NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT 1103-40-11625
9 June 1961
NBS REPORT 7167

ON THE "SYNTHETIC RECORD" PROBLEM

(Autoregressive Models for Stream Discharge)

by

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Technical Report No. 3
to
Water Resources Division
U. S. Geological Survey
Department of Interior

NBS

U. S. DEPARTMENT OF COMMERCE
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1. Introduction

This report is supplementary to NBS Report 6513 (July 1959), in which the background for the problem and a simple probabilistic model are described. The model stated in that report is now modified to incorporate autocorrelations in the time series of stream discharges.

In this study, the correlation structure for a bivariate time series is derived from an autoregressive model (Section 2). A given correlation structure does not, however, uniquely determine a bivariate autoregressive scheme. At the end of Section 2, the results are given of a brief study of possible autoregressive schemes consistent with the correlation structure derived from the model assumed here. This may give some insight into the interpretation of the assumptions and their applicability in hydrologic studies.

The main results of this investigation describe the properties of a procedure for estimating the mean μ_y of the distribution of discharge for a stream for which there is a "short record", the estimation procedure making use of the correlation between the discharge from this stream and the discharge from a nearby stream for which there is a "long record". Estimation of the mean discharge μ_y is discussed in Section 3. Derivation of the variance formula is outlined in an appendix.

2. Autoregressive Models for Stream Discharge

2.1 Derivation of Correlation Structure

It is assumed, as in the earlier report, that the simultaneous discharges (X_t, Y_t) from two streams at time t have a joint normal distribution with parameters μ_x, μ_y, σ_x^2, σ_y^2, $\rho = \beta\sigma_x/\sigma_y$. Stream discharge is assumed to be stationary; i.e., these parameters (among others) are the same for any $t = 0, \pm 1, \pm 2, \ldots$.

Instead of assuming independence among pairs (X_t, Y_t) at different times, it is assumed that each of the two discharge series $\{X_t\}$ and $\{Y_t\}$ is dependent, with autocorrelations arising from an autoregressive structure of the form:

$$X_t - \mu_x = \rho_x(X_{t-1} - \mu_x) + U_t ,$$

$$Y_t - \mu_y = \rho_y(Y_{t-1} - \mu_y) + V_t ,$$

where $\{(U_t, V_t)\}$ is a sequence of bivariate random variables which are assumed to be independent and identically distributed with joint normal distribution. Also, any U_t or V_t is independent of any X_{t-s} or Y_{t-s} $(s>0)$.

From the assumptions about the joint distribution of (X_t, Y_t), it follows that U_t and V_t have zero means, variances

$$\sigma_u^2 = (1 - \rho_x^2)\, \sigma_x^2 ,$$

$$\sigma_v^2 = (1 - \rho_y^2)\, \sigma_y^2 ,$$

and correlation

$$\rho_{uv} = \rho(1-\rho_x\rho_y)/ \sqrt{1-\rho_x^2}\, \sqrt{1-\rho_y^2} .$$

In order that the above formulas make sense, it is necessary that $\rho_x^2 \leq 1$, $\rho_y^2 \leq 1$, and $\rho_{uv}^2 \leq 1$, i.e.,

$$\rho^2(1-\rho_x\rho_y)^2 \leq (1-\rho_x^2)(1-\rho_y^2).$$

The last inequality is evidently satisfied if $\rho_x = \rho_y$, but places some restrictions on possible values of ρ_x, ρ_y, and ρ when $\rho_x \neq \rho_y$.

We now calculate the lagged correlations:

$$\text{Corr.}\ (X_t, X_{t-s}) = \rho_x^s, \quad s > 0,$$

$$\text{Corr.}\ (Y_t, Y_{t-s}) = \rho_y^s, \quad s > 0,$$

$$\text{Corr.}\ (X_t, Y_{t-s}) = \rho\rho_x^s, \quad s > 0,$$

$$\text{Corr.}\ (X_t, Y_{t+s}) = \rho\rho_y^s, \quad s > 0.$$

This completes the derivation of the correlation structure for a sequence of pairs $\{(X_t, Y_t)\}$.

One further specializing assumption will be made in the derivations given in Section 3. That is, it will be assumed that

$$\rho_x = \rho_y = \tau.$$

This greatly simplifies the calculations.

2.2. Alternative Autoregressive Models

It was remarked in the introduction that the autoregressive structure assumed above is not the only one which might lead to this set of correlations. For a brief general discussion of this question, see for example Quenouille (1957).

As an illustration, the bivariate autoregressive series generated by the following equations would have the same correlation structure:

$$X_t - \mu_x = \rho_x(X_{t-1} - \mu_x)$$
$$+ \sigma_x \sqrt{1-\rho_x^2} \sqrt{\frac{1 + \rho_{uv}}{2}} \epsilon_t$$
$$+ \sigma_x \sqrt{1-\rho_x^2} \sqrt{\frac{1 - \rho_{uv}}{2}} \delta_t ,$$
$$Y_t - \mu_y = \rho_y(Y_{t-1} - \mu_y)$$
$$+ \sigma_y \sqrt{1 - \rho_y^2} \sqrt{\frac{1 + \rho_{uv}}{2}} \epsilon_t$$
$$- \sigma_y \sqrt{1 - \rho_y^2} \sqrt{\frac{1 - \rho_{uv}}{2}} \delta_t ,$$

where ϵ_t , δ_t are all mutually independent normal random variables with mean zero and variance unity.

The postulated correlation structure would also be generated by the following set of equations (with the same assumptions about ϵ_t , δ_t):

$$X_t - \mu_x = \rho_x(X_{t-1} - \mu_x) + \sigma_x \sqrt{1-\rho_x^2} \epsilon_t ,$$
$$Y_t - \mu_y = \rho_y(Y_{t-1} - \mu_y) + \rho_{uv}\sigma_y \sqrt{1-\rho_y^2} \epsilon_t$$
$$+ \sqrt{1 - \rho_{uv}^2} \sigma_y \sqrt{1 - \rho_y^2} \delta_t .$$

The point of these examples is that there would be many ways of writing (U_t , V_t) as linear functions of (ϵ_t , δ_t), where ϵ_t , δ_t are independent and U_t , V_t have the prescribed variances

and covariance. Unless there are physical reasons for postulating a particular autoregressive structure, the model employed here has certain ambiguities.

3. Estimation of the Mean

Suppose that X_t denotes discharge from the stream with long record, and Y_t denotes discharge from the stream with short record. The data given, then, are

$$(X_1 , Y_1) , \ldots , (X_{n_1} , Y_{n_1}) ,$$

and

$$X_{n_1 + 1} , \ldots , X_{n_1 + n_2} .$$

Let $N = n_1 + n_2$ denote the number of time periods covered by the long record.

Estimation of the mean discharge μ_y for the stream with a short record is assumed (as in the earlier report) to employ the "synthetic" values

$$\hat{Y}_{n_1 + 1} , \ldots , \hat{Y}_{n_1 + n_2} ,$$

obtained from the estimated regression of Y_t on X_t . Thus, the estimation for μ_y is

$$U = \bar{Y}_1 + \frac{n_2}{N} b(\bar{X}_2 - \bar{X}_1),$$

where

$$\bar{X}_1 = \sum_{i=1}^{n_1} X_i/n_1 ,$$

$$\bar{X}_2 = \sum_{i=1}^{n_2} X_{n_1+i} / n_2 ,$$

$$\overline{Y}_1 = \sum_{i=1}^{n_1} Y_i / n_1 \quad ,$$

and

$$b = \sum_{i=1}^{n_1} (X_i - \overline{X}_1)(Y_i - \overline{Y}_1) / \sum_{i=1}^{n_1} (X_i - \overline{X}\ )^2 \quad .$$

The estimator U is an unbiased estimator for μ_y ; that is,

$$EU = \mu_y \quad ,$$

where E denotes mathematical expectation.

An approximate formula for the variance of U , including terms up to order $1/n_1^2$ is:

$$\text{Var } U \doteq \frac{\sigma_y^2}{n_1} \Bigg[\left(\frac{1+\tau}{1-\tau} - \frac{2\tau}{(1-\tau)^2} \; \frac{1}{n_1} \right) - \frac{n_2}{N} \left\{ \rho^2 \left(\frac{1+\tau}{1-\tau} - \frac{2\tau}{(1-\tau)^2} \cdot \frac{N+n_1}{N\, n_1} \right) - \frac{(1-\rho^2)}{n_1} \cdot \frac{1+\tau^2}{(1-\tau)^2} \right\} \Bigg]$$

This is written in a form which makes it evident that, for $\tau = 0$, this formula agrees with the formula derived by H. A. Thomas (up to the same order of approximation).

Acknowledgement

This investigation was made in response to a request by Dr. N. C. Matalas of the USGS Water Resources Division, who has made numerous helpful suggestions during discussions of the work.

Reference

Quenouille, M. H. (1957), The Analysis of Multiple Time-Series, New York: Hafner.

Appendix

The basic approach to the derivation of Var U is given, together with the matrix notation employed. The derivation itself is sketched, with lengthy calculations suppressed.

A-1. Transformation into a representation in terms of mutually independent random variables.

For t = 2, 3, ... , we have the autoregressive relations

$$X_t - \mu_x = \tau(X_{t-1} - \mu_x) + U_t ,$$

$$Y_t - \mu_y = \tau(Y_{t-1} - \mu_y) + V_t ,$$

where the (U_t , V_t) are mutually independent bivariate random variables with zero means, Variances

$$\sigma_u^2 = (1 - \tau^2)\, \sigma_x^2 , \quad \sigma_v{}^2 = (1 - \tau^2)\, \sigma_y^2 ,$$

and correlation $\rho_{uv} = \rho$.

Now let

$$\xi_1 = (X_1 - \mu_x)/\sigma_x , \quad \eta_1 = (Y_1 - \mu_y)/\sigma_y ,$$

and, for $t \geq 2$,

$$\xi_t = U_t \,/\, \sigma_x \sqrt{1 - \tau^2} , \quad \eta_t = V_t \,/\, \sigma_y \sqrt{1 - \tau^2} .$$

By repeated use of the autoregressive equations, we derive the transformation

$$\begin{cases} X_1 - \mu_x = \sigma_x \xi_1 , \\ X_t - \mu_x = \tau^{t-1} \sigma_x \xi_1 + \sqrt{1 - \tau^2} \sum_{i=2}^{t} \tau^{t-j} \sigma_x \xi_j , \quad t \geq 2, \end{cases}$$

$$\begin{cases} Y_1 - \mu_y = \sigma_y \eta_1 , \\ Y_t - \mu_y = \tau^{t-1} \sigma_y \eta_1 + \sqrt{1 - \tau^2} \sum_{j=2}^{t} \tau^{t-j} \sigma_y \eta_j , \quad t \geq 2, \end{cases}$$

through which the X_t and Y_t are expressed in terms of independent bivariate random variables (ξ_t, η_t) having joint normal distribution with zero means, unit variances, and correlation ρ.

We now make the further transformation

$$\eta_t = \rho\, \xi_t + \sqrt{1 - \rho^2}\, \theta_t , \quad t = 1, 2, \ldots , n_1 ,$$

where the θ_t are normal with zero means and unit variances, and the θ_t and ξ_t are all mutually independent.

A-2. Matrix notation and some preliminary remarks.

The derivations are most efficiently conducted in matrix notation. This section assembles all the definitions of notation.

I_m - identity matrix of order m

e_m - column vector of m ones

J_m - matrix of order m all of whose elements are equal to one

J_{mn} - (m x n) rectangular matrix of ones

A' - transpose of the matrix A

tr A - trace of the (square) matrix A, i.e., the sum of the elements on the diagonal.

Extensive use is made of the fact that tr(AB) = tr(BA), whenever the products AB and BA are both possible. In particular, any quadratic form $x' Ax$ may be written $tr(A x x')$.

We now introduce the matrix notation for the random variables X_t, Y_t, and for the transformations given in the preceding section. Let

$$x_1 = (X_1 , \ldots , X_{n_1})' ,$$

$$x_2 = (X_{n_1+1}, \ldots , X_{n_1+n_2})' ,$$

$$y_1 = (Y_1 , \ldots , Y_{n_1})' ,$$

each being a column vector. The transformation may be written

$$\begin{bmatrix} x_1 - \mu_x e_{n_1} \\ x_2 - \mu_x e_{n_2} \end{bmatrix} = \sigma_x \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix},$$

$$y_1 - \mu_y e_{n_1} = \sigma_y G_{11} \eta_1 ,$$

where (at the risk of minor confusion),

$$\xi_1 = (\xi_1 , \ldots , \xi_{n_1})' ,$$

$$\xi_2 = (\xi_{n_1+1} , \ldots , \xi_{n_1+n_2})' ,$$

$$\eta_1 = (\eta_1 , \ldots , \eta_{n_1})' .$$

Also,

$$\eta_1 = \rho \xi_1 + \sqrt{1 - \rho^2} \theta_1 ,$$

with

$$\theta_1 = (\theta_1 , \ldots , \theta_{n_1})' .$$

Now, define

$$S = \sum_{i=1}^{n_1} (X_i - \bar{X}_1)^2 .$$

In the matrix notation, this becomes

$$S = x_1' (n_1 I_{n_1} - J_{n_1}) x_1$$

$$= \sigma_x^2 \xi_1' G_{11}' (n_1 I_{n_1} - J_{n_1}) G_{11} \xi_1 ,$$

and it will be convenient to define

$$T = G_{11}' (n_1 I_{n_1} - J_{n_1}) G_{11} ,$$

so that

$$S = \sigma_x^2 \xi_1' T \xi_1 .$$

Now the estimated regression coefficient may be written (recalling $\beta = \rho \sigma_y/\sigma_x$) ,

$$b = \frac{1}{S} x_1' (n_1 I_{n_1} - J_{n_1}) Y_1 = \frac{\sigma_x \sigma_y}{S} \xi_1' T \eta_1$$

$$= \frac{\rho \sigma_x \sigma_y}{S} \xi_1' T \xi_1 + \sqrt{1 - \rho^2} \frac{\sigma_x \sigma_y}{S} \xi_1' T \theta_1$$

$$= \beta + \sqrt{1 - \rho^2} \frac{\sigma_x \sigma_y}{S} \xi_1' T \theta_1 .$$

Finally, the estimator for μ_y may be written

$$U = \bar{Y}_1 + \frac{n_2}{N} b(\bar{X}_2 - \bar{X}_1)$$

$$= \bar{Y}_1 + \frac{n_2}{N} \beta(\bar{X}_2 - \bar{X}_1) + \frac{n_2}{N} (b - \beta)(\bar{X}_2 - \bar{X}_1) .$$

To verify that $EU = \mu_y$, we observe that $E \bar{Y}_1 = \mu_y$, and $E(\bar{X}_2 - \bar{X}_1) = 0$. Also, it has been established above that $(b - \beta)$ is a linear function of the θ_i , which are independent of the ξ_i ; and $(\bar{X}_2 - \bar{X}_1)$ is a linear function of the ξ_i . Hence $E(b - \beta)(\bar{X}_2 - \bar{X}_1) = 0$.

A-3. Derivation of the variance of U.

The derivation involves some lengthy calculations, which will not be given in full. Referring to the expression for U given at the end of the preceding section, we write

$$U - \mu_y = (\bar{Y}_1 - \mu_y) + \frac{n_2}{N} \beta (\bar{X}_2 - \bar{X}_1)$$

$$+ \frac{n_2}{N} (b - \beta)(\bar{X}_2 - \bar{X}_1) ,$$

and

$$(U - \mu_y)^2 = (\bar{Y}_1 - \mu_y)^2 + \frac{n_2^2}{N^2} \beta^2 (\bar{X}_2 - \bar{X}_1)^2$$

$$+ 2 \frac{n_2}{N} \beta (\bar{X}_2 - \bar{X}_1)(\bar{Y}_1 - \mu_y)$$

$$+ 2 \frac{n_2}{N} (b - \beta)(\bar{X}_2 - \bar{X}_1)(\bar{Y}_1 - \mu_y)$$

$$+ \frac{n_2^2}{N^2} (b - \beta)^2 (\bar{X}_2 - \bar{X}_1)^2 + \frac{2 n_2^2}{N^2} \beta(b-\beta)(\bar{X}_2-\bar{X}_1)^2 .$$

The last of the six terms in the expression above is a linear function of the θ_i which enter only in $(b - \beta)$, and hence has expected value zero. The expected values of the first three terms are easily obtained, and are

$$E\,(\bar{Y}_1 - \mu_y)^2 = \frac{\sigma_y^2}{n_1^2}\; e'_{n_1}\; G_{11}\, G_{11}'\; e_{n_1}\,,$$

$$\frac{n_2^2}{N^2}\;\beta^2\;E(\bar{X}_2 - \bar{X}_1)^2 = \frac{\beta^2\,\sigma_x^2}{N^2}\;\left\{\frac{n_2^2}{n_1^2}\; e'_{n_1}\; G_{11}\, G_{11}'\, e_{n_1} + e'_{n_2}\; G_{21}\, G_{21}'\, e_{n_2} + e'_{n_2}\; G_{22}\, G_{22}'\, e_{n_2} - 2\,\frac{n_2}{n_1}\; e'_{n_2}\; G_{21}\, G_{11}'\, e_{n_1}\right\},$$

$$2\,\frac{n_2}{N}\;\beta\,E(\bar{X}_2 - \bar{X}_1)(\bar{Y}_1 - \mu_y) = \frac{2\,\beta^2\,\sigma_x^2}{N}\;\left\{\frac{1}{n_1}\; e'_{n_2}\, G_{21}\, G'_{11}\, e_{n_1} - \frac{n_2}{n_1^2}\; e'_{n_1}\; G_{11}\, G_{11}'\, e_{n_1}\right\}.$$

If we write

$$G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix},$$

then, after some rearranging, the sum of the three terms given above is:

$$\frac{\rho^2\,\sigma_y^2}{N^2}\; e'_N\; G\,G'\, e_N + (1-\rho^2)\,\frac{\sigma_y^2}{n_1^2}\, e'_{n_1}\, G_{11}\, G_{11}'\, e_{n_1}$$

Next, consider the fourth and fifth terms in the expression for $(U - \mu_y)^2$. These involve S and S^2 in their denominators, and their expected values will be evaluated approximately. First, however, observe that the vectors θ_1 and ξ_2 appear only in the numerators. Holding the vector ξ_1 fixed, and taking

expectations with respect to the random variables in θ_1 and ξ_2, the calculations for the fourth and fifth terms are reduced to the following:

$$\frac{2n_2}{N} E(b-\beta)(\bar{X}_2 - \bar{X}_1)(\bar{Y}_1 - \mu_y) = \frac{2}{N} \sigma_y^2 (1-\rho^2) \frac{n_2}{n_1} \sigma_x^2$$

$$E \frac{1}{S} \left[\frac{1}{n_2} \xi_1' G_{21}' J_{n_2 n_1} G_{11} T \xi_1 - \frac{1}{n_1} \xi_1' G_{11}' J_{n_1} G_{11} T \xi_1 \right]$$

$$\frac{n_2^2}{N^2} E (b-\beta)^2 (\bar{X}_2 - \bar{X}_1)^2 = \frac{n_2^2}{N^2} \sigma_y^2 (1-\rho^2) \sigma_x^4$$

$$E \frac{1}{S^2} \xi_1' T^2 \xi_1 \left[\frac{1}{n_2^2} e'_{n_2} G_{22} G_{22}' e_{n_2} + \frac{1}{n_1^2} \xi_1' G_{11}' J_{n_1} G_{11} \xi_1 + \frac{1}{n_2^2} \xi_1' G_{21}' J_{n_2} G_{21} \xi_1 - \frac{2}{n_1 n_2} \xi_1' G_{11} J_{n_1 n_2} G_{21} \xi_1 \right]$$

There are six terms to be evaluated here, each involving a ratio of random variables. For a first-order approximation, each random variable is replaced by its expected value. Further approximation involves deletion of certain terms having the factor τ^{n_1} (or τ^{n_2}), which is assumed to be negligible. Finally, only terms of order up to $1/n_1^2$ are retained.

Evaluation of the first term will be sketched, and the values of the other five terms will be listed. For the first term, we require

$$E \frac{1}{S} \xi' G_{21}' J_{n_2 n_1} G_{11} T \xi_1 .$$

Now

$$\frac{1}{\sigma_x^2} E S = E \xi_1' T \xi_1 = E \operatorname{tr}(T \xi_1 \xi_1')$$

$$= \operatorname{tr}[T \cdot E(\xi_1 \xi_1')] = \operatorname{tr} T ,$$

and

$$\operatorname{tr} T = \operatorname{tr}[G_{11}' (n_1 I_{n_1} - J_{n_1}) G_{11}]$$

$$= n_1 \operatorname{tr} G_{11}' G_{11} - \operatorname{tr} G_{11}' J_{n_1} G_{11} ;$$

but we can write $J_{n_1} = e_{n_1} e'_{n_1}$, whence

$$\text{tr } T = n_1 \text{ tr } G_{11} G_{11}' - e'_{n_1} G_{11} G_{11}' e_{n_1} .$$

Now it is easily verified that

$$G_{11} G_{11}' = \begin{bmatrix} 1 & \tau & \tau^2 & \cdots & \tau^{n_1-1} \\ \tau & 1 & \tau & \cdots & \tau^{n_1-2} \\ & \cdots & & & \\ \tau^{n_1-1} & \tau^{n_1-2} & \tau^{n_1-3} & \cdots & 1 \end{bmatrix} .$$

Thus the trace of this matrix is n_1 and the sum of all the elements of this matrix is

$$n_1 \frac{1+\tau}{1-\tau} - \frac{2\tau}{(1-\tau)^2} (1 - \tau^{n_1}) .$$

Finally, then, deleting the negligible term in τ^{n_1},

$$E\ S \doteq \left[n_1^2 - n_1 \frac{1+\tau}{1-\tau} + \frac{2\tau}{(1-\tau)^2} \right] \sigma_x^2$$

Next, in a similar way, we obtain

$$E\ \xi' G_{21}' J_{n_2 n_1} G_{11} T \xi_1 = \text{tr}(G_{21}' J_{n_2 n_1} G_{11} T)$$

$$= e'_{n_1} G_{11} T G_{21}' e_{n_2}$$

$$= n_1 e'_{n_1} G_{11} G_{11}' G_{11} G_{21}' e_{n_2}$$

$$- (e'_{n_1} G_{11} G_{11}' e_{n_1})(e'_{n_1} G_{11} G_{21}' e_{n_2})$$

$$\doteq \frac{2\tau^2}{(1-\tau)^4} - \frac{n_1 \tau^2}{(1-\tau)^3 (1+\tau)} ,$$

after deletion of terms involving τ^{n_1} and τ^{n_2}.

The ratio of these two expected values is of order $1/n_1$, and this ratio enters a term which is already of order $1/n_1^2$ (actually $1/n_1 n_2$); hence this term will not appear in the

first-order approximation.

The second term and also the last two terms turn out similarly to make no contribution to an approximation up to order $1/n_1^2$.

For the third term, we require terms up to order $1/n_1^2$ in:

$$\frac{1}{n_2^2} \; e'_{n_2} \; G_{22} \; G_{22}' \; e_{n_2} \quad \cdot \; E \, \frac{1}{S^2} \; \xi_1' \; T^2 \; \xi_1$$

$$\doteq \frac{1}{n_2^2} \left[n_2 \frac{1+\tau}{1-\tau} - \frac{2\tau}{(1-\tau)^2} \right]$$

$$\left[\frac{1+\tau^2}{1-\tau^2} \; n_1^3 + O(n_1^2) \right] \Big/ \sigma_x^4 \left[n_1^4 + O(n_1^3) \right]$$

$$\doteq \frac{1}{\sigma_x^4} \; \frac{1+\tau^2}{(1-\tau)^2} \; \frac{1}{n_1 n_2} \; .$$

For the fourth term, we have similarly:

$$\frac{1}{n_1^2} \; E \; \frac{1}{S^2} \; \xi_1' \; T^2 \xi_1 \; \cdot \; \xi_1' \; G_{11}' \; J_{n_1} \; G_{11} \; \xi_1$$

$$\doteq \frac{1}{\sigma_x^4} \; \frac{1+\tau^2}{(1-\tau)^2} \; \frac{1}{n_1^2} \; .$$

Assembling all these terms together with the terms evaluated previously without approximations, we obtain the variance formula given in Section 3 of this report.