NATIONAL BUREAU OF STANDARDS REPORT

5850

APPLICATION OF THE DIRECT PRODUCT OF MATRICES TO THE ANALYSIS OF FRACTIONAL FACTORIALS OF THE 2^m3^n SERIES

by

W. S. Connor

A Report
to
Bureau of Ships
Department of the Navy

NBS

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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APPLICATION OF THE DIRECT PRODUCT
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W. S. Connor
Statistical Engineering Laboratory
Applied Mathematics Division

A Report
to
Assistant Chief of Bureau for Nuclear Propulsion
Bureau of Ships
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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

PREFACE

This report describes a method that greatly facilitates the analysis of many factorial designs and is expected to be especially helpful in developing a catalogue of fractional factorial experiment designs for use in experiments in which some factors are to be studied at two conditions and others at three conditions. This work is under the immediate direction of W. S. Connor and M. Zelen. Professor R. C. Bose of the University of North Carolina is serving as consultant.

Introduction.

The problem of constructing fractional factorial designs for the 2^m3^n series has not received much attention in the statistical literature (see, e.g., [1]). Accordingly, before preparing a catalogue, it is desirable to study methods of construction.

Such a study is now in progress. The first major problem encountered was the formation of the normal equations which correspond to any particular design. Although it is known from the general theory of least squares how to form the equations of expectation, and from them the normal equations, we were still faced with the difficult problem of implementing the theory. We recognized that progress would be very slow indeed unless some short cuts could be found.

Fortunately, we have found a very effective method for forming the normal equations. The method is described in this report, and is applied to a 1/2 replicate of the 2^33^2 design. The use of this fractional replicate is illustrated by application to some real experimental data.

The method is so general that it applies to the analysis of all factorial experiments. It can be used for analyzing fractional factorials of any mixed series - not merely the 2^m3^n series. Another use is for complete or fractional factorials with repetitions of some treatment combinations.

We are now engaged in using the method to study various designs for several fractions of the 2^m3^n series. Certain considerations of symmetry suggest that particular designs are optimum in the sense of providing estimates which are not highly correlated. However, these estimates may be arithmetically more complicated than the estimates which are provided by other less symmetrical arrangements. Thus, designs which are optimum in one sense are not necessarily optimum in some other sense.

We plan to program the method for use on the National Bureau of Standards electronic computer. We then shall be able to progress rapidly in the evaluation of the comparative merits of various contending designs. From these we shall select "optimum" designs for inclusion in the catalogue.

The method of forming the design

We partition the N factors into two collections, C_1 and C_2, containing c_1 and c_2 factors, respectively, $(c_1+c_2 = N)$. Then, for each collection separately, we form w sets of treatment combinations. It is not required that all of the treatment combinations be included in some set, nor is there any restriction as to the number of sets which contain any particular treatment combination. Also, a treatment combination may occur more than once in a set.

We denote the sets for collection C_1 by $m_1, m_2, \ldots, m_w$ and their respective numbers of treatment combinations by $u_1, u_2, \ldots, u_w$. Similarly, we denote the sets for collection C_2 by $n_1, n_2, \ldots, n_w$ and their respective numbers of treatment combinations by $v_1, v_2, \ldots, v_w$.

The design consists of adjoining every treatment combination of set m_i to every treatment combination of set n_i, $(i=1, \ldots, w)$. This produces sets p_i which contain $u_i v_i$ treatment combinations for all N factors.

To illustrate, let there be N=3 factors A, B, and α, all having two levels. Let C_1 consist of A and B; and C_2 consist of α. We may choose the sets m_i and n_i as follows:

	A B			α	
m_1:	0 0 0 0	$(u_1=2)$	n_1:	0	$(v_1=1)$
m_2:	0 0 1 1	$(u_2=2)$	n_2:	0 1	$(v_2=2)$
m_3:	1 0	$(u_3=1)$	n_3:	1	$(v_3=1)$

The design D consists of the following treatment combinations:

	A	B	α	
p_1:	0 0	0 0	0 0	$(u_1v_1=2)$
p_2:	0 0 1 1	0 0 1 1	0 1 0 1	$(u_2v_2=4)$
P_3:	1	0	1	$(u_3v_3=1)$

Equations of expectation and normal equations

We introduce column vectors of expected responses $\eta(m_i)$ and $\eta(n_i)$, and write the equations of expectation as

$$\eta(m_i) = M_i \begin{bmatrix} g \\ \underline{p} \end{bmatrix}$$

and

$$\eta(n_i) = N_i \begin{bmatrix} g \\ \underline{q} \end{bmatrix}$$

where g denotes the grand mean, $\underline{p}$ is a column vector of s parameters, $\underline{q}$ is a column vector of t parameters: and M_i is a $u_i \times (s+1)$ matrix of coefficients and N_i is a $v_i \times (t+1)$ matrix of coefficients.

We now are ready to consider the equations of expectation and the normal equations for the design D. For this purpose, we introduce

$$y(D) = \begin{bmatrix} y(p_1) \\ \vdots \\ y(p_w) \end{bmatrix}$$

a column vector of the responses of the treatment combinations of D, $\eta(D)$, a column vector of the corresponding expected responses, and $\underline{r}$, a column vector of st parameters.

We shall be concerned with the following equations of expectation for the treatment combinations of D:

$$\eta(D) = \begin{bmatrix} M_1 \otimes N_1 \\ M_2 \otimes N_2 \\ \cdot \\ \cdot \\ \cdot \\ M_w \otimes N_w \end{bmatrix} \begin{bmatrix} g \\ \underline{p} \\ \underline{q} \\ \underline{r} \end{bmatrix}$$

where $\otimes$ denotes the right direct product* The normal equations are as follows:

$$[(M_1 \otimes N_1)^T \ldots (M_w \otimes N_w)^T] \begin{bmatrix} (M_1 \otimes N_1) \\ \cdot \\ \cdot \\ (M_w \otimes N_w) \end{bmatrix} \begin{bmatrix} g \\ \underline{p} \\ \underline{q} \\ \underline{r} \end{bmatrix}$$

$$= [(M_1 \otimes N_1)^T \ldots (M_w \otimes N_w)^T] \; y(D) \; .$$

By examination of the indicated operations it can be verified that

$$(M_i \otimes N_i)^T = M_i^T \otimes N_i^T$$

and

$$(M_i \otimes N_i)^T (M_i \otimes N_i) = M_i^T M_i \otimes N_i^T N_i \; .$$

Hence, the normal equations become

$$[\sum_i (M_i^T M_i \otimes N_i^T N_i)] \begin{bmatrix} g \\ \underline{p} \\ \underline{q} \\ \underline{r} \end{bmatrix} = \sum_i (M_i \otimes N_i)^T \; y(p_i)$$

* If $A=(a_{ij})$ and $B=(b_{kl})$, then $A \otimes B=(a_{ij}B)$. See, e.g.,[4]

Definition of the parameters.

The parameters g, p, q and r will be defined in terms of the expected responses in the complete factorial.

Let the equations of expectation for the complete factorial (CF) be written as

$$\eta(CF) = E\, \underline{p}(CF) ,$$

where η (CF) is a column vector containing (s+1)(t+1) expected responses, E is a square matrix of order (s+1)(t+1), and p(CF) is a column vector containing (s+1)(t+1) parameters. If E is non-singular, then the parameters p(CF) are defined by

$$\underline{p}(CF) = E^{-1}\eta(CF)$$

We shall define the matrix E as follows. The vector of coefficients of g (a column of E) is the unit vector. For any factor F with i levels, we introduce i-1 parameters, which may be referred to as the linear effect, the quadratic effect, the cubic effect, etc. The elements in the vector of coefficients (a column of E) of each of these effects are the values of the corresponding orthogonal polynomial [2].

For example, if a factor has three levels, 0, 1 and 2, then there are two parameters: the linear effect and the quadratic effect. The coefficient of the linear effect is -1, 0, or 1 and of the quadratic effect is 1, -2 or 1, depending on whether the factor is at level 0, 1 or 2.

Other parameters are associated with several factors simultaneously, and may be called interaction effects. The vector of coefficients of such an effect (a column of E) is obtained by taking the element by element products of the several corresponding vectors. For example, consider factors F_1, F_2 and F_3 which have 2, 2 and 3 levels, respectively.

The vector of coefficients of the linear by linear by quadratic effect is obtained as indicated below:

Level			Effect			
F_1	F_2	F_3	Linear, F_1	Linear, F_2	Quad., F_3	L×L×Q
0	0	0	-1	-1	1	1
0	1	0	-1	1	1	-1
1	0	0	1	-1	1	-1
1	1	0	1	1	1	1
0	0	1	-1	-1	-2	-2
0	1	1	-1	1	-2	2
1	0	1	1	-1	-2	2
1	1	1	1	1	-2	-2
0	0	2	-1	-1	1	1
0	1	2	-1	1	1	-1
1	0	2	1	-1	1	-1
1	1	2	1	1	1	1

We have defined E, and now shall examine the definitions of the parameters. We may write

$$E(dI) = C$$

where (dI) is a diagonal matrix and C is an orthogonal matrix. The element d_{jj} of (dI) is $(\sum_i e_{ij}^2)^{-1/2}$, where e_{ij} is the element in the ith row and jth column of E.

Because E(dI) is orthogonal,

$$[E(dI)]^{-1} = [E(dI)]^T,$$

whence

$$E^{-1} = (dI)^2 E^T$$

Therefore,

$$\underline{p}(CF) = (dI)^2 E^T \underline{\eta}(CF).$$

This equation defines the effects in terms of the expected responses.

A one-half replicate of the $2^3 3^2$.

The following set of tables, Tables 1, ..., 8 traces the application of the method to a particular 1/2 replicate of the $2^3 3^2$ complete factorial.

Table 1 contains the 36 treatment combinations which comprise the design, together with data for an illustrative example. There are three factors, A, B, and C which have two levels, and two factors, α and β, which have three levels. The collection C_1 contains A, B, C and the collection C_2 contains α and β. The sets of C_1 are

m_1:

A	B	C
0	0	0
1	1	0
1	0	1
0	1	1

and m_2:

A	B	C
1	0	0
0	1	0
0	0	1
1	1	1

and the sets of C_2 are

n_1:

α	β
0	0
0	2
1	1
2	0
2	2

and n_2:

α	β
0	1
1	0
1	2
2	1

The treatment combinations of m_1 are adjoined to those of n_1, and those of m_2 to those of n_2 to form the $4 \cdot 5 + 4 \cdot 4 = 36$ treatment combinations in Table 1.

TABLE 1

Treatment Combinations*
(Observed values in parentheses)

A	0	1	1	0	1	0	0	1	0	1	1	0
B	0	1	0	1	0	1	0	1	0	1	0	1
C	0	0	1	1	0	0	1	1	0	0	1	1
α	0	0	0	0	0	0	0	0	0	0	0	0
β	0	0	0	0	1	1	1	1	2	2	2	2
	(85.9)	(115.5)	(119.8)	(99.3)	(42.0)	(78.4)	(88.9)	(142.0)	(164.8)	(190.3)	(203.9)	(123.2)
A	1	0	0	1	0	1	1	0	1	0	0	1
B	0	1	0	1	0	1	0	1	0	1	0	1
C	0	0	1	1	0	0	1	1	0	0	1	1
α	1	1	1	1	1	1	1	1	1	1	1	1
β	0	0	0	0	1	1	1	1	2	2	2	2
	(92.8)	(110.4)	(94.9)	(167.2)	(54.8)	(128.2)	(144.6)	(144.4)	(110.2)	(99.3)	(104.4)	(205.3)
A	0	1	1	0	1	0	0	1	0	1	1	0
B	0	1	0	1	0	1	0	1	0	1	0	1
C	0	0	1	1	0	0	1	1	0	0	1	1
α	2	2	2	2	2	2	2	2	2	2	2	2
β	0	0	0	0	1	1	1	1	2	2	2	2
	(80.6)	(110.2)	(178.5)	(145.9)	(121.6)	(127.7)	(141.0)	(189.2)	(178.4)	(168.0)	(197.5)	(172.1)

* The entries are read vertically, in groups of five. For example, the first entry indicates that all of the factors are at their zero level.

TABLE 2

Expected Values of the Responses in a 2^3 Factorial

Response at Combination	Effect: E(000,00)	E(100,00)	E(010,00)	E(001,00)	E(110,00)	E(101,00)	E(011,00)	E(111,00)
				M_1				
000	+	-	-	-	+	+	+	-
110	+	+	+	-	+	-	-	-
101	+	+	-	+	-	+	-	-
011	+	-	+	+	-	-	+	-
				M_2				
100	+	+	-	-	-	-	+	+
010	+	-	+	-	-	+	-	+
001	+	-	-	+	+	-	-	+
111	+	+	+	+	+	+	+	+

TABLE 3

Expected Values of the Responses
in a 3^2 Factorial

Effect

Response at Combination	E(000,00)	E(000,10)	E(000,20)	E(000,01)	E(000,02)	E(000,11)	E(000,12)	E(000,21)	E(000,22)
					N_1				
00	1	-1	1	-1	1	1	-1	-1	1
02	1	-1	1	1	1	-1	-1	1	1
11	1	0	-2	0	-2	0	0	0	4
20	1	1	1	-1	1	-1	1	-1	1
22	1	1	1	1	1	1	1	1	1
					N_2				
01	1	-1	1	0	-2	0	2	0	-2
10	1	0	-2	-1	1	0	0	2	-2
12	1	0	-2	1	1	0	0	-2	-2
21	1	1	1	0	-2	0	-2	0	-2

TABLE 4

The Matrices $M_1^T M_1$ and $M_2^T M_2$

$$M_1^T M_1 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ & 4 & 0 & 0 & 0 & 0 & -4 & 0 \\ & & 4 & 0 & 0 & -4 & 0 & 0 \\ & & & 4 & -4 & 0 & 0 & 0 \\ & & & & 4 & 0 & 0 & 0 \\ & & & & & 4 & 0 & 0 \\ & \text{Symmetric} & & & & & 4 & 0 \\ & & & & & & & 4 \end{bmatrix}$$

$$M_2^T M_2 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ & 4 & 0 & 0 & 0 & 0 & 4 & 0 \\ & & 4 & 0 & 0 & 4 & 0 & 0 \\ & & & 4 & 4 & 0 & 0 & 0 \\ & & & & 4 & 0 & 0 & 0 \\ & & & & & 4 & 0 & 0 \\ & \text{Symmetric} & & & & & 4 & 0 \\ & & & & & & & 4 \end{bmatrix}$$

TABLE 5

The Matrices $N_1^T N_1$ and $N_2^T N_2$

$$N_1^T N_1 = \begin{bmatrix} 5 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 8 \\ & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ & & 8 & 0 & 8 & 0 & 0 & 0 & -4 \\ & & & 4 & 0 & 0 & 0 & 4 & 0 \\ & & & & 8 & 0 & 0 & 0 & -4 \\ & & & & & 4 & 0 & 0 & 0 \\ & & & & & & 4 & 0 & 0 \\ & \text{Symmetric} & & & & & & 4 & 0 \\ & & & & & & & & 20 \end{bmatrix}$$

$$N_2^T N_2 = \begin{bmatrix} 4 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & -8 \\ & 2 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ & & 10 & 0 & -8 & 0 & 0 & 0 & 4 \\ & & & 2 & 0 & 0 & 0 & -4 & 0 \\ & & & & 10 & 0 & 0 & 0 & 4 \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & & 8 & 0 & 0 \\ & \text{Symmetric} & & & & & & 8 & 0 \\ & & & & & & & & 16 \end{bmatrix}$$

TABLE 6

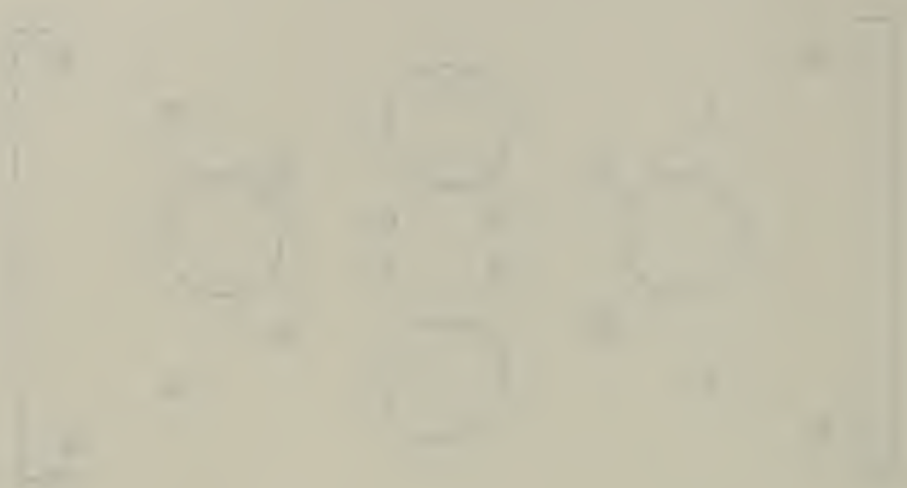
The Coefficients in the Normal Equations

$$(M_1^T M_1 \otimes N_1^T N_1) + (M_2^T M_2 \otimes N_2^T N_2)$$

$$= \begin{bmatrix} A & & & & & & & B \\ & A & & & \bigcirc & & B & \\ & & A & & & B & & \\ & \bigcirc & & A & B & & \bigcirc & \\ & & & B & A & & & \\ & & B & & & A & & \\ & B & & & \bigcirc & & A & \\ B & & & & & & & A \end{bmatrix}_{72 \times 72}$$

where $A = 4(N_1^T N_1 + N_2^T N_2)$

$$= 4 \begin{bmatrix} 9 & & & & & & & & \\ & 6 & & & & & & & \\ & & 18 & & & & \bigcirc & & \\ & & & 6 & & & & & \\ & & & & 18 & & & & \\ & & & & & 4 & & & \\ & & \bigcirc & & & & 12 & & \\ & & & & & & & 12 & \\ & & & & & & & & 36 \end{bmatrix}_{9 \times 9}$$



and $B = -4(N_1^T N_1 \; - \; N_2^T N_2)$

$$
= -4 \begin{bmatrix}
1 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 16 \\
 & 2 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\
 & & -2 & 0 & 16 & 0 & 0 & 0 & -8 \\
 & & & 2 & 0 & 0 & 0 & 8 & 0 \\
 & & & & -2 & 0 & 0 & 0 & -8 \\
 & & & & & 4 & 0 & 0 & 0 \\
 & & & & & & -4 & 0 & 0 \\
 & \text{Symmetric} & & & & & & -4 & 0 \\
 & & & & & & & & 4
\end{bmatrix}_{9 \times 9}
$$

TABLE 7

Effects Which Occur in Each Subset of the Normal Equations

Subset 1	Subset 2	Subset 3	Subset 4
E(000,00)*	E(100,00)*	E(010,00)*	E(001,00)*
E(000,10)*	E(100,10)*	E(010,10)*	E(001,10)*
E(000,20)*	E(100,20)*	E(010,20)*	E(001,20)*
E(000,01)*	E(100,01)*	E(010,01)*	E(001,01)*
E(000,02)*	E(100,02)*	E(010,02)*	E(001,02)*
E(000,11)*	E(100,11)	E(010,11)	E(001,11)
E(000,12)*	E(100,12)	E(010,12)	E(001,12)
E(000,21)*	E(100,21)	E(010,21)	E(001,21)
E(000,22)*	E(100,22)	E(010,22)	E(001,22)
E(111,00)	E(011,00)*	E(101,00)*	E(110,00)*
E(111,10)	E(011,10)	E(101,10)	E(110,10)
E(111,20)	E(011,20)	E(101,20)	E(110,20)
E(111,01)	E(011,01)	E(101,01)	E(110,01)
E(111,02)	E(011,02)	E(101,02)	E(110,02)
E(111,11)	E(011,11)	E(101,11)	E(110,11)
E(111,12)	E(011,12)	E(101,12)	E(110,12)
E(111,21)	E(011,21)	E(101,21)	E(110,21)
E(111,22)	E(011,22)	E(101,22)	E(110,22)

*) Indicates the grand mean, the main effects, and the two-factor interaction effects.

TABLE 8

Estimation of the Main and Two-Factor Interaction Effects in Subset 2

$$4\begin{bmatrix} 9 & & & & & -1 \\ & 6 & & & \bigcirc & 0 \\ & & 18 & & & -4 \\ & \bigcirc & & 6 & & 0 \\ & & & & 18 & -4 \\ -1 & 0 & -4 & 0 & -4 & 9 \end{bmatrix} \begin{bmatrix} \hat{E}(100,00) \\ \hat{E}(100,10) \\ \hat{E}(100,20) \\ \hat{E}(100,01) \\ \hat{E}(100,02) \\ \hat{E}(011,00) \end{bmatrix} = \begin{bmatrix} Y(100,00) \\ Y(100,10) \\ Y(100,20) \\ Y(100,01) \\ Y(100,02) \\ Y(011,00) \end{bmatrix}$$

$$= \frac{1}{2^8 3^2}\begin{bmatrix} \hat{E}(100,00) \\ \hat{E}(100,10) \\ \hat{E}(100,20) \\ \hat{E}(100,01) \\ \hat{E}(100,02) \\ \hat{E}(011,00) \end{bmatrix} = \frac{1}{2^8 3^2}\begin{bmatrix} 65 & 0 & 2 & 0 & 2 & +9 \\ & 96 & 0 & 0 & 0 & 0 \\ & & 36 & 0 & 4 & +18 \\ & & & 96 & 0 & 0 \\ & & & & 36 & +18 \\ & \text{Symmetric} & & & & 81 \end{bmatrix} \begin{bmatrix} Y(100,00) \\ Y(100,10) \\ Y(100,20) \\ Y(100,01) \\ Y(100,02) \\ Y(011,00) \end{bmatrix}$$

Tables 2 and 3 contain sketchs of the equations of expectation. At the left are the expected responses $\eta(000)$, $\eta(110)$, etc., with $\eta(\ \ )$ omitted. The equality signs have been omitted and the parameters (effects) have been indicated at the top, instead of as column vectors. In Table 2 it is understood that the + and - are to be read as +1 and -1. The matrices M_1, M_2, N_1 and N_2 have been indicated.

The notation for the effects is as follows: The symbol E denotes "effect", and the five positions correspond, in order, to A, B, C, α and β. The comma separates A, B, and C from α and β. Zero (0) in a position indicates that no effect of the corresponding factor is involved in the definition of E( ). One (1) in a position indicates that the linear effect of the corresponding factor is involved, and two (2) that the quadratic effect is involved.

The following typical examples should suffice to explain the notation:

Grand average	E(000,00)
Main effect of A	E(100,00)
Interaction, A with B	E(110,00)
Linear effect of α	E(000,10)
Quadratic effect of α	E(000,20)
Linear α by linear β	E(000,11)
Linear α by quadratic β	E(000,12)
Linear A by linear α	E(100,10)
Linear A by quadratic α	E(100,20)

Tables 4 and 5 exhibit the matrices $M_1{}^T M_1$, $M_2{}^T M_2$, $N_1{}^T N_1$, and $N_2{}^T N_2$; and Table 6 displays the matrix of coefficients of the normal equations. This matrix is of order 72, and decomposes into four identical submatrices of order 18. These submatrices are of rank 9, which implies that it is impossible to uniquely estimate the 72 effects.

At this point it is reasonable to equate certain effects to zero, and to solve for the remaining effects. A selection of effects to be estimated is indicated by asterisks in Table 7. (The unstarred effects are put equal to zero.) The effects to be estimated are all of the main effects and two-factor interactions.

The subsets in Table 7 correspond to the submatrices of Table 6. Solutions for the effects in subset 1 are particularly easy, because the corresponding matrix of coefficients is a diagonal matrix.

Table 8 displays the solution for the effects in subsets 2, 3 and 4. The symbol $Y(i_1 i_2 i_3, j_1 j_2)$ - where i_1, i_2, and i_3 take on the values 0 and 1; and j_1 and j_2 take on the values 0, 1 and 2 - denotes a linear combination of the observed responses. It occurs in the normal equation which corresponds to $E(i_1 i_2 i_3, j_1 j_2)$; and is the inner product of the vector of responses with the vector of coefficients of $E(i_1 i_2 i_3, j_1 j_2)$ in the equations of expectation.

From the inverse matrix, one may read the variances and covariances of the estimates. The maximum correlation between estimates is $18/[(36)(81)]^{1/2} = 1/3$.

There is very little correlation among the estimates. In fact, of the $27(26)/2 = 351$ pairs of estimates, there is correlation between the estimates in only 18 pairs.

If, instead of using the present design, one used the design* which is formed by adjoining the treatments of m_1 to those of n_1 and n_2, only 21 of the chosen effects would be estimable (the effects E(100,00), E(011,00), E(010,00), E(101,00), E(001,00), and E(110,00) would not be estimable).

It is interesting to compare the variances of the estimates obtained from the two designs. They are identical, except in six instances. For $\hat{E}(100,20)$, $\hat{E}(100,02)$, $\hat{E}(010,20)$, $\hat{E}(010,02)$, $\hat{E}(001,20)$, and $\hat{E}(001,02)$ the variance for the alternative design is 8/9 as large as for the present design.

* As suggested in [1].

An illustrative example

The following example is taken from a paper by Youden and Zimmerman [3]. Youden was concerned with comparing various methods of producing tomato plant seedlings prior to transplanting in the field. Comparison was made by planting in the field and then weighing the ripe produce. Thus, the observations were pounds of tomatoes.

Although Youden used five methods of production, we shall select only three: flats, fibre pots, and fibre pots soaked in one percent sodium nitrate solution. Other factors considered were different soil conditions, different sizes of pots, different varieties of tomato, and different locations on the field. The factors and their levels are recorded below:

Factor	Levels	
Soil condition, A	Field Soil Plus fertilizer	0 1
Size of pot, B	Three-inch Four-inch	0 1
Variety of tomato, C	Bonny Best Marglobe	0 1
Method of production, α	Flat Fibre Fibre + NO_3	0 1 2
Location on field, β		0,1,2

The object of the experiment was to evaluate the effects of these factors on the yield of tomatoes. The observations have been recorded, along with the experiment design, in Table 1.

The first stage in the analysis is the same as for a complete factorial. We form a number of summary tables, which are given below:

TABLE 9

Summary Tables

		Soil Condition (A)		
		0	1	Total
Size of Pot (B)	0:	993.7	1210.9	2204.6
	1:	1100.7	1415.9	2516.6
	Total:	2094.4	2626.8	4721.2

		Soil Condition (A)		
		0	1	Total
Variety (C)	0:	980.3	1078.8	2059.1
	1:	1114.1	1548.0	2662.1
	Total:	2094.4	2626.8	4721.2

		Size of Pot (B)		
		0	1	Total
Variety (C)	0:	931.1	1128.0	2059.1
	1:	1273.5	1388.6	2662.1
	Total:	2204.6	2516.6	4721.2

		Method of Production (α)			
		0	1	2	Total
Soil Cond. (A)	0:	640.5	608.2	845.7	2094.4
	1:	813.5	848.3	965.0	2626.8
	Total:	1454.0	1456.5	1810.7	4721.2

Table 9 (Continued)

		Method of Production (α)			
		0	1	2	Total
Size of Pot (B)	0:	705.3	601.7	897.6	2204.6
	1:	748.7	854.8	913.1	2516.6
	Total:	1454.0	1456.5	1810.7	4721.2

		Method of Production (α)			
		0	1	2	Total
Variety (C)	0:	676.9	595.7	786.5	2059.1
	1:	777.1	860.8	1024.2	2662.1
	Total:	1454.0	1456.5	1810.7	4721.2

		Location (β)			
		0	1	2	Total
Soil Cond. (A)	0:	617.0	635.2	842.2	2094.4
	1:	784.0	767.6	1075.2	2626.8
	Total:	1401.0	1402.8	1917.4	4721.2

		Location (β)			
		0	1	2	Total
Size of Pot (B)	0:	652.5	592.9	959.2	2204.6
	1:	748.5	809.9	958.2	2516.6
	Total:	1401.0	1402.8	1917.4	4721.2

		Location (β)			
		0	1	2	Total
Variety (C)	0:	595.4	552.7	911.0	2059.1
	1:	805.6	850.1	1006.4	2662.1
	Total:	1401.0	1402.8	1917.4	4721.2

		Method of Production (α)			
		0	1	2	Total
Location (β)	0:	420.5	465.3	515.2	1401.0
	1:	351.3	472.0	579.5	1402.8
	2:	682.2	519.2	716.0	1917.4
	Total:	1454.0	1456.5	1810.7	4721.2

The entry in any cell is the sum of the observations which have the row and column designations associated with the cell. For example, in the last table, the first entry is the sum of the observations in Table 1 which have $\alpha = 0$ and $\beta = 0$, i.e., the first four observations.

We calculate the Y's from the summary tables. For example, from the first summary table we compute

$$Y(100,00) = -993.7 - 1100.7 + 1210.9 + 1415.9 = 532.4$$
$$Y(010,00) = -993.7 + 1100.7 - 1210.9 + 1415.9 = 312.0$$

and $$Y(110,00) = 993.7 - 1100.7 - 1210.9 + 1415.9 = 98.0$$

We observe that there is some repetition of Y's from the various tables. For instance, Y(100,00) is obtained from all four tables which have A as a factor. This repetition is unnecessary and in practice would be avoided. Having already calculated a Y from a previous table, we would not calculate it again.

The complete list of 27 distinct Y's is given in Table 10.

TABLE 10

Values of Y's

Y_1 = Y(000,00) =	4721.2	Y_{15} = Y(001,20) =	-192.3
Y_2 = Y(100,00) =	532.4	Y_{16} = Y(000,01) =	516.4
Y_3 = Y(010,00) =	312.0	Y_{17} = Y(000,02) =	512.8
Y_4 = Y(110,00) =	98.0	Y_{18} = Y(100,01) =	66.0
Y_5 = Y(001,00) =	603.0	Y_{19} = Y(100,02) =	135.2
Y_6 = Y(101,00) =	335.4	Y_{20} = Y(010,01) =	-97.0
Y_7 = Y(011,00) =	-81.8	Y_{21} = Y(010,02) =	-339.0
Y_8 = Y(000,10) =	356.7	Y_{22} = Y(001,01) =	-114.8
Y_9 = Y(000,20) =	351.7	Y_{23} = Y(001,02) =	-289.2
Y_{10} = Y(100,10) =	-53.7	Y_{24} = Y(000,11) =	-60.9
Y_{11} = Y(100,20) =	-187.9	Y_{25} = Y(000,21) =	354.7
Y_{12} = Y(010,10) =	-27.9	Y_{26} = Y(000,12) =	-327.9
Y_{13} = Y(010,20) =	-447.3	Y_{27} = Y(000,22) =	391.3
Y_{14} = Y(001,10) =	137.5		

The formulae and estimates of the main and interaction effects are given in Table 11.

TABLE 11

Estimated Effects

$\hat{E}(000,00) = Y_1/36 = 131.1$	$\hat{E}(100,10) = Y_{10}/12 = -4.5$
$\hat{E}(000,10) = Y_8/12 = 29.7$	$\hat{E}(100,01) = Y_{18}/12 = 5.5$
$\hat{E}(000,01) = Y_{16}/12 = 43.0$	$\hat{E}(010,10) = Y_{12}/12 = -2.3$
$\hat{E}(000,20) = Y_9/24 = 14.7$	$\hat{E}(010,01) = Y_{20}/12 = -8.1$
$\hat{E}(000,02) = Y_{17}/24 = 21.4$	$\hat{E}(001,10) = Y_{14}/12 = 11.5$
$\hat{E}(000,11) = Y_{24}/8 = -7.6$	$\hat{E}(001,01) = Y_{22}/12 = -9.6$
$\hat{E}(000,21) = Y_{25}/16 = 22.2$	
$\hat{E}(000,12) = Y_{26}/16 = -20.5$	
$\hat{E}(000,22) = Y_{27}/32 = 12.2$	

$$\hat{E}(100,00) = (65Y_2 + 2Y_{11} + 2Y_{19} + 9Y_7)/1152 = 29.3$$
$$\hat{E}(100,20) = (Y_2 + 18Y_{11} + 2Y_{19} + 9Y_7)/384 = -8.6$$
$$\hat{E}(100,02) = (Y_2 + 2Y_{11} + 18Y_{19} + 9Y_7)/384 = 4.8$$
$$\hat{E}(011,00) = (Y_2 + 2Y_{11} + 2Y_{19} + 9Y_7)/128 = -2.4$$

$$\hat{E}(010,00) = (65Y_3 + 2Y_{13} + 2Y_{21} + 9Y_6)/1152 = 18.9$$
$$\hat{E}(010,20) = (Y_3 + 18Y_{13} + 2Y_{21} + 9Y_6)/384 = -14.1$$
$$\hat{E}(010,02) = (Y_3 + 2Y_{13} + 18Y_{21} + 9Y_6)/384 = -9.5$$
$$\hat{E}(101,00) = (Y_3 + 2Y_{13} + 2Y_{21} + 9Y_6)/128 = +13.7$$

$$\hat{E}(001,00) = (65Y_5 + 2Y_{15} + 2Y_{23} + 9Y_4)/1152 = 34.0$$
$$\hat{E}(001,20) = (Y_5 + 18Y_{15} + 2Y_{23} + 9Y_4)/384 = -6.7$$
$$\hat{E}(001,02) = (Y_5 + 2Y_{15} + 18Y_{23} + 9Y_4)/384 = -10.7$$
$$\hat{E}(110,00) = (Y_5 + 2Y_{15} + 2Y_{23} + 9Y_4)/128 = 4.1$$

These formulae have been arranged into four sets. All of the estimated effects in the first set are uncorrelated. In addition the estimated effects within any set are uncorrelated with the estimated effects in any other set.

The analysis of variance is given in Table 12.

Table 12

Analysis of Variance

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square
Effects	26	58975.51	2268
Residual	9	4150.08	461
Total	35	63125.59	

In order to evaluate the effects, t-tests were carried out. It was found that $\hat{E}(100,00)$, $\hat{E}(001,00)$, $\hat{E}(000,10)$ and $\hat{E}(000,01)$ are significant at the .01 level of significance. There were no other significant effects.

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