s + v_0 + v_1$, where v_1 is of order k^2, and since A_{21} and A_{22} are of orders k and k^2 respectively, it follows that

$$\sum_{n=1}^{2} A_{2n}\sin(n\,M_s + n\,v_0) = \sum_{n=1}^{2} A_{2n}\sin nv \tag{93}$$

to order k^2. Thus, to order k^2,

$$\psi = \beta_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}} + \alpha_2\left(1 + \frac{c^2}{a_0\,p_0}\right)^{\frac{1}{2}}(-2\alpha_1)^{-\frac{1}{2}}\left(A_2 v + \sum_{n=1}^{2} A_{2n}\sin nv\right) \tag{94}$$

On inserting (94) into (86), we then find

$$\phi = \beta_3 + \beta_2 \operatorname{sgn} \alpha_3 + \alpha_3(-2\alpha_1)^{-\frac{1}{2}}(A_2 v + \sum_{n=1}^{2} A_{2n} \sin nv)$$

$$-c^2 \alpha_3(-2\alpha_1)^{-\frac{1}{2}}(A_3 v + \sum_{n=1}^{4} A_{3n} \sin nv) \qquad (95)$$

It is a simple exercise to check these results for a purely equatorial orbit. To do so, let X and Y be the usual Cartesian coordinates, define ρ and ϕ by

$$X + iY = (\rho^2 + c^2)^{\frac{1}{2}} \exp i\phi \quad , \qquad (96)$$

write down the kinetic energy $\frac{1}{2}(\dot{X}^2 + \dot{Y}^2)$ in terms of ρ and ϕ and their time derivatives, write the potential as $-\mu\rho^{-1}$, construct the Hamiltonian, and then write down the Hamilton-Jacobi equation. Separate the latter, to obtain the solution

$$W = \alpha_3\phi \pm \int_{\rho_1}^{\rho} (\rho^2 + c^2)^{-1} F(\rho)^{\frac{1}{2}} d\rho, \qquad (97)$$

where

$$F(\rho) \equiv c^2 \alpha_3^2 + (\rho^2 + c^2)(-\alpha_3^2 + 2\mu\rho + 2\alpha_1\rho^2) \qquad (98)$$

The kinetic equations are then

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1 = \pm \int_{\rho_1}^{\rho} \rho^2 F(\rho)^{-\frac{1}{2}} d\rho \qquad (99)$$

$$\frac{\partial W}{\partial \alpha} = \beta_3' = \phi \mp \alpha_3 \int_{\rho_1}^{\rho} (\rho^2 + c^2)^{-1} \rho^2 F(\rho)^{-\frac{1}{2}} d\rho \qquad (100)$$

$$= \phi \mp \alpha_3 \int_{\rho_1}^{\rho} F(\rho)^{-\frac{1}{2}} d\rho \pm c^2\alpha_3 \int_{\rho_1}^{\rho} (\rho^2 + c^2)^{-1} F(\rho)^{-\frac{1}{2}} d\rho \qquad (101)$$

On then following the procedure in (A), we find that ρ is given by just the results of section 5 of the present paper, with η_0 placed equal to zero, and that ϕ is given by

$$\phi = \beta_3' + \alpha_3(-2\alpha_1)^{-\frac{1}{2}} (A_2 v + \sum_{n=1}^{2} A_{2n} \sin nv)$$

$$-c^2 \alpha_3(-2\alpha_1)^{-\frac{1}{2}} (A_3 v + \sum_{n=1}^{4} A_{3n} \sin nv) \qquad (102)$$

Comparison of (102) with (95) shows that the results agree if

$$\beta_3' = \beta_3 + \beta_2 \operatorname{sgn} \alpha_3 \qquad (103)$$

In the equatorial plane $r^2 = \rho^2 + c^2$, so that r is at minimum whenever ρ is at minimum. That is, the satellite is at perigee whenever $v = 2\pi \tau$, $\tau = 0,1,2,\ldots$. By (102) the right ascension ϕ_p thus changes value from one perigee to the next. For an equatorial orbit about a planet of zero oblateness, however, the coefficient of v in (102), viz., $\alpha_3(-2\alpha_1)^{-\frac{1}{2}} (A_2 - c^2 A_3)$, would reduce to sgn α_3, so that in such a limiting case

$$\phi_p = \beta_3' + 2\pi \tau \operatorname{sgn} \alpha_3, \; (\tau = 0,1,2,\ldots) \qquad (104)$$

so that the actual position of perigee would remain fixed and β_3' would be its right ascension.

But for the case of a non-equatorial orbit around a planet of zero oblateness β_3 is simply the right ascension Ω of the ascending node and β_2 is the argument ω of perigee. Thus for the limiting case of a purely equatorial orbit about a planet of zero oblateness (103) would take the expected form

$$\text{R.A. of perigee} = \Omega \pm \omega , \qquad (105)$$

where the sign would be plus for a direct orbit and minus for a retrograde orbit. Thus the result (103) is reasonable.

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