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MEAN MOTIONS IN CONDITIONALLY PERIODIC

SEPARABLE SYSTEMS

by

John P. Vinti



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# NATIONAL BUREAU OF STANDARDS REPORT

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MEAN MOTIONS IN CONDITIONALLY PERIODIC SEPARABLE SYSTEMS*

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John P. Vinti
Mathematical Physics Section

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U. S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

# MEAN MOTIONS IN CONDITIONALLY PERIODIC SEPARABLE SYSTEMS[1]

by

John P. Vinti

A search of the literature failed to disclose any general statement or proof of a theorem informally current among dynamical astronomers. The present paper gives a proof of the theorem, which states that, in any conditionally-periodic separable system, the mean frequency $n_k$ of any separation coordinate $q_k$ is equal to $\nu_k \equiv \partial\alpha_1/\partial J_k$. Here $\alpha_1$ is the energy and $J_k$ is the k'th action variable. The proof is carried out for non-singular Staeckel systems, so that it is applicable to any non-polar orbit of an artificial satellite, when the potential leads to separability.

## 1. Introduction

Conditionally periodic separable systems are commonly illustrated in works on advanced dynamics by the motion of a particle under the joint action of harmonic oscillator forces at right angles. This happens to be a very special case, for which each rectangular coordinate $q_k$ has a constant frequency, equal to the corresponding "fundamental frequency" $\nu_k \equiv \partial\alpha_1/\partial J_k$, where $\alpha_1$ is the energy and $J_k$ the corresponding action variable.

In a more general system of this type, each generalized coordinate $q_k$ may have a variable frequency, but it appears to be generally believed among dynamical astronomers that the mean frequency of $q_k$ must be equal to $\nu_k$, if the conditionally periodic system is separable. Needing to refer to such a theorem in solving a specific problem, I have searched the literature but have found no explicit statement or proof of it. The present paper is an attempt to furnish such a reference, with a proof sufficiently general to be applicable to all the separable problems that

1. This work was supported by the U. S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

may arise in the gravitational theory of the orbit of a satellite of an oblate planet.

Little of the analysis in this paper can claim to be really new. Much of the pertinent material in the literature, however, is discursive, relatively unavailable, and expressed in notations now unfamiliar to most mathematical physicists. Some of it is inadequate, if not incorrect, especially in the treatment of the periodicity of the q's as functions of the angle variables. Moreover, none of it seems to have been carried out in the shortest and most appropriate way to prove the theorem in question. The present paper attempts to give a concise and correct treatment that will serve this purpose.

It is easy to see why such a theorem should have escaped formal statement and proof. Physicists have not been concerned with mean frequencies of this kind. Dynamical astronomers have been, but ordinarily for non-separable systems. Until 1957 their only separable problems were the Kepler problem [2][2] for which the coordinate frequencies are all equal to the constant value $\nu_1 = \nu_2 = \nu_3$, and the problem of two centers, which has remained a curiosity to be found in Charlier's famous book [1], but without application. Since 1957 various potentials have been suggested, by Sterne [3], by Garfinkel [4], and by the author [5], for the gravitational field of an oblate planet, all of which lead to separability and to intermediary satellite orbits with $\nu_1 \neq \nu_2 \neq \nu_3$. Since the solution for these orbits is greatly facilitated by knowledge of the mean coordinate frequencies, it now becames desirable to have a formal and general proof of the theorem.

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2. Figures in brackets indicate the literature references at the end of the paper.

It is now convenient to discuss briefly the general plan of the paper, without definitions. The difficulty in proving the theorem arises only when the fundamental frequencies $\nu_1 \ldots \nu_n$ are incommensurable. If $w_1 \ldots w_n$ are the angle variables, the plan is first to show that for a conditionally periodic system there exist infinitely many values of the time, with no upper bound, at which the orbit in w-space passes arbitrarily closely to points separated from the initial point $w_1(0) \ldots w_n(0)$ by integer intervals $\Delta w_k = m_k$, $k = 1 \ldots n$. This fact follows directly from a theorem of Dirichlet, which is easy to understand and to apply.

To convert this result to q-space, it is necessary to know that $q_k$ is a single-valued, continuous, periodic function of the w's. To show this the author restricts considerations to non-singular Staeckel systems, proving that for them each $q_k$ is a single-valued differentiable function of $v_k \equiv \int dq_k/p_k$ and each $v_k$ of the angle variables $w_1 \ldots w_n$. With a careful use of the single-valuedness, the periodic property then follows.

Application of these properties of the q's as functions of the w's shows that, at the values of the time mentioned above, the orbit in q-space then passes arbitrarily closely to points where each $q_k$ would have gone through exactly $m_k$ cycles. The proof of the theorem then follows.

## 2. Staeckel Systems.

If $q_1 \ldots q_n$ and $p_1 \ldots p_n$ denote the generalized coordinates and momenta of a dynamical system of n degrees of freedom, the system is said to be of the Staeckel type [1, 6, 7, 8, 9] if the Hamiltonian H is given by

$$H = \tfrac{1}{2} \sum_{k=1}^{n} A_k(q_1 \ldots q_n) p_k^2 + V(q_1 \ldots q_n), \quad A_k > 0, \quad (k = 1 \ldots n) \tag{1}$$

and if there exist functions $\phi_{ij}(q_i)$, $\psi_i(q_i)$, $i, j = 1 \ldots n$, such that

$$A_k = M_{k1} / \det(\phi_{ij}) \tag{2}$$

$$V = \sum_{k=1}^{n} \psi_k(q_k)\, A_k, \tag{3}$$

$M_{k1}$ being the cofactor of $\phi_{k1}(q_k)$ in the determinant $\det(\phi_{ij})$.

Conditions (2) and (3) are necessary and sufficient for the separability of a system with such a Hamiltonian.

If we next define the domain Q of the q's as the totality of real values of $q_1 \ldots q_n$ for which $p_k^2 \geqq 0$, $k = 1 \ldots n$, we may then define a non-singular Staeckel system as one for which $\psi_k(q_k)$ and $\phi_{ki}(q_k)$, i, $k = 1 \ldots n$, exist and are single-valued and for which $\det(\phi_{ij}) \neq 0$ anywhere in Q. If we put

$$\Phi \equiv (\phi_{ij}(q_i)) \tag{4}$$

for the Staeckel matrix, then $\Phi^{-1}$ exists and is single-valued anywhere in Q; in particular

$$(\Phi^{-1})_{1k} = A_k \quad (k = 1 \ldots n) \tag{5}$$

all exist anywhere in Q. (This restriction thus rules out polar orbits from consideration if the right ascension $\phi$ is one of the coordinates, since $A_3$ then becomes infinite on the polar axis.)

The momenta $p_k$ are then given by

$$p_k^2 = -2\psi_k(q_k) + 2 \sum_{i=1}^{n} \phi_{ki}(q_i)\alpha_i, \quad (k = 1 \ldots n) \tag{6}$$

where the $\alpha$'s are separation constants, $\alpha_1$ being the energy. (For satellite problems, where n = 3, $2\alpha_2$ and $2\alpha_3$ are usually denoted by $\alpha_2^2$ and $\alpha_3^2$.)

The Hamilton-Jacobi function W is then given by

$$W = \sum_{k=1}^{n} \int_{q_{ko}}^{q_k} p_k dq_k = \sum_{k=1}^{n} (\pm) \int_{q_{ko}}^{q_k} [-2\psi_k(q_k) + 2 \sum_{i=1}^{n} \phi_{ki}(q_i)\alpha_i]^{\frac{1}{2}} dq_k, \tag{7}$$

where the sign is $\pm$ respectively as $dq_k \gtrless 0$.

## 3. Conditionally Periodic Staeckel Systems.

We call a Staeckel system conditionally periodic if each coordinate is either rotational or librational. A coordinate $q_k$ is rotational if: (i) it is an angle, (ii) with $p_k^2 \equiv F_k(q_k)$ there exist positive real numbers $c_{1k}$ and $c_{2k}$ such that $c_{2k} \geqq F_k(q_k) \geqq c_{1k} > 0$ for all real values of $q_k$, and (iii) if

$$F_k(q_k + 2\pi) = F_k(q_k). \tag{8}$$

Note that $c_{1k} > 0$ rules out asymptotic motions and that the periodicity implied by (8) may reduce to simple constancy. The latter holds, e.g., when $q_k$ is the right ascension $\phi$ of an artificial satellite, since there does not exist any potential, depending on the right ascension $\phi$, which both leads to separability and remains finite on the polar axis.[3] For such a rotational coordinate $q_k$ either $p_k > c_{1k}^{\frac{1}{2}}$ for all $q_k$ or $p_k < -c_{1k}^{\frac{1}{2}}$ for all $q_k$. In either case

$$v_k(q_k) \equiv \int_{q_{ko}}^{q_k} dq_k/p_k \tag{9}$$

is a single-valued function of $q_k$, with derivative $dv_k/dq_k$ existing and differing from zero for all values of $q_k$. Thus $q_k$ is a single-valued differentiable[4] function of $v_k$.

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3. See the tables in [9], pp. 656, 658, 660

4. Hereafter abbreviated to "s.v.d."

A coordinate $q_i$ is librational if there exist real numbers $a_i$, $b_i$, $C_{1i}$, and $C_{2i}$ and a real function $G_i(q_i)$ such that

$$p_i^2 = (q_i - a_i)(b_i - q_i)G_i(q_i), \tag{10.1}$$

with

$$C_{2i} \geqq G_i(q_i) \geqq C_{1i} > 0 \quad (a_i \leqq q_i \leqq b_i) \tag{10.2}$$

and

$$a_i \leqq q_i(0) \leqq b_i, \tag{10.3}$$

$q_i(0)$ being the initial value of $q_i$. If we again define $v_i$ by (9), then

$$v_i(q_i) = \pm \int_{q_{io}}^{q_i} [(q_i - a_i)(b_i - q_i)G_i(q_i)]^{-\frac{1}{2}} dq_i, \tag{11}$$

where the sign is $\pm$ accordingly as $dq_i \gtrless 0$, respectively. Then

$$v_i = \int_{E_{io}}^{E_i} G_i^{-\frac{1}{2}} dE_i, \tag{12}$$

where the uniformizing variable $E_i$ is defined by the equation

$$2q_i = a_i + b_i + (a_i - b_i) \cos E_i \tag{13}$$

and the requirement that $E_i$ shall always increase as $q_i$ varies. By (12) and (10.2) $v_i$ is then a single-valued function of $E_i$, with derivative existing and non-vanishing for all $E_i$, so that $E_i$ must be a s.v.d. function of $v_i$. By (13), however, $q_i$ is a s.v.d. function of $E_i$, so that, finally, $q_i$ is a s.v.d. function of $v_i$. Thus in a conditionally periodic Staeckel system any coordinate $q_k$ is a s.v.d. function of the corresponding $v_k$.

4. The $v_k$'s as Functions of the Angle-Variables $w_k$.

If we now let an increase of $2\pi$ be one cycle of a rotational coordinate and a single round trip from $a_k$ to $b_k$ be one cycle of a librational coordinate, we may define the action and angle variables $J_k$ and $w_k$ by

$$J_k \equiv \oint p_k dq_k \tag{14}$$

and

$$w_k \equiv \partial W/\partial J_k, \tag{15}$$

where W is now to be considered a function of the q's and the J's, rather than of the q's and the $\alpha$'s. It is well known [10, 11] that $J_k$ and $w_k$ are canonically conjugate and that

$$\dot{w}_k = \partial\alpha_1/\partial J_k = \nu_k . \tag{16}$$

If we also define the Jacobi variables $B_i$ by

$$B_i \equiv \partial W/\partial\alpha_i, \tag{17}$$

we obtain

$$B_i = \sum_{k=1}^{n} \frac{\partial W}{\partial J_k} \frac{\partial J_k}{\partial \alpha} = \sum_{k=1}^{n} w_k \omega_{ki}, \tag{18}$$

where

$$\omega_{ki} = \partial J_k/\partial\alpha_i \tag{19}$$

Increments $dw_1 \ldots dw_n$ then lead to

$$dB_i = \sum_{k=1}^{n} (dw_k)\,\omega_{ki} \qquad (i = 1 \ldots n). \tag{20}$$

But, by (7) and (17)

$$dB_i = \sum_{k=1}^{n} \frac{\partial p_k}{\partial\alpha_i} dq_k = \tfrac{1}{2} \sum_{k=1}^{n} \frac{\partial p_k^2}{\partial\alpha_i} dq_k/p_k \tag{21}$$

or

$$dB_i = \sum_{k=1}^{n} \phi_{ki}(q_k)dv_k, \tag{22}$$

by (21), (6), and (9). Also, by (19), (14), (6), and (9)

$$\omega_{ki} = \oint \phi_{ki}(q_i)dv_k . \tag{23}$$

If we now introduce the matrix $\Phi$, the matrix $\Omega \equiv (\omega_{ki})$, and the row matrices $dv \equiv (dv_1, \dots dv_n)$ and $dw = (dw_1, \dots dw_n)$, we find from (20) and (22)

$$dv\,\Phi = dw\,\Omega . \tag{24}$$

For a non-singular Staeckel system[5] $\Phi$, $\Phi^{-1}$, and $\Omega$ all exist at every point of Q, so that

$$dv = dw\,\Omega\,\Phi^{-1} \tag{25}$$

or

$$\partial v_i/\partial w_k = (\Omega\,\Phi^{-1})_{ki} . \tag{26}$$

Thus each derivative $\partial v_i/\partial w_k$ exists and is single-valued everywhere in Q. Now the J's are all real, by (14). Thus, by (7) and (15), if the p's are all real, then W and the w's are all real; if some of the p's are non-real, then W is non-real and so are some of the w's. If all the w's are real, it then follows that all the p's are real, else we should have a contradiction. Thus the domain Q, corresponding to the totality of all real values of the q's for which the p's are all real, also corresponds exactly to the set of all possible real values for all the w's. It therefore follows that each derivative $\partial v_i/\partial w_k$ exists and is single-valued at any point in w-space. Thus each $v_k$ must be a s.v.d. function of $w_1 \dots w_n$. In Section 3, however, we showed that each $q_k$ is a s.v.d. function of the corresponding $v_k$. Thus for a conditionally periodic non-singular Staeckel system each $q_k$ is a s.v.d. function $f_k(w_1 \dots w_n)$.

## 5. The Periodic Properties of $q_k = f_k(w_1 \dots w_n)$.

By (7) and (15)

$$dw_k = \sum_{i=1}^{n} \frac{\partial p_i}{\partial J_k} dq_i . \tag{27}$$

5. See Appendix for examples.

If now each coordinate $q_i$ goes through an integral number $m_i$ of cycles, then by a familiar argument

$$\Delta w_k = \sum_{i=1}^{n} m_i \oint (\partial p_i/\partial J_k) dq_i = \frac{\partial}{\partial J_k} \sum_{i=1}^{n} m_i \oint p_i dq_i = m_k \quad . \quad . \qquad (28)$$

Thus if each $q_k$ goes through exactly $m_k$ cycles, each $w_k$ increases by the integer $m_k$. (Note, however, that such simultaneous increases are not always physically possible: this section is thus concerned only with the mathematical properties of the functions $f_k(w_1 \ldots w_n)$.)

But we are really interested in the inverse problem where each $w_k$ has increased by an integer $m_k$ and we ask what has happened to the q's. Now the q's are uniquely determined by the w's, because of the single-valued property. In the situation of the preceding paragraph where each librational coordinate returns to its initial value and each rotational coordinate $q_i$ increases by $2\pi m_i$, each angle variable $w_k$ increases by $m_k$. Since the w's determine the q's uniquely, this has the result that whenever $\Delta w_k = m_k$, $k = 1 \ldots n$, each librational coordinate returns to its initial value and each rotational coordinate $q_i$ increases by $2\pi m_i$.

Thus, in the inverse problem, whenever we are given $\Delta w_k = m_k$, $k = 1 \ldots n$, we find that each rotational coordinate $q_i$ must go through exactly $m_i$ cycles and that each librational coordinate $q_j$ must go through some integral number of cycles, $\tau_j$, say. By (28), however, we then find $\tau_j = m_j$. Thus whenever the angle variables $w_k$ are all increased by integer amounts $\Delta w_k = m_k$, each of the functions $q_k = f_k(w_1 \ldots w_n)$ must go through exactly $m_k$ cycles.

## 6. Mean Motions.

If in a time interval T the number of complete cycles passed through by any coordinate $q_k$ is $N_k$, the corresponding mean frequency $n_k$ is, by definition

$$n_k = \lim_{T \to \infty} (N_k/T), \qquad (29)$$

if the limit exists. We shall now prove that $n_k = \nu_k = \partial\alpha_1/\partial J_k$, $k = 1 \ldots n$, for any conditionally periodic non-singular Staeckel system.

To do so, note that if $\nu_1 \ldots \nu_n$ are all commensurable, there exist a positive $\nu_o$ and positive integers $m_1 \ldots m_n$ such that

$$\nu_k = m_k \nu_o, \quad (k = 1 \ldots n) \tag{30}$$

where we may choose $\nu_o$ to be the greatest common divisor of the $\nu_k$'s. Then, from (16) and (30), during the actual motion,

$$w_k = w_k(0) + m_k \nu_o t \quad (k = 1 \ldots n) \tag{31}$$

and in the time interval $T \equiv 1/\nu_o$ we have

$$\Delta w_k = m_k \quad (k = 1 \ldots n) \tag{32}$$

By Section (5) each $q_k$ goes through exactly $m_k$ cycles in this time, so that in this case the motion is truly periodic, with period $1/\nu_o$. The mean frequency of $q_k$ is thus

$$n_k \equiv m_k/T = m_k \nu_o = \nu_k . \tag{33}$$

If the frequencies $\nu_1 \ldots \nu_n$ are not all commensurable, we may let

$$\xi_k \equiv \nu_k/\nu_1 \quad (k = 1 \ldots n) \tag{34}$$

and then at least one of the $\xi$'s will be irrational. Then by (16) and (34), during the actual motion

$$w_k = w_k(0) + \xi_k \nu_1 t . \tag{35}$$

We now use a theorem of Dirichlet [12], which states that if the set of real numbers $\xi_1 \ldots \xi_n$ has at least one irrational member, then the system of inequalities

$$|\xi_k - m_k/P| < P^{-1-\frac{1}{n}} \quad (k = 1 \ldots n) \tag{36}$$

has an infinite number of integer solutions for P and the m's. Note that the solutions for P have no upper bound.

To apply this theorem, consider only those values of the time interval T such that $\nu_1 T = P$, where P is an integer that satisfies (36). In this time each $w_k$ increases from its initial value $w_k(0)$ to a final value given by

$$w_k(T) = w_k(0) + P\xi_k, \tag{37}$$

by (35). But by (36)

$$P\xi_k = m_k + \eta_k, \qquad |\eta_k| < P^{-\frac{1}{n}} \tag{38}$$

so that

$$w_k(T) = w_k(0) + m_k + \eta_k. \tag{39}$$

As $\nu_1 T = P$ takes on those larger and larger integer values corresponding to solutions of (36), each $\eta_k$ approaches zero, by (38). Then, by (39), there exist infinitely many values of T, with no upper bound, at which the orbit in w-space passes arbitrarily closely to points where $\Delta w_k = m_k$, $k = 1 \ldots n$, the $m_k$'s being solutions of (36).

If the initial q's are given by

$$q_k(0) = f_k[w_1(0), \ldots w_n(0)], \qquad (k = 1 \ldots n) \tag{40}$$

then the values of the q's at any of these times T are given by

$$q_k(T) = f_k[w_1(0) + m_1 + \eta_1, \ldots w_n(0) + m_n + \eta_n] \quad (k = 1 \ldots n). \tag{41}$$

As we let $T = P/\nu_1$ assume those larger and larger values already referred to, the q's then approach arbitrarily closely to the values

$$q_k^*(T) = f_k[w_1(0) + m_1, \ldots w_n(0) + m_n] \qquad (k = 1 \ldots n). \tag{42}$$

This conclusion follows from (38) and the single-valuedness and differentiability of the functions $f_k$.

Comparison of (40) and (42) then shows that the values $q_k^*(T)$ correspond to $\Delta w_k = m_k$, $k = 1 \ldots n$, and are thus, by Section 5, the values that would be reached after each $q_k$ had gone through exactly $m_k$ cycles. Now, by the definition (29), it follows that the mean frequency

$$n_k = \lim_{T \to \infty} (m_k/T), \tag{43}$$

if the limit exists. But $m_k/T = \nu_1 m_k/P$ and, as $T \to \infty$, $\lim(m_k/P) = \xi_k$, by (36). Thus

$$n_k = \nu_1 \xi_k = \nu_k \tag{44}$$

by (43) and (34).

Thus, for each coordinate $q_k$ of a conditionally periodic non-singular Staeckel system, the mean frequency $n_k$ is equal to the corresponding fundamental frequency $\nu_k \equiv \partial \alpha_1/\partial J_k$.

## APPENDIX

For theories of satellite orbits, appropriate coordinates are spherical or oblate spheroidal. The corresponding Staeckel matrices and their inverses are, if $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ :

### Spherical

$$\Phi = \begin{pmatrix} 1 & -r^{-2} & 0 \\ 0 & 1 & -\csc^2\theta \\ 0 & 0 & 1 \end{pmatrix} \qquad \Phi^{-1} = \begin{pmatrix} 1 & r^{-2} & r^{-2}\csc^2\theta \\ 0 & 1 & \csc^2\theta \\ 0 & 0 & 1 \end{pmatrix}$$

or, if $x = c[(\xi^2+1)(1-\eta^2)]^{\frac{1}{2}}\cos\phi$, $y = c[(\xi^2+1)(1-\eta^2]^{\frac{1}{2}}\sin\phi$, $z = c\xi\eta$ :

### Oblate Spheroidal

$$\Phi = \begin{pmatrix} c^2\xi^2(\xi^2+1)^{-1} & -(\xi^2+1)^{-1} & (\xi^2+1)^{-2} \\ c^2\eta^2(1-\eta^2)^{-1} & (1-\eta^2)^{-1} & -(1-\eta^2)^{-2} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\Phi^{-1} = \begin{pmatrix} c^{-2}(\xi^2+1)(\xi^2+\eta^2)^{-1} & c^{-2}(1-\eta^2)(\xi^2+\eta^2)^{-1} & c^{-2}(\xi^2+1)^{-1}(1-\eta^2)^{-1} \\ -\eta^2(\xi^2+1)(\xi^2+\eta^2)^{-1} & \xi^2(1-\eta^2)(\xi^2+\eta^2)^{-1} & (\xi^2+\eta^2)^{-1}[\xi^2(1-\eta^2)^{-1} + \eta^2(1+\xi^2)^{-1}] \\ 0 & 0 & 1 \end{pmatrix}.$$

For spherical coordinates $\Phi$ is most easily written down from the expressions for $p_r^2$, $p_\theta^2$, and $p_\phi^2$ in the Kepler problem [2], with replacement of $\alpha_2^2$ and $\alpha_3^2$ by $2\alpha_2$ and $2\alpha_3$. For oblate spheroidal coordinates $\Phi$ may be found by comparing equations (53) and (59.1) of [5] with Eq. (6) of the present paper.

Note that $\Phi$ or $\Phi^{-1}$ could fail to exist only when $\sin\theta = 0$ or when $\eta^2 = 1$. This could happen only when the satellite goes over a pole and thus only in a polar orbit. Such a singularity in a polar orbit, however, is to be expected, since non-singularity of a Staeckel system leads to the q's being differentiable functions of the w's and thus of the time. In a polar orbit, on the other hand, the right ascension $q_3 \equiv \phi$ is a discontinuous function of time, being constant except at polar crossings, where it changes by $\pi$.

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# THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards at its major laboratories in Washington, D.C., and Boulder, Colo., is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section carries out specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant publications, appears on the inside of the front cover.

WASHINGTON, D.C.

ELECTRICITY. Resistance and Reactance. Electrochemistry. Electrical Instruments. Magnetic Measurements. Dielectrics.

METROLOGY. Photometry and Colorimetry. Refractometry. Photographic Research. Length. Engineering Metrology. Mass and Scale. Volumetry and Densimetry.

HEAT. Temperature Physics. Heat Measurements. Cryogenic Physics. Rheology. Molecular Kinetics. Free Radicals Research. Equation of State. Statistical Physics. Molecular Spectroscopy.

RADIATION PHYSICS. X-Ray. Radioactivity. Radiation Theory. High Energy Radiation. Radiological Equipment. Nucleonic Instrumentation. Neutron Physics.

CHEMISTRY. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Molecular Structure and Properties of Gases. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

MECHANICS. Sound. Pressure and Vacuum. Fluid Mechanics. Engineering Mechanics. Combustion Controls.

ORGANIC AND FIBROUS MATERIALS. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Plastics. Dental Research.

METALLURGY. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion. Metal Physics.

MINERAL PRODUCTS. Engineering Ceramics. Glass. Refractories. Enameled Metals. Constitution and Microstructure.

BUILDING RESEARCH. Structural Engineering. Fire Research. Mechanical Systems. Organic Building Materials. Codes and Safety Standards. Heat Transfer. Inorganic Building Materials.

APPLIED MATHEMATICS. Numerical Analysis. Computation. Statistical Engineering. Mathematical Physics.

DATA PROCESSING SYSTEMS. Components and Techniques. Digital Circuitry. Digital Systems. Analog Systems. Applications Engineering.

ATOMIC PHYSICS. Spectroscopy. Radiometry. Mass Spectrometry. Solid State Physics. Electron Physics. Atomic Physics.

INSTRUMENTATION. Engineering Electronics. Electron Devices. Electronic Instrumentation. Mechanical Instruments. Basic Instrumentation.

Office of Weights and Measures.

BOULDER, COLO.

CRYOGENIC ENGINEERING. Cryogenic Equipment. Cryogenic Processes. Properties of Materials. Gas Liquefaction.

IONOSPHERE RESEARCH AND PROPAGATION. Low Frequency and Very Low Frequency Research. Ionosphere Research. Prediction Services. Sun-Earth Relationships. Field Engineering. Radio Warning Services.

RADIO PROPAGATION ENGINEERING. Data Reduction Instrumentation. Radio Noise. Tropospheric Measurements. Tropospheric Analysis. Propagation-Terrain Effects. Radio-Meteorology. Lower Atmosphere Physics.

RADIO STANDARDS. High frequency Electrical Standards. Radio Broadcast Service. Radio and Microwave Materials. Atomic Frequency and Time Standards. Electronic Calibration Center. Millimeter-Wave Research. Microwave Circuit Standards.

RADIO SYSTEMS. High Frequency and Very High Frequency Research. Modulation Research. Antenna Research. Navigation Systems. Space Telecommunications.

UPPER ATMOSPHERE AND SPACE PHYSICS. Upper Atmosphere and Plasma Physics. Ionosphere and Exosphere Scatter. Airglow and Aurora. Ionospheric Radio Astronomy.

NBS