INDEX TO THE DISTRIBUTIONS OF MATHEMATICAL STATISTICS

by

Frank A. Haight
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INDEX TO THE DISTRIBUTIONS OF

MATHEMATICAL STATISTICS

by

Frank A. Haight

Statistical Engineering Laboratory

IMPORTANT NOTICE

Approved for public release by the director of the National Institute of Standards and Technology (NIST) on October 9, 2015

U.S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS
This index of distribution functions was prepared under the Statistical Engineering Laboratory's program of developing aids for the application of modern statistical methods in the physical sciences. The index is a fairly complete summary of published results on statistical distributions, and should serve to eliminate unnecessary derivation of results already in the literature.
PREFACE

This index was started in April, 1954 with the limited intention of supplying my students at Auckland University College with a small reference pamphlet. In their study of mathematical statistics it appeared that no textbook contained a complete treatment of all the distributions which a student might encounter; my index was supposed to facilitate a quick selection of the appropriate book.

Once started, it was not difficult to continue noting information. I even persuaded myself to spend the summer of 1954-55 in a systematic search of journals. By the beginning of 1955 my interest in the project faltered, and simultaneously the supply of statistical journals available in New Zealand ([a] -- [o] of Table 1) failed. I typed the collected results on stencils and published the index in mimeographed form. During the three years following, I sent out several hundred copies of the index in response to requests; finally the stencils wore out.
At the invitation of the National Bureau of Standards, I spent the summer of 1958 at the Statistical Engineering Laboratory supplementing and editing the index for publication. This work has included:

(a) Extending the range of journals covered,
(b) Bringing these up to the end of 1957,
(c) Collecting items from several additional books,
(d) Adding information supplied me by readers of the original version, and
(e) Correcting various mistakes found in the original version.

I wish to thank Dr. Churchill Eisenhart for making possible the invitation to the N. B. S., and Mrs. Lola Deming for helpful advice on the typescript. Also, I am grateful to Dean L. M. K. Boelter (acting on behalf of the Regents of the University of California) for granting me two months' leave from my work at the Institute of Transportation and Traffic Engineering, even though I was newly employed by them.

Frank A. Haight
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INTRODUCTION

1. Organization.

The material given under each distribution consists of a number of entries, most of which are provided with one or more references. In the case of the normal distribution with mean m and variance \( v \) (No. 1.1) the number of entries is fairly large, and therefore the standard order is most easily seen:

I Functions and constants which characterize the distribution

II Derived distributions
   (a) of linear quantities
   (b) of quadratic quantities

III Estimation
   (a) point
   (b) interval

IV Testing statistical hypotheses
   (a) by linear statistics
   (b) by quadratic statistics

V Miscellaneous

VI "See also"

These categories are by no means always used for less important distributions. With the limited information available a complete listing of the headings in such cases would be wasteful since the majority would be empty. Keeping in mind the above order, it should not be difficult to find
the required entry.

Occasionally an entry will be indented; such an entry should be read as a continuation of the preceding one.

2. References.

The references to the literature are of the following types: coded, uncoded, reviews.

I Coded

(a) Journals, e.g. [c]4:17, which refers to the 17th page of the 4th volume of the journal designated as [c] in Table 1.

(b) Books, e.g. [12]53, which refers to the 53rd page of the book designated as [12] in Table 2.

II Uncoded, e.g. Trans. Am. Math. Soc. 17:382, conforming to the usual volume and page reference style.

III Reviews

(a) Mathematical Reviews is designated by MR,

(b) Zentralblatt fur Mathematik is designated by Z.

MR and Z references will in no case offer a review of a paper appearing in coded journals and therefore may be considered to indicate publications in obscure (from the point of view of the present work) sources. Moreover, every effort has been made to avoid a MR or Z references to an uncoded paper quoted and very few duplications of this sort should be found.
The choice between direct (i.e. coded or uncoded) and indirect (i.e. review) references is frequently available. The one given is the one which was actually inspected, with all direct references collected before the search of MR and Z. Consequently each entry corresponding to a direct references is based on the paper, and never its review, and each entry corresponding to an indirect reference is based on the review and never the paper.

Since it is difficult to distinguish priority in a large number of references, a chronological table of the coded and review references is provided. This table also exhibits which volumes have been systematically searched in the preparation of this index.

3. Criteria for Inclusion

I. Distributions. As a general principle, a distribution is included if its density (or probability) function is a known, explicit function. The following exceptions may be noted:

(a) Certain families of distributions are mentioned, e.g. Pearson and Koopman, whose densities are specified only implicitly.

(b) Certain distributions are mentioned in terms of their cumulative probability function or characteristic function.
II. Entries. The general principle governing the selection of entries is this: that it must exhibit a property of the distribution in question. Exceptions to this rule are generally of one of the following forms:

(a) Historical information about well known distributions, although not systematically sought, may in some circumstances be included.

(b) Important applications, such as those which led to the discovery of the distribution are usually supplied.

(c) Bibliographies

Reference to tables has been excluded in almost every case.

It is clear that applications must be severely limited. With a slight exaggeration, several whole branches of statistics may be considered applications of some particular distribution, as exhibited for example in the following table:

<table>
<thead>
<tr>
<th>DISTRIBUTION</th>
<th>APPLICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial</td>
<td>Quality control</td>
</tr>
<tr>
<td>Normal</td>
<td>Analysis of variance</td>
</tr>
<tr>
<td>Lognormal</td>
<td>Probit analysis</td>
</tr>
<tr>
<td>Poisson</td>
<td>Random processes</td>
</tr>
<tr>
<td>Deterministic</td>
<td>Applied mathematics</td>
</tr>
</tbody>
</table>
4. Relationship between distributions
   
   I. Mention. In some cases (such as 2.1 and 2.3) the relationship between two distributions is asserted in their designation. In others (such as 5.3 and 5.15) a very close connection is not pointed out. In the majority of cases, however, known relationships are simply listed among the miscellaneous properties of both.

   In choosing between these alternatives, an attempt has been made to reproduce current statistical usage and terminology.

   II. Inclusion. Very similar principles have been used to decide for or against independent listing. If one distribution is relatively important and its equivalent much less so (for example Chi-square and Erlang) inclusion has been practiced. In other cases independent reputation seems to justify independent categories.

   It must certainly be supposed that many of the trivial distributions of Chapter VIII could be included in some larger category, or even combined with each other. The production of such a systematic classification which would exhibit all connections, even if worth doing, is certainly removed from the purpose of this book, and has hardly been attempted.

   For example, it is well known that No. 8.1 contains as special cases all the distributions of Chapters I and II;
very likely it also contains dozens of others listed. Nevertheless to indicate this by a system of sub-headings, applied to all entries, would quickly undermine the utility of the whole work, since it is the special cases rather than the general principle which occur in statistical practice.

5. **Notation and Terminology**

In univariate distributions the stochastic variable is always denoted by \( x \), in bivariate by \( x \) and \( y \) and in multivariate by \( x_1, \ldots, x_k \), quite regardless of the domain of definition. This departs from the usage of certain authors in two respects:

(a) The letter \( n \) is not used for a discrete variable.

(b) The statistic obeying a particular distribution is not used in the density. For example in Student's "t" distribution we write

\[
\left(1 + \frac{x^2}{r}\right)^{-\frac{1}{2}(r+1)}
\]

rather than

\[
\left(1 + \frac{t^2}{r}\right)^{-\frac{1}{2}(r+1)}
\]

This practice is justified not only by the need for uniformity but by the belief that the alternative is wasteful of the alphabet: \( t, F, Z, D, \ldots \). Similarly we prefer to call distributions by the names of their discoverers (or reputed discoverers) rather than by the symbol used to denote some
statistic found to satisfy them. Of course all known designations will be found in the final index.

In many books the expression \( f(x) \) is employed to denote a probability density. However \( f \) is commonly used in mathematics for an arbitrary function, and therefore we prefer to adopt something more distinctive for this special function, and have selected \( D(x) \) for the purpose.

In the discrete case this replaces the probability distribution, which is often written \( p_n \). \( C(x) \) is the cumulative function.

When we come to the characteristic function the situation is a little more complicated. Using \( t \) for the variable, statistical works generally have to define several symbols for characteristic functions of various quantities, for example:

\[
\begin{align*}
\chi(t) &= \text{characteristic function of distribution of } x \\
\varphi(t) &= \text{characteristic function of distribution of } n\bar{x} \\
\xi(t) &= \text{characteristic function of distribution of } \bar{x}, \text{ etc.}
\end{align*}
\]

Since we will be dealing with many different statistics and possibly their characteristic functions, it is more economical and systematic simply to abbreviate by the following system: \( \text{Ch}(x) \), \( \text{Ch}(n\bar{x}) \), \( \text{Ch}(\bar{x}) \), etc. Thus it is not necessary to select a new letter to denote the characteristic function of each new statistic.
However, this practice leads to equations like

$$\text{Ch}(x) = e^{-\frac{1}{2}vt^2}$$

which may be offensive to some, however clear the meaning. Such readers are advised to interpret the equality sign as an abbreviation for the verb "is".

This interpretation has another important connection with the notation being used. A variety of verbs have been employed to describe the relation between a stochastic variable and its distribution, for example:

- $x$ obeys the normal distribution with mean $m$ and variance $v$,
- $x$ follows the normal distribution with mean $m$ and variance $v$,
- $x$ is a normal variable with mean $m$ and variance $v$.

It seems equally felicitous: to assert this relationship by the convenient abbreviation

$$D(x) = N(m,v)$$

which may, if advantageous for any reason, be regarded not as a mathematical equation but as shorthand. In any case it makes possible an unambiguous condensation of the facts.

Similar remarks apply to the expressions $\text{MGF}(x)$, $\text{FD}(p)$ which are used to mean moment generating function of the distribution of $x$ and fiducial distribution of the parameter $p$. 
Another application of this use of the equality sign relates to the symbols C.-R.(p), MLE(p), MME(p), UMVUE(p), BANE(p), and is exemplified by the following:

\[ \text{C.-R.} \left( \sigma \right) = \frac{v}{2n} \]

\[ \text{MLE}(v) = s^2. \]

For the meaning of these and other abbreviations, the reader is referred to the List of Abbreviations.
LIST OF ABBREVIATIONS

D(x)  Density or probability function of a stochastic variable x
C(x)  Cumulative distribution function of x
Ch(x) Characteristic function of distribution of x
MGF(x) Moment generating function of distribution of x
PGF(x) Probability generating function of x
FD(p) Fiducial distribution of a parameter p
m  Mean of a population
\bar{x}  Mean of a sample
\nu = \sigma^2  Variance of a population
\nu^2  Variance of a sample
\mu_k  k^{th} central moment of a population
\alpha_k  k^{th} moment about the origin for a population
k^k  k^{th} cumulant
r  Correlation coefficient in a sample
\rho  Correlation coefficient in a population
\xi  Median
GM  Geometric mean
HM  Harmonic mean
n  Number of items in a sample
\beta  Slope of regression line in a population
b  Slope of regression line in a sample
\sim  Asymptotic (= large sample)
\beta_1, \beta_2  Pearson's betas
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>MLE(p)</td>
<td>Maximum likelihood estimate of the parameter p</td>
</tr>
<tr>
<td>MME(p)</td>
<td>Minimax estimate of the parameter p</td>
</tr>
<tr>
<td>( M_\chi^2E(p) )</td>
<td>Minimum Chi-square estimate of the parameter p</td>
</tr>
<tr>
<td>UMRVUE(p)</td>
<td>Uniformly minimum variance unbiased estimate of the parameter p</td>
</tr>
<tr>
<td>BANE(p)</td>
<td>Best asymptotically normal estimate of the parameter p</td>
</tr>
<tr>
<td>LR</td>
<td>Likelihood ratio</td>
</tr>
<tr>
<td>L</td>
<td>The likelihood function ( \prod D(x_j) )</td>
</tr>
<tr>
<td>Seq</td>
<td>Sequential</td>
</tr>
<tr>
<td>OC</td>
<td>Operating characteristic</td>
</tr>
<tr>
<td>BCR</td>
<td>Best critical region</td>
</tr>
<tr>
<td>Q</td>
<td>A quadratic form</td>
</tr>
<tr>
<td>E</td>
<td>Expectation</td>
</tr>
</tbody>
</table>
1.1 NORMAL (m,v)

I. Functions and parameters

\[ D(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{(x-m)^2}{2v}} \]

\[ Ch(x) = \exp(-\frac{1}{2}vt^2 + mit) \]

\[ MGF(x) = \exp(\frac{1}{2}vt^2 - tm) \]

Derivatives etc.

Transformations

Obtained from Pearson's differential equation

Called Type VII

Limit of binomial

Variance of \( \overline{x} \) and \( s^2 \)

\[ \text{Var}(m_3) = 6\nu^3n^{-1}, \text{Var}(m_4) = 96\nu^4n^{-1} \text{ and} \]

many other constants

Calculation of constants and numerical examples

Mean deviation \( E|x-m| = (2v/\pi)^{\frac{3}{2}} = .79788\sigma \)
Probable error = 0.6745

\[ \alpha_{2k} = (2k - 1)v^k \]

Quasi-range

II. Derived distributions

\[ D(\bar{x}) = N(m, \frac{v}{n-1}) \]

\[ D(\bar{x}/s) \]

\[ D[(n-1) \frac{s^{-1}}{n} (\bar{x} - m)] = \text{Student}(n-1) \]

\[ D[s^{-1}(n-1)^{\frac{3}{2}} (\bar{x} - m_1)] \text{, where } m_1 \text{ not all equal} \]

\[ D[(\bar{x} - m)/\text{range}] \]

D (range) etc.

\[ D(\Sigma k_i x_i) = N(\Sigma k_i m_i, \Sigma k_i v_i) \]

\[ (\frac{x-m}{\sigma})^2 \text{ is Chi-square, } (\frac{x}{\sigma})^2 \text{ is non-central Chi-square, } (x-m)^2 \text{ is Type III, product of normal variables is Bessel, quotient } \sim \text{ normal for large } \frac{m}{\sigma} \]

[4] 58


[d] 24:603


[3] 139


[d] 19:406

[d] 22:469

MR 13:762


[18] 1-150
$$D \left[ \frac{m_1 - m_2 (x_1 / x_2)}{\sqrt{v_1 + v_2 (x_1^2 / x_2^2)}} \right] = N(0,1), \quad m_2 \gg \sigma_2 \quad [2]253$$

$$D(x_1 - x_2) = N(m_1 - m_2, v_1/n_1 + v_2/n_2) \quad [4]100$$

$$D \left[ \frac{x_1 - x_2}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right] = N(0,1) \quad [6]263$$

$$D \left[ \frac{(x_1 - x_2) \sqrt{n_1 + n_2 - 2} \sqrt{n_1 n_2}}{\sqrt{n_1 + n_2} \sqrt{n_1 s_1^2 + n_2 s_2^2}} \right] = \text{Student (} n_1 + n_2 - 2 \text{)} \quad [4]112, [3]109, 112, [5]98, [c]29:350, [c]33:252, \text{MR8:42}$$

$$D \left[ \frac{(n-2) \frac{n_1 (x_1 - m_1)^2 + n_2 (x_2 - m_2)^2}{s_1 + s_2}}{s_1 + s_2} \right] \quad [4]132$$

- Snedecor (2, n-2), confidence ellipse

$$D(\text{k}^{\text{th}} \text{ value from top}) \quad [1]374, [d]25:565$$

$$D(\text{smallest sample value}) = \text{No. 8.40} \quad [g]42:408$$

$$D[ \frac{1}{2} v^{-1}(x-a)] = \text{Gamma (} \frac{1}{2} \text{)} \quad [10]150$$

$$D(\text{HM}) \quad \text{MR4:164}$$

$$D(\sqrt{x^2 + y^2}), D(\sqrt{x^2 + y^2 + z^2}) \text{ in special circumstances} \quad \text{MR16:377}$$
\[ D(\sum x_i^2) \]
\[ D(x^2) = D\left( \sum \frac{x_i - m_i}{\sigma_i} \right)^2 = \chi^2(n) \]
\[ D(s^2) = \text{Type III} \left( \frac{n}{2\nu}, \frac{1}{\nu} n \right) \]
\[ D \left( \frac{1}{n} \sum (x_i - \bar{x})^2 \right) \]
\[ = \text{Type III} \left( \frac{1}{\nu} n^{-1}, \frac{1}{\nu}(n-1) \right) \]
\[ D(s) \text{ for } n = 2, 3 \]
\[ D(s/\bar{x}), \text{Coefficient of variation} \]
\[ D(s/\text{range}) \]
\[ D(2s^2)^{-\frac{1}{2}}, \text{"precision constant"} \]
\[ = \text{Type V, moments, etc.} \]
\[ D(\log s^2) \]
\[ D(ns^2v^{-1}) = \chi^2(n-1), \text{For unequal } v, \]
\[ D(s_1/s_2) = \text{generalized Student} \]
\[ (n_2v_1/n_1v_2 + 1, n_1 + n_2 - 1) \]
\[ D(s_1^2v_2/s_2^2v_1) = \text{Snedecor} (n_1 - 1, n_2 - 1) \]
\[ \text{testing and confidence intervals} \]
\[ \text{power function} \]
D(n_1 s_1^2 v^{-1} + n_2 s_2^2 v^{-1}) = \chi^2 (n_1 + n_2 - 2),

D \left( \frac{1}{2} \log \frac{n_1 (n_2 - 1)s_1^2}{n_2 (n_1 - 1)s_2^2} \right) = \text{Fisher} \left( \frac{n_1 - 1}{n_2 - 1} \right) \quad [10] 198

D(v^{-1} \sum (n_1 - 1)s_1^2) = \chi^2 \quad [4] 116

D(\text{variance ratio})

\text{Distribution of various statistics from k normal populations with common variance}

\text{Ranking variances}

\text{Distribution of many quantities in a wide variety of cases}


D(\text{various Q}) \quad \text{MR13:142}

D(\overline{x}, s) \quad [2] 238, \text{ MR8:161}

D(b_2) \text{ for n=4 is hypergeometric} \quad [c] 25:411

D(b_1) \text{ for n=4 is hypergeometric} \quad [c] 25:207, [c] 33:68

D(\text{midrange}) \quad [d] 21:100


\text{Quasi-range} \quad [d] 28:179

D(r) \quad [b] 15:193
\[ D(\xi) \]

\[ D(Q) \]

\[ D(x_1x_2) \]

\[ D(x_1, \ldots, x_n) \]

\[ FD(m) = N(\bar{x}, \text{vn}^{-1}) \]

\[ FD[n^{\frac{1}{2}}s^{-1}(\bar{x} - m)] = \text{Student (n - 1)} \]

Tests

\[ FD(m, \sigma) \]

\[ FD(1/\sigma) = \text{Helmert } \left( n-1, \frac{1}{s\sqrt{n}} \right) \]

A priori distributions of \( m \) and \( 1/\sigma \)

Ranking means

III. Estimation

\[ \text{C.-R.}(m) = \text{vn}^{-1}, \text{i.e. } \bar{x} \text{ efficient} \]

\[ \text{Var}(\bar{x}) \leq \text{var}(\xi) \]
UMVUE(m) = \bar{x} \\
\bar{x} unbiased \\
Minimax estimates of m \\
Best "density unbiased" estimate of m \\
\sim efficiency of \xi is .6366 \\
Estimation of m when it must be integral, etc. \\
Mean of k^{th} values from top and bottom has \sim efficiency zero \\
C.-R. (\sigma) = v(2n)^{-1}, hence s is efficient \\
Efficiency of estimates of \sigma \\
In estimating \sigma, \\
\sqrt{\frac{1}{\tilde{3}n}} \frac{\Gamma \left( \tilde{3}n \right)}{\Gamma \left[ \frac{\tilde{3}}{2}(n+1) \right]} \sqrt{\frac{1}{n} \Sigma (x_i - m)^2} \text{ is more efficient than} \\
\sqrt{\frac{1}{\tilde{3}n}} \frac{\Gamma \left[ \frac{\tilde{3}}{2}(n-1) \right]}{\Gamma \left( \frac{\tilde{3}}{2}n \right)} s \\
C.-R. (\sigma/m) \\
\frac{n}{n-1} s^2 unbiased for v \\
[b]12:192
[c]37:182
[e]8:204
[p]7:152
Estimation of $\sigma$ for industrial quality control

Estimation of $\sigma$ from percentiles

Estimation of $v$ and $\sigma$

$C.-R.(v) = 2v^2n^{-1}$, hence $s^2$ efficient

$\text{MLE}(v) = s^2$

$\text{MLE}(v)$ depends on whether $m$ known

Estimation of $v$

In estimating $v$, $\sim$ efficiency of $s^2 = +1$

If $m$ constant, $s^2$ not sufficient

In estimating $v$, $\sim$ efficiency of mean deviation $= .876$

Unbiased estimates

Closest estimates of $m,v$

Unbiased estimation of mean absolute deviation

In estimating $m,v$ $\sim$ variance-covariance matrix

A.M.S. Translation No. 98

MR13:367

[g]49:375

[i]40:85

[d]18:584, [b]1:78

[1]484

[6]10.3

[3]34

[e]12:57, [i]40:85

[3]7


[3]7

A.M.S. Translation No. 98

MR13:367

[4]142
$\bar{x}$, $s^2$ are moments estimates of $m,v$

$\bar{x}$, $\frac{n}{n-1} s^2$ are $\sim$ efficient

$\text{MLE}(v,m) = \bar{x}, s^2$

$\bar{x}$, $s^2$ joint sufficient

Estimation of $m, \sigma$

Estimation

Minimum $\chi^2$ estimation

MLE from censored sample

Censored sample

Estimation of $m^2$

If $v$ known, confidence regions for $m$ are $\bar{x} + k_p v^{\frac{1}{2}} n^{-\frac{1}{2}}$, where $k_p$ are the $p^{\circ}/o$ values of the normal

Confidence intervals for $m$

[1]498
[1]494
[c]35:186
[c]11:262
[i]32:124
[u]45:214
[1]514
Seq confidence intervals for $m$

Confidence intervals for $v$

Interval estimation of $v$ and $\sigma$

Confidence limits for $v_1/v_2$

Confidence limits for $m_1 - m_2$ with same $v$

Interval estimation of $m_1 - m_2$
(Behren's Problem)

Confidence limits on $m$ and $s$

Confidence intervals for $m$, $v$

Tolerance limits

($\sim$)

IV. Testing

Testing $m_1 > m_2$

Tests on $m$

Unbiased regions for testing $m_1 = m_2$
Power function of \(m \geq m_0\) [3]305

Test of \(m\) using range in place of \(s\) [d]17:71

Hypotheses on \(m\) [1]533, [4]149

Tests based on (rectangular a priori) distributions of \(m\) and \(v\) Z18:158


Control charts on \(m\) [e]17:80

Testing \(m_1\) against \(m_2\) and \(\sigma_0\) against \(\sigma_1\) by quick counting methods [d]20:502, [g]40:303

Seq hypotheses on \(m\) [g]31:318


Student's is best for testing \(m_1 = m_2\) [4]150

LR test of \(m = m_0\) is Student [d]17:192

Power function for Student test [c]37:326

Seq Student test

Comparison of k means

Testing whether variance is constant

Three decision seq. test of \( m \)

Testing whether many means are all zero
Testing \( \bar{x}_1 - \bar{x}_2 \)

Testing \( \bar{x}_1 - \bar{x}_2 \) without assuming \( \sigma_1 = \sigma_2 \)

Linear hypotheses

Joint tests
Testing outlying observations
H: \( \sigma = \sigma_0 \)

OC for \( \chi^2 \) test of \( \sigma = \sigma_0 \)

Seq. test on \( \sigma \)

Seq. test on \( \sigma_1 = \sigma_2 \)

Tests on \( \nu \)

Fisher is best for testing \( \nu_1 = \nu_2 \)
Testing homogeneity of variances

The most powerful test of

\[
\begin{align*}
H: \sigma &= \sigma_0, \text{ } m \\
\text{Alt: } \sigma &= \sigma^o, \text{ } m = m_1
\end{align*}
\]

is \( \Sigma(x_i - \bar{x})^2 > c \)

Hypothesis of equality of many normal variances

Significance of smallest of set of variances

Critical regions for \( m \) and \( v \)

Bibliography of testing equality of variances

Seq. ratio test terminates

OC function

Power functions of tests

Whether two samples are from the same normal population

Tests for normality
Tests of various composite hypotheses

Decision problems

V. Miscellaneous

Independence of $\bar{x}$ and $s$ (Student-Fisher Theorem)

$s^2$, $\bar{x}$ independent

Normality if and only if $\bar{x}$ and $s^2$ independent

Normality if and only if $D(\bar{x}, s) = \frac{L}{s} sn^{-2}$

Normality if $D(x) D(y) = \phi \sqrt{x^2 + y^2}$

Various characterizations of normality

Generalization of Student-Fisher Theorem
Independence of quadratic forms

Bayes theorem

Cochran's theorem

More generally

History of Normal

History of distribution of $s$

Distributions which converge to normality

Discrete Analogue

Regression of $x$ and $t$, where $m = a - be^{-kt}$

Comparing percentage points of two normals

$k$ samples

Truncated sample

Sampling from $N(\sum a_k m_k', v)$
Max $\Sigma x_i$, min $\Sigma x_i$  \hspace{1cm} [c]40:35

1.2 NORMAL: \( N(0, v) \)

\[
D(x) = (2\pi v)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} x^2 v^{-1}\right)
\]

\[
\text{Ch}(x) = \exp\left(-\frac{1}{2} vt^2\right)
\]

\[
\text{MGF}(x) = \exp\left(\frac{1}{2} vt^2\right)
\]

\[
C(x)
\]

\[
\alpha_{2k} = \frac{\sigma^{2k}(2k)!}{2^k k!}
\]

2nd cumulant = \( v \), others zero

Pearsonian type

\[
D(\bar{x}) = N(0, v/n)
\]

\[
D(x/y) = \text{Cauchy}
\]

\[
D(x^2) = \text{Type III}
\]

\[
D(s^2)
\]

\[
D(\Sigma x_i^2/2v) = \text{Gamma} \left( \frac{3}{2} n \right)
\]

\[
D(\Sigma x_i^2) \text{ etc.}
\]

\[
D(\Sigma x_i^2), D(Q_1/Q_2)
\]

\[
D(Q/v) = \chi^2(r), Q \text{ of rank } r; \; \dagger \; 0
\]
eigenvalues all \( +1 \)
\[ \text{Ch}(Q) = \left[ \Pi (1-2\text{itv}_{k_1})^{\frac{1}{2}} \right]^{-1} \]

\[ \text{D}[x \ n^{\frac{1}{2}} (\Sigma x_i^2)^{-\frac{1}{2}}] = \text{Type II} \] \[ \text{[i]29:13} \]

\[ \text{D}(x) \text{ assuming } v \text{ is Type III} \] \[ \text{[d]28:510} \]

\[ \text{D}(x/s) = \text{Student (testing)} \] \[ \text{[c]37:65} \]

\[ \text{D(range) for } n=3, \text{ unbiased critical region} \] \[ \text{[3]327} \]

\[ \text{FD}(v) = \text{Type V} [\Sigma x_i^2/n-2, \ \frac{1}{2}(n-2)] \] \[ \text{[p]7:226} \]

\[ \text{C.-R.}(v) = 2n^{-1}v^2 \] \[ \text{[1]484, [p]7:159} \]

Suppose \( n \) obeys No. 3.5 \[ \text{MR14:391} \]

\[ \text{MLE}(v) = \hat{n}^{-1} \Sigma x_i^2 \] \[ \text{[4]141} \]

\( s^2 \) UMVUE of \( v \), but \( s \) not of \( \sigma \) \[ \text{[3]52,54} \]

Neyman-Pearson on hypothesis testing \[ \text{[c]20:178} \]

Most powerful test of \[ \left\{ \begin{array}{c} \text{H: } \sigma = \sigma_0 \\ \text{Alt: } \sigma = \sigma_1 \end{array} \right\} \] \[ \text{[3]275} \]

is \( \bar{x}^2 + s^2 \lesssim c \)

Completeness \[ \text{[e]10:313} \]

Unbiased critical regions \[ \text{[3]212} \]

Testing \( s_1^2/s_2^2 \) etc. \[ \text{[e]5:157} \]
Testing serial correlation

Various devices for showing area = +1

Inference

As "Maxwell-Bolzmann" distribution

Variance of mean deviation is
\( \sim vn^{-1}(0.8068)^2 \)

Mean difference

Properties of \( f(x) \), where \( x \) is \( N(0,v) \)


1.3 NORMAL: \( N(m,1) \)

\[
D(x) = (2\pi)^{-\frac{1}{2}} \exp[-\frac{1}{2} (x - m)^2]
\]

\[
E(x_1^2 + x_2^2) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}}, \ Var(x_1^2 + x_2^2)
\]

\[
= 2 - \frac{1}{2\pi}, \ E|x| = (2/\pi)^{\frac{1}{2}}, \ E(e^{ax})
\]

\[
= e^\frac{1}{2}a^2, \ var(e^{ax}) = e^{2a^2} - e^a^2
\]
D(\bar{x})

D(x^2), D(x_1^2 + x_2^2), D(x_1^2 + x_2^2)^{1/3}

D(xy) = \text{Bessel}

FD(m) = N(\bar{x}, n^{-1})

Bayes Distribution \(m\)

\(\bar{x}\) sufficient

\(\bar{x}\) is consistent

\(\bar{x}\) is MLE

\(\bar{x}\) is minimax

Efficiency of \(\xi = .637\)

Confidence intervals for \(m\)

MLE for \(\chi^2\) test

Remarks on testing

Most powerful test of

\[
\begin{cases} 
\text{H: } m \leq m_0 \\
\text{Alt: } m = m_1m_0 
\end{cases}
\]

is \(\bar{x} > c\)

UMP test of \(m = 0\)
Seq test

Completeness

Unbiased critical regions

Testing equality of several means

Testing; called "Laplace"

Peculiar composite hypothesis on m

Inference

Pitman's method

Range

See also: [c]27:466, [c]31:202, [c]36:460, [g]7:95

3rd Berkeley Symposium 1:197.
1.4 NORMAL: $N(0,1)$ (Gaussian)

$$D(x) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} x^2\right)$$

[7]129

General expose

Acta Math 77:1

C(x) as continued fraction

[2]130

C(x) as a series

[c]19:13

Bounds on C(x)

[c]42:263

Property of C(x)

Math. Zeit 41:405

C(x)

MR10:267

$\sim C(x)$

MR16:628

$\text{MGF}(x) = \exp\left(\frac{1}{2} t^2\right)$


$\text{Ch}(x) = \exp\left(-\frac{1}{2} t^2\right)$

[d]5:32, [d]11:353,

[h]1:13, 193, [1]208

$\alpha_{2k} = \frac{(2k)!}{2^k k!}$

Absolute moments

Z1:26

Median and quartiles

[c]25:79

$D(\bar{x}) = \text{Normal}$

[c]19:227

$D(s^2)$

[e]5:138
\[ D(x^2) \]

\[ D \left\{ \frac{x}{\pi x_i^2} \right\} \text{ etc} \]

\[ D(\Sigma x_i^2) = \chi^2(n) \]

\[ D(\Sigma k_i x_i) \]

\[ C(n^{\frac{1}{2}} s^{-1} \bar{x}) = \text{Student} \]

\[ D(Q), D(Q_1 / Q_2) \]

\[ D[\Sigma(x_i - \bar{x})^2] = \chi^2(n-1) \]

\[ C(r) \text{ expressed as an integral} \]

\[ D(x_1 x_2) \text{ by Mellin Transformation} \]

\[ D(x_1 / x_2) = \text{Cauchy} \]

\[ D(\bar{x}, s) \]

\[ D(\text{range}) \text{ for } n=3, \sim D(\text{range}) \]

Moments of sample median

\[ D(\text{extreme deviate}) \]

\[ C(\text{range}) \]

\[ D(\Sigma x_i^2 / \Sigma y_j^2) = \text{Snedecor} \]
Ch fcn's of estimates of \( v \)  
Generating functions  
Estimation of dispersion  
Estimation of mean deviation  
Variance of median  
Testing \( N(0,1) \) against various alternatives  
Multivariate analysis  
Limit of binomial  
Central Limit Theorem  
\( k^\text{th} \) value from the top  
Censored samples  
Ordered samples  
Stratified sampling  
Variance in two samples  
Ratio of two ranges  
Tetrachloric functions
Approximations

Sheppard's tables

Grouping

Moments of order statistics

Occasionally called Laplace-Gauss, or even Laplace


1.5 TRUNCATED NORMAL

C(x)

Introduction, estimation, examples

D(Σ x_i)

Fitting

Estimating m and v

[d]17:363

[c]2:174

[i]32:135

[d]41:200


[15]144

[b]8:223

[c]39:252

Distribution of estimate of $\sigma$ \[ [15]316 \]

Censored sample \[ [c]42:516 \]

MLE \[ [i]32:119 \]


1.6 GENERALIZED NORMAL (Kapetyn)

\[
(2\pi v)^{-\frac{1}{2}} \exp\left[-\frac{1}{2} v^{-1} (f(x) - m)^2\right] df(x) \]  \[ [5]93 \]

C.-R.($m$) = $v/n$, C.-R.($v$) = $2v^2/n$ \[ [5]139 \]

MLE($\sigma$) = $\left[\frac{1}{n} \Sigma (f(x_i) - m)^2\right]^{\frac{1}{2}}$

MLE($m$) = $\frac{1}{n} \Sigma f(x_i)$.

See also: [c]5:168.
1.7 NORMALS ADDED

\[ D(x) = (1+k)^{-1} \{ (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} (x + m_1)^2 \right) \]
\[ + kv^{-\frac{1}{2}} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} (x - m_2)^2 \right) \}

\[ \text{Var}(x) = (1+k)^{-1} (1 + m_1^2 + k(v + m_2^2)) \]

Method for partition with example

Semi-invariants

D(\bar{x})

Three normals added

More generally

Sampling theory

Many normals added

Called "compound normal"

Bivariates

1.8 LOGNORMAL (a, m, v)

\[ D(x) = (x - a)^{-1}(2\pi v)^{-\frac{1}{2}} \exp\left[-\frac{1}{2} v^{-1} \right] \]

\[ \log(x-a) - m^2 \], parameters and moments

Graphical determination of parameters

Mean = \( a + e^m + \frac{1}{2} v \), \( \text{var} = e^{2m} + v(e^v - 1) \)

Another form

\[ D(x) = \frac{1}{\sqrt{2\pi c(x-a)}} \exp \left\{ -\frac{1}{2c^2} \left[ \log \frac{x-a}{b} \right]^2 \right\} \]

\( m = be^{\frac{1}{2}c^2} + a \), mode = \( be^{-c^2} + a \), GM = \( e^m \)

\( a + b \), moments, tables, regression, examples, bibliography

Moments, transformations

Complete treatment with bibliography

Estimation of m

MLE

Regression

~ Tests on m
Called "Galton-Macalister"
Called Gibrat

Used to approximate Fisher Distribution
Deduced from hypothesis about errors etc.

\[ \frac{x-a}{\ln x} \]

lognormal

Transformation

Vs. Normal

Discrete lognormal

Compared with normal by means of Galton-Kapetyn apparatus
Truncated lognormal

Lognormal \((0,0,1), \frac{1}{2}

\[ E(x) = e^{\frac{x^2}{2}}, \quad \nu = e^2 - 1 \]


1.9 WRAPPED-UP NORMAL

\[ D(x) = k\Sigma e^{-c(x+j)^2} \]

1.10 GRAM-CHARLIER

Two-term $D(x) = (2\pi)^{-\frac{1}{2}} [1 - k/6(3x-x^2)]$
$\exp(-\frac{1}{2}x^2)$

General Gram-Charlier

$D(\overline{x})$

$D(x^2)$

$D(s)$

$t$-test

$D(x) = f(x) N(m, v)$

$F$(various statistics)

Log Gram-Charlier

Type B Gram-Charlier

MGF factorial moments,

1.11 BIVARIATE NORMAL $N\left(\begin{array}{l} m_1 \\ m_2 \end{array}, \begin{array}{l} v_1 \\ v_2 \end{array}, \rho \right)$

$$D(x,y) = \left[2\pi \sigma_1 \sigma_2 (1-\rho^2)^{\frac{1}{2}} \right]^{-1} \exp \left\{ -\frac{1}{2} \left[ \begin{array}{c} v_1 v_2 (1-\rho^2) \end{array} \right]^{-1} \left[ (x-m_1)^2 v_2 - 2\rho \sigma_1 \sigma_2 (x-m_1)(y-m_2) + (y-m_2)^2 v_1 \right] \right\}$$

Introduction, properties, examples

Another form (Koopman-Darmois)

$$Ch(x,y) = \exp \left\{ i(m_1 s + m_2 t) - \frac{1}{2} (v_1 s^2 + 2\rho \sigma_1 \sigma_2 s t + v_2 t) \right\}$$

MGF

$$D(x_1, x_2) = \text{Normal bivariate}$$

$$D(\bar{x} - \bar{y})$$

$$D(y/x)$$

$$f(xy) \text{ and } f(x/y)$$

$$Ch(xy)$$

$$D(r)$$

If $\rho=0$, $D(r^2) = B(1, n-2)$
If \( \rho = 0 \), \( D(b) = B(1, n-1) \)

\[ D(\text{correlation ratio}) = \text{No. 5.3} \]

\( \sim D(r) \)

\[ D(rs_1s_2, s_1^2, s_2^2) \]

Distribution of various statistics

\[ \text{MLE}(m_1, m_2, v_1, v_2, \rho \sigma_1 \sigma_2) \]

\[ = \bar{x}, \bar{y}, s_1^2, s_2^2, rs_1s_2 \]

\[ \text{Var}(\bar{x}) = \frac{v_1}{n}, \text{Var}(s_1^2) = \frac{v_1}{2n}, \]

\[ \text{var}(r) = n^{-1}(1-\rho^2)^2, \text{cov}(s_1, s_2) \]

\[ = \rho \sigma_1 \sigma_2 / 2n, \text{cov}(r, s_1) \]

\[ = \rho \sigma_1 (1-\rho^2) / 2n, \text{cov}(\bar{x}, \bar{y}) = \]

\[ \rho \sigma_1 \sigma_2 / n \]

MLE from fragmentary information

\( \bar{x}, \bar{y} \) are joint efficient in estimating \( m_1 \) and \( m_2 \);

\[ \bar{x}, \bar{y}, \frac{ns_1^2}{n-1}, \frac{ns_2^2}{n-1}, \frac{n}{n-1} rs_1s_2 \] have

joint efficiency \((n-1/n)^3\)

\[ \frac{x-m_1}{\sigma_1} + \frac{y-m_2}{\sigma_2} \text{ and } \frac{x-m_1}{\sigma_1} - \frac{y-m_2}{\sigma_2} \]

are independent and \( N(0, 2(1+\rho)), N(0, 2(1-\rho)) \)

respectively
Estimation

Estimation, testing

Censored samples

Dist. ratio standard deviations

Confidence limits for r

Confidence limits for \( m_1/m_2 \)

Sufficient statistics

Comparison of two correlations

Tests of seven hypotheses on the parameters

Testing equality of two r's

Some tests

Testing equality of variances

\[ r_1 - r_2 \]

Seq. tests of \( \rho \)

Truncation

Fisher's original work on \( r \) and \( \rho \)

Sufficient conditions for normal bivariate

[d]17:395, [e]8:322

[15]606, [g]50:884

[x]6:83

[x]6:93


[e]17:212

[10]203

[d]11:410

[d]12:279

[w]7:46

[e]1:13, bibliography

[a]109:462

[c]25:102

[w]8:202

MR2:231

[n]1-4:1

[e]6:399, MR15:805
k samples

\[
x / \sigma_1 + y / \sigma_2 \text{ and } x / \sigma_1 - y / \sigma_2 \text{ are independent normal variables}
\]

If \( \rho = 0 \) then \( \frac{v_1/v_2}{v_1/v_2 + s_1^2/s_2^2} \) is Beta

Called Bravais distribution

Properties


1.12 BIVARIATE NORMAL \( N(0, v_1, \rho) \)

\[
D(x,y) = 2 \pi M^{\frac{1}{2}} \exp[-\frac{1}{2} M^{-1}(x^2 v_2 - 2 \rho \sigma_1 \sigma_2 xy + y^2 v_1)]
\]

where \( M = \begin{vmatrix} v_1 & \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & v_2 \end{vmatrix} = v_1v_2(1-\rho^2) \)

\[
Ch(x,y) = \exp(-\frac{1}{2} n^{-1}(v_1 s^2 + 2 \rho \sigma_1 \sigma_2 st + t^2 v_2))
\]
\[ \alpha_{40} = 3v_1^2, \alpha_{31} = 3\rho \sigma_1^3 \sigma_2, \alpha_{22} = (1+2\rho^2)v_1 v_2 \]

Moments

Central moments

Incomplete moments

Product-moments

As limit of binomial

If \( \rho = 0, \sigma_1 = \sigma_2 \) called "circular normal", \( D(r) \), properties

\[ C(x,y) \] with other properties

Cumulants

\[ \text{Var}(r) = n^{-1}(1-\rho^2) \]

\[ \text{Var}(b) = n^{-1}\sigma_1/\sigma_2(1-\rho^2) \]

Marginal and conditional distributions

Regression

Correlation and regression with generalization

Bilinear forms
Quadratic forms

D(x^2, y^2)

D(variance-covariance) = Wishart

D(r)

D(r) for n=4

D(b) = a Type VII

D(s_1/s_2) = No.8.3 if \( \sigma_1 = \sigma_2 \)

D(x^k, y^l)

When, further, \( \rho = 0 \) generalized Student

Simple function of x/y is normal

D(y/x)

D(xy), D(x/y)

Joint distribution of Pearson betas

D(s_1^2, s_2^2, rs_1s_2) etc.

D(s_1^2, s_2^2)

\[
D \left( \frac{(n-1) ^ {\frac{1}{2}} (b - \beta) \sigma_1}{\sqrt{v_2 - b^2 v_1}} \right) = \text{Student } (n-1)
\]
\[
\tilde{D} \left( \frac{\frac{1}{\sqrt{n-2}} (b_2 - \beta) s_1}{\sqrt{b_2 - b_1}} \right) = \text{Student}(n-2)
\]

Moments of dist. of covariance from \( N(0, \frac{1}{1}, \rho) \)

\[
D(\text{radial standard deviation}) = D\left[ n^{-1} \sum (x_i - \bar{x})^2 + \sum (y_i - \bar{y}) \right]^{\frac{1}{2}}
\]

\( D(\text{sample covariance}) \)

MLE(\( \rho \))

Confidence intervals for \( \rho \)

MLE(\( v_1, v_2, \rho \))

Estimation of \( \rho \) when \( v_1 = v_2 \)

Estimation of \( \rho \) by rank correlation

Truncation

and estimation

Testing \( v_1 / v_2 \)

Testing \( \sigma_1 / \sigma_2 \) and \( \rho \)

Test whether two samples are from same population


[3] 334

[3] 15:75


[3] 33

[3] 81


[e] 12:277, MR16:498

[3] 138

[e] 5:151

[3] 140
Hotelling's generalized T applied to tolerance limits

Odd fact for $\rho = 0$


1.13 TRIVARIATE NORMAL

$D(x)$

Moments

Correlation

Partial correlation

Yielding 2 x 2 x 2 table

Student test for partial correlation

Snedecor test for multiple correlation

1.14 MULTIVARIATE NORMAL

\[ D(x) = C e^{-Q} = [(2\pi)^{\frac{1}{2}} \sigma_1 \ldots \sigma_n R^{\frac{1}{2}}]^{-1} \exp\left\{-\frac{1}{2} \sum \frac{R_{ij}}{\sigma_i \sigma_j} (x_i - m_i)(x_j - m_j)\right\} \]

where \( R = |r_{ij}| \)

\( C(x) \)

\( Ch(x) = \exp \left\{ i \sum m_j t_j - \frac{1}{2} \sum r_{ij} \sigma_i \sigma_j t_i t_j \right\} \)

Moments

Marginal distributions, conditional distributions, regression

Independence of quadratic forms

Distributions of moments, partial and multiple correlations

\( D(\text{product moment}) = \text{Wishart} \)

Cumulants of logarithmic generalized variance

Independence of distribution of means and second order moments
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[d]27:174

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[c]31:31, [c]34:311

Truncation  
[x]5:17

Various tests  

Quadrivariate  
[c]43:206

Bibliography of tests of hypothesis of equality of variances called 'Bipolar' distribution  
[e]14:61

Variables separated into two sets  
[c]30:295

Many samples  
[c]31:221

Seq analysis  
[e]12:328

Central limit theorem  
[i]28:109

Generalizations of N(m,v) theorems  
[e]17:221

II. TYPE III DISTRIBUTIONS

2.1 TYPE III (p,q)

\[ D(x) = \frac{p^q}{\Gamma(q)} x^{q-1} e^{-px}, \quad (0, \infty) \]

\[ D(x) = \frac{x^b e^{-x/a}}{b! \ a^{b+1}} \quad ("\text{gamma")} \quad [6]112 \]

\[ \text{Ch}(x) = (1 - it/p)^{-q} \quad [2]55, [1]126 \]

\[ \text{MGF}(x) = (1 - at)^{-b-1} \quad [6]115, [4]74 \]

\[ \alpha_1 = q/p, \quad \alpha_2 = p^{-2}q(q+1), \quad \alpha_3 = p^{-3}(q+1)(q+2) \quad [2]55 \]

\[ E(x) = a(b+1), \quad \nu = a^2(b+1) \]

\[ r^{th} \text{ cumulant} = q(r-1)!p^{-r} \quad [2]67 \]

\[ \mu_2 = q p^{-2}, \quad \sigma^2 = q/p^2, \quad \mu_3 = 2q p^{-3}, \quad [2]433 \]

\[ \mu_4 = 3q(q+2)p^{-4} \]

Arithmetic, geometric means

Math. Student 13:11

Type III (p/q, p) ("Eulerian")

\[ C(x) \quad [c]25:379 \]

\[ f(\frac{x}{x+y}) = \text{Beta} \quad [14]41 \]

\[ \sim C(x) \quad \text{MR13:553} \]
Transformations \( y = x^n \), \( y = e^x \)

Normalizing Transform

Transformation \( y = (x + k)^{\frac{1}{3}} \)

Moments \( \text{Ch}(x) \), cumulants when \( x-y-c \)

Type III \((p, p+1)\)

\( \overline{D(x)} = \text{Type III} (np, nq) \)

\( D[(nq-1)n^{-1} \overline{x}^{-1}] = \text{Reciprocal Type III}(p,nq-1) \)

\( \sim D(\frac{1}{\sqrt{x}}) \)

\( D(x/y) = \text{Beta of Second Kind} (p=q) \)

\( FD(p^{-1}) = \text{Type V} (n \overline{x}/nq-1, nq-1) \)

Bayes \( D(p^{-1}) = \text{Rectangular} \)

\( D(xy) \) where \( x \) is Type III and \( y \) Type V

\( D(HM) \)

\( D(xy), \text{Ch}(xy) \)

MLE \( (p) = \text{Moments} (p) = q\sqrt{x} \), correcting for bias \( = \frac{nq-1}{n\overline{x}} \), sufficient \( \sim \) efficient, not efficient

\( \text{Var} (\frac{nq-1}{n\overline{x}}) = p^2(nq - 2)^{-1} \)
MLE(1/p) = \bar{x}/q

\text{Var}(\bar{x}/q) = p^{-2}n^{-1}q^{-1}, \text{sufficient}

Sufficient statistics for p

Ordered LSE is MLE for 1/p

MLE(p,q), variance-covariance of estimates

UMVUE(1/p) = \bar{x}/q, \text{ with}
\text{Var} = p^{-2}n^{-1}q^{-1}

There is a sufficient estimate of q

Estimation

Minimax

Gauging

Closest estimate

Testing n such populations

Slippage tests for p

Testing equality of 1/p

Confidence intervals for 1/p

Koninkl. Nederl. Akad. (A)59:329

[3]21

[e]17:212, 219

[d]25:315


[3]53

[3]26

[e]8:324

[16]64, C.R. [e]14:57

[e]15:192

[u]33:217

[3]325

[c]31:205

Truncated distribution

Estimation from Truncated Type III

Truncated samples

Relation with Poisson

Characterized by independence of sum and quotient

Discrete Type III

Mills ratio

Normal limit

Renewal theory

Trivariate

2.2 TYPE III (p, l)

\[ D(x) = pe^{-px} \]  

Type X

\[ \alpha_I = \frac{1}{p} \]  

\[ v = p^{-2} \]  

Moments


\[ [5]67 \]


Characterization

\[ [t]7:60, 3rd Berkeley Symp. 2:195 \]

Cumulants


\[ C(x) \]

\[ [c]25:379 \]

Mean difference

\[ [c]28:432 \]

\[ MGF(x) = (1-x/p)^{-1} \]

\[ [10]37 \]

\[ MGF(\log x) \]

\[ [v]7:296 \]

Grouping corrections

\[ [c]39:433 \]

A priori distributions of p

\[ [i]27:36 \]

\[ D(x + y) \]

\[ [8]95 \]
Examples and applications

D(\bar{x})
D(\xi) = No. 8.9, MGF(\xi)
\sum x_i \text{ where } x_i \text{ is Type III}(p_i,1)

Rank variates

Estimation

Censored sample

Moments of D(s)

Variance of mean deviation \approx \frac{4}{3np^2}

Testing \( p = p_0 \)

Sequential test

Testing against four other possible dist.

Confidence intervals

Estimation from truncated exponential

Relation with Poisson

See also: \([d]7:19, [d]25:555, [g]48:488, [g]50:904\)
If \( x = y - c \).

\[
\text{Mean} = c + \frac{1}{p}, \quad \text{var} = \frac{1}{p^2}, \quad \text{skewness} = 2, \quad \text{kurtosis} = 6, \quad \text{Ch}(x), \quad \text{cumulants}
\]

MLE

LR test for hypothesis that \( n \) such populations are identical, etc.

\( k \) samples

Best linear estimates of \( m \) and \( \sigma \)

\[
D = \left\{ \frac{1}{n-1} \sum_{i=1}^{n-1} | x_{i+1} - x_i | \right\}
\]

Life testing

Original Neyman-Pearson paper on hypothesis testing

Censored samples

Estimation by order statistics

Quasi-range

LR tests

Seq. testing

Confidence intervals
\[
\begin{align*}
H: & \quad p = p_0, \quad c = c_0 \\
\text{Alt:} & \quad p = p_1, \quad c = c_1
\end{align*}
\]


**Maxwell-Bolzmann**

\(x^2\) is Type III \((p, \frac{3}{2})\) [5]39,60

Connection with \(N(0, v)\) [12]40

\(D(\bar{x})\) [n]10-3:90

See also: [n]17-1:125.

### 2.3 TYPE III \((1,q)\)

\[
D(x) = \frac{1}{\Gamma(q)} e^{-x} x^{q-1}, \text{ "Gamma"}
\]

\[
\alpha_r = \frac{\Gamma(q+r)}{\Gamma(q)}
\]

\[
k_r = q(r-1)^\prime!
\]

\(Ch(x) = (1-ix)^{-q}\) [17]No. 34

Skewness \(= q^{-\frac{1}{3}}\) [10]161
$HM = q-1$

$D(nx) = \text{Type III} \ (l,nq)$

$D(\Pi x_i)$

$D(x_1 - x_2)$

$D(x_1 + x_2) = \text{Type III} \ (l,q_1 + q_2)$

$D(GM) \text{ as a series}$

(with generalization)

$D\left( \frac{x_1}{x_1 + x_2} \right) = \Gamma\left( \frac{1}{2} q_1, \frac{1}{2} q_2 \right)$

$D(x_1/x_2) = \text{Beta of second kind}$

$D(x_1 - x_2) \text{ involves a Bessel}$

function if $x_1$ and $x_2$ are from two separate distributions, and $D(x_1/x_2)$ is Fisher

$\sim D(\bar{x})$

$D(\bar{x}, GM/\bar{x}) = D(\bar{x}) \ D(GM/\bar{x})$

$C.-R.(q) = [n \frac{d^2}{dq^2} \log \Gamma \ (q)]^{-1}$

$\bar{x}$ is moments estimate of $q$, not sufficient
Closest estimate

\[ \text{MLE}(q) = \log \text{GM} \]

\[ E(\bar{x}) = q, \quad \text{var}(\bar{x}) = q/n, \quad \text{efficiency} \]

\[ (\bar{x}) \rightarrow 0 \]

\[ \sim \text{D(} \log \text{GM)} = \text{Normal} \]

Confidence intervals

Mellin transform

Log \[ \log \frac{1}{x} \]

\[ A + B \log x \]

Multivariate generalization

Tetrachoric functions

Fermi-Dirac, \( x - c \) is Type III \((3/2,\text{const.})\)

2.4 TYPE III (1,1)

\[ D(x) = e^{-x}, \text{"exponential"} \]

\[ Ch(x) = \frac{1}{1-it} \]

\[ D(\overline{x}, s) \]

\[ x = y - c, \text{Type X, confidence intervals} \]

\[ D(\Sigma x_i) = \text{Type III} \]

by convolutions

\[ D(\Sigma x_i/i) = D(\max x_i) \]

Doubly truncated, \( Ch(x) \)

Ratio of two ranges

C.-R. theorem false for \( x = y - c \)

2.5 CHI-SQUARE (k)

\[ D(x) = \frac{1}{2^\frac{k}{2} \Gamma \left(\frac{k}{2}\right)} x^{\frac{k}{2} - 1} e^{-\frac{1}{2} x} \] 

\[ \text{(Type III)} \]

\( \left(\frac{1}{2}, \frac{1}{2} k\right) \]

\( a_1 = k, \nu = 2k \)

\( a_s = k(k+2)\ldots(k+2s-2) \)

\( \mu_2 = 2k, \mu_3 = 8k, \mu_4 = 48k + 12k^2, \)

\( \mu_5 = 32k(5k+12) \)

Cumulants

\( \text{Ch}(x) = (1 - 2it)^{-\frac{1}{2} k} \)

\[ \text{C}(x), \text{relation with Poisson} \]

Introduction, properties, examples

Obtained as dist. of normal variance

\( D(x_1 + x_2) \)

\( D(\log x) \)

\( D(2 \sqrt{xy}) = \text{Chi square}(2n-2) \) if \( D(x) = \text{chi-square} (n) \) and \( D(y) = \text{chi-square} (n-1) \)
D(x,s) for k = 2,3,4, n = 3,4

D(x_1/x_2) = Beta of second kind
(\(\frac{1}{2}k_1, \frac{1}{2}k_2\))

D(\(\frac{x_1}{x_1 + x_2}\)) = Beta (k_1,k_2)

Obtained as D(\(\sum x_i^2\))

\(\sim D[(x-k)(2k)^{-\frac{1}{2}}] = N[k,(2k)^{-\frac{1}{2}}]\),

\(\sim D(2x)^{\frac{1}{2}} = N[(2k)^{\frac{1}{2}}, 1]\)

\(\sim D(-2 \log LR)\)

Reproductive property

Normal approximation

Large parameter

Convolution for a pair of Chi-square variables

Percentage points

\(\sim\) Significance levels

Elderton's tables

MR12:345

[10]177

[10]177

[10]169

[1]251

[d]9:60


[d]17:216, [d]27:786

6th Symp. in Appl.
Math., p. 251

[8]134

[c]41:313, [i]33:168

[d]14:57,93

[c]1:155
Minimax estimation

Connection between \( \chi^2 \) test and \( \chi^2 \) distribution

Original Neyman-Pearson paper on hypothesis testing

Queueing

Called Erlang's Distribution

Approximation for small sample


2.6 NON-CENTRAL CHI-SQUARE

\[
D(x) = e^{-\frac{1}{2}x} e^{-\frac{1}{2}k} 2^{-\frac{1}{2}n} \quad \quad [c]36:204
\]

\[
\sum_{j=0}^{\infty} \frac{x^{\frac{1}{2}n+j-1} k^j}{\Gamma(\frac{1}{2}n+j) 2^{2j} j!}
\]

\[
Ch(x) = \frac{\exp[k/(1-2it)]}{(1-2it)^{n/2}} \quad \quad [18]1-162
\]

\[
k^r_r = 2^{r-1}(r-1)!(n+rk), \sim \text{form}
\]

Logarithmic Non-central Chi-square

2.7 HELMERT \((p,q)\)

\[
D(x) = \frac{(x/q)^{p-1} e^{-\frac{1}{2} x^2/q^2}}{\Gamma(\frac{1}{2} p) \frac{1}{2} p - 2} \quad [5]94
\]

For \(2q^2 - k, \ p=2\), called Rayleigh \[12]39

\[
D(x^2/q^2) = \text{Chi-square}
\]

Called semi-normal \[i]20:61

Refs, Ch(x) \[17]No. 42

\[
\alpha_1 = \left[\frac{1}{2}(p-1)\right]! \left[\frac{1}{2}(p-2)\right]! 2^{\frac{1}{2}} q, \ v=pq^2 - \alpha_1^2
\]

Non-central \[d]23:467


2.8 RECIPROCAL TYPE III \((p,q)\)

\[
D(x) = \frac{p^{q+1} q^{q+1}}{\Gamma(q+1)} x^{-q-2} e^{-pq/x}, \ pq > 0
\]

Mean = \(p\), \(\text{var} = \frac{p^2}{q-1}\), Type V

\[
\alpha_r = \frac{(pq)^r \Gamma(q-r+1)}{\Gamma(q+1)} \quad [2]86, 142
\]
Obtained from Pearson equation

Various constants, with an example

Ch(x) in special case

If \( q = -\frac{1}{2}, \ p = -1 \),

\[ \text{Ch}(x) = \exp \left( (-1+i) \sqrt{x} \right) \]

More generally

MLE

As dist. of precision const. in normal sample

C(x)

See also: \([c]26:388, [c]36:165, [o]8:55\)

2.9 GENERALIZED TYPE III

\[ D(x) = C(1+x/a)^{pa} e^{-px}, \text{ with moments of } D(s) \]

\[ D(x) = \frac{p^p e^{-p}}{q^p \Gamma(p)} (q + x)^{p-1} e^{-px/q}, \text{ with semi-invariants, } D(\bar{x}) \text{etc.} \]

\[ \text{Ch}(x) = e^{-aix} (1 - \frac{ix}{p})^{-ap-1} \]
\[ D(x) = \frac{1}{p R(q)} \left( \frac{x-c}{p} \right)^{q-1} \exp \left( \frac{c-x}{p} \right), \ c \leq x < \infty \]

MLE(c,p,q) \quad [3]39

Variance of estimates \quad [3]42

Tables \quad [d]1:191

Estimation \quad [g]48:336


\[ D(x) \]

\[ D(x) = A(x-c)^{q-1} e^{-p(x-c)}, \ x > c, \ p > 0, \ q > 0 \] \quad [1]249

Various constants, with an example \quad [11]66

One root of quadratic in Pearson equation is \infty \quad [11]44

\[ D(x_1/x_2) \] where each is Generalized Type III, or one is Generalized Type III and the other N(m,v) \quad [2]253

Bayes' Theorem \quad [n]16-1:114

Counting radioactive particles \quad [d]18:260
Two Generalized Type III distributions added


2.10 WISHART UNIVARIATE

\[ D(x) = \frac{a^{\frac{1}{2}(n-1)} x^{\frac{1}{2}(n-3)} e^{-ax}}{\Gamma\left[\frac{1}{2}(n-1)\right]}, \text{ Type III} \quad [1]391 \]

[a, \frac{1}{2}(n-1)]
III. BINOMIAL DISTRIBUTIONS

3.1 BINOMIAL \((k,p)\)

\[ D(x) = \binom{k}{x} p^x (1-p)^{k-x}, \ x=0,1,\ldots,k, \ "Bernoulli" \]

\[ \alpha_1 = kp, \ \nu = kpq, \ (where \ p+q=1) \]
\[ \alpha_3 = kpq(q-p), \ \alpha_4 = 2k^2p^2q^2 \]
\[ + pq(1-6pq) \]
\[ \mu_{r+1} = pq(k\mu_{r-1} + \frac{d\mu_r}{dp}) \]
\[ \beta_1 = \frac{q-p}{\sqrt{kpq}} \quad \beta_2 = 3 + \frac{1-6pq}{kpq} \]

Factorial moments \[ \alpha[i] = k[i]p^i \]

Long introductory article

Moments

Cumulants \[ k_{r+1} = pq \frac{dk_r}{dp} \]

Mean deviation

Semi-invariants
Several formulas for moments about the mean

C(x) as a Beta integral, recursion formula for moments

Moments in general

Moments and series

Arcsin transform

Another form of D(x)

\[ \beta_1 = \frac{(q-p)^2}{kpq}, \quad \beta_2 = \frac{1+3pq(k-2)}{kpq} \] etc.

\[ kp - q \leq \text{mode} \leq kp + p \]

\[ \text{Ch}(x) = (pe^{it} + 1 - p)^k \]

\[ \text{MGF}(x) = (q + pe^{it})^k \]

C(x)

\[ \text{PGF} = (q+px)^n \]

C(x) in terms of incomplete Beta integral


[14]33


[i]14:168

[14]210

[d]8:116

[12]52, [d]4:216

[6]57


[c]38:423

[18]1–146

[18]1–152
\[ C(x) \]

\[ E\left(\frac{1}{x}\right) \]

\[ D(n\bar{x}) = \text{Binomial } (nk, p) \]

\[ D(s^2) \]

\[ FD(p) \]

\[ D(x-y) \text{ in terms of Legendre functions} \]

Reproductive property by convolutions

A convolution with respect to \( p \)

\[ C.-R.(p) = \frac{pq}{kn} \]

\( \bar{x} \) is sufficient

\( (kn)^{-1} \sum x_i \) is efficient and unbiased

Is MLE

\[ \sim \text{BCR for } k = k_0 \]

Minimax estimate

Minimax and Bayes

Minimax estimation

Modified Bayes
BANE(\(p\)) [d]21:402

Biased and unbiased statistics [t]5:149

MLE(\(k, p\)) [k]18:117

Estimation of \(p\) based on runs MR14:1102

LR comparison of two binomials [c]37:140


Sequential [d]17:288, 489,


Chi-square test [k]7:207

UMP Test [c]43:465

Sampling inspection [j]8:626

Multiple sampling [d]14:363

Analysis of variance [d]11:335

Order statistics MR16:729, [s]8:62
Approximate formulas

Normal approximation

Using generating functions

Inequalities for tails

Asymptotic behavior

Convergent sequences of binomials

n binomials

Normalizing transform \( y = k^{\frac{1}{2}} \sin^{-1}(x+a/k) \)
and other transforms

\[ \log \frac{x}{1-x}, -2 \tanh^{-1}x \]

Other transforms

Choosing between several binomials

If \( p \) not constant (called "Lexian")

Transformations, approximations, applications
Chain binomials

Gambler's ruin

Connection with Beta distribution

Actuarial application

Binomials added

Generalization

Generalized binomial

\[ \text{Ch}(x) = \prod (q_i + p_i e^{ix}) \]

A modification

3.2 BINOMIAL (1,p)

\[ D(x) = p^x(1-p)^{1-x}, \quad x=0,1 \]

\[ D(x) = \text{Binomial} (n,p) \]

Chi-square test

Confidence intervals for p

Completeness

\[ \text{UMVUE}(p) = \bar{x}, \quad \text{UMVUE}(pq) = \frac{n\bar{x}(1-\bar{x})}{n-1} \]


3.3 TRUNCATED BINOMIAL (k,p)

\[ D(x) = \binom{k}{x} p^x q^{k-x}, \quad x=1, \ldots, k \]

\[ E(x) = \frac{kp}{1-q}, \quad v = \frac{kpq}{1-q} \]

moments of \( x^{-p} \)

Estimation

Tables

With an application

See also: [b]11:2, MR15:969
(1.3) Derivative of ln x:

\[ \frac{d}{dx} \ln x = \frac{1}{x} \]

(1.4) Limit of \( \frac{\sin x}{x} \) as \( x \) approaches 0:

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1 \]

(1.5) Taylor series for \( e^x \):

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

(1.6) Maclaurin series for \( \sin x \):

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \]

(1.7) Taylor series for \( \ln(1+x) \):

\[ \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \]
### 3.4 NEGATIVE BINOMIAL \((k, p)\)

<table>
<thead>
<tr>
<th>(D(x))</th>
<th>(p)</th>
<th>(q)</th>
<th>Mean</th>
<th>Variance</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-k\choose x)p^x(1-p)^{-k-x})</td>
<td>(p)</td>
<td>(q)</td>
<td>(-kp)</td>
<td>(-kpq)</td>
<td></td>
</tr>
<tr>
<td>((-1/a)^k\left(-k\choose x\right)\frac{(-1)^x}{(1+a)^x})</td>
<td>(-1/a)</td>
<td>(\frac{a+1}{a})</td>
<td>(k/a)</td>
<td>(k/a + k/a^2)</td>
<td>[1]259, [a]83:255</td>
</tr>
<tr>
<td>((a_{1+a})k \left(1 - \frac{t}{1+a}\right)^{-k})</td>
<td>(-1/a)</td>
<td>(\frac{a+1}{a})</td>
<td>(k/a)</td>
<td>(k/a + k/a^2)</td>
<td>[2]125</td>
</tr>
<tr>
<td>(p^k \left(x+k-l\choose k-l\right)(1-p)^x)</td>
<td>(\frac{p-l}{p})</td>
<td>(1/p)</td>
<td>(qk/p)</td>
<td>(qk/p^2)</td>
<td>[d]17:53, [6]61, [18]1-158, [17]No. 6, [7]218</td>
</tr>
<tr>
<td>(\left(\frac{x+k-l}{k-l}\right)\frac{p^x}{(1+p)^{k+x}})</td>
<td>(-p)</td>
<td>(l+p)</td>
<td>(p)</td>
<td>(p + p^2)</td>
<td>[f]9:176, [4]54</td>
</tr>
<tr>
<td>(\left(\frac{m}{1+bm}\right)^x \left(\frac{1}{1+bm}\right)^{-1/b} \frac{1}{x!} \prod_{j=1}^{x-1} \left(1+jb\right))</td>
<td>(-bm)</td>
<td>(l+bm)</td>
<td>(m)</td>
<td>(m(l+bm))</td>
<td>[5]32</td>
</tr>
<tr>
<td>(\left(\frac{n}{n+km}\right)^n \left(x+n-l\choose n-l\right)\left(\frac{km}{n+km}\right)^x)</td>
<td>(-km/n)</td>
<td>(\frac{n+km}{n})</td>
<td>(km)</td>
<td>(km + \frac{k^2m^2}{n})</td>
<td>[c]41:78</td>
</tr>
</tbody>
</table>
If \( Qp - 1, Q(1-p) = P \) then

\[
\beta_1 = \frac{p+Q}{\sqrt{kPQ}}, \quad \beta_2 = 3 + \frac{1+6PQ}{kPQ}
\]

Obtained by assuming a Poisson parameter to be Type III

Some derivations, with interesting properties

Moments

If \( k = h/p \), called Polyga-Eggenburger

\[
Ch(x), \quad [4]55
\]

A "contagious" distribution

Skewness, Kurtosis, Cumulants

\[
C(x) = (p + q)^{-n}
\]

\[
Ch(x) = [1 + bm(l-e^{it})]^{-1/b}
\]

Called compound Poisson

\[
Ch(x) \quad [15]727
\]

Limit of contagious

Limited by Poisson and Pascal

PGF

Problem leading to Negative Binomial, with generalization

[c]41:78, [a]110:132,
[f]5:162

[18]1-159, [c]44:530

Z13:70

[17]No. 7, extension
Mem. Fac. Sci. Kyusku
Imp. Univ. (Ser A)
1:178

[7]128

[18]1-136, 1-144

[c]37:209

[5]62

[15]727

[v]4:9

[c]41:269

[7]233

[18]1-146

[d]17:53
Generalization

Paper by Fisher

Recurrence formula for cumulants,

Moments

Formulas for tails

MLE

$M_{\chi^2 E}$

UMVUE(p)

Estimation

Sequential

Sampling

Fitting

Truncated

Moments estimation, MLE

Bhattacharyya bounds

Transformation $\sinh^{-1} x$
Transformations

Called "Pascal", satisfies

\[ D(x+1) = \frac{(1-p)(x+k)}{(x+1)} D(x) \text{, etc.} \]

Accident proneness

Telephone traffic

Bibliography


3.5 NEGATIVE BINOMIAL \((1, -m)\)

\[ D(x) = \frac{1}{1+m} \left( \frac{m}{1+m} \right)^x \text{, "Pascal", or "Furry",} \]

\[ \alpha_1 = m, \quad \nu = m^2 + m, \]

\[ Ch(x) = \left[ 1 + m(1-e^{it}) \right]^{-1} \]

Cumulants

PGF

[c]44:265
3.6 DISCRETE LEXIAN

\[ D(x) = \sum f(p) \binom{k}{x} p^x(1-p)^{k-x} \text{, moments} \]

etc.

"Generalized Binomial"

Poisson-Lexian

\[ Ch(x) = [p\phi(t) + q]^k \]

If a priori distribution of p is Beta

[1]26:34

[1]26:57

[17] No. 69


3.7 DETERMINISTIC

\[ D(x) = \begin{cases} 0, & x \neq c \\ 1, & x = c \end{cases} \]

\[ Ch(x) = e^{ict} \]

For c = 1

Bibliography

Moments, cumulants, \(Ch(x)\), PGF

See also: [c]44:366, Z9:363, [w]1:9
3.8 RECIPROCAL TRUNCATED BINOMIAL

\[ D(x) = \frac{e^{-m} x^m}{x!}, \quad x=0,1,\ldots, \text{Law of small numbers} \]

IV. DISCRETE DISTRIBUTIONS

4.1 POISSON (m)

\[ D(x) = e^{-m} \exp \left( \frac{m e^t - 1}{t} \right) \]

\[ \text{Ch(x)} = \exp \left[ m(e^{it} - 1) \right] \]

\[ \text{MGF(x)} = e^{-m} \exp (me^t) \]

\[ \alpha_1 = m, \quad \alpha_2 = m^2 + m, \quad \nu = m \]

\[ \alpha_3 = m[(m+1)^2 + m], \]

\[ \alpha_4 = m(m^3 + 6m^2 + 7m + 1) \]

\[ \mu_2 = m, \quad \mu_3 = m, \quad \mu_4 = m(1 + 3m), \]

\[ \mu_5 = m(1 + 10m), \quad \mu_6 = m(1 + 25m + 15m^2) \]

Skewness, Kurtosis

PGF
\[
\mu_{r+1} = r\mu_r + \frac{d\mu_r}{dm}
\]

All cumulants = \(m\)

Factorial moments \(\alpha[i] = m^i\)

Moments in general

Recursion formula for moments, correction with multinomial, \(C(x)\) as a \(\Gamma\) integral

\[
\beta_1 = \frac{1}{m}, \quad \beta_2 = 3 + \frac{1}{m}, \quad \gamma = \frac{1}{m}
\]

\(C(x)\)

Transform \(y = \sqrt{x}\)

\[
D(\overline{x}) = \frac{e^{-nm}(nm)^{n\overline{x}}}{(n\overline{x})!}
\]

\(D(x + y)\)

\[
\sim D\left(\frac{x-m}{m}\right) = N(m, \ m^{\frac{1}{x}})
\]

\(D(x - y)\)


\[\text{[2]}\text{66}\]

\[\text{[1]}\text{257}\]

\[\text{[d]}\text{8:103, [i]14:173}\]

\[\text{[14]}\text{36-8}\]

\[\text{[12]}\text{52}\]

\[\text{[j]}\text{5:604, MR4:194, [c]}\text{37:313}\]

\[\text{[14]}\text{209}\]

\[\text{[1]}\text{379, [2]}\text{243, [6]}\text{208, [15]}\text{219}\]

\[\text{[10]}\text{59}\]

\[\text{[1]}\text{250}\]

\[\text{[2]}\text{251, MR14:566, [a]}\text{109:296, [a]}\text{100:415, [v]}\text{7:175}\]
\[ D(x - y) \]

\[ D(\text{gap between two Poisson events}) = \text{exponential} \]

\[ D[\bar{x}^{-1} \Sigma (x_i - \bar{x})], \text{i.e., Chi-square test} \]

Various a priori distributions of \( m \), in particular Type III

\[ E(x^2) = E(x + 1) \]

\[ E(1/x) \]

\[ \text{Var } (x^{\frac{1}{n}}) \]

Reproductive property by convolutions by Ch. functions

\[ C.-R.(m) = m/n \]

\[ C.-R.(m^2) = 4m^3n^{-1} \]

BANE

Estimation when \( m \) must be integral

Estimation from censored samples
Estimation of bacteria population

Estimation of m or 1/m

$M^2E$  

Approximation, estimation, application

$\text{MLE}(m) = \bar{x}$

$\bar{x}$ is sufficient

$\bar{x}$ is efficient

$\bar{x}$ is unbiased

Completeness

Confidence intervals, tail

C.-R.

Confidence intervals

Order statistics

Approximate moments of ordered variables

Testing whether two Poissons are the same
Two Poissons, etc.

Whether k Poissons are the same

Analysis of variance

Testing m

Testing ratios of means

Chi-square test

Testing against contagious

Sequential testing m

Small sample tests

Monograph on Poisson testing and estimation

Obtained from a difference equation analogous to Pearson's differential equation

Obtained from postulates

Early discussion with numerical examples
Transform $y = (x + k)^{\frac{1}{3}}$

Transformations

Domain of attraction

As limit of binomial

Generalizations

A modification

Convergent sequence of Poissons

Connection with hypergeometric

If $m$ Poisson, called double Poisson

Normal approximation

Compounded with binomial

As approximation to Beta

Normal approximation

An approximation

[c]41:312
MR3:2
[e]15:237,251
Am. Math. Monthly 50:97
[c]25:300
Kendall and Buckland, A Dictionary of Stat. Terms
[7]128,221
[j]20:19
[r]4:37
MR18:423
Connection with Gram-Charlier

Connection with Type X

Connection with Type III

Limiting theorems

Characterization

Generalizations

Possibility of a continuous analogue

Traffic control

Accident causation

Poisson as a limiting distribution in five different ways, relation with multinomial, exponential

Accident proneness

Pedestrian delay

[2]154

[0]2:13

[0]3:123

[8]148


[i]14:43

[b]7:65

[b]7:89, [a]90:487

[18]1-156

[c]37:24

[c]38:383
Insurance risk

Frequency of war

Nomograph for acceptance inspection

Telephone switchboards

An early treatment, with the famous example of the Prussian horse-kicks

Other applications

4.2 TRUNCATED POISSON

\[ D(x) = \frac{m^x e^{-m}}{x!} \cdot \frac{1}{1-e^{-m}}, \quad x=1,2,... \]

Estimation

UMVUE

Tables

Servicing machines

\[ D(1/x) \]

Doubly truncated, \( D(x) \) etc.

4.3 COMPOUND POISSON

\[ D(x) = k_1 \frac{e^{-m} x^m}{x!} + k_2 \frac{e^{-n} x^n}{x!} \]

double Poisson

If \( k_1 = k_2 = \frac{1}{2} \)

[5]151
Compound Poisson $D(x) = \frac{1}{x!} \sum_{i}^{x} e^{-m} i^{k}$

4.4 UNIFORM

$D(x) = \frac{1}{k}, \text{ for } x=1, \ldots, k$

Discrete rectangular

MGF and moments

Sampling from

Estimation of range

Range and quotient of ranges

See also: MR16:376.
4.5 HYPERGEOMETRIC

\[ D(x) = \frac{\binom{m}{x} \binom{n}{r-x}}{\binom{m+n}{r}} \]

\( D(x) \) in another form, moments, etc.

In the form of a hypergeometric series, asymptotic forms

Various forms

\( C(x) \) as a power series

\( C(x) \), refs.

\[ E(x) = \frac{mr}{m+n}, \quad v = \frac{mnr (m+n-r)}{(m+n)^2 (m+n-1)} \]

Difference equation, moments, etc.

Factorial moments

Skewness

PGF

Moments in general


\[ [2]126 \]


\[ [c]41:317 \]

\[ [17] \text{No. 8} \]


\[ [i]14:178 \]


\[ [i]6:79 \]

\[ [18]1-136, [18]1-146 \]

Binomial and Poisson as limits
Binomial as limit
Poisson as limit
Normal as limit
Normal, Poisson, Binomial approximations
Minimax estimation
Completeness
"Confidence Limits for the Hypergeometric Distribution"
Generalization
Satisfying difference equation
Linguistic application
Truncated hypergeometric, moments

[15]690
[7]47
[7]146
[d]21:191
[e]10:315
Chung and DeLury
Univ. of Toronto Press
1950, reviewed,
[a]115:286
[7]39, [c]41:266,
see also No. 8.59,
[b]18:202
[w]3:5
[b]12:27
[d]16:59
Double hypergeometric


4.6 CONTAGIOUS

Obtained by considering \( \int (\text{Poisson}) \, dF(m) \)

\[ F \text{ step function yields} \]

\[ D(x) = \frac{1}{x!} \sum p_i e^{-a_i} a_i^x \]

\[ F \text{ Type III yields Polya-Eggenberger} \]

\[ D(x) = \frac{1}{x!} \frac{\Gamma(x+h/d)}{\Gamma(h/d)} (1+d)^{-d/h} (1+d)^{-x} \]

\( m \) itself also Poisson yields

\[ D(x) = e^{-k} \frac{c^x}{x!} \sum \frac{1^x}{1!} (e^{-c}k)^i \]

Neyman contagious Type A

Fitting

Ch. fcns.
Testing against Poisson

Rutherford contagious

\[ \Sigma \text{(Poisson)} f_j(x), \text{Ch}(x), \text{sp. cases} \]

\[ \text{Bivariate} \]


4.7 POLLACZEK-GEIRINGER

\[ f(x), \text{Ch}(x), \]

multiple occurrence of rate events

4.8 BOREL-TANNER

\[ f(x) = \frac{e^{-\alpha x} x^{-r} x^{-r-1}}{(x-r)!}, x=r,r+1,..., \]

\[ 0 \leq \alpha \leq 1, \quad r = 1,2,... \]

\[ m = \frac{r}{1-\alpha}, \quad v = \frac{r^\alpha}{(1-\alpha)^3} \]

\[ \text{[s]214:452, [c]40:58} \]
4.9 POLYA

\[ f(x) = \binom{N}{x} \prod_{i=0}^{x-1} \frac{N-x-1}{(m+iR)} \prod_{j=0}^{N-x-1} \frac{1}{(n-m+jR)} \prod_{k=0}^{n-1} \frac{1}{(n+kR)} \]

Contains Polya-Eggenburger No. 3.4, and Exceedance No. 4.10

4.10 EXCEEDANCE

\[ f(x) = \binom{n}{m} \binom{N}{x} \frac{(N+x-N)}{(n+N)(m+x-1)} , \quad x=0,1,\ldots,N \]

Moments

4.11 INVERSE HYPERGEOMETRIC

\[ f(x) = \frac{(n-m)(N-n)(n-m+1)}{(x-1)(N-x+1)} , \quad \text{estimation, truncation} \]

Kungl. Lantbuck. Ann. 18:123
V. DISTRIBUTIONS ON (a, b)

5.1 SERIAL CORRELATION

\[ D(x) = \frac{\Gamma\left(\frac{1}{2}k+1\right)}{\Gamma\left(\frac{1}{2}k+\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)} \left(1-x^2\right)^\frac{1}{2} \left(1+r^2-2rx\right)^{-\frac{1}{2}k} \]

\[ \gamma_1 = \frac{rk}{k+2} \]

\[ \nu = (k+2)^{-1} \left[1-r^2k(k+1)(k+2)^{-1}(k+4)^{-1}\right] \]

Moments, called "Leipnik"  

[c]44:270


5.2 TYPE I

\[ D(x) = C(x-a)^{p-1}(b-x)^{q-1}, \]

a < x < b, p > 0, q > 0

Beta for a=0, b=1  

\[ \text{normal for } p=q=\frac{1}{2}b^2, a=-b, b\to\infty \]

Type III for b\to\infty, q = \alpha b  

Obtained by assuming roots in quadratic of Pearson differential equation real with different sign  

[1]249
Relations of various constants  

Bayes Theorem  

Fitting to observations  

Early volumes of [c] give many examples with $D(x) = C(1+x/a)^m(1-x/b)^n$


5.3 BETA $(p,q)$

$$D(x) = \frac{\Gamma \left[ \frac{1}{2}(p+q) \right] }{\Gamma \left( \frac{1}{2}p \right) \Gamma \left( \frac{1}{2}q \right) } x^{\frac{1}{2}p-1} (1-x)^{\frac{1}{2}q-1},$$

$0 < x < 1, \; p < -2, \; q < -2$

Called Beta distribution of the first kind

$$\alpha_1 = \frac{p}{p+q}, \; \nu = \frac{pq}{(p+q)^2(\frac{1}{2}p+\frac{1}{2}q+1)}$$

3rd Moments

Mode = $\frac{p-2}{p+q-4}, \; \text{etc.}$$

References:


[n]16-1:115  


[2]139, [17]No. 21  


[d]25:401  


[2]419  

[m]2:128
\[ \alpha_r = \frac{B(p+r, q)}{B(p, q)} \]

\[ \text{HM} = \frac{p-2}{p+q-2} \]

\[ \text{GM} = \exp \left\{ \frac{\gamma}{\sqrt{2\pi}} \left[ \log \Gamma \left( \frac{p}{2} \right) - \log \frac{1}{2}(p+q) \right] \right\} \]

C(x)

Moments

Obtained as a Pearson Type

From an example

C(x)

Moments

Obtained as a Pearson Type

From an example

C(x)

Moments

Obtained as a Pearson Type

From an example

C(x)

Moments

Obtained as a Pearson Type

From an example

D(x)

When \( p = q \)

Special cases and variants

\( x^2 \) is Beta

Beta (\( n-k, k \))

Beta(2n+2, 4) connected with tolerance limits

Correlation ratio in samples from uncorrelated bivariate normal is

Beta (\( k-1, n-k \))


[10]163

[d]20:451

[3]21

[4]74

[10]45


[c]19:230

[2]26

[17]No. 20

[1]409


D(1-x) - Beta (q,p)  
If x is Beta (p,q), y is Type III  
(l, p+q) then xy is Type III (l,p)  

Standardized Beta variable is N(0,1) as  
p→∞, q→∞  

D(xy)  
No sufficient statistic for (½ p-l)  

Mellin transform  
more generally  

log log 1/x, arcsin √x,  

Generalized  
Transform  

Connection with Fisher and Snedecor  

Non-central Beta  

Connection with binomial  

Fitting straight lines  

Approximated by Poisson
Range of rectangular is

\[ D(x) = n(n-1)a^{-n}x^{n-2}(a-x) \]

In trivariate normal analysis

\[ \text{GM } (x_1, \ldots, x_n) \text{ and GM } (1-x_1, \ldots, 1-x_n) \]

are joint sufficient

In rank correlation


5.4 TYPE II

\[ D(x) = \frac{(1-(x-m)^2a^{-2})^p}{aB(\frac{1}{2}, p+1)}, m-a \leq x \leq m+a \]

Properties

\[ \text{MR10:131} \]

Transform to Student

\[ \text{[c]28:308} \]

\[ \text{D(X)} \]

\[ \text{[n]10-3:91} \]

\[ \text{C(x)} \]

\[ \text{[c]19:12, [c]25:379} \]

\[ \sim \text{Dist. of rank correlation coefficient (Pitman)} \]

\[ \text{[c]30:259} \]
Called Thompson's distribution, relation with Student's, normal approximation

m location, a scaling, p shape

Likelihood function

C.-R.(m)

Tables

$$v = \frac{a^2}{2p+3}$$

If m=0

Dist. of Spearman's ρ for large n

A numerical example

Estimation of center

From Pearson system


5.5 PARTIAL CORRELATION

$$u(x) = \pi^{-\frac{1}{2}} \frac{\Gamma\left[\frac{1}{2}(n-1+1)\right]}{\Gamma\left[\frac{1}{2}(n-k)\right]} \left(1-x^2\right)^{\frac{1}{2}(n-k-2)}$$

-1<x<1

Type II with m=0, a=1, p= \(\frac{1}{2}\)(n-k-2)
Ch(x)

If $k=2$, transform to Student

If corresponding population parameter is zero

As No. 5.1 with $r=0$,

If population is non-normal


5.6 PARABOLIC

$$D(x) = \frac{3(a^2-x^2)}{4a^3}, -a<x<a, \ \frac{v}{a^2} = 5$$

grouping corrections

Estimation

5.7 TYPE IX

$$D(x) = \frac{m+1}{a} (1+x/a)^m, -a\leq x < 0$$

See also: [d]7:26, [c]24:234,240,263.

Type VIII for negative $m$
5.8 TYPE XII

\[ D(x) = (p/q)^m \frac{1}{(p+q)\Gamma(1+m',1-m)} (1+x/p)^m (1-x/q)^{-m} \]

\[ \frac{1}{(p+q)\Gamma(1+m',1-m)} \]

|m|<1, -p<x<q


5.9 CORRELATION DETERMINANT

\[ D(x) = \frac{\Gamma \left[ \frac{1}{2}(n-1) \right]}{\pi^k(k-1)} \frac{k-1}{\Gamma \left[ \frac{1}{2}(n-2) \right]} \cdot \ldots \cdot \frac{\Gamma \left[ \frac{1}{2}(n-k) \right]}{\Gamma \left[ \frac{1}{2}(n-3) \right]} \]

\[ \alpha_1 = (n-1)^{1-k}(n-2)(n-3) \cdot \ldots \cdot (n-k), \]

\[ v = k(k-1)n^{-2} + 0(n^{-3}) \]

Downton calls this "Geometric", and mentions the following special cases:

I. \[ D(x) = px^{p-1}, \quad 0 \leq x < 1, \]

\[ C(x) = x^p, \quad \alpha_1 = p/p-1, \quad v = p(p+2)^{-1}(p+1)^{-2} \]

II. \[ D(x) = pb^{-p}(x+a)^{p-1}, \quad -a \leq x < b-a \]

III. \[ D(x) = pv^{-\frac{1}{2}} b^{-p}(x-m/v^{\frac{1}{2}} + a)^{p-1}, \]

\[ m - av^{\frac{1}{2}} \leq x < m+(b-a)v^{\frac{1}{2}}, \quad \alpha_1 = m, \]

\[ a = p^{\frac{1}{2}} (p+2)^{\frac{1}{2}}, \quad b = p^{-\frac{1}{2}} (p+1)^{3/2}, \quad p \geq 1 \]

\[ \log \log (x) \]
5.10 TRIANGULAR

\[ D(x) = 1 - |1 - x|, \quad D(x) \text{ from rectangular} \quad [c]25:417, [y]24:22, \]
when \( n = 2, \) D(range) \quad MR 3:171

\[ D(x) = \frac{2x}{2k+1}, \quad k \leq x \leq k + 1 \quad [3]47, [8]32 \]
Stratified sampling \quad [c]13:48

\[ D(x) = (9\sigma)^{-1}[\frac{x-m}{\sigma} + 2 \sqrt{2}], \quad \text{right triangular} \quad [d]4:256, [d]25:308 \]
\[ m - 2 \sqrt{2} \sigma \leq x < m + \sqrt{2} \sigma \]

\[ D(x) = 4R^{-2}(\frac{3}{2}R - |x-m|), \quad |x-m| \leq \frac{3}{2}R, \quad [d]25:318 \]
best linear estimate of \( m \) and \( \sigma \)

Testing \quad [d]25:695
See Also \quad [d]2:48, [d]28:179

\[ D(x) = \frac{1}{a} [1 + k - \frac{2k}{a(a-x)}], \quad 0 \leq x \leq a, \quad [d]4:244 \]
-1 \leq k \leq 1, called "linear"

\[ D(x) = \begin{cases} 1 + x, & -1 \leq x < 0 \\ 1 - x, & 0 \leq x < 1 \end{cases}, \quad \text{called "Tine"} \quad [d]5:33 \]

\[ Ch(x) = (2/t^2)(1 - \cos t) \quad [17]\text{No.14} \]

\[ D(x) = \frac{2}{a^2}(a - x), \quad \text{called "semi-triangular"} \quad [c]39:432 \]
m = a/3, \( v = a^2/18, \) grouping corrections

\[ Ch(x) \quad [17]\text{No.13} \]

Triangular on \((a,b)).\) If \( x \) and \( y \) are extreme values of the sample, then
107

\[ \begin{align*}
E\left[ \frac{1}{2} (x + y) \right] &= \frac{1}{2} (a + b) \\
\text{Var}\left[ \frac{1}{2} (x + y) \right] &= \frac{4 - \pi}{16n} (b - a)^2 + O(n^{-2}) \\
E[(x - y)] &= [1 - \frac{\sqrt{n}}{2n}] (b - a) + O(n^{-3/2}) \\
\text{Var}(x - y) &= \frac{4 - \pi}{4n} (b - a)^2 + O(n^{-2})
\end{align*} \]

*See Also:* [y]16:16

5.11 **RECTANGULAR** (a-h, a+h)

\[ \begin{align*}
D(x) &= \frac{1}{2h}, \quad m = a, \quad v = \frac{h^2}{3} \quad \text{[1]244, [5]34} \\
\text{Skewness} &= 0, \quad \text{kurtosis} = -6/5, \quad K_k, \quad \text{Ch}(x) \quad \text{[18]1-136,1-144} \\
\text{Ch}(x) &= \frac{\sin ht}{ht} e^{ait} \quad \text{[1]259} \\
\text{Special case of Type II} \quad \text{[2]142} \\
\text{Ch}(x), \quad \text{bibliography for rectangular over} \quad \text{[17]Nos.11,12} \\
(a,b), (-a,a) \quad \text{[17]Nos.11,12} \\
C(x) &= \frac{1}{2} + \frac{x}{a} \quad \text{[15]93} \\
\text{If} \ x \text{ and} \ y \text{ are the} k^{th} \text{values from the top and bottom of sample,} \\
E(x) &= a + h - \frac{k}{n+1} (2h) \quad \text{[1]372} \\
E[\frac{1}{2}(x+y)] &= a \\
E(x - y) &= (1 - \frac{2k}{n+1})(2h)
\end{align*} \]
\[ \text{Var}(x) = \frac{k(n-k+1)}{(n+1)^2(n+2)} (2h)^2 \]

\[ \text{Var}\left[ \frac{1}{2} (x+y) \right] = \frac{4kh^2}{2(n+1)(n+2)} \]

\[ \text{Var}(x - y) = \frac{2k(n-k+1)}{(n+1)^2(n+2)} (2h)^2 \]

\[ \sim D(\bar{x}) = N(0, \frac{h^2}{3n}), \sim D(c) = \text{Laplace}(0, \frac{h}{n}), \quad [3]48 \]

where \( c = \frac{1}{2} (\text{max} + \text{min}), n \text{ var}(c) = 6 \text{ var}(\bar{x}) \)

\[ \text{D}(\zeta) \]

Moments of max and min

\[ \text{D}(\bar{x}) \text{ given incorrectly} \]

\[ \text{D}(\bar{x}) = \text{No. 5.16} \]

\[ \text{D}(-2\log \prod x_j) \]

\[ \text{FD}(a) = \text{Rectangular}(\text{max} - h, \text{min} + h) \]

\[ \text{D}(q^{th} \text{ ranking item}) = \text{Type I} \]

Testing against simple unimodal distribution

Grouping corrections

Transformation to Cauchy

C.-R. Theorem may not hold

\[ \text{MLE}(a-h, a+h) = (\text{max}, \text{min}) \]

Best linear estimate of \( m \) and \( \sigma \)  

Estimation  

Location and scaling, closest estimate  

Minimax estimate of \( a \)  

UMVUE  

Bayes Theorem  

Variance of estimates of \( a \)  

Testing \( a \)  

Critical regions  

See also: Archiv. der Math. 3:3, MR 6:235  

5.12 RECTANGULAR(0,a)  

\[
D(x) = \frac{1}{a}, \quad Ch(x) = \frac{e^{ait} - 1}{ait} \quad [2]245
\]

\[
MGF(x) = \frac{\sinh \frac{1}{2}at}{\frac{1}{2}at} \quad [10]38
\]

\[
a^{2r} = \frac{(\frac{1}{2}a)^{2r}}{2r + 1} \quad [10]14
\]

Cumulants  

\[
\text{Var(mean deviation)} \approx \frac{a^2}{45n} \quad [2]217
\]
Mean difference

\[ \text{Ch}(\Sigma x_j), C(\Sigma x_j) \]

\[ D(GM) = \frac{n^n a^{n-1}}{a^n n (n)} (\log a/x)^{n-1} \]

\[ D(\text{range}) = n(n-1) a^{-n} x^{n-2}(a - x) \]

\[ D(\sum^n_{j=1} \log x_j) = \text{Type III} \]

\[ D(\bar{x}, s) \text{ for } n = 2, 3 \]

\[ D(\text{quotient of ranges}) \]

\[ D(\max_1/\max_2) = \text{No. 8.63} \]

\[ FD(a) = k x^{-n-1} \]

\[ \text{MGF}(\log \log \frac{1}{x}), \text{MGF}(\arcsin \sqrt{x}) \]

Completeness

\[ \text{Estimation by order statistics} \]

Quasi-range

Best linear estimate

\[ \text{UMVUE}(a) = (1 + \frac{1}{n})(\max), \max \text{ is sufficient} \]
Sufficient statistics

Confidence intervals for $a$

Example

Estimation of dispersion

UMP test of $a = 1$


5.13 RECTANGULAR ($0, 1$)

$D(x) = 1, \quad 0 \leq x \leq 1$

$D(-\log x) = e^{-x}, \quad Ch(- \log x) = (1 - it)^{-1}$

$D(-\Sigma \log x_j) = \text{Type III } (1, k)$

$D(\Sigma x_j) = \frac{1}{(n-1)!} \left[ x^{n-1} - (\frac{n}{1}) (x-1)^{n-1} + (\frac{n}{2}) (x-2)^{n-1} - \ldots \right]$

"Irwin-Hall" distribution

Convolutions

Sheppard's corrections
\( D(\text{value from top of sample}) = \text{Type I} \)
\[ m = 1 - \frac{q}{n+1}, \quad v = \frac{q(n-q+1)}{(n+1)^2(n+2)} \]

\[ \text{Var}(\xi) = \frac{1}{4(n+2)} \]

\( \sim D(\bar{x}) \)

Mellin transform

Order statistics

\( D(\text{range}) = \text{Type I} \)

\( \text{MGF}(\log \log \frac{1}{x}) = (1 + t) \)

\( \text{MGF}(\arcsin x) = 2(e^{-\pi/4}t + 1)/(t^2 + 4) \)

Ratio of two ranges

Moments of the range

C.-R. Theorem may not hold

Censored sample

Stratified sample

Significance levels for \( \bar{x} \)

\( D(\text{GM}) = \text{No. 8.12} \)
Estimation of center, $D(\bar{x})$, $D(\xi)$

$D(\text{max}_1\text{max}_2) =$ No. 8.64

Two rectangulars added

Hypothesis testing


5.14 CORRELATION

\[
D(x) = \frac{(1-x^2)^{\frac{1}{2}(n-4)}}{\pi (n-3)!} \left(1 - \rho^2\right)^{\frac{1}{2}(n-1)} 2^{n-3} \sum_{i=0}^{\infty} \frac{(2x\rho)^i}{i!} \int 2 \left(\frac{n+i-1}{2}\right)
\]

\[\text{[1]}398\]

\[
D(x) = \frac{(1-\rho^2)^{\frac{1}{2}(n-1)}}{\pi (n-2)!} \frac{(1-x^2)^{\frac{1}{2}(n-4)}}{d(\rho x)^{n-2}} \frac{\cos^{-1}(-\rho x)}{1-\rho^2 x^2}
\]

\[\text{[2]}342, \text{[10]}200\]

Special cases $n = 2, 3, 4$, moments

$n = 4$

$C(x)$

$\sim D(x)$

If $\rho = 0$, $D(x) =$ No. 5.5

\[\text{[6]}314, \text{[4]}120, \text{[g]}26:129\]
If \( \rho = 0 \), \( D \left( \frac{x}{\sqrt{1 - x^2}} \right) = \text{Student}(n-2) \) \([2]343\)

Transform \( x = \tanh z \), \( \rho = \tanh \zeta \) \([10]200, [c]21:358\)

If \( \rho = 0 \), \( D(x^2) = \text{Beta} \left[ \frac{1}{4}, \frac{1}{4}(n-2) \right] \) \([10]160,192\)

Bayes distribution of \( \rho \) is No. 5.5(0,1) \([3]91, [c]41:278\)

Moments \([n]5:3\)

Papers dealing with this distribution generally \([c]10:507, [c]11:328\)

Interval estimation \([e]7:415\)

Confidence limits for \( \rho \) \([c]29:157\)

Stratified sampling \([i]36(\text{Suppl.}):87\)


5.15 MULTIPLE CORRELATION

\[
D(x) = \frac{\gamma}{B\left[\frac{1}{2}(n-k), (k-1)\right]} (1 - R^2)^{\frac{1}{2}(n-1)} x^{\frac{1}{2}(k-3)} (1-x)^{\frac{1}{2}(n-k-2)} \]

where \( \gamma = F\left(\frac{n-1}{2}, \frac{n-1}{2}, \frac{k-1}{2}, R^2x\right) \) and

\[
F(a, b, c, x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \ldots \]

If $R = 0$, $D(x^2) = \text{Beta}(k, n-k)$

Testing

Another form

Limiting form when $n \to \infty$

When $R = 0$, $D(x) = \text{Snedecor}$

Mean, variance

Moments

More generally


5.16 RECTANGULAR MEAN

$$D(x) = \frac{1}{(n-1)!} \sum_{j=0}^{\lfloor x \rfloor} (-1)^j \binom{n}{j} (x - j)^{n-1}$$

Generalization

Called Irwin-Hall, cf. No. 5.13

Compare No. 8.47 and No. 8.70

See Also: London P. O. Res. Rep. 13443, Archiv der Math. 3:3,
VI. DISTRIBUTIONS ON \((0, \infty)\)

### 6.1 TYPE VI

\[ D(x) = C(x - a)^{p-1}(x - b)^{q-1}, \quad x > b, \ a < b, \ q > 0, \ p + q < 1 \]

If \(b = 0\), \(C = \frac{a^{1-p-q}}{B(1-p-q, \ p)}\)

Roots of quadratic in Pearson equation real and same sign

Truncation

Various constants and an example


### 6.2 SNEDECOR \((p,q)\)

\[ D(x) = \left(\frac{p}{q}\right)^{\frac{1}{2}p} x^{\frac{1}{2} p-1} (1 + \frac{px}{q})^{-\frac{1}{2}(p+q)}, \quad x > 0 \]

"\(F\)" distribution

\(m, v, k_3, k_4, \rho_1, \rho_2\)

Derivation, properties, examples

Area unity if \(p, q\) both even

Obtained as distribution of ratio of two Chi-square variables
D(\sqrt{x})

D\left(\frac{px}{q+px}\right) = \text{Beta}(p-1, q-1)

Therefore called "inverted Beta"

Various properties

m = \frac{q}{q-2}

\alpha_r = \frac{\Gamma\left(\frac{1}{2}p + r\right) \Gamma\left(\frac{1}{2}q - r\right)}{\Gamma\left(\frac{1}{2}p\right) \Gamma\left(\frac{1}{2}q\right)} (q/p)^r

Mode = \frac{pq - 2q}{pq + 2p}

Approximated by normal distribution

If x, y each Snedecor (n-1,n), then

D(\sqrt{x/y}) = \text{Snedecor}(2n-2, 2n-2)

Testing

Used to test multiple correlation coefficient

6.3 BETA OF SECOND KIND (p, q)

\[ D(x) = \frac{x^{p-1}}{B(p, q) (1+x)^{p+q}} , \quad m = \frac{p}{q-1} , \]

\[ v = \frac{p(p+q-1)}{(q-1)^2(q-2)} , \quad \text{mode} = \frac{p-1}{q+1} , \quad \text{HM} = \frac{p-1}{q} \]

if \( r < q \), \( a_r = \frac{p(p+1)\ldots(p+r-1)}{(q-1)(q-2)\ldots(q-r)} \)

\[ D(1/x) = \text{No. 6.3(q,p)}, \quad D\left(\frac{1}{1+x}\right) = \text{Beta} \]

\( p = q, \ x \geq 1 \) \[ \text{[d]22:418} \]

Called Fisher's F \[ \text{[w]1:9} \]

C(x) \[ \text{[p]7:102} \]

6.4 HOTELLING

\[ D(x) = \frac{2}{B\left[\frac{1}{2}(p-q), \frac{1}{2}q\right]} \left(\frac{x^{q-1}}{[1 + \frac{x^2}{p-1}]^{\frac{1}{2}p}}\right) \]

For \( q=1 \), this is positive half of Student distribution, hence called generalized Student \[ \text{[1]409, [d]2:375} \]

\[ D(x^2) \]

Mellin transform \[ \text{[c]32:70} \]

Percentage points, relation with Chi-square \[ \text{[d]27:1091} \]

6.5 PARETO

\[ D(x) = \frac{p}{q} \left( \frac{q}{x} \right)^{p+1} \]

\[ \alpha_1 = \frac{p}{p-1} q, \quad \xi = 2^{1/p_q} \]

More generally

Testing, location and dispersion

As Type XI

Ranking

Double Pareto

Kendall and Buckland, 

See also: [l]19:174, [h']1:149, [g]48:537, [i]8:76, [t]3:77,

6.6 KENDALL

\[ D(x) = \frac{re^{-(x-r)/\alpha(x-r)^{x-1}}}{\alpha^x \Gamma(x+1)} \]

\[ 0 < r < x < \infty, \quad 0 \leq \alpha \leq 1, \]

\[ m = \frac{r}{1-\alpha}, \quad v = \frac{r \alpha^2}{(1-\alpha)^3} \]
6.7  INVERSE GAUSSIAN

\[ D(x) = \exp \left[ -\lambda (x-\mu)^2 / 2\mu^2 x \right] \left[ \lambda / 2\pi x^3 \right]^{\frac{1}{2}} \]

Introduction, moments, estimation

[\text{d}]28:362,696

VII. DISTRIBUTIONS ON \((-\infty, \infty)\)

7.1  TYPE VII

\[ D(x) = \frac{(1+x^2/a^2)^{-m}}{aB(\frac{1}{2},m-\frac{1}{2})} , \ m > \frac{1}{2} \]

Estimation

[\text{c}]36:412, [\text{t}]4:35

See also:  [\text{c}]15:401, [\text{c}]36:412,167

7.2  STUDENT \((r)\)

\[ D(x) = \frac{\Gamma \left[ \frac{1}{2}(r+1) \right]}{\Gamma \left( \frac{1}{2} r \right) (\pi r)} \left( 1+x^2/r \right)^{-\frac{1}{2} (r+1)} \]

Introduction, properties, examples

[\text{15}]388

\[ v = \frac{r}{r-2} \]

[\text{1}]239
Type VII with \( m = \frac{1}{2} (r+1) \), \( \alpha^2 = r \\
"
"t" distribution

\[
\alpha_{2k} = \frac{1.3 \cdot \ldots \cdot (2k-1)r^k}{(r-2)(r-4)\cdot \ldots \cdot (r-2k)}
\]

Original paper in which this distribution was discovered

Ch(x), refs for \( r=3 \)

Ch(x)

Distribution of the ratio of a Chi-square variable to a normal variable

In bivariate normal samples

\[
D\left( \frac{\sigma_1 (n-1)^{\frac{1}{2}}}{\sigma_2 (1-\rho^2)^{\frac{1}{2}}} (b-\beta) \right) = \text{Student} (n-1)
\]

\[
D\left( \frac{s_1 (n-2)^{\frac{1}{2}}}{s_2 (1-\rho^2)^{\frac{1}{2}}} (b-\beta) \right) = \text{Student} (n-2)
\]

if \( \rho=0 \)

\[
D\left[ (n-2)^{\frac{1}{2}} \frac{r}{(1-r^2)^{\frac{1}{2}}} \right] = \text{Student} (n-2)
\]

C(x)

D(x^2) = Snedecor
Transform to Type II

\[ D(\bar{x}) \]

As \( r \to \infty \), Student \( \to N(0,1) \)


Approximations


\[ D(\log x) \]

[1]34:176

Two Student variables


Used to test partial correlation

[10]256

\( \sim \) significance levels

[1]14:60

Generalizations


7.3 NORMAL REGRESSION SLOPE

\[
D(x) = \frac{[\sigma_1^2 \sigma_2^2 (1-\rho^2)]^{\frac{1}{2}} (n-1) \Gamma \left( \frac{1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{1}{2} (n-1) \right) \sigma_1^{n-2} (\sigma_2^2 - 2\rho \sigma_1 \sigma_2 x + \sigma_1^2 x^2)^{\frac{1}{2}} n} [1]\402
\]

\[
v = \frac{1}{n-3} \frac{\sigma_2^2}{\sigma_1^2} (1-\rho^2) [2]\365, [e]\1:432
\]

For \( \rho = 0 \), can use Student distribution to test \( x \)

Stratified sampling [1]\36:96

7.4 CAUCHY \((p,q)\)

\[
D(x) = \frac{1}{\pi} \frac{p}{p^2 + (x-q)^2} [1]\246, [5]\35,
[18]
\]

\[
Ch(x) = \exp \left[ qit - p |t| \right] [5]\60
\]

\( q \) is the mode and median

there is no mean, or any moment

Quartiles are \( q + p \) [5]\58

\( \bar{x} \) is not a consistent estimate of \( q \) [5]\105

\( \bar{x} \) is a "density unbiased" estimate of \( q \) [d]\25:400
There are no sufficient estimators

C.-R. (q), C.-R. (p), C.-R. (p/q)

Dist. of $t$ and $F$ statistics

Mean and variance of $\frac{1}{2} (x+y)$, where $x$ and $y$ are respectively the kth values from the top and bottom of the sample

$\frac{1}{2} (x+y)$ is not a consistent estimate of $q$

See also: [d]17:2, [d]21:133

7.5 CAUCHY ($\alpha, q$)

$$D(x) = \frac{1}{\pi} \frac{1}{1 + (x-q)^2}$$

$q$ incorrectly asserted to be the mean $E(x)$, $E(x^2)$, $D(x+y)$

C.-R. (q) = $2/n$

$$\text{Var} (\xi) \approx \frac{2}{4n}$$

$D(\bar{x}) = D(x)$, hence $\bar{x}$ not consistent

$$D(\xi)$$

MLE $\neq$ minimax
MLE is solution of $\sum \frac{2(x_i - q)}{1 + (x_i - q)^2} = 0$ [3]24, [p]7:169

Gauging [e]15:194


There is no UMVUE [3]51

Information and estimation [e]8:315

Loss of information [3]32

Testing $m = m_0$ [d]9:83, [d]13:65

Cauchy's added MR17:863


7.6 CAUCHY $(p, 0)$

$D(x) = \frac{P}{\pi} \frac{1}{p^2 + x^2}$, $Ch(x) = \exp -p|t|$ [9]275

Reproductive property [9]276

Information and estimation [e]8:316

Completeness [e]10:314

$D(\bar{x}) = D(x)$ [n]10-3:91
Truncated to \((-p, p)\)

7.7 CAUCHY \((1, 0)\)

\[
D(x) = \frac{1}{\pi} \frac{1}{1 + x^2}
\]

\[
Ch(x) = \exp(-|t|)
\]

Sample median

\[D(\bar{x}) = D(x)\]

From an example

Moments

As distribution of ratio of two normal variables

\[
C(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}x
\]

Censored sample

Wrapped-up Cauchy

7.8 LAPLACE $(m,v)$

\[ D(x) = (2v)^{-\frac{1}{2}} \exp \left\{ -\frac{|x - m|}{\frac{1}{2} (2v)^{\frac{1}{2}}} \right\} \]

\[ Ch(x) = \exp (mit) \cdot (1 + \frac{3}{8} vt^2)^{-1} \]

Mean and variance of average of greatest and least sample values

A priori distributions of $m, v$

"Best" estimates of $m$ and $\sigma$ are $\xi$ and $1/n\Sigma |x_i - \xi|$, but $\xi$ not sufficient

Distribution of smallest sample value

\[ D(\xi) \]

Quasi-range

"Double exponential" distribution

Convolution, estimation, generalization

[18]
[5]62
[1]375
MR9:294
[5]147-8
[g]43:408
[d]26:115
[d]28:179
Proc. Roy. Soc. Lond. (Series A)
154:124, [v]7:164
LAPLACE \((0,v)\), also called "Poisson's first [d]6:102 law of error", \(D(|x|), D(|x| - |y|)\), 
\(D(\log |x| - \log |y|), D(\Sigma x^2)\), 
\(D(GM), D(HM)\)

Laplace if \(D(x)D(y) = \phi(|x| + |y|)\) \(\text{MR10:125}\)

See also: [n]10-3:80

LAPLACE \((m,1)\), MLE \(\text{[3]45}\)

LAPLACE \((0,1)\), \(\text{[8]120, [9]279, [1]100}\)

Sample Median \(\alpha_{2k} = (2k)!\) \(\text{[d]26:599}\)

See also: [d]22:425, [17]No. 38, "Laplace (1774), "Memoire sur la Probabilité des causes par les évènemens"

7.9 FISHER \((p,q)\)

\[ D(x) = \frac{2^{\frac{1}{2}p} q^{\frac{1}{2}q} e^{px}}{B(\frac{1}{2}p, \frac{1}{2}q)(q+pe^{2x})^{\frac{1}{2}(p+q)}} \]  

'\(z'\) distribution \(\text{[1]3:355}\)

C(x) as a power series \(\text{[1]3:360}\)

Cumulants \(\text{MR9:48,735}\)

Moments, cumulants \(\text{[y]5:317}\)
Transform to Snedecor by

\[ y = e^{2x}; \quad x = \frac{1}{2} \log \frac{q\chi_1^2}{p\chi_2^2} \]

Various properties

\[ \text{Ch}(x) = \left(\frac{q}{p}\right)^{\frac{1}{2}it} \frac{\Gamma\left(\frac{1}{2}(q-it)\right)\Gamma\left(\frac{1}{2}(p+it)\right)}{\Gamma\left(\frac{1}{2}p\right)\Gamma\left(\frac{1}{2}q\right)} \]

\[ \sim \mathcal{C}(x) \]

Obtained from two Chi-square variables

Approximate significance levels, transformation to Normal

Normal limit

Am. Math. Monthly, 50:100

Generalization

Non-central

7.10 TYPE IV

\[ D(x) = C (1 + x^2/a^2)^{-m} \exp \left( -p \tan^{-1} \frac{x}{a} \right) \]

\( (x - h) \) Type IV

Various constants with an example

Roots of quadratic in Pearson equation complex

\[ \alpha_r = \frac{a}{2m-r-1} \left[ (r-1)a \alpha_{r-2} - p\alpha_{r-1} \right] \]

\[ \alpha_1 = 1 \]


VIII. MISCELLANEOUS UNIVARIATE

8.1 PEARSON

\[ \frac{1}{D(x)} \frac{dD(x)}{dx} = \frac{a_0 + a_1x}{b_0 + b_1x + b_2x^2}, \]

types listed with associated parameters

Differential equation for Ch(x)

Ch(x)

Original paper:

Philosophical Trans., 1895.
First seven types treated in [11]

New classification

\[ D(\bar{x}) \]

Truncation, estimation

Bivariate

Flexes equidistant from mode, etc.

Bivariate generalization

Orthogonal Polynomials

Generalizations

Log Pearson distributions

Romanovsky's generalization

See also: [d]8:18, [d]8:206, [d]20:461, [e]6:415, [c]7:127,
8.2 BESSEL FUNCTION


Wilk's distribution of dispersion determinant, etc. [e]3:26

\[ D(x) \]

Distribution of vector correlation [c]28:353

\[ D(x) , Ch(x) , \text{Moments in a special case} \]

Bivariate Gamma distribution [e]5:140

Distribution of the range [d]18:384, [d]21:133

Marginal total of Elfving's distribution [c]36:142

8.3 VARIANCE RATIO

\[ D(x) = \frac{2(1-p^2) \frac{1}{2}(n-1)x_n^2(1-4p^2x^2(1+x^2))^{-\frac{1}{2}}}{B[\frac{1}{2}(n-1), \frac{1}{2}(n-1)](1+x^2)^{n-1}} \]

\[ B[\frac{1}{2}(n-1), \frac{1}{2}(n-1)] \]

For \( p = 0 \), \( D[(2n-3)x^2] = \) Hotelling \((2n-2,n-1)\)


8.4 KULLBACH

\[ D(x) = \frac{nx^{np-1}}{\Gamma(n)[\Gamma(p)]^2} \sum_{j=0}^{\infty} (-1)^{n+j+1} \left( \frac{d^{n-1}}{dt^{n-1}} \frac{x^nt}{\Gamma(t+1)} \right) t^j \]

Distribution of GM from Type III \((1,p)\) \[2\]251

8.5 NONCENTRAL STUDENT

\[ D(x) \]

\[ [c]31:362, [18]1-162, \]

\[ [v]4:173,307 \]

Multivariate

\[ [v]4:331 \]

Application

\[ [c]43:219 \]

8.6 CONTINUOUS LEXIAN

\[ D(x) = \int_0^1 f(p) \left( \frac{n}{x} \right) p^x (1-p)^{n-x} \, dp; \text{ parameters; } [i]31:1, [i]34:197 \]

if \( f(p) \) is Beta, \( D(x) \) is hypergeometric

8.7 NONCENTRAL SNEDECOR

\[ D(x) \quad [c]36:220, [18]1-163, [e]15:321 \]

See also: [c]38:112, [i]36(Suppl.):33, [r]3:33

8.8 FISHER'S LOGARITHMIC SERIES

\[ D(x) = \frac{k^x}{p \log \frac{1}{1-k}}, \quad x=1,2,\ldots \quad [c]35:6, \]

( asserted to be multivariate ) Cf. [c]37:358

negative binomial \((1,-m), \text{No.3.5}\)

8.9 RANK VARIATE

\[ D(x) = \frac{N}{\sigma} \frac{n!}{(q-1)!(n-q)!} \left[ \exp - \frac{x}{\sigma} (n-q+1) \right] (1-e^{-x/\sigma})^{q-1} \quad [c]24:231, 239, [c]25:79 \]

In special case called Yule's distribution, [c]42:23,425, MLE,

A Distribution of Type III median [p]7:153

8.10 GENERALIZED PARETO

\[ D(x) = ax^{-n} \frac{1}{e^{bx}/x - 1} \quad [1]6:184 \]

\( b=1, \quad n=5, \quad a=15/\pi^4, \) Planck's radiation [17]No. 56

function
8.11 GHOSH

\[ D(x) = \frac{2}{F([k] + 1)} x^k e^{-x^2}, \text{ where } [k] \text{ is} \]

the largest integer \( \leq k. \)

Furnishes counterexample to theorem \( [e]8:330 \)
on similar regions

8.12 RECTANGULAR GEOMETRIC MEAN

\[ D(x) = \frac{n^n x^{n-1}}{a^n \Gamma(n)} (\log a/x)^{n-1} \]


\[ [w]1:73 \]

8.13 CAUCHY MEDIAN

\[ D(x) = \frac{(2m + 1)!}{(m!)^2 \pi^{2m+1}} \left( \frac{\pi^2}{4} - [\tan^{-1}(x-k)]^2 \right)^m \frac{1}{1+(x-k)^2} \]

\[ [3]46 \]

8.14 SPEARMAN'S RANK CORRELATION

\[ D(x) \]

\[ [c]30:256, [c]34:183, \]

\[ [c]38:131 \]

\( \sim D(x) = \text{Type II, moments} \)

\[ [c]40:409 \]

8.15 CIRCULAR NORMAL CORRELATION

\[ D(x) = \frac{n(n-1) e^{-x^2/\nu}}{(1-e^{-x^2/\nu})^{n-2}} x \]

\[ [c]39:139, [g]48:496 \]
8.16 KOOPMAN

\[ D(x) = Q(k) R(x) \exp k H(x), \text{ most general distribution admitting a sufficient estimate} \]
\[ \text{of } k \]

\[ \text{Trans Am. Math. Soc. 39:399,} \]
\[ [p]7:162 \]

8.17 VON MISES

\[ f(x) = C \exp k \cos (x-a) \]

\[ \text{Physikal. Zeitschr} \]
\[ 19:490 \]
\[ [g]48:131, [g]49:53,268, \]
\[ [c]43:344, [d]26:233, \]
\[ [c]43:344 \]

8.18

Family of distributions having all moments equal

\[ D(x) = \frac{1}{6} e^{-\frac{x^2}{6}} (1-p \sin \frac{x}{4}) 0 \leq x \leq \infty, 0 \leq p \leq 1 \]
\[ \alpha_k = \frac{1}{6} (4k+3)! \]

8.19

Family of distributions having all moments equal

\[ D(x) = e^{-\frac{x}{6}} \pi^{-\frac{1}{2}} x^{-\log x} [1-p \sin (2\pi \log x)] \]
\[ \alpha_k = \exp \left[ \frac{1}{4} k(k+2) \right] \]
8.20

"Non-null $t^2$ distribution", involving a hyper- \[14\] geometric function

8.21

$$D(x) = \frac{\text{sech}^{k-2} x}{B(1,k-2)}$$, the distribution of \[b\] in samples from a bivariate normal distribution with zero means and zero correlation

$\text{tanh}^{-1} r$ in samples from a bivariate normal distribution with zero means and zero correlation

$\text{Ch}(x)$, also special cases and refs \[17\] Nos. 53-5

$D(3x_i)$ for $k=3$ \[z\] 219

8.22

$$D(x) = \frac{1}{2a} \text{sech}^2 \left( \frac{x-m}{a} \right)$$, connected with \[e\] lognormal

Called 'logistic', $\text{Ch}(x) = \pi x \text{sech} \pi x$ when $m=0$, $a=2$

Distribution of $t$ and $F$ statistics \[v\] 163

8.23

Distribution of the correlation ratio, involving series \[a\] 121, \[c\] 441
Various distributions of the form \( \exp(-\text{quartic polynomial}) \)

Giving an example where no minimum variance estimator exists

\[
\text{MLE}
\]

\[
k(1 + x^2)^{-m}
\]

\[
\chi(x)
\]

\[
D(x) = b \sin 2(a + bx)
\]

\[
\text{Normal multiplied by an eighth degree polynomial}
\]

\[
D(x) = \left(\frac{1}{4} h + \frac{1}{6} h^2 |x| \right) e^{-h|x|}, D(\bar{x})
\]

For \( h=1 \), \( \chi(x) = \left(\frac{1}{1+x^2}\right)^2 \)

- 138 -

8.24

[e]12:43
[c]31:188, [a]98:114
[2]52
[2]67
[b]9:61
[a]106:361
[n]10-3:90
Miscellaneous distributions given in terms of \( C(x) \)

\[ D(x) = \frac{1}{2} k (1 + |x|)^{-k-1} \]

\( k \) negative

\( k = 1 + p \)

\[ D(x) = k \exp(-ax^p)[1 + q \sin(bx^p)] \]

Various distributions formed from rational functions of \( x \), rational functions multiplied by \( e^{-1/x} \), \( e^{-x} \) and \( \exp(-\tan^{-1}x) \)

\[ D(x) = (e^2 + |x|)^{-1}[\log(e^2 + |x|)]^2, \]

having a pathologically long tail


[1]225 No. 2,

[c]33:126

[c]36:93, [17] No. 17, generalization No. 18

[d]1:137

[d]17:11
8.34 WEIBULL

\[ D(x) = ab x^{b-1} \exp(-ax^b), \]
\[ x \geq 0, \ a > 0, \ b > 1 \]

Moments of order statistics

8.35

\[ D(x) = k \sin^m x \cos^n x \]
\[ n=0, \ \text{value of } k, \ \text{Ch(cos } x) \]

8.36

\[ D(x) = 2h \pi^{-1}(1 + h^2x^2)^{-2} \]

8.37

\[ D(x) = 2h \pi^{-1}(e^{hx} + e^{-hx})^{-1}, \]
\[ D(\bar{x}) \]

'PERKS'

8.38

\[ D(x) = (2 \pi)^{-\frac{1}{2}} \frac{x}{k} \left[ \exp(-\frac{1}{2}(x-k)^2) - \exp(-\frac{1}{2}(x+k)^2) \right] \]

8.39

Four distributions formed by multiplying the normal distribution by a polynomial, used to illustrate kurtosis

[g]40:259
8.40 EXTREME VALUE

D(x) = a exp [-a(x-m)] exp [-exp(-a(x-m))] \[d]17:299

Gumbel Distribution
Determination of constants

Estimation, MLE

Bias

Moments called "Fisher-Tippett Type I", Ch(x), cumulants,

For m=0, a=1, Ch(x) = \Gamma (1-ix),

References
Connection with No. 2.3

Special cases

\~D(x)

With slight modification

\~D(x) is distribution of log survival time

Application,
8.41

\[ D(x) = \frac{2}{\pi} \frac{x^2}{(1+x^2)^2}, \quad Ch(x) = e^{-|x|}(1-|x|), \quad [n]10:75, [17]No. 30 \]

8.42

\( D(x+y) \), where \( x \) and \( y \) obey various trivial distributions [d]5:16

8.43

\[ D(x) = C x^{-1}(1 + p/x)^{-2} \] [d]6:106

8.44

\[ D(x) = \frac{a+1}{2a} (1-|x|^a), \quad -1<x<1 \] [17]No. 15

8.45

\[ D(x) = \left( \frac{\lambda}{1 + \frac{c}{\lambda}} \right)^n e^{-\lambda x} x^{a-1} \sum_{j=0}^{n} \binom{n}{j} \frac{c^j x^b j}{\Gamma(bj+a)}, \quad [17]No. 36 \]

\( 0<x<\infty, \lambda>0, c\geq0, b\geq0, a>0, n=1,2,..., \)

Ch(x), References

8.46

\[ D(x) = (x-k)x^n e^{-ax} \] [g]42:572

8.47 STEVENS-FISHER

\[ D(x) = \sum \binom{n}{j} (-1)^j (1-jx)^{n-1} \] [k]9:315, [k]10:14, [4]203

Compare No. 5.16 and No. 8.70
8.48
\[ D(x) = C \exp \left( ax^b - cx \right), \quad 0 < x < 1 \]  

8.49
\[ D(x) = C \exp \left[ -a(b - x)^c \right] \]  

8.50
\[ D(x) = \frac{a}{2 [\Gamma(1/a)]} e^{-|x|^a}, \quad -\infty < x < \infty, \quad a > 0 \]  

8.51
Distribution of non-normal correlation  

8.52
\[ D(x) = C \left[ \frac{p - x^2}{q + x^2} \right]^m \frac{1}{p+q}, \quad -\sqrt{p} < x < \sqrt{p} \]  

Value of C, references

Hansmann's distributions, obtained from a generalized Pearson differential equation  

8.53
\[ D(x) = \left( \frac{k}{x} \right) \exp \left[ -ax - \frac{b}{x} \right] \]  

Called "Type Harmonique"  

C.R. Acad Sci. Paris 213:634
8.54

\[ D(x) = \frac{a^2}{\ln x} e^{2a\sqrt{b}} \frac{1}{x^y} e^{-bx} - \frac{a^2}{x} \]

\[ \text{Ch}(x) = \exp [2a(\sqrt{b} - \sqrt{b-1}t)] \]

8.55

\[ D(x) = \frac{a^p |r|}{r^p} x^{p-1} e^{-ax^r}, 0 < x < \infty, a>0, \]

\[ p>0 \]

8.56

\[ D(x) = (a - 1)^2 \frac{\log x}{x^a}, 1 < x < \infty, 1<a \]

8.57

\[ D(x) = -[L(b^{1-a}) x^a \log x]^{-1}, b < x < \infty, \]

\[ b > 1, b > 1, \text{ and} \]

\[ L(u) = \int_0^u \frac{dv}{\log v}, u \geq 0 \]

8.58

\[ D(x) = -(a + 1)^2 x^a \log x, 0 < x < 1, a>-1 \]
A generalization of the hypergeometric distribution based on the Whittaker function

\[ x^{m+\frac{1}{2}} e^{-\frac{1}{2} x} \cdot _1 F_1(m+\frac{1}{2} - k, 2m+x; x) \]

\[ D(x), \text{Ch}(x), \text{references} \]

\[ D(x) = C \frac{e^{-(x^2/2a^2)}}{b^2 + x^2}, \text{moments, Ch}(x), \quad [v]2:293, [v]3:139 \]

Limiting cases (Cauchy, Normal)

\[ D(x) = \frac{1}{\pi} \frac{1 - \cos x}{x^2} \quad [v]2:328 \]

\[ D(x) \text{ where } A + Bf(x) \text{ is (a) Normal, or } \quad [c]36:149, [v]5:283 \]

(b) Laplace and \( f(x) \) is (i) \( \log x \),
(ii) \( \log \frac{x}{1-x} \), (iii) \( \text{arcsinh} x \)

\[ D(x) = \begin{cases} 
1 - e^{-\mu NT}, & x = N \\
(1 - e^{-\mu T}) \exp[-\mu T(x-1)], & x = N+1, N+2, \ldots
\end{cases} \]

\[ N = 1, 2, \ldots, \mu > 0, T > 0 \]
Garwood's distribution of length of gaps in traffic

8.65 MATCHING

\[ D(x) = \frac{1}{x^x}[1 - \frac{1}{1^x} + \frac{1}{2^x} - \ldots + \frac{(-1)^{n-x}}{(n-x)^x}] \]

Generalization

8.66

Cigarette card distribution


8.67

Cubic polynomial over a finite range, estimation

\[ D(x) = \begin{cases} \frac{mn}{m+n} x^{m-1}, & 0 < x < 1 \\ \frac{mn}{m+n} x^{-n-1}, & 1 \leq x < \infty \end{cases} \]

Moments, etc.

\[ [g]50:196, [d]26:505,591 \]

\[ [g]50:1137 \]
8.69

\[ \begin{cases} \frac{mn}{(m-n)-1} x^{n-1} (1 - x^{m-n}), & m \neq n \\ n^2 x^{n-1} \log(1/x), & m = n \end{cases} \]

Moments

8.70

\[ D(x) = \frac{n^n}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( x - \frac{j}{n} \right)^{n-1}, \quad 0 \leq x \leq 1 \]

Compare No. 5.16 and 8.47

8.71

\[ D(x) = \frac{(r-1)! \ (r-x)}{x! \ r^{r-x}}, \quad x = 0, 1, \ldots, (r-1) \]

Moments, approximations

8.72 ARFWEDSON

\[ D(x) = \sum_{j=0}^{\infty} (x - j)^n (-1)^j \binom{x}{j} \]

8.73 STEVENS - CRAIG

\[ D(x) = C n(x) \sigma_s^x, \quad \sigma_s^x \text{ being Stirling's number of second kind} \]

Generalization

8.74

\[ D(x) = \frac{1}{x} + x^4, \quad -1 < x < 1 \]
8.75 ISING–STEVENS

\[ D(x) = \frac{(m-1)(m+1)}{x(x-1)(m+n)} \]


8.76

\[ D(x) = \frac{1}{\pi \sqrt{m^2 - x^2}} \]

Am. of Math. 27:18

8.77

\[ D(x) = \frac{x^{-2/3}}{3 \sqrt[3]{2\pi}} \cdot e^{-\frac{1}{2} (x^{1/3} - b)^2} \]

Ann. of Math. 27:19

8.78 NEGATIVE HYPERGEOMETRIC

\[ D(x) = \binom{n}{x} \frac{B(p+x, q+n-x)}{B(p, q)} \]


Obtained by assuming binomial probability to obey Beta

8.79

\[ D(x) = k(1 + x)^{-2} \text{ over various ranges} \]

[d]22:425

8.80

\[ D(x) = (n-1)(1 - \frac{1}{x})^{n-2} \frac{1}{x^2}, \ x \geq 1 \]

[d]22:425
8.81

\[ D(x) = \frac{-\log x^2}{\pi^2(1-x^2)}, \quad -\infty < x < \infty \]  
\[ [d]22:425 \]

8.82

\[ D(x) = \frac{a-(a-bx)^2/2cx}{\sqrt{2\pi}cx^3} \]  
\[ \text{Called 'inverse Gaussian'} \]

\[ \text{Written} \]

\[ D(x) = \exp\left(\frac{-c(x-m)^2}{2m^2x}\right) - \frac{c}{2x^3} \]  
\[ [d]28:362,696 \]

\[ D(x) = k x^n e^{-x^2+ax} \]  
\[ \text{called "Halphen",} \]
\[ \text{Publ. Inst. Statist. Univ. Paris 4:38} \]

8.83

\[ D(x) = a(\alpha,\beta)/ [\exp(\alpha^2x^2) - \beta] \]  
\[ \text{MR16:381} \]

8.84

\[ D(x) = a \exp [-k^2 \log^2 \rho] \]  
\[ \text{Z10:313} \]

where

\[ \rho = \frac{(x-x_0)(x_2-x_1)}{(x_2-x)(x_1-x_0)} \]
\[ \text{Ch}(x) = \frac{1}{\cosh t}, \frac{t}{\sinh t}, \frac{1}{\cosh^2 t} \]

8.86

\[ D(x) = \frac{2\lambda (\lambda+1)}{(x+\lambda-1)(x+\lambda)(x+\lambda+1)} \]

\( \lambda > 0, \ x=1,2,\ldots, \ m=1+\lambda, \ v=\infty \)

8.87

\[ D(x) = \frac{N!n!}{\prod_{i=0}^{n} x_i! (i!)^{x_i}} \]

8.88

\[ D(x) = \frac{c e^{cx}}{(1+e^{cx})^2} \]

Hyperbolic Error distribution
A discrete distribution from an urn model

IX. MISCELLANEOUS BIVARIATE

9.1 CAUCHY BIVARIATE

\[ D(x,y) = \frac{1}{\pi} \frac{1}{(1+x^2+y^2)^2} \]

\[ D(x,y) = \frac{1}{2\pi} (1+x^2+y^2)^{-3/2} \]

9.2 STUDENT BIVARIATE

\[ D(x,y) = (2\pi)^{-1}(1-r^2)^{-\frac{1}{2}} \left(1 + \frac{x^2-2rxy+y^2}{n(1-r^2)}\right)^{-\frac{1}{2}(n+2)} \]

Tables

If \( x,y \) independent
9.3 POISSON BIVARIATE

Discussion

Special case obtained from binomial

9.4

Lognormal bivariate

9.5

Normal-lognormal

9.6 BINOMIAL BIVARIATE

\[ Ch(x,y) = (a \exp (ix+iy) + be^{ix} + ce^{iy} + d)^n \]

If \( a=0 \), \( D(x,y) = \frac{k!}{x! y! (k-x-y)!} p^x q^y (1-p-q)^{k-x-y} \), etc.


9.7 GAMMA BIVARIATE

\[ MGF(x,y) = [ (1+x)(1-y) - xyr^2 ]^{-p} \]
9.8 GAMA–NORMAL

\[ \text{MGF}(x,y) = (1-x)^{-\frac{1}{2}} \exp\left\{ \frac{1}{2} y^2 (1 + \frac{x^2}{1-x}) \right\} \]

9.9 HYPERGEOMETRIC BIVARIATE

\[ D(x,y) = \frac{(a \binom{b}{x} \frac{c}{y})}{(a+b+c \binom{k}{k-x-y})} , \text{ various properties} \]

Moments

\[ \text{Ganita} 5:97, \text{Koninkl. Nederl. Akad (A)} 60:124 \]

9.10 NEGATIVE BINOMIAL BIVARIATE

Various properties

\[ D(x,y) = \frac{m^p}{(p+2m)^p} \frac{1}{\Gamma(p)} \frac{\Gamma(x+y+p)}{\Gamma(x+1)\Gamma(y+1)} \left[ \frac{m^r}{p+2m} \right] x+y \]

\[ \text{correlation} = \frac{m}{p+m} , \text{regression etc.} \]

Polya–Eggenberger

\[ \text{MR11:605} \]

9.11 ELFVING

\[ D(x,y) = \frac{1}{2} x \exp (-x \cosh y) , \text{ connected} \]

with \( \mathcal{N}D(\text{range}) \)

9.12

\[ D(x,y) = C e^{-ax-by} (1-x+y)^p (1+x-y)^q \]

Rhodes surface

\[ \text{[c]22:134, [c]41:550} \]
\[ D(x,y) = (1+x/a)^m(1+y/b)^n(1 - \frac{x+y}{c})^q, \]

Filon-Isserlis surface  

\[ D(x,y) = (xy)^k(x-y)(1-x)(1-y)^n \]

\[ D(x,y) = \frac{x^{n-2}(1-y^2)^{\frac{1}{2}}(n-4)}{(1-2rxy+y^2)^{n-1}} \]

\[ D(x,y) = n^{-2} \frac{n!}{[(k-1)!]^2(n-2k)!} (x/n)^{k-1}(y/n)^{k-1}(1-x/n-y/n)^{n-2k} \]

\[ x > 0, \ y > 0, \ x+y<n, \ 2k < n \]

As \( n \to \infty, \ x,y \to \) independent Type III(1,k)

Special case, \( k=0 \)

\[ D(x,y) = C \times \frac{1}{\pi}(n-3)(y-x)^{-\frac{1}{2}} \exp -\frac{1}{2} (k-1)y, \]

\[ D(x/y) = \text{Beta} \]

Uniform bivariate, triangular bivariate
Gram-Charlier bivariate

The fifteen constant surface, (quartic polynomial). $e^{-Q}$

Pearson's Student-like surfaces

$D(x,y) = x + y$, $D(x + y)$

Normal-negative binomial

Edgeworth surface

Raleigh bivariate:

Discussion of 'possible' bivariate distributions, Narumi's system

Generalization
9.27 VON MISES–FISHER DISTRIBUTION

Generalization of No. 8.17

\[ D(x,y) = k \frac{e^{-\frac{1}{2}(ax^2+2bxy+cy^2)}}{m^2 + (ax^2+2bxy+cy^2)} \]

9.28

Beta Bivariate

\[ D(x,y) = k [1-a^2x^2 - b^2y^2 + 2abrxy]^n, \]
\[ n+1>0, \ r^2<1, \]

9.30

\[ D(x,y) = k \exp [-Q(x,y)] \cdot [h^2+Q(x,y)]^n \]

9.32

Defined over \((0,0)(0,1)(1,0)\) from urn model

9.33

'Correlation by common factor' surface
9.34
Johnson's system; ten surfaces obtained by Translation

9.35
Nine surfaces with Pearson or Bessel marginal distributions

9.36
\[ D(x,y) = C[(x-1)!(h-x)!(y-1)!(k-y)!]^{-1} \]

\[ D(x,y) = \frac{1}{4} (1+kxy), \quad |k| \leq 1, -1 \leq x, y \leq 1 \]

9.37
Type III bivariate, with discussion and calculation of \( D(r) \)

9.38
[\text{e}] 7:159

X. MISCELLANEOUS MULTIVARIATE

10.1 WISHART TRIVARIATE

\[ D(x,y,z) = \frac{n^{n-1}(xy-z^2)^{\frac{1}{2}(n-4)}}{4 \pi \Gamma(n-2) M^{\frac{1}{2}(n-1)}} \exp \left( -\frac{n}{2M} (v_2 x - 2\mu z + v_1 y) \right), \]

\[ \mu = \rho \sigma_1 \sigma_2, \quad M = v_1 v_2 (1-\rho^2) \]

[\text{[1]} 397, \text{[3]} 330, \text{[4]} 226]
\[ 
\text{Ch}(x,y,z) = \left( \frac{A}{A^*} \right)^{\frac{1}{2}} (n-1) 
\]

where \( A = \begin{vmatrix} \frac{\nu_2}{2M} & \frac{\mu_n}{2M} \\ \frac{\mu_n}{2M} & \frac{v_{1n}}{2M} \end{vmatrix} \) and \( A^* = \begin{vmatrix} \frac{\nu_2}{2M} & -is \\ -is & \frac{\nu_2}{2M} + it \end{vmatrix} \)

\begin{align*}
\text{Moments and cumulants} & \quad [3]\text{334} \\
\text{As distribution of normal bivariate variance-covariance} & \quad [c]\text{10:510}, [c]\text{21:164}, [c]\text{27:230} \\
\end{align*}

\textbf{10.2 WISHART MULTIVARIATE}

\[ 
D(x_{ij}) = K_{kn} A_{n-k-2} \exp(- \sum a_{ij} x_{ij}), 
\]

where \( X = |x_{ij}|, \quad A = |a_{ij}| \) and

\[ 
K_{kn} = \pi^{\frac{k(k-1)}{2}} (n_{-1})^{\frac{k}{2}} \cdots (n_{-k})^{\frac{k}{2}}\]

\[ 
\text{Ch}(x_{ij}) = \left( \frac{A}{A^*} \right)^{\frac{1}{2}} (n-1) 
\]

where \( A^* = |a_{ij} - i\epsilon_{ij}^t i| \) and \( \epsilon_{ij} = \begin{cases} 1, & i=j \\ \frac{1}{2}, & i\neq j \end{cases} \)
Reproductive property

Various properties

Non-central Wishart


10.3 MULTINOMIAL

\[ D(x_1, \ldots, x_k) = \frac{n!}{\Pi (x_1^i)} \Pi (p_i x_1) \]

\[ \text{MGF}(x_1, \ldots, x_k) = (p_1 e^{x_1} + \ldots + p_k e^{x_k})^n \]

\[ E(x_i) = np_i, \quad \text{Var}(x_i) = np_i (1-p_i) \]

Moments

Introductory article with applications

PGF

Chi-square test

Information and estimation
MLE

Distinguishing between two multinomials, asymptotic form

Trivariate

Bivariate multinomial


10.4 TYPE X MULTIVARIATE

\[ D(x_1, \ldots, x_n) = C \, e^{-x/b} \text{ where } x = \sum x_i^2 \]

10.5

Gamma multivariate

See also: [w]9:143

10.6 STUDENT MULTIVARIATE

\[ D(x_1, \ldots, x_p) = \frac{A^{\frac{1}{2}} \Gamma\left[ \frac{1}{2} (n+p) \right]}{(n\pi)^{\frac{1}{2}} p^{\frac{1}{2}} \Gamma\left( \frac{1}{2} n \right)} \left[ 1 + \frac{1}{n} \sum a_{ij} x_i x_j \right]^{-\frac{1}{2} (n+p)} \]

Student for \( p=1 \)

See also: [c]41:153, MR16:602

\[ \text{Student for } p=1 \]
10.7 CAUCHY MULTIVARIATE

\[ D(x_1, \ldots, x_n) = C(a^2 + x^2)^{-\frac{1}{2}}(s^2)^{-\frac{1}{2}}, \text{ where } x^2 = \sum x_i^2 \]  
\[ D(x_1, \ldots, x_n) = C_n(1 + \sum_{j=1}^{n} x_j^2)^{-\left(\frac{n+1}{2}\right)} \]

10.8 SPHERICAL

\[ D(x_1, \ldots, x_n) = (2\pi)^{-\frac{1}{2} n} r^{-\frac{1}{2} n+1} \int_0^\infty \frac{\rho^{\frac{1}{2} n}}{J_{\frac{1}{2} n-1}(r \rho)} (r \rho)^{\frac{1}{2} n-1} \rho \]  
where \( r = \sqrt{\sum x_i^2} \), \( \rho = \sqrt{\sum t_j^2} \)

10.9 POISSON MULTIVARIATE

Derivation  
\[ e^{11:120}, d^{28:466}, i^{37:1} \]

Without correlation, Multiple Poisson

\[ D(x_1, \ldots, x_n) = \exp - (k_1 + \ldots + k_r) \frac{k_1 x_1 \ldots k_n x_n}{x_1! \ldots x_n!} \]  
\[ e^{19:210}, z^{12:113,410} \]

10.10 BINOMIAL MULTIVARIATE

\[ MGF(x_1, \ldots, x_n) = (1 + \sum p_i x_i + \sum p_{ij} x_i x_j + \ldots)^N \]  
\[ e^{11:119}, z^{12:113,410} \]

\[ D(x), \text{ etc., in special case} \]  
\[ i^{18:271} \]

10.11

Negative binomial multivariate  
\[ i^{18:274}, i^{19:211}, \text{Koninkl. Nederl. Akad.} \]
\[ (A)^{60:121} \]
Multinomial multivariate

Hotelling Multivariate

Multivariate distributions obtained from the Normal multivariate

Generalization of No. 9.14

Generalization of No. 8.3

Generalization of No. 8.60, No. 9.29

Hypergeometric multivariate

Gram-Charlier multivariate
10.20  RUN LENGTH

\[ D(x) \]

[\{d\}11:367, \{4\}202,206, 
\{s\}5:143]

10.21  BETA MULTIVARIATE

Tolerance limits

[\{4\}94]
TABLE 1

Journals

[a] Journal of the Royal Statistical Society, Series A
[c] Biometrika
[d] Annals of Mathematical Statistics
[e] Sankhya
[f] Biometrics
[g] Journal of the American Statistical Association
[h] Nordisk Statistisk Tidskrift
[h'] Nordic Statistical Journal
[i] Skandinavisk Aktuarietidskrift
[j] Bell System Technical Journal
[l] Econometrica
[m] Applied Statistics
[n] Metron
[o] Annals of the Institute of Statistical Mathematics
[p] Journal of the Institute of Actuaries Students' Society
[q] Bulletin of Mathematical Statistics
[r] Reports of Statistical Application Research, Union of Japanese Scientists and Engineers
[s] Statistica (Neerlandica)
[t] Calcutta Statistical Association Bulletin
[u] Proceedings of the Cambridge Philosophical Society
[v] Trabajos de Estadistica
[w] Mitteilungsblatt fur Mathematische Statistik
[y] Revue de l'Institut International de Statistique
### TABLE 2

**Books**


TABLE 2 (Continued)


TABLE 2 (Continued)

### TABLE 3

**Chronological**

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The scope of activities of the National Bureau of Standards at its headquarters in Washington, D. C., and its major laboratories in Boulder, Colo., is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section carries out specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant publications, appears on the inside front cover.

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