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Mathematical Physics Section

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I. Introduction.

In previous papers [1,2], the authors have solved the Stokes flow problem for certain axially symmetric bodies, with the velocity at infinity uniform and parallel to the axis of symmetry. Each of the bodies considered possessed the property that the meridional section intercepted a segment of the axis of symmetry. In the present paper this assumption is removed; in addition, we shall consider the particular case of the Stokes flow about a torus.

With the introduction of the Stokes stream function ψ, the Stokes flow (or "slow flow") problem becomes a boundary value problem for ψ in a meridional plane, and it is found that ψ must reduce to a constant $\varkappa$ (in general, different $\varkappa$ for different profiles) on each profile in the flow. But whereas in [1,2] it was possible to determine á priori the value of $\varkappa$ from the form of the stream function at infinity, in the case of profiles which nowhere intersect the axis of symmetry this is not longer the case. It will be shown below that the value of $\varkappa$ for each profile can be determined by requiring that the pressure be single-valued in the flow. This requirement can be expressed in the form of an integral condition similar to that imposed in inviscid flows to render a motion acylic [7,3,4], or to that employed to eliminate dislocations in certain problems of elasticity [5,6].

2. Statement of the Problem; Determination of the Boundary Constants.

We consider a collection of m bodies ($m \geq 1$), each of which has an axis of symmetry, and arranged collectively in such a way that the aggregate also has an axis of symmetry. Let this configuration be immersed in a uniform flow of a viscous fluid, the axis of symmetry of the configuration

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of bodies being taken parallel to the direction of the uniform flow. If we assume that U, the speed of the uniform flow, is so small that inertial effects of the motion are negligible in comparison with those of viscosity, then we obtain what is referred to as a Stokes flow [1].

We introduce cylindrical coordinates x, r, θ, where the x-axis is taken along the axis of symmetry with the positive x-direction the same as that of the uniform stream, r is radial distance from the x-axis, and θ an azimuthal angle. We assume that the flow is axi-symmetric, in which case θ plays no role in our problem, and we may restrict our attention to any meridional half-plane $r \geq 0$. The meridional cross-section of the configuration will consist of closed contours C_i, $i = 1,2,\ldots,m$, each lying entirely above the x-axis. (Fig. 1)

By a well-known procedure [1,7], the Stokes flow problem with axial symmetry outlined above is reduced to the determination of a stream function which satisfies in the region of flow the equation

$$L_{-1}^{2} \psi = 0, \tag{2.1}$$

where

$$L_{-1} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}, \tag{2.2}$$

and is such that

$$\left.\begin{aligned} \psi &= \varkappa_i, & (2.3)\\ \frac{\partial \psi}{\partial n} &= 0, & (2.4) \end{aligned}\right\} \text{ on } C_i,\ i = 1,2,\ldots,m$$

and

$$\lim_{\rho\to\infty} \psi = \tfrac{1}{2} r^2 U + O(\rho), \qquad (\rho^2 = r^2 + x^2) \tag{2.5}$$

In problems of classical hydrodynamics the values of the constants $\varkappa_i$ are made determinate by a condition of irrotationality, but this procedure is not available to us. Their determination in the present instance is made in the following way. The equations of motion for a Stokes flow may be written in the form [7]

$$(\text{div. grad})\ \underset{\sim}{u} = \frac{1}{\mu}\ \text{grad}\ p \tag{2.6}$$

where $\underset{\sim}{u}$ is the velocity of flow, p the thermodynamic pressure, and μ is the coefficient of viscosity. In the present instance $\underset{\sim}{u} = (u_x, u_r)$ and grad p $= (\partial p/\partial x, \partial p/\partial r)$. The vorticity is defined as $\underset{\sim}{\zeta} =$ curl $\underset{\sim}{u}$, and it is well known that in the case of axi-symmetric flow the only non-vanishing component is that normal to the meridian plane, so that we may deal solely with $|\underset{\sim}{\zeta}| = \zeta$, given by

$$\zeta = \frac{\partial u_r}{\partial x} - \frac{\partial u_x}{\partial r} \tag{2.7}$$

It is easily verified by the use of (2.6) and (2.7) that

$$\frac{1}{\mu}\frac{\partial p}{\partial r} = \frac{1}{r}\frac{\partial}{\partial x}(r\zeta), \qquad \frac{1}{\mu}\frac{\partial p}{\partial x} = -\frac{1}{r}\frac{\partial}{\partial r}(r\zeta), \tag{2.8}$$

which have precisely the form of the Stokes-Beltrami equations which relate the potential and stream functions in the classical potential flow theory [7].

From these it follows that we may write

$$\frac{\partial p}{\partial \sigma} = \frac{\mu}{r}\frac{\partial}{\partial n}\ (r\zeta) \tag{2.9}$$

or, since [1,7]

$$\zeta = -\frac{1}{r}\ L_{-1}\psi \tag{2.10}$$

that

$$\frac{\partial p}{\partial \sigma} = -\frac{\mu}{r}\frac{\partial}{\partial n}\ [L_{-1}\psi]. \tag{2.11}$$

The unit normals $\underset{\sim}{n}$ and $\underset{\sim}{\sigma}$ defining the direction of the differentiations in (2.11) are related so that counterclockwise rotation of n through a right angle brings it into coincidence with $\underset{\sim}{\sigma}$. Eq. (2.11) has precisely the form of the Stokes-Beltrami equations which relate the potential and stream functions in the classical potential flow theory [7]. On physical grounds it is reasonable to assume that p is a single-valued function of the coordinates (x,r), and the condition that this be so may be expressed in the integral form

$$\oint_C \frac{\partial p}{\partial \sigma}\, d\sigma = 0, \tag{2.12}$$

where C is any closed contour which lies entirely in the flow region. But (2.11) allows us to rewrite this condition of single-valuedness alternatively as

$$\oint_C \frac{1}{r}\frac{\partial}{\partial n}\,[L_{-1}\psi]\; d\sigma = 0 \tag{2.13}$$

In particular, this condition must hold when C is chosen to be a C_i, hence

$$\oint_{C_i} \frac{1}{r}\frac{\partial}{\partial n}\,[L_{-1}\psi] ds = 0, \qquad i = 1, 2, \ldots, m. \tag{2.14}$$

It will be convenient to write the stream function in the form

$$\psi = \frac{1}{2}\, Ur^2 - \psi' + \sum_{i=1}^{m} \varkappa_i \psi_i \tag{2.15}$$

The boundary condition (2.5) is satisfied at once if we require that

$$\lim_{\rho\to\infty} \frac{1}{r}\text{grad}\,\psi' = 0, \tag{2.16}$$

$$\lim_{\rho\to\infty} \frac{1}{r}\text{grad}\,\psi_i = 0, \qquad i = 1,2,\ldots,m \tag{2.17}$$

Eq. (2.3) is satisfied if we require that

$$\psi' = \tfrac{1}{2}\, Ur^2 \qquad \text{on } C_i,\ i=1,2,\ldots,m, \tag{2.18}$$

$$\psi_i = \begin{cases} 1 & \text{on } C_i \\ 0 & \text{on all other profiles,} \end{cases} \tag{2.19}$$

and (2.4) is satisfied provided that

$$\left.\begin{aligned} \frac{\partial \psi'}{\partial n} &= Ur\,\frac{\partial r}{\partial n} \qquad &(2.20)\\ \frac{\partial \psi_i}{\partial n} &= 0 \qquad &(2.21) \end{aligned}\right\}\ \text{on } C_i,\ i=1,2,\ldots,m.$$

Finally, we note that since $L_{-1}(r^2) = 0$ the functions ψ' and ψ_i must satisfy the same differential equation as ψ, i.e.,

$$L_{-1}^2 \psi' = 0 \qquad (2.22)$$

$$L_{-1}^2 \psi_i = 0, \quad i = 1,2,\ldots,m. \qquad (2.23)$$

We have now replaced the original problem for ψ by m + 1 problems for ψ' and ψ_i, i = 1,2,...,m as specified above, provided the $\varkappa_i$ are known. If (2.15) is inserted in (2.14) we obtain

$$\sum_{i=1}^{m} \varkappa_i \oint_{C_k} \frac{1}{r}\frac{\partial}{\partial n} L_{-1}\psi_i \, ds - \oint_{C_k} \frac{1}{r}\frac{\partial}{\partial n} L_{-1}\psi' ds = 0,$$

$$k = 1,2,\ldots,m \qquad (2.24)$$

This constitutes a set of m linear non-homogeneous equations for the determination of the m constants $\varkappa_i$.

It is interesting to note that if we leave the $\varkappa_i$ undetermined in (2.15), and demand that the rate of dissipation of energy of the flow calculated from this ψ be a minimum, we obtain precisely the same $\varkappa_i$ as are given by the condition that p be single-valued.

The drag is given by

$$P = \frac{2\pi\mu}{U}\iint_D r\zeta^2 drdx = \frac{2\pi\mu}{U}\iint_D \frac{1}{r}\left[L_{-1}\psi\right]^2 drdx$$

for any axially symmetric configuration of bodies [1,7], and it was shown in [1] that if the region of flow is simply connected then P has the representation

$$\frac{P}{8\pi\mu} = \lim_{\rho\to\infty} \frac{\rho\psi'}{r^2} \qquad (2.25)$$

In the case at hand, bodies not intersecting r = 0 occur, so that the flow region is multiply connected, but by a slight alteration of the procedure of

[1], Sect. 4, the result (2.25) can be shown to hold once more.* In [1], Eq. (4.4), the portion of the boundary integral in the right hand member which is evaluated on C_i (C_i' in the notation of [1]) vanishes because of (2.13) and the conditions (2.3-.4). Thus (2.25) remains valid when the boundary of the flow region contains profiles of type C_i' and/or C_j.

We shall now consider a problem in which m=1, and the single profile C_1 which occurs is a circle; i.e., we consider the Stokes flow about a torus.

3. The Flow About a Torus.

In order to calculate the flow about a torus we introduce toroidal coordinates [8] (ξ, η) in a meridional (x,r) plane (Fig. 2) by the transformation

$$x = \frac{b \sin \xi}{s - t} \qquad r = \frac{b \sinh \eta}{s - t} \tag{3.1}$$

where

$$s = \cosh \eta \qquad t = \cos \xi \tag{3.2}$$

The curves η = const. in $r \geq 0$ are circles which nest about the point (0,b). Hence any curve $\eta = \eta_o$ = const. defines the boundary (profile) of a torus whose exterior is given by

$$\eta_o \geq \eta \geq 0, \qquad 0 \leq \xi < 2\pi \tag{3.3}$$

We have here an instance of the general problem discussed in Sec. 2 in which m = 1. It is convenient to abandon the notation used there, and refer to the profile as C, and to the value ψ must assume on C as $\varkappa$. Thus, in order that the velocity components be single-valued, we seek solutions of

$$L_{-1}^2 \psi = 0 \tag{3.4}$$

which are periodic (of period 2π) in ξ, and which satisfy the conditions

*The ψ' of equation (2.25) is to be understood here in the sense in which it was used in [1], i.e., $\psi' = \frac{1}{2}Ur^2 - \psi$. It should not be confused with the ψ' in equation (2.15).

$$\left.\begin{aligned} \psi &= \varkappa \\ \frac{\partial \psi}{\partial n} &= 0 \end{aligned}\right\} \quad \text{on C } (\eta = \eta_o) \tag{3.5}$$

where the constant $\varkappa$ must now be chosen as outlined in Sec. 2.* We return to this point later.

In accord with (2.15) we write

$$\psi = \tfrac{1}{2}\, Ur^2 - \psi' + \varkappa\psi_1 \tag{3.6}$$

where ψ' and ψ_1 give rise to zero velocity at infinity and by (2.18-2.19) satisfy on $\eta = \eta_o$ the conditions

$$\psi' = \tfrac{1}{2}Ur^2, \qquad \frac{\partial \psi'}{\partial n} = Ur\frac{\partial r}{\partial n}; \tag{3.7}$$

$$\psi_1 = 1, \qquad \frac{\partial \psi_1}{\partial n} = 0. \tag{3.8}$$

We first determine ψ'. To this end we choose to represent it in the following way (see Payne [9])

$$\psi' = \frac{Ur^2}{2}[\psi^1 + (\rho^2 + b^2)\psi^3] \tag{3.9}$$

where $\psi^k(x,r)$ represent any solution of

$$L_k(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{k}{r}\frac{\partial v}{\partial r} = 0 \tag{3.10}$$

i.e., ψ^k is a generalized axially symmetric potential function in the terminology of A. Weinstein [10]. From the relation $\psi^k = r^{1-k}\psi^{2-k}$, due to Weinstein, we see that

$$\psi^3 = r^{-2}\psi^{-1} \tag{3.11}$$

where ψ^{-1} is the stream function of inviscid incompressible hydrodynamics. From [6] we then obtain as functions suitable for use in toroidal coordinates

*The flow about a torus was considered previously by S. Ghosh [11]. His solution is for the case $\varkappa = 0$ and (see Sec. 2) consequently is not of physical interest.

$$\psi^1 = (s-t)^{\frac{1}{2}} \sum_{n=o}^{\infty}{}' B_n P_{n-\frac{1}{2}}(s) \cos n\xi$$

$$\psi^3 = \frac{(s-t)^{3/2}}{2b^2} \sum_{n=o}^{\infty}{}' A_n P^{(1)}_{n-\frac{1}{2}}(s) \cos n\xi$$

and we have then the representation

$$\psi' = \tfrac{1}{2} Ur^2 (s-t)^{\frac{1}{2}} \sum_{n=o}^{\infty}{}' [A_n s P^{(1)}_{n-\frac{1}{2}}(s) + B_n P_{n-\frac{1}{2}}(s)] \cos n\xi \tag{3.12}$$

where $\sum'$ indicates that the term for n = 0 is to be multiplied by 1/2, $P^{(1)}_{n-\frac{1}{2}}(s) = dP_{n-\frac{1}{2}}/ds$, and the A_n, B_n are undetermined coefficients.

With the insertion of (3.12) the first of conditions (3.7) becomes

$$(s_o - t)^{-\frac{1}{2}} = \sum_{n=o}^{\infty}{}' [A_n s_o P^{(1)}_{n-\frac{1}{2}}(s_o) + B_n P_{n-\frac{1}{2}}(s_o)] \cos n\xi \tag{3.13}$$

where $s_o = \cosh \eta_o$. But (see [8], p. 443)

$$(s_o - t)^{-\frac{1}{2}} = \frac{2\sqrt{2}}{\pi} \sum_{n=o}^{\infty}{}' Q_{n-\frac{1}{2}}(s_o) \cos n\xi \ , \tag{3.14}$$

and thus one is led to the condition

$$A_n s_o P^{(1)}_{n-\frac{1}{2}}(s_o) + B_n P_{n-\frac{1}{2}}(s_o) = \frac{2\sqrt{2}}{\pi} Q_{n-\frac{1}{2}}(s_o) \tag{3.15}$$

to be satisfied by A_n and B_n if the first equation of (3.7) is to hold. The second condition (3.7), under the assumption that the first is satisfied and that ψ' is given by (3.12), reduces to

$$\sum_{n=o}^{\infty} \frac{d}{ds_o} [A_n s_o P^{(1)}_{n-\frac{1}{2}}(s_o) + B_n P_{n-\frac{1}{2}}(s_o)] \cos n\xi = \frac{d}{ds_o}(s_o - t)^{-\frac{1}{2}} \tag{3.16}$$

The series (3.14) is now inserted on the right, and the permissible (in $s_o > 1$) interchange of operations made there, yielding the relation

$$A_n \frac{d}{ds_o}\left[s_o P^{(1)}_{n-\frac{1}{2}}(s_o)\right] + B_n P^{(1)}_{n-\frac{1}{2}}(s_o) = \frac{2\sqrt{2}}{\pi} Q^{(1)}_{n-\frac{1}{2}}(s_o) \tag{3.17}$$

to be satisfied if the second relation (3.7) is to hold. If we let

$$F_n(s_o) = s_o P^{(1)}_{n-\frac{1}{2}}(s_o) \qquad G_n(s_o) = P_{n-\frac{1}{2}}(s_o)$$

$$H_n(s_o) = \frac{2\sqrt{2}}{\pi} Q_{n-\frac{1}{2}}(s_o) \tag{3.18}$$

then A_n and B_n are found from (3.15) and (3.17) to be given by

$$A_n = \frac{1}{\Delta_n(s_o)}\left[G_n(s_o)H'_n(s_o) - G'_n(s_o)H_n(s_o)\right]$$
$$B_n = \frac{1}{\Delta_n(s_o)}\left[F'_n(s_o)H_n(s_o) - F_n(s_o)H'_n(s_o)\right] \tag{3.19}$$

where

$$\Delta_n(s_o) = F'_n(s_o)G_n(s_o) - F_n(s_o)G'_n(s_o) \tag{3.20}$$

and $(\ )' = d(\ )/d\,s_o$.

By a rather lengthy procedure similar to that used in [2] we obtain the identities

$$\frac{d}{ds_o}\left[s_o P^{(1)}_{n-\frac{1}{2}}(s_o)\right] Q_{n-\frac{1}{2}}(s_o) - s_o P^{(1)}_{n-\frac{1}{2}}(s_o)\, Q^{(1)}_{n-\frac{1}{2}}(s_o) = -\frac{2}{s_o^2-1}\int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)Q_{n-\frac{1}{2}}(\tau)d\tau$$

$$\frac{d}{ds_o}\left[s_o P^{(1)}_{n-\frac{1}{2}}(s_o)\right]P_{n-\frac{1}{2}}(s_o) - s_o\left[P^{(1)}_{n-\frac{1}{2}}(s_o)\right]^2 = -\frac{2}{s_o^2-1}\int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)P_{n-\frac{1}{2}}(\tau)d\tau \tag{3.21}$$

These relations, together with the well-known identity ([8],p.233),

$$Q_{n-\frac{1}{2}}(s_o)P^{(1)}_{n-\frac{1}{2}}(s_o) - P_{n-\frac{1}{2}}(s_o)Q^{(1)}_{n-\frac{1}{2}}(s_o) = \frac{1}{s_o^2-1} \tag{3.22}$$

yield as alternate representations to (3.19)

$$A_n = \frac{\sqrt{2}}{\pi}\left(\int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)P_{n-\frac{1}{2}}(\tau)\,d\tau\right)^{-1}$$

$$B_n = 2A_n \int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)\,Q_{n-\frac{1}{2}}(\tau)\,d\tau \tag{3.23}$$

We turn now to the determination of ψ_1. In this case we choose the representation

$$\psi_1 = \frac{r^2}{b^2}\,[\psi_1^1 + (\rho^2 + b^2)\psi_1^3]$$

$$= \frac{s^2-1}{(s-t)^{3/2}} \sum_{n=o}^{\infty}{}' [C_n s\, P^{(1)}_{n-\frac{1}{2}}(s) + D_n P_{n-\frac{1}{2}}(s)] \cos n\xi \tag{3.24}$$

where ψ_1^1 and ψ_1^3 are the same as ψ_1 and ψ_3 in (3.12), with A_n and B_n replaced by C_n and D_n, respectively. If we substitute this expression into the first equation of (3.8) the result is

$$\sum_{m=o}^{\infty}{}' [C_m s_o P^{(1)}_{m-\frac{1}{2}}(s_o) + D_m P_{m-\frac{1}{2}}(s_o)] \cos m\xi = \frac{(s_o - \cos\xi)^{3/2}}{s_o^2 - 1} \tag{3.25}$$

If we multiply both members by cos $n\xi$ and integrate with respect to ξ from 0 to 2π, we find that C_n and D_n must satisfy

$$C_n s_o\, P^{(1)}_{n-\frac{1}{2}}(s_o) + D_n P_{n-\frac{1}{2}}(s_o) = \frac{2}{\pi(s_o^2-1)}\int_o^{\pi} (s_o - \cos\xi)^{3/2} \cos n\xi\, d\xi \tag{3.26}$$

if the first equation (3.8) is to hold. Reference to [8],p. 248 shows, however, that

$$\frac{2}{\pi(s_o^2-1)}\int_o^{\pi} (s_o - \cos\xi)^{3/2} \cos n\xi\, d\xi = \frac{2(-1)^n}{(s_o^2-1)^{1/4}}\,\frac{\Gamma(5/2)}{\Gamma(n+\frac{5}{2})}\,P^n_{3/2}\left(\frac{s_o}{\sqrt{s_o^2-1}}\right) \tag{3.27}$$

If the right side of (3.26) is replaced by this and use made of Whipple's relation ([8],p.245), we obtain

$$C_n s_o P^{(1)}_{n-\frac{1}{2}}(s_o) + D_n P_{n-\frac{1}{2}}(s_o) = \frac{3}{\pi\sqrt{2}} Q^{-2}_{n-\frac{1}{2}}(s_o) \tag{3.28}$$

to be satisfied by C_n and D_n.

The second condition (3.8) is handled in much the same way as (3.7), and leads finally to

$$C_n \frac{d}{ds_o}\left[s_o P^{(1)}_{n-\frac{1}{2}}(s_o)\right] + D_n P^{(1)}_{n-\frac{1}{2}}(s_o) = \frac{3}{\pi\sqrt{2}} \frac{d}{ds_o} Q^{-2}_{n-\frac{1}{2}}(s_o) \tag{3.29}$$

to be satisfied by C_n and D_n. Now let $F_n(s_o)$ and $G_n(s_o)$ be defined as in (3.18) and let

$$M_n(s_o) = \frac{3}{\pi\sqrt{2}} Q^{-2}_{n-\frac{1}{2}}(s_o) \tag{3.30}$$

The solution of (3.28-.29) for C_n and D_n then gives

$$\begin{aligned} C_n &= \frac{1}{\Delta_n(s_o)}\left[M_n'(s_o)G_n(s_o) - M_n(s_o)G_n'(s_o)\right] \\ D_n &= \frac{1}{\Delta_n(s_o)}\left[M_n(s_o)F_n'(s_o) - M_n'(s_o)F_n(s_o)\right] \end{aligned} \tag{3.31}$$

The repeated use of the differential equations satisfied by the Legendre functions which appear in (3.31) allows us to derive identitites similar to those of (3.21), and so finally to show that

$$C_n = \frac{3\sqrt{2}}{4(n^2-\frac{9}{4})(n^2-\frac{1}{4})} \cdot \frac{(n+\frac{3}{2})(n+\frac{1}{2}) + 4\int_{s_o}^{\infty} P_{n-\frac{1}{2}}(\tau)Q^{(2)}_{n-\frac{1}{2}}(\tau)d\tau}{\int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)\, P_{n-\frac{1}{2}}(\tau)d\tau} \tag{3.32}$$

$$D_n = \frac{3\sqrt{2}}{4(n^2 - \frac{9}{4})} \frac{(n+\frac{3}{2}) + 2\int_{s_o}^{\infty} P_{n-\frac{1}{2}}(\tau)Q^{(2)}_{n-\frac{1}{2}}(\tau)d\tau}{\int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)P_{n-\frac{1}{2}}(\tau)d\tau}$$

or as

$$C_n = \frac{3\sqrt{2}}{4(n^2-\frac{9}{4})(n^2-\frac{1}{4})} \frac{(n^2-\frac{9}{4}) + \frac{4s_o^2}{s_o^2-1} - 4\int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)Q_{n-\frac{1}{2}}(\tau)d\tau}{\int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)\, P_{n-\frac{1}{2}}(\tau)d\tau}$$

$$D_n = \frac{3\sqrt{2}}{2(n^2-\frac{9}{4})} \frac{\frac{s_o^2}{s_o^2-1} - \int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)\, Q_{n-\frac{1}{2}}(\tau)d\tau}{\int_1^{s_o} P^{(2)}_{n-\frac{1}{2}}(\tau)\, P_{n-\frac{1}{2}}(\tau)d\tau} \tag{3.33}$$

Accordingly, the complete solution ψ is given by

$$\psi(x,r) = \tfrac{1}{2}Ur^2\left\{1-(s-t)^{\frac{1}{2}}\sum_{n=o}^{\infty}{}' \left[A_n s\, P^{(1)}_{n-\frac{1}{2}}(s) + B_n P_{n-\frac{1}{2}}(s)\right]\cos n\xi \right. \tag{3.34}$$

$$\left. + \varkappa \frac{s^2-1}{(s-t)^{\frac{3}{2}}} \sum_{n=o}^{\infty}{}' \left[C_n s\, P^{(1)}_{n-\frac{1}{2}}(s) + D_n P_{n-\frac{1}{2}}(s)\right]\cos n\xi\right\}$$

where the A_n and B_n are given by (3.23) and the C_n and D_n by (3.33).

4. The Calculation of $\varkappa$.

A scheme for the determination of the values taken on by the stream-function on the profiles in a flow was outlined in Sec. 2. For the particular problem we are considering the value of the single constant $\varkappa$ (i.e., $\varkappa_1$) is given by the single equation (2.24) obtained when m = k = 1. The use of that equation requires that ψ and ψ_1 given by (3.12) and (3.24) respectively, be inserted therein, and the indicated integrations carried out. This is a formidable procedure, and one which, fortunately, need not be undertaken. We shall use an alternative method for determining $\varkappa$ which makes use of the fact noted in Sec. 2 that the contour used in the integrations just mentioned may be any which encloses the body profile. We choose for this contour one composed of the segment of the x-axis $-a \leq x \leq a$ and the semi-circle $\rho = a$ which joins its end points $(\pm a, 0)$ (where a is arbitrary save that the profile C must lie within the region enclosed by these arcs), and let a tend to infinity. Denoting these arcs by Γ_1 and Γ_2 respectively, (2.13) yields

$$\int_{\Gamma_1} \frac{1}{r}\frac{\partial}{\partial n}(L_{-1}\psi)ds + \int_{\Gamma_2} \frac{1}{r}\frac{\partial}{\partial n}(L_{-1}\psi)ds = 0. \tag{4.1}$$

From (3.9) and (3.24), letting $\gamma = 2\varkappa/b^2U$,

$$\psi = \frac{Ur^2}{2}\left[1-\psi^1-(\rho^2+b^2)\psi^3\right] + \gamma\left[\psi_1^1 + (\rho^2+b^2)\psi_1^3\right] \tag{4.2}$$

and hence

$$\begin{aligned} L_{-1}\psi = U\Big\{ &-\left[r\frac{\partial\psi^1}{\partial r} + r^2\psi^3 + 2\rho\frac{\partial}{\partial\rho}(r^2\psi^3)\right] \\ &+ 2\gamma\left[r\frac{\partial\psi_1^1}{\partial r} + r^2\psi_1^3 + 2\rho\frac{\partial}{\partial\rho}(r^2\psi_1^3)\right]\Big\} \end{aligned} \tag{4.3}$$

On Γ_1,

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial n}(L_{-1}\psi) = -\frac{1}{r}\frac{\partial}{\partial r}(L_{-1}\psi) = U\Big\{ &-\left[\frac{\partial^2\psi^1}{\partial x^2} - 10\psi^3 - 4x\frac{\partial\psi^3}{\partial x}\right] \\ &+2\gamma\left[\frac{\partial^2\psi_1^1}{\partial x^2} - 10\psi_1^3 - 4x\frac{\partial\psi_1^3}{\partial x}\right]\Big\}_{r=o} \end{aligned} \tag{4.4}$$

and if we insert this in (4.1), and note that it follows from the remarks of [1], Sec. 4 that the integral along ρ =a tends to zero as a $\rightarrow \infty$, we obtain

$$\int_{-\infty}^{\infty}\left\{\left[\frac{\partial^2\psi^1}{\partial x^2} - 10\,\psi^3 - 4x\,\frac{\partial\psi^3}{\partial x}\right] - \gamma\left[\frac{\partial^2\psi_1^1}{\partial x^2} - 10\psi_1^3 - 4x\,\frac{\partial\psi_1^3}{\partial x}\right]\right\}_{r=0} dx=0 \tag{4.5}$$

From the properties of $\psi^1(x,r)$ and $\psi^3(x,r)$ it follows that

$$\lim_{\rho\to\infty}\frac{\partial\psi^1}{\partial x} = 0, \qquad \lim_{\rho\to\infty}\,(x\psi^3) = 0 \tag{4.6}$$

These, and the fact that ψ^3 and ψ^1 are even functions in x, permit us to write

$$\int_0^{\infty}\left\{\psi^3 - \gamma\psi_1^3\right\}_{r=0} dx = 0 \tag{4.7}$$

or

$$\gamma = \int_0^{\infty}\psi^3\,(0,x)dx \Bigg/ \int_0^{\infty}\psi_1^3(0,x)dx \tag{4.8}$$

The integrands ψ^3 and ψ_1^3 are to be obtained from (3.12) and (3.24) respectively. (See the remarks preceding (3.12).) Inserting these in (4.8), and noting that $dx = (1-t)^{-1}d\xi$ for r=0, the result is

$$\gamma = \frac{\displaystyle\int_0^{\pi}(1-t)^{\frac{1}{2}}\sum_{n=0}^{\infty}{}' A_n P_{n-\frac{1}{2}}^{(1)}(1)\cos n\xi\; d\xi}{\displaystyle\int_0^{\pi}(1-t)^{\frac{1}{2}}\sum_{n=0}^{\infty}{}' C_n P_{n-\frac{1}{2}}^{(1)}(1)\cos n\xi\; d\xi}$$

$$= \frac{\sum_{n=0}^{\infty}{}' A_n (n^2 - \frac{1}{4}) \int_0^{\pi} (1-\cos\tau)^{\frac{1}{2}} \cos n\tau \, d\tau}{\sum_{n=0}^{\infty}{}' C_n (n^2 - \frac{1}{4}) \int_0^{\pi} (1-\cos\tau)^{\frac{1}{2}} \cos n\tau \, d\tau} \tag{4.9}$$

The integrals are easily evaluated. We find that

$$\begin{aligned} \int_0^{\pi} (1-\cos\tau)^{\frac{1}{2}} \cos n\tau \, d\tau &= \sqrt{2} \int_0^{\pi} \sin \frac{\tau}{2} \cos n\tau \, d\tau \\ &= - \frac{1}{\sqrt{2}(n^2 - \frac{1}{4})} \end{aligned} \tag{4.10}$$

so that we have

$$\gamma = \frac{\sum_{n=0}^{\infty}{}' A_n}{\sum_{n=0}^{\infty}{}' C_n} \quad \text{and} \quad \varkappa = \frac{b^2 U}{2} \frac{\sum_{n=0}^{\infty}{}' A_n}{\sum_{n=0}^{\infty}{}' C_n} . \tag{4.11}$$

5. Determination of the Drag.

Since in the present case, ψ' of (2.25)* is represented by the last two terms on the right of (2.15) , the drag P of the torus is given by

$$\frac{P}{8\pi\mu} = \lim_{\rho \to \infty} \frac{\rho}{r^2} (\psi' - \varkappa \psi_1) . \tag{5.1}$$

The expressions (3.12) and (3.24) for ψ' and ψ_1, respectively, are now substituted in this, as well as ρ/r^2 from (3.1). We then obtain

$$\frac{P}{2\pi\mu} = \sqrt{2}Ub \sum_{n=0}^{\infty}{}' \left\{ (A_n - \gamma C_n)(n^2 - \frac{1}{4}) + 2(B_n - \gamma D_n) \right\} \tag{5.2}$$

*As previously noted, ψ' is used in different senses in [1] and the present paper.

where we have employed

$$P^{(1)}_{n-\frac{1}{2}}(1) = \tfrac{1}{2}(n^2 - \tfrac{1}{4}), \quad P_{n-\frac{1}{2}}(1) = 1.$$

6. The Normal Pressure.

Once the function ψ of (2.15) has been found, the integration of (2.11) yields the normal pressure p. If p is desired at (x', r'), we integrate (2.11) from the point at infinity along any curve joining it to (x', r'), thus obtaining

$$p(x',r') = p_\infty + \mu \int_{(x',r')}^{\infty} \frac{1}{r}\frac{\partial}{\partial n}(L_{-1}\psi)\, d\sigma \tag{6.1}$$

where p_∞ is the free stream pressure of the uniform flow, and σ is used to indicate arc-length.

The integration is usually best carried out along coordinate curves. Thus, in the case of the torus, if (ξ', η') are the bipolar coordinates of (x', r'), the integration is taken along the x-axis from $x = \infty$ to the point T at which $\xi = \xi'$, cuts the x-axis, and then along the curve $\xi = \xi'$ until $\eta = \eta'$ is reached. (Fig 2.) It is easy to show that (6.1) then becomes

$$p(\xi',\eta') = p_\infty + \int_{(0,0)}^{(\xi',0)} \frac{1}{r}\frac{\partial}{\partial n}(L_{-1}\psi)\, d\sigma - \mu \int_{(\xi',0)}^{(\xi',\eta')} \frac{\partial}{\partial n} L_{-1}\psi \frac{d\eta}{\sinh\eta} \tag{6.2}$$

The third term is not easy to evaluate, and since we have found the quantity of primary interest associated with the stresses, viz., the drag, we shall not carry (6.2) further for arbitray (ξ', η'). On the x-axis, however, the expansion of p becomes fairly simple. Noting that $[\partial(\ )/\partial n]_{r=0} = [\partial(\ )/\partial r]_{r=0}$, (6.1) becomes

$$p(\alpha,0) = p_\infty + \mu \int_{x'}^{\infty} \left[\frac{1}{r}\frac{\partial}{\partial r} L_{-1}\psi\right]_{r=0} dx \tag{6.3}$$

for $0 \leq \alpha \leq \infty$, where the arguments of p are now x and r. Now if we insert (4.4) for the integrand, and take into account (4.6), the result is

$$p(\alpha,0) = p_{\infty} - \mu U\Big\{4\alpha\psi^3(\alpha,0) - \frac{\partial\psi'(\alpha,0)}{\partial x} - 6\int_{\alpha}^{\infty}\psi^3(\lambda,0)d\lambda$$

$$-\gamma\Big(4\alpha\psi_1^3(\alpha,0) - \frac{\partial\psi_1^1(\alpha,0)}{\partial x} - 6\int_{\alpha}^{\infty}\psi_1^3(\lambda,0)d\lambda\Big)\Big\} \tag{6.4}$$

Where ψ^1, ψ^3, ψ_1^1, ψ_1^3 are as given before. If we set $\alpha = 0$ in (6.4) it is easy to show that

$$p(0,0) = p_{\infty} + 6\mu U\Big\{\int_0^{\infty}\psi^3(\lambda,0)d\lambda - \gamma\int_0^{\infty}\psi_1^3(\lambda,0)d\lambda\Big\}$$

In view of (4.8), however, the bracketed term vanishes, and we have the interesting result that

$$p(0,0) = p_{\infty}\,. \tag{6.5}$$

Along the r-axis between the origin and the surface of the torus, (6.1) yields

$$p(0,r) = p_{\infty} + \mu\int_0^r\Big[\frac{1}{r}\frac{\partial}{\partial n}L_{-1}\psi\Big]_{x=0}dr - \mu\int_0^{\infty}\Big[\frac{1}{r}\frac{\partial}{\partial n}L_{-1}\psi\Big]_{r=0}dx \tag{6.6}$$

We have just shown, however, that the last term of (6.6) vanishes. Moreover, the expressions for ψ^3, etc., in (ξ,η) coordinates are inserted in (4.4), it is found that every term of the integrand of the second term above contains either $\sin\xi$ or $\sin n\xi$. But $\xi = \pi$ on $x = 0$ (r b), and hence it follows that $[1/r\cdot\partial(L_{-1}\ \ )/\partial n]_{x=0}=0$ in the second term of (6.6). Accordingly, not only is $p(0,0) = p_{\infty}$, but p retains this value on the segment $0 \leq \eta \leq \eta_0$ $(\xi = \pi)$ of the r-axis, i.e., at all points of the equatorial diaphragm across the "hole" of the torus. It is obvious that in fact $p = p_{\infty}$ along the entire r-axis outside the torus.

7. The Stress Components.

In an axially symmetric flow referred to as x,r coordinate system in a meridional plane, the only stress components of interest are p_{xx}, p_{rr}, and

$p_{xr} = p_{rx}$. That is to say, at each point of a meridional plane we need consider only the stress tensor associated with elements of fluid surface which are normal to this plane, and hence are determined by elements of arc lying in it.

From [7], p. 518 and the expressions for the velocity components in terms of ψ the stress components of the flow are

$$
\begin{aligned}
p_{xx} &= -p + \frac{2\mu}{r}\psi_{xr} \\
p_{rr} &= -p - \frac{2\mu}{r}\psi_{xr} \\
p_{xr} = p_{rx} &= \frac{\mu}{r}\left(\psi_{rr} - \psi_{xx} - \frac{1}{r}\psi_r\right) \\
&= \frac{\mu}{r}\left(L_{-1}\psi - 2\psi_{xx}\right)
\end{aligned}
\tag{7.1}
$$

For an arbitrary element of arc determined by its unit normal $\underset{\sim}{n} = (n_x, n_r)$ (or by the unit tangent vector σ oriented with respect to n as in Fig. 2) it is natural to define the stress vector on the element in terms of p_{nn} and p_{ns}, the normal and tangential components of stress, respectively. We may write

$$
\begin{aligned}
p_{nn} &= p_{xn}n_x + p_{rn}n_r \\
p_{ns} &= -p_{xn}n_r + p_{rn}n_x
\end{aligned}
\tag{7.2}
$$

where

$$
\begin{aligned}
p_{xn} &= p_{xx}n_x + p_{xr}n_r \\
p_{rn} &= p_{xr}n_x + p_{rr}n_r
\end{aligned}
\tag{7.3}
$$

Using (7.1) and (7.3), it is easy to show that

$$
\begin{aligned}
p_{xn} &= -pn_x + \frac{\mu n_r}{r} L_{-1}\psi + \frac{2\mu}{r}\frac{d}{d}\psi_x \\
p_{rn} &= -pn_r - \frac{\mu n_x}{r}\left(L_{-1}\psi + \frac{2}{r}\psi_r \right) + \frac{2\mu}{r}\frac{d}{d}\psi_r
\end{aligned}
\tag{7.4}
$$

We are principally interested in p_{nn} and p_{ns} on the complete boundary C of a body in the flow — and hence on each boundary of type C_i' and C_j of [1]. From (2.3) and (2.4) it follows that $\psi_x = 0$ on C, and from this that $d(\psi_x)/d\sigma = 0$ on C. Similarly, $\psi_r = 0$ on C, and it follows that $d(\psi_r)/d\sigma = 0$ on C. When these results are applied to (7.4), we find that

$$
\begin{aligned}
p_{xn} &= -pn_x + \frac{\mu n_r}{r} L_{-1}\psi \\
p_{rn} &= -pn_r - \frac{\mu n_x}{r} L_{-1}\psi
\end{aligned}
\tag{7.5}
$$

If the expression (6.1) for p is now introduced into (7.5), and the result inserted in (7.2), then the normal and tangential stress components on the element defined by n at a point (x,r) of the flow are found to be

$$
\begin{aligned}
p_{nn} &= -p_{\infty} - \mu \int_{(x,r)}^{\infty} \frac{1}{r}\frac{\partial}{\partial n} L_{-1}\psi \, d\sigma \\
p_{ns} &= -\frac{\mu}{r} L_{-1}\psi = \mu \zeta
\end{aligned}
\tag{7.6}
$$

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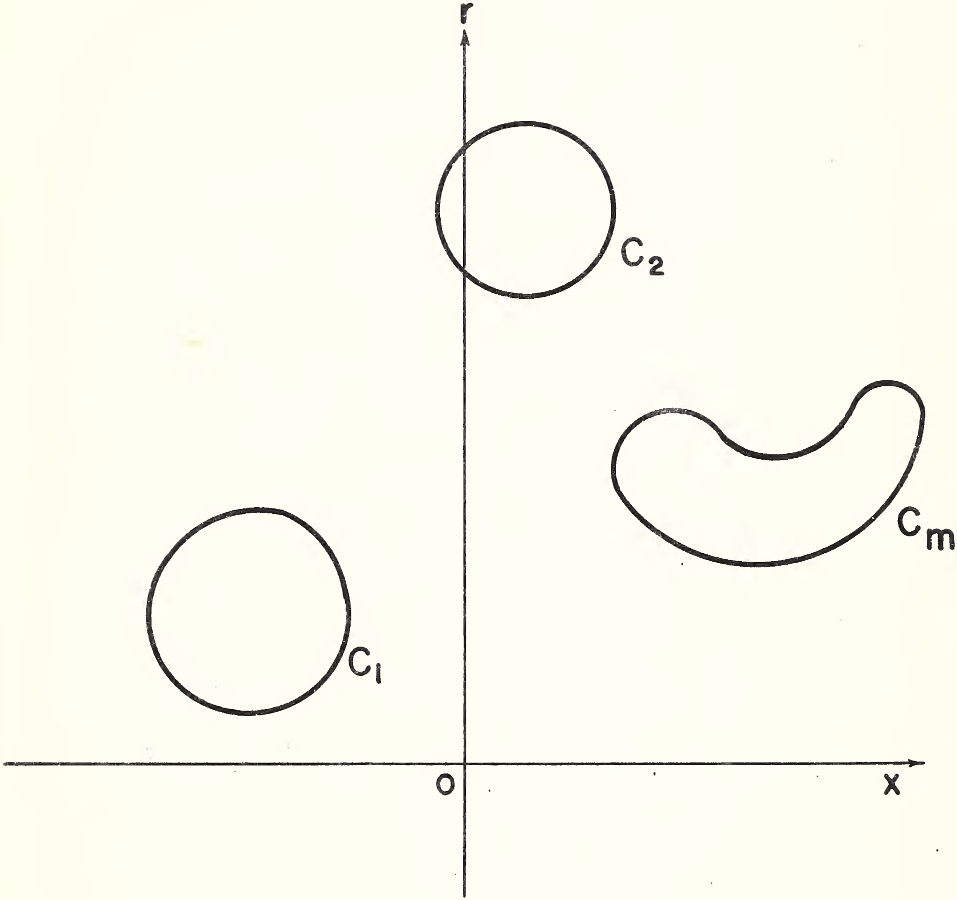

Fig. I. THE GENERAL CONFIGURATION

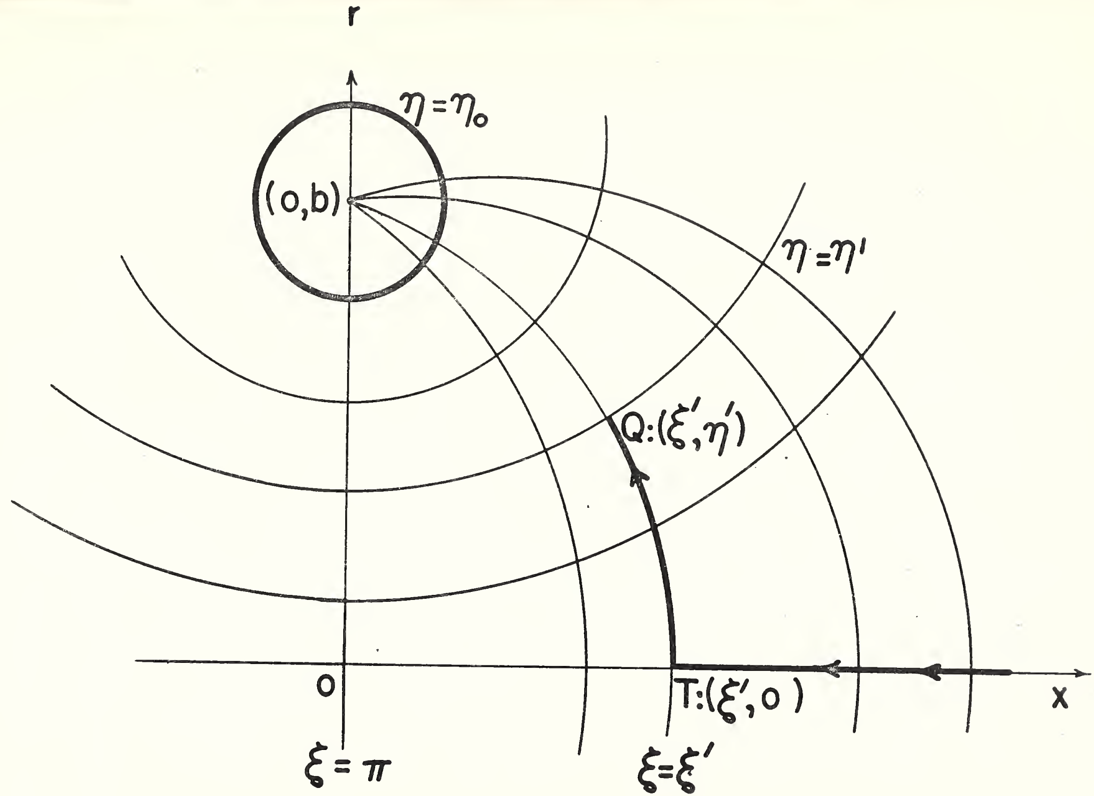

Fig. 2. THE TORUS

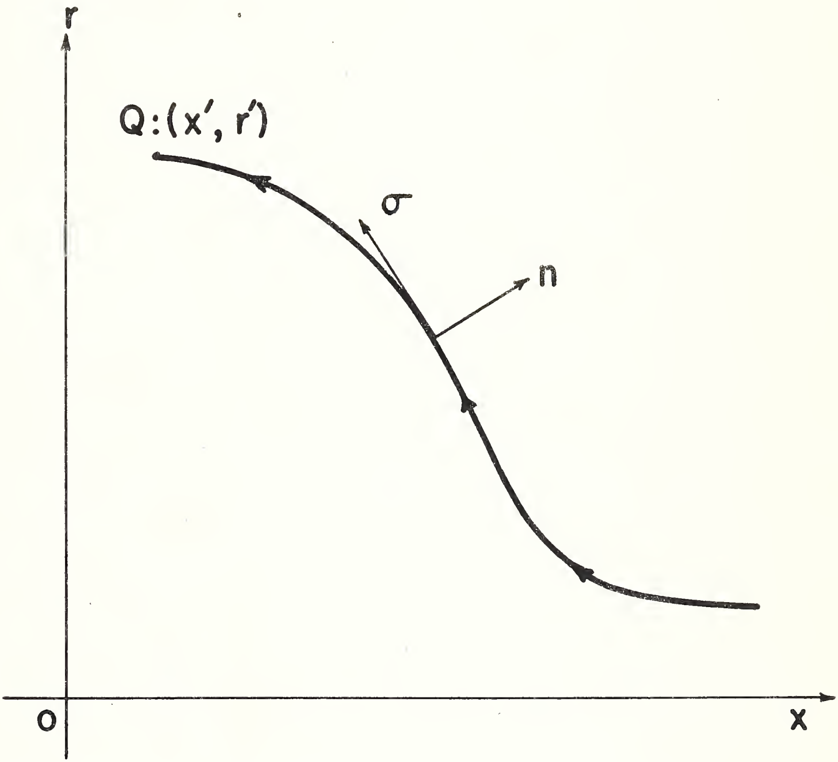

Fig. 3. PATH OF INTEGRATION FOR DETERMINING p