NATIONAL BUREAU OF STANDARDS REPORT

6884

A BRIEF SURVEY
OF MATHEMATICAL FORMULATIONS AND SOLUTION METHODS
FOR OPTIMAL DISTRIBUTION AND STORAGE
OF A FROZEN-FOOD COMMODITY
FOR VETERANS ADMINISTRATION HOSPITALS

by

Lambert S. Joel

Technical Report
to
Veterans Administration

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

THE NATIONAL BUREAU OF STANDARDS

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Computation Laboratory
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PREFACE

The problem considered in this report is that of optimally selecting the locations and employment of intermediate storage facilities (warehouses) in the delivery of a single frozen-food commodity from a single supplier to a group of Veterans Administration hospitals. The mathematical features of the problem are analyzed, the features rendering it more difficult than standard transportation problems are identified, and related computation techniques are cited. The assigned scope of this study includes neither the development of new mathematical techniques nor the detailed adaptation of exisisting ones for the purposes of the problem.

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1. Introduction. In fulfilling its responsibility for supplying frozen foods to hospitals under its jurisdiction, the Veterans Administration must maintain (i.e., rent, purchase or build) sufficient warehouse space to store foods which are purchased seasonally and are requested periodically by the hospitals throughout the year. The transportation and storage costs involved are in general subject to volume discounts.

The problem can be conveniently segmented for the purposes of the following discussion of difficulties and computational techniques. Unfortunately this splitting of the problem into parts, though useful for exposition, is not especially helpful in arriving at numerical solutions.

2. The Classical Transportation Problem. Initially we will ignore the possibility of economies of scale and assume that costs are linear; that is, a constant $c_{ij} \geq 0$ is associated with each combination of a shipment-origin (the i-th, say) and a shipment-destination (the j-th, say), and the cost of shipping the amount x_{ij} from the i-th origin to the j-th destination is simply the product $c_{ij}x_{ij}$.

The "classical" or Hitchcock [(1)] transportation problem is that of finding the optimal , (cost minimizing) pattern of shipments of a single commodity stored at m fixed warehouses (the origins) to a set of n hospitals (the destinations) with known requirements. To formulate this problem algebraically, we set

a_i = amount stored for shipment at the i-th warehouse, $(i = 1,2,\ldots,m)$

b_j = amount required at the j-th hospital, $(j = 1,2,\ldots,n)$;

c_{ij} and x_{ij} are as defined above. The formulation is then given by:

(1) $$\sum_{i=1}^{m} x_{ij} = b_j \qquad (j = 1,2,\ldots,n)$$

(2) $$\sum_{j=1}^{n} x_{ij} = a_i \qquad (i = 1,2,\ldots,m)$$

(3) $$x_{ij} \geq 0 \qquad (i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$$

(4) $$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij} \quad \text{is to be minimized.}$$

Here equations (1) state that the shipments (from all warehouses) to each hospital add up to the hospital's requirement. The equations (2) assert that the shipments (to all hospitals) from each warehouse add up to the amount stored at the warehouse for shipment. Conditions (3) forbid shipments in the "wrong direction," since $x_{ij} < 0$ would correspond to a shipment from the j-th hospital to the i-th warehouse. Display (4) defines the objective function, i.e., the function whose minimization is the objective in the solving of the problem. This minimization is to be attained by a proper selection of values for the shipment-variables x_{ij}, among those sets of values satisfying the "feasibility conditions" (1), (2) and (3).

It can be deduced from (1) and (2) above that

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j,$$

which expresses a perfect balance between total supply and total demand. If

$$\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j ,$$

then the hospitals' requirements cannot be met. If there is an excess of total supply over demand

$$(\text{i.e., } \sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j)$$

then a warehouse need not ship all that it has on hand, so that (2) should really be replaced by

$$(2') \qquad \sum_{j=1}^{m} x_{ij} \leq a_i \qquad (i = 1,2,\ldots,m).$$

It is simpler, however, to reduce this case to that of "perfect balance" by introducing a fictitious (n+1)-st hospital whose fictitious requirement is exactly equal to the excess of total supply over demand; zero transportation costs are associated with this hospital,

$$c_{i,n+1} = 0 \qquad (i = 1,2,\ldots,m),$$

and shipments to the fictitious hospital are ignored in interpreting the solution of the "enlarged" problem.

Generally the commodity being shipped comes in certain indivisible units, and each shipment must consist of an integral number of these units (no fractions). Thus only a finite number of shipment patterns exist which satisfy the feasibility conditions, and in theory the problem could be solved by enumerating these patterns one by one and finally selecting a pattern whose cost (the value of the objective function C) is lowest. The total number of patterns is too large, however, for this procedure to be possible in practice. This can be seen by examining the simplest type of transportation problem, which is called the <u>assignment problem</u>(2). Here the numbers of origins and destinations (i.e., warehouses and hospitals) are equal, m = n, while each origin has one unit to be shipped and each destination requires one unit. Thus the problem is simply that of selecting a warehouse to ship to each hospital, i.e., of pairing off warehouses and hospitals so as to minimize total transportation costs. The number of possible shipment-patterns is

$$n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times (n),$$

which for n = 10 is 518,400. Examination of a single pattern requires ten additions (one for each warehouse-hospital combination in the pattern) plus some "bookkeeping" operations, so that current medium speed digital computers like the IBM 650 could examine roughly 400 patterns per second and would therefore require about 20 minutes to examine all patterns when n = 10. If n = 50, however, then even a hypothetical computer which could examine 1,000,000 patterns per second would require more than 10^{51} years to examine all patterns! (A current estimate for the age of the earth is 10^{10} years.) Thus even for the simplest type of transportation problem, solution of a fairly large problem by examining all possibilities is totally out of the question.

Before the development of linear programming and the growth of interest in allocation problems by mathematicians and mathematical economists, all practical schemes for solving transportation problems consisted essentially of enumeration with "skips," i.e., rules of thumb for bypassing large classes of patterns which obeyed the feasibility conditions (1)-(3) but could reasonably be assumed to be nonoptimal; these rules of thumb were refined by generally efficient methods of "scorekeeping." It is a tribute to the acumen of experienced but mathematically untrained personnel in logistic organizations that these methods proved adequate so long as the size of the problem (i.e., the numbers of origins and destinations) remained moderately small and there was no statutory or legal requirement for attaining the minimum cost exactly.

For example, tests were made in 1952 to compare solution using traditional trial methods by experienced Quartermaster Corps personnel, with solution by computer using precise mathematical methods. The problems involved 20-30 origins, roughly the same number of destinations and thousands of commodity-pounds; the computer used, the NBS SEAC, has a speed roughly comparable to that of the IBM 650. It was found that experienced personnel could generally (but not invariably) come within 0.1% of the true minimum cost in only four or five hours more than the computer required to find the exact solution.(3)

The application of geometry and algebra to logistics problems led to the mathematical theory of linear programming and its most widely used computational technique, the simplex method of G. B. Dantzig(4) which has been subsequently developed further by several authors(5). Most computer programs for solving transportation problems are based on a modification of the simplex method(6) which takes advantage of the fact that these problems are of a "combinatorial" nature permitting their solution without use of multiplication or division. Recently an alternate method, a generalization of H. W. Kuhn's application(7) of a theorem by the Hungarian mathematician Egervary(8) to the assignment problem, has been developed(9) and translated into a computer program. Gerstenhaber's "threshold method" should also be mentioned(10). The avoidance of multiplication and division is quite desirable because these operations are substantially slower than addition or subtraction on a digital computer.

The salient characteristics of these theoretically-based computational methods are

(i) "economy," in the sense that they include formal rules for bypassing huge subsets of the possible shipment patterns,

(ii) "progressive optimization," in the sense that the partial solution at each stage in the computation is objectively better (11) than that at the preceding stage, and

(iii) "dependability," in that there is a rigorous guarantee that the absolute minimum cost will be reached plus an estimate (generally extremely conservative) of the greatest possible number of steps required to reach it.

In spite of the ingenuity of these methods, and of the current advances in technology leading to substantially more rapid computers, there is a real possibility that still more powerful methods will be required in order to deal with the ever-longer transportation problems (i.e., more origins and/or destinations) presented for solution [11a]. Some sources of these enormous problems will be mentioned below. Among recent work which may possibly lead to the necessary improvements, we mention that of Rosen [12] and Warga [13].

3. Linear Generalizations of the Transportation Problem

Thoughout this section, we continue to assume that costs are linear, i.e. to consider only situations in which there are no economies of scale.

One generalization of the classical transportation problem which frequently occurs in practice is known as the *transshipment* or *multistage transportation problem*. Here the commodity need not be shipped directly from an origin to a destination, but instead passes through certain intermediate points. The two-stage problem is fairly representative of problems with more than two stages, and in addition represents the particular situation described to us by the Veterans Administration; therefore we will confine attention to this case. Here we have "original sources" or *suppliers*, intermediate storage points (the warehouses), and final destinations (the hospitals). As in the previous section, we use the notation:

x_{ij} = amount shipped from i-th warehouse to j-th hospital,

c_{ij} = cost of shipping a unit amount from i-th warehouse to j-th hospital,

b_j = requirement at j-th hospital,

m = the number of warehouses,

Fortunately we need not consider the two-stage transportation problem in full generality. In the particular situation under study here, all the suppliers happen to be located so close together that the transportation costs from any two suppliers to the same warehouse are very nearly equal. We can therefore conceptually lump the individual suppliers together into "one big supplier," so that we are dealing with a two-stage transportation problem with only <u>one original source</u>. We use the notation

c_i = cost of shipping a unit amount from supplier to i-th warehouse,

c'_{ij} = cost of storing a unit amount at i-th warehouse for appropriate period[14] associated with j-th hospital.

If we assume that everything shipped to a warehouse is ultimately transmitted to the hospitals, then the amount shipped <u>to</u> the i-th warehouse is simply the sum $\sum_{j=1}^{m} x_{ij}$ of amounts shipped <u>from</u> this warehouse to the various hospitals. Thus the objective function (i.e., the total cost, which is to be minimized) is no longer given by equation (4), but rather by

$$C = \sum_{i=1}^{m} (c_i \sum_{j=1}^{n} x_{ij}) + \sum_{i=1}^{m} \sum_{j=1}^{n} c'_{ij} x_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

the first term representing the costs of supplier-to-warehouse transportation, the second giving the storage costs, while the third accounts for the warehouse-to-hospital transportation costs. This expression can be rewritten as

$$C = \sum_{j=1}^{n} \sum_{i=1}^{m} (c_i + c'_{ij} + c_{ij}) x_{ij}$$

The problem can now be formulated algebraically as

(5) $$\sum_{i=1}^{m} x_{ij} = b_j \qquad (j=1,2,\ldots,n)$$

(6) $$x_{ij} \geq 0 \qquad (i=1,2,\ldots,m;\ j=1,2,\ldots,n)$$

(7) $$C = \sum_{j=1}^{n} \sum_{j=1}^{m} (c_i + c'_{ij} + c_{ij})\, x_{ij} \text{ is to be minimized.}$$

The explanations of conditions (5) and (6) are like those for (1) and (3) in the preceding section. To use linear programming techniques for this problem would be like using an elephant gun to kill a mouse; purely elementary methods suffice for the solution. The point is that the n hospitals play independent roles in the problem.[15] Therefore for each hospital, say the j-th, it is only necessary to find a warehouse for which the cost $c_i + c'_{ij} + c_{ij}$ of sending each unit of the commodity through its supplier-to-warehouse-to hospital history is smallest. The entire requirement b_j for the j-th hospital is then shipped to this warehouse and thence to the hospital; of course different hospitals may use the same warehouse. The minimized cost is then

(8) $$C_{min} = \sum_{j=1}^{n} b_j \min_{1 \leq i \leq m} (c_i + c'_{ij} + c_{ij})$$

where the "min," standing for "minimum," recalls how the warehouse associated with the j-th hospital was selected.

Before proceeding further, we must bring up a topic outside the assigned scope of this study. In this report we are dealing solely with a single commodity. Any reasonably large warehouse can hold a year's supply for all the hospitals (actually eight in number) of this one commodity, and this is one of the reasons why we were specifically permitted to ignore possible restrictions imposed by the finite capacities of the warehouses. A more complete analysis of the Veterans Administration's transportation and storage of goods for its hospitals will necessarily have to take account of the many different commodities involved, and of the almost inevitable interdepency of the shipment patterns for them.[16] Commodities may be treated independently[17] if they reqire totally separate warehouse facilities, but it may be found that foods and other (e.g., medical) supplies "compete" for the same (limited) warehouse space. It is therefore of interest to see how the preceding problem (defined by (5) through (7)) is affected if warehouse capacities are taken into account. The answer is that a classical transportation problem (the type described in the preceding section) is obtained; if

$$a_i = \text{capacity of the i-th warehouse}$$

then the algebraic formulation given by (1), (2'), (3), and (4) of Section 2, with $c_i + c'_{ij} + c_{ij}$ replacing c_{ij} in (4); the inequalities (2') can be replaced with the equations (2) by introducing a fictitious extra hospital as explained earlier.[17a] From now on we shall respect the restricted scope of the study and ignore warehouse capacity restrictions; the remaining difficulties are still formidable.

So far the warehouse locations have been treated as fixed. Actually, the selection of these locations is part of the problem faced by the Veterans Administration. The simplest case of this problem is that in which a list of possible sites for the warehouses is specified, and the actual warehouses to be used must be selected from this list. Here one imagines that a warehouse is available at each possible site and solves the resulting problem; in the solution some of the warehouses will prove to be unused ($x_{ij} = 0$ for $j = 1, 2, \ldots, n$) and the remaining ones form an optimal selection from the list. This method also yields the optimal pattern of shipments. For the problem defined by (5) through (7) above, the computation offers no difficulties. If there are additional complications, however (e.g. volume discounts or warehouse capacity limitations), and if the list of possible warehouses is much larger than the actual number of warehouses to be selected, then the number of (hypothetical) warehouses considered in this method may be so great as to make the problem intractable to current computational techniques.[18] One would expect, though, that an intelligently-selected list of "reasonable" possible sites would not contain more than say double the number of warehouses to be actually used.

If warehouses are not actually present at some of the "possible sites" but might be built there, then appropriately-amortized construction costs must be added to the storage costs.

Here consideration of the proper capacities for these new warehouses enters the problem; if relatively few capacities are to be considered, one approach would be to treat these capacities as parameters and use the methods of parametric programming [19] to explore systematically the results of varying these parameters.

A different version of the "warehouse location" problem arises if the warehouse can be located "almost anywhere"; i.e., if no finite list of possible sites is given in advance. First, combinatorial methods are not applicable since the location of a given warehouse is a continuous variable (combinatorial problems involved only variables restricted to a discrete set of values). Second, the transportation cost rates c_{ij} and c_i become functions $c_{ij}(L_i)$, $c_i(L_i)$ of the location of the i-th warehouse. These costs probably depend in a fairly complicated way on the existing network of transportation facilities, though in especially favorable cases they might depend simply on the warehouse-hospital and supplier-warehouse distances. For the situation described by (5) - (7) the problem is apparently that of selecting the warehouse-locations $L_1, \ldots, L_m$ so as to minimize (cf.(8))

$$(9) \qquad C_{min}(L_1, \ldots, L_m) = \sum_{j=1}^{n} b_j \min_{1 \leq i \leq m} [c_i(L_i) + c_{ij} + c_{ij}(L_i)].$$

Such problems have appeared in the literature [19a], but no general computational methods are available; we believe, however, that developing a computation technique for minimizing the expression (9) is well within the current "state of the art." Such a technique would be inadequate for the Veterans Administration problem, however, both because of the additional complications (to be discussed later) of that problem and because the number of warehouses (i.e., the number of variables in (9)) is itself a variable to be determined rather than a datum.

Still another aspect of the problem ensues if the warehouse sites are to be picked more or less permanently, but the relative requirements at the hospitals may fluctuate in the future in a not entirely predictable way. Since the Veterans Administration has indicated that these fluctuations are not expected to be of significance in the hospital subsistence program, we will only remark that this area of "stochastic programming" or "programming under uncertainty," which deals in part with the optimization of supply systems whose demands are only predictable on a probabilistic basis, has only recently been studied intensively[20]; theoretical understanding in this field seems inadequate for the creation of fully satisfactory computation methods.

4. Nonlinearities. The greatest single difficulty in determining the optimal system of warehouse locations and shipments arises from the nonlinearity of the costs, i.e., the volume discounts. A very simply example will illustrate the nature of these complications. If John is selling apples at 5 cent/lb. and Mary is selling them at 10 cents/lb., then

(assuming all other things equal) one need make no calculation before deciding to do business with John. If Mary now adopts a "nonlinear" price structure and offers to sell at 10 cents/lb. up to four pounds but at 3 cents/lb. thereafter, then it is clear that (a) our optimal purchasing policy now depends strongly on the size of our requirement, and (b) if we want more than four pounds then our choice between the two entrepeneurs involves enormously more analysis, relatively speaking, than what was required before. The confusion is evidently compounded as the number of apple-sellers, and the number of breakpoints in their prices, increases.

Strangely enough, it is only volume discounts that are so troublesome. Adequate computational techniques exist for problems whose nonlinearities arise only from volume penalties (e.g., where a warehouse-owner cannot store an assigned quantity without building an extra wing and therefore charges more per unit for the overflow).

Until recently no method at all existed for guaranteeing a solution to a nonlinear optimization problem involving volume discounts. In 1957 Markowitz and Manne [21] presented a useful reformulation of the problem, which avoided the nonlinearities at the price of a different kind of complication. First, the problem was no longer of the multistage transportation type, but rather of the more general linear programming type, so that its solution would require the relatively slow operations of multiplication and division. Second, the reformulated problem is of mixed integer type; that is, some of its variables are restricted to be integers. The recent adaptation by Gomory [22] of Dantzig's simplex method to linear programming problems involving such "integrality constraints" marks a significant advance in this area of mathematics and provides hope of achieving the computational technique needed to implement the Markowitz-Mannes formulation. [22a]

Gomory's method has not been tested on any large-scale problems. Although its theoretical correctness has been proved (i.e., it is guaranteed to arrive at the correct solution in finitely many steps), its computational usefulness has not been tested on any large-scale problems and so it cannot yet be considered a dependable "packaged product" like the simplex method. Furthermore, Gomory's method applies to pure integer problems (in which all variables are restricted to integer values), and even less is known about its extensions (one by Gomory, one by Beale [23]) to the mixed-integer case presented by the Markowitz-Manne problem.

In spite of these uncertainties, it is generally felt that Gomory's work will lead, in the next few years, to computer programs capable of dealing with moderately large mixed integer linear programming problems. Upon applying the Markowitz-Manne reformulation to the Veterans Administration situation, however, we find that resulting problem is much more than "moderately" large. The presence of seven breakpoints in the shipping and storage cost rates leads, even with a fairly small number (say eight) of possible warehouse sites, to a problem so large as to defy solution in a reasonable amount of time on present-day computers by Gomory-type methods.

5. Conclusions Progress in mathematical knowledge, computational techniques and digital computer rapidity during the past three years have moved the frontiers of the "state of the art" enormously closer to the point where the Veterans Administration problem can be coped with, but that point has not yet been reached. No neatly packaged computer program for the complete solution of such problems now exists, and a substantial research effort would be required before one could be created. It may of course prove possible to devise *ad hoc* techniques which yield more or less satisfactory approximations so long as the terms of the problem do not change appreciably.[24] In any case, we suggest that the problem is inherently so big and so complicated as to require a larger faster computer than the IBM 650 now available to the VA.

(1) F. L. Hitchcock, "The Distribution of a Product from Several Sources to Numerous Localities," Journal of Mathematics and Physics (MIT) 20, 1941, pp. 224-230. Note that our literature references are intended to be illustrative rather than exhaustive.

(2) J. von Neumann, "A Certain Zero-Sum Two-Person Game Equivalent to the Optimal Assignment Problem," Princeton Annals of Mathematical Study 28 (1953), pp. 1-12.

(3) The real object of these experiments was to convince skeptics that the computer solutions could be believed. The problems treated were bid evaluations, which were decomposed into a series of transportation problems. See L. Gainen, "Linear Programming in Bid Evaluations," Proceedings of the Second Symposium in Linear Programming (NBS and USAF), 1955, pp. 29-38 of Vol. 1.

(4) G. B. Dantzig, "Maximization of a Linear Function of Variables Subject to Linear Inequalities," pp. 339-347 of Cowles Commission Monograph 13 on "Activity Analysis of Production and Allocation" (1951).

(5) A survey of work through the end of 1954 is given by A. J. Hoffman, "How to Solve a Linear Programming Problem," Proceedings of the Second Symposium on Linear Programming (NBS and USAF), 1955, pp. 397-424 of Vol. 2. See also G. B. Dantzig, A. Orden and P. Wolfe, "Method for Minimizing a Linear Form under Linear Inequality Restraints," Pacific Journal of Mathematics 5 (1955), and G.B. Dantzig, L. R. Ford and D.R.Fulkerson, "A Primal-Dual Algorithm for Linear Programs," Princeton Annals of Mathematics Study 38 (1956), pp. 171-181.

(6) G. B. Dantzig, "Application of the Simplex Method to a Transportation Problem," Cowles Commission Monograph 13 (1951), pp. 359-373.

(7) H. W. Kuhn, "The Hungarian Method for the Assignment Problem," Naval Logistics Research Quarterly 2 (1955), pp. 83-97, and "Variants of the Hungarian Method," Naval Logistics Research Quarterly 3 (1956), pp. 253-258.

(8) E. Egervary, "Matrixok Kombinatorius Tulajdonsagairol," Mat. es Fiz. Lapok 38 (1931), pp. 16-28.

(9) L. R. Ford and D. R. Fulkerson, "A Simple Algorithm for Finding Maximal Network Flows and an Application to the Hitchcock Problem," Canadian J. Math. 9 (1957), pp. 210-218, and "Solving the Transportation Problem," Management Science 3 (1957), pp. 24-32. J. Munkres, "Algorithms for the Assignment and Transportation Problems," J. Soc. Indust. Appl. Math. 5 (1957), pp. 32-38.

(10) M. Gerstenhaber, "A Solution Method for the Transportation Problem," J. Soc. Indust. Math. 6 (1958), pp. 321-334.

(11) Either because fewer of the conditions (1) through (3)are violated or because the value of the objective function C has been reduced.

(11a) The "standard" computer program for the solution of the transportation problem on the IBM 704 (32,768 words of fast access memory) uses the stepping stone method. It is feasible for solution of systems up to about 600 x 1000 using tapes. Problems of approximately 100 x 100 and smaller can be solved very rapidly because they don't require the use of tapes during the computation. Recent computations at NBS of allocation problems for the Navy averaged 3 to 4 hours for systems 250 x 350. It is interesting to note here that the computer program doesn't eliminate the utility of human expertise. When several problems which are sufficiently similar are to be solved, ingenious rearrangement of the data for one problem based on the solution to a previous one can sometimes save up to 60% of the expected computing time.

(12) J. B. Rosen, "The Gradient Projection Method for Nonlinear Programming," J. Soc. Indust. Appl. Math. 8 (1960), pp. 181-217.

(13) J. Warga, "Convex Minimization Problems II; The Transportation Problem," AVCO RAD Tech-Memo TM-59-21 (1959).

(14) The "appropriate period" depends on how often the hospital orders from the warehouse, and may differ for different hospitals.

(15) This is no longer true when volume discounts are taken into account.

(16) A recent paper by P. Wolfe and G. B. Dantzig," Decomposition Principle for Linear Programs," Operations Research 8, No. 1, Jan. - Feb. 1960, pp. 101-111, treats the problem of synthesizing an overall solution to an interrelated system of linear programs. Their techniques, which will probably be translated into a computer program in the near future, appear at present to provide the most promising approach to multi-commodity problems.

(17) I.e., shipping and storage decisions may be made for each without affecting the costs of the others.

(17a) More generally, any multistage transportation problem, with possible capacity limitations, any number of original sources, and any number of stages, can be transformed into a classical transportation problem. This was recently shown by D. R. Fulkerson, "On the Equivalence of the Capacity-Constrained Transhipment Problem and the Hitchcock Problem," RAND Memo, RM-2480 (1960). The resulting transportation problem is very large, however.

(18) If warehouse capacities are considered, for example, then (assuming linear costs) one would have a Hitchcock problem with a great many warehouses.

(19) S. I. Gass and T. L. Saaty, "The Computational Algorithm for the Parametric Objective Function," Naval Research Logistics Quarterly 2 (1955), "Parametric Objective Function, Part II. Generalization," Operations Research 3 (1955).

(19a) E. L. Brink and J. S. de Cani, "An Analogue Solution of the Generalized Transportation Problem with Specific Application to Marketing Location," Proc. First Int. Conf. on Operational Research, Operations Research Society of America (1957), pp. 123-136.

(20) A Madansky, "Inequalities for Stochastic Linear Programming Problems," Management Science 6 (1960), pp. 197-204. S. Elmaghraby, "Allocation under Uncertainty when the Demand has Continuous d.f.," Management Science 6 (1960), pp. 270-294.

(21) H. M. Markowitz and A. S. Manne, "On the Solution of Discrete Programming Problems," Econometrica 25 (1957), pp. 84-110.

(22) R. E. Gomory, "Outline of an Algorithm for Integer Solutions to Linear Programs," Bull. Amer. Math. Soc. 64 (1958), pp. 275-278.

(22a) G. B. Danzig has remarked that there is another extremely elegant operational formulation of the integrality condition, which however, lacks the virtue of demonstrable convergence to a solution: "Note on Solving Linear Programs in Integers," Naval Logistics Research Quarterly, Vol. 6, No. 1, (March 1959), pp. 75-76.

(23) E. M. Beale, "A Method of Solving Linear Programming Problems When Some but not all of the Variables must take Integral Values," Princeton. Stat. Tech. Research Group Tech. Rep. 19 (1958).

(24) It is known that sufficiently small changes in the coefficients in a linear program will not alter the solution. Studies of the effects of "tolerances" of this sort should prove fruitful in the development of approximate methods, and in fact were the motivation for the technique of Parametric Programming cited in footnote 19, above.

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THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards at its major laboratories in Washington, D.C., and Boulder, Colorado, is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section carries out specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant publications, appears on the inside of the front cover.

WASHINGTON, D.C.

Electricity and Electronics. Resistance and Reactance. Electron Devices. Electrical Instruments. Magnetic Measurements. Dielectrics. Engineering Electronics. Electronic Instrumentation. Electrochemistry.

Optics and Metrology. Photometry and Colorimetry. Photographic Technology. Length. Engineering Metrology.

Heat. Temperature Physics. Thermodynamics. Cryogenic Physics. Rheology. Molecular Kinetics. Free Radicals Research.

Atomic and Radiation Physics. Spectroscopy. Radiometry. Mass Spectrometry. Solid State Physics. Electron Physics. Atomic Physics. Neutron Physics. Radiation Theory. Radioactivity. X-rays. High Energy Radiation. Nucleonic Instrumentation. Radiological Equipment.

Chemistry. Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Molecular Structure and Properties of Gases. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

Mechanics. Sound. Mechanical Instruments. Fluid Mechanics. Engineering Mechanics. Mass and Scale. Capacity, Density, and Fluid Meters. Combustion Controls.

Organic and Fibrous Materials. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Plastics. Dental Research.

Metallurgy. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion. Metal Physics.

Mineral Products. Engineering Ceramics. Glass. Refractories. Enameled Metals. Constitution and Microstructure.

Building Technology. Structural Engineering. Fire Protection. Air Conditioning, Heating, and Refrigeration. Floor, Roof, and Wall Coverings. Codes and Safety Standards. Heat Transfer. Concreting Materials.

Applied Mathematics. Numerical Analysis. Computation. Statistical Engineering. Mathematical Physics.

Data Processing Systems. SEAC Engineering Group. Components and Techniques. Digital Circuitry. Digital Systems. Analog Systems. Application Engineering.

- Office of Basic Instrumentation.
- Office of Weights and Measures.

BOULDER, COLORADO

Cryogenic Engineering. Cryogenic Equipment. Cryogenic Processes. Properties of Materials. Gas Liquefaction.

Radio Propagation Physics. Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Sun-Earth Relationships. VHF Research. Radio Warning Services. Airglow and Aurora. Radio Astronomy and Arctic Propagation.

Radio Propagation Engineering. Data Reduction Instrumentation. Modulation Research. Radio Noise. Tropospheric Measurements. Tropospheric Analysis. Propagation Obstacles Engineering. Radio-Meteorology. Lower Atmosphere Physics.

Radio Standards. High Frequency Electrical Standards. Radio Broadcast Service. High Frequency Impedance Standards. Electronic Calibration Center. Microwave Physics. Microwave Circuit Standards.

Radio Communication and Systems. Low Frequency and Very Low Frequency Research. High Frequency and Very High Frequency Research. Ultra High Frequency and Super High Frequency Research. Modulation Research. Antenna Research. Navigation Systems. Systems Analysis. Field Operations.

NBS