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# NATIONAL BUREAU OF STANDARDS REPORT

5850

APPLICATION OF THE DIRECT PRODUCT  
OF MATRICES TO THE ANALYSIS  
OF FRACTIONAL FACTORIALS  
OF THE  $2^m 3^n$  SERIES

by

W. S. Connor

A Report  
to  
Bureau of Ships  
Department of the Navy



U. S. DEPARTMENT OF COMMERCE  
NATIONAL BUREAU OF STANDARDS

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# NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT

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April 9, 1958

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by

W. S. Connor  
Statistical Engineering Laboratory  
Applied Mathematics Division

A Report  
to

Assistant Chief of Bureau for Nuclear Propulsion  
Bureau of Ships  
Department of the Navy

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U. S. DEPARTMENT OF COMMERCE  
NATIONAL BUREAU OF STANDARDS



## PREFACE

This report describes a method that greatly facilitates the analysis of many factorial designs and is expected to be especially helpful in developing a catalogue of fractional factorial experiment designs for use in experiments in which some factors are to be studied at two conditions and others at three conditions. This work is under the immediate direction of W. S. Connor and M. Zelen. Professor R. C. Bose of the University of North Carolina is serving as consultant.

# History

The history of the United States is a story of discovery, exploration, and settlement. From the first Native American inhabitants to the arrival of European explorers, the land has been shaped by the actions of many different peoples. The story of the United States is a story of the struggle for freedom, the pursuit of the American dream, and the building of a nation.

The first people to live in the United States were Native Americans. They lived in small, nomadic groups and hunted, fished, and gathered for food. They had a deep knowledge of the land and its resources. In the 15th century, European explorers arrived in the Americas. They were looking for new lands to settle and new sources of wealth. The explorers found a land of great beauty and potential. They began to settle in the Americas and to build a new society.

The story of the United States is a story of the struggle for freedom. The first settlers in the Americas were seeking freedom from the oppression of their native lands. They wanted to live in a land where they could be free to worship as they saw fit and to govern themselves. The struggle for freedom was a long and difficult one. It was a struggle that shaped the character of the United States and that continues to this day.

The story of the United States is a story of the pursuit of the American dream. The American dream is the belief that anyone can achieve success and prosperity through hard work and determination. The American dream is a dream that has inspired millions of people to come to the United States and to build a new life for themselves. The American dream is a dream that is still alive and well in the hearts of the American people.

The story of the United States is a story of the building of a nation. The United States is a nation of many different peoples and cultures. It is a nation that has been built by the actions of many different people. The United States is a nation that is still growing and changing. The story of the United States is a story that is still being written.

Continued on next page

## Introduction.

The problem of constructing fractional factorial designs for the  $2^m 3^n$  series has not received much attention in the statistical literature (see, e.g., [1]). Accordingly, before preparing a catalogue, it is desirable to study methods of construction.

Such a study is now in progress. The first major problem encountered was the formation of the normal equations which correspond to any particular design. Although it is known from the general theory of least squares how to form the equations of expectation, and from them the normal equations, we were still faced with the difficult problem of implementing the theory. We recognized that progress would be very slow indeed unless some short cuts could be found.

Fortunately, we have found a very effective method for forming the normal equations. The method is described in this report, and is applied to a  $1/2$  replicate of the  $2^3 3^2$  design. The use of this fractional replicate is illustrated by application to some real experimental data.

The method is so general that it applies to the analysis of all factorial experiments. It can be used for analyzing fractional factorials of any mixed series - not merely the  $2^m 3^n$  series. Another use is for complete or fractional factorials with repetitions of some treatment combinations.



We are now engaged in using the method to study various designs for several fractions of the  $2^m 3^n$  series. Certain considerations of symmetry suggest that particular designs are optimum in the sense of providing estimates which are not highly correlated. However, these estimates may be arithmetically more complicated than the estimates which are provided by other less symmetrical arrangements. Thus, designs which are optimum in one sense are not necessarily optimum in some other sense.

We plan to program the method for use on the National Bureau of Standards electronic computer. We then shall be able to progress rapidly in the evaluation of the comparative merits of various contending designs. From these we shall select "optimum" designs for inclusion in the catalogue.



The method of forming the design

We partition the  $N$  factors into two collections,  $C_1$  and  $C_2$ , containing  $c_1$  and  $c_2$  factors, respectively, ( $c_1 + c_2 = N$ ). Then, for each collection separately, we form  $w$  sets of treatment combinations. It is not required that all of the treatment combinations be included in some set, nor is there any restriction as to the number of sets which contain any particular treatment combination. Also, a treatment combination may occur more than once in a set.

We denote the sets for collection  $C_1$  by  $m_1, m_2, \dots, m_w$  and their respective numbers of treatment combinations by  $u_1, u_2, \dots, u_w$ . Similarly, we denote the sets for collection  $C_2$  by  $n_1, n_2, \dots, n_w$  and their respective numbers of treatment combinations by  $v_1, v_2, \dots, v_w$ .

The design consists of adjoining every treatment combination of set  $m_i$  to every treatment combination of set  $n_i$ , ( $i=1, \dots, w$ ). This produces sets  $p_i$  which contain  $u_i v_i$  treatment combinations for all  $N$  factors.

To illustrate, let there be  $N=3$  factors  $A, B$ , and  $\alpha$ , all having two levels. Let  $C_1$  consist of  $A$  and  $B$ ; and  $C_2$  consist of  $\alpha$ . We may choose the sets  $m_i$  and  $n_i$  as follows:

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$$m_1: \begin{array}{cc} \underline{A} & \underline{B} \\ 0 & 0 \\ 0 & 0 \end{array} (u_1=2) \quad n_1: \begin{array}{c} \underline{\alpha} \\ 0 \end{array} (v_1=1)$$

$$m_2: \begin{array}{cc} \underline{A} & \underline{B} \\ 0 & 0 \\ 1 & 1 \end{array} (u_2=2) \quad n_2: \begin{array}{c} \underline{\alpha} \\ 0 \\ 1 \end{array} (v_2=2)$$

$$m_3: \begin{array}{cc} \underline{A} & \underline{B} \\ 1 & 0 \end{array} (u_3=1) \quad n_3: \begin{array}{c} \underline{\alpha} \\ 1 \end{array} (v_3=1)$$

The design D consists of the following treatment combinations:

$$p_1: \begin{array}{ccc} \underline{A} & \underline{B} & \underline{\alpha} \\ 0 & 0 & 0 \end{array} (u_1 v_1=2)$$

$$p_2: \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array} (u_2 v_2=4)$$

$$p_3: \begin{array}{ccc} 1 & 0 & 1 \end{array} (u_3 v_3=1)$$

1875-1876 1877-1878 1879-1880

1881-1882 1883-1884 1885-1886

1887-1888 1889-1890 1891-1892

1893-1894 1895-1896 1897-1898 1899-1900

1901-1902 1903-1904 1905-1906 1907-1908

1909-1910 1911-1912 1913-1914 1915-1916

1917-1918 1919-1920 1921-1922 1923-1924

1925-1926 1927-1928 1929-1930 1931-1932

1933-1934 1935-1936 1937-1938 1939-1940

1941-1942 1943-1944 1945-1946 1947-1948

1949-1950 1951-1952 1953-1954 1955-1956

1957-1958 1959-1960 1961-1962 1963-1964

1965-1966 1967-1968 1969-1970 1971-1972

### Equations of expectation and normal equations

We introduce column vectors of expected responses  $\eta(m_i)$  and  $\eta(n_i)$ , and write the equations of expectation as

$$\eta(m_i) = M_i \begin{bmatrix} g \\ \underline{p} \end{bmatrix}$$

and 
$$\eta(n_i) = N_i \begin{bmatrix} g \\ \underline{q} \end{bmatrix}$$

where  $g$  denotes the grand mean,  $\underline{p}$  is a column vector of  $s$  parameters,  $\underline{q}$  is a column vector of  $t$  parameters: and  $M_i$  is a  $u_i \times (s+1)$  matrix of coefficients and  $N_i$  is a  $v_i \times (t+1)$  matrix of coefficients.

We now are ready to consider the equations of expectation and the normal equations for the design  $D$ . For this purpose, we introduce

$$y(D) = \begin{bmatrix} y(p_1) \\ \vdots \\ y(p_w) \end{bmatrix}$$

a column vector of the responses of the treatment combinations of  $D$ ,  $\eta(D)$ , a column vector of the corresponding expected responses, and  $\underline{r}$ , a column vector of  $st$  parameters.

We shall be concerned with the following equations of expectation for the treatment combinations of  $D$ :

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$$\eta(D) = \begin{bmatrix} M_1 \otimes N_1 \\ M_2 \otimes N_2 \\ \vdots \\ M_w \otimes N_w \end{bmatrix} \begin{bmatrix} \underline{g} \\ \underline{p} \\ \underline{q} \\ \underline{r} \end{bmatrix}$$

where  $\otimes$  denotes the right direct product\*. The normal equations are as follows:

$$\begin{aligned} & \left[ (M_1 \otimes N_1)^T \dots (M_w \otimes N_w)^T \right] \begin{bmatrix} (M_1 \otimes N_1) \\ \vdots \\ (M_w \otimes N_w) \end{bmatrix} \begin{bmatrix} \underline{g} \\ \underline{p} \\ \underline{q} \\ \underline{r} \end{bmatrix} \\ &= \left[ (M_1 \otimes N_1)^T \dots (M_w \otimes N_w)^T \right] y(D) . \end{aligned}$$

By examination of the indicated operations it can be verified that

$$(M_i \otimes N_i)^T = M_i^T \otimes N_i^T$$

and

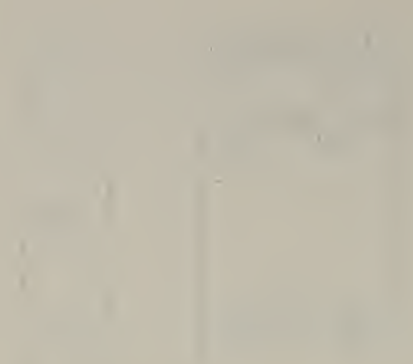
$$(M_i \otimes N_i)^T (M_i \otimes N_i) = M_i^T M_i \otimes N_i^T N_i .$$

Hence, the normal equations become

$$\left[ \sum_i (M_i^T M_i \otimes N_i^T N_i) \right] \begin{bmatrix} \underline{g} \\ \underline{p} \\ \underline{q} \\ \underline{r} \end{bmatrix} = \sum_i (M_i \otimes N_i)^T y(p_i)$$

---

\* If  $A=(a_{ij})$  and  $B=(b_{kl})$ , then  $A \otimes B=(a_{ij}B)$ . See, e.g., [4]



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Definition of the parameters.

The parameters  $g$ ,  $\underline{p}$ ,  $\underline{q}$  and  $\underline{r}$  will be defined in terms of the expected responses in the complete factorial.

Let the equations of expectation for the complete factorial (CF) be written as

$$\eta(\text{CF}) = E \underline{p}(\text{CF}) ,$$

where  $\eta(\text{CF})$  is a column vector containing  $(s+1)(t+1)$  expected responses,  $E$  is a square matrix of order  $(s+1)(t+1)$ , and  $\underline{p}(\text{CF})$  is a column vector containing  $(s+1)(t+1)$  parameters. If  $E$  is non-singular, then the parameters  $\underline{p}(\text{CF})$  are defined by

$$\underline{p}(\text{CF}) = E^{-1} \eta(\text{CF})$$

We shall define the matrix  $E$  as follows. The vector of coefficients of  $g$  (a column of  $E$ ) is the unit vector. For any factor  $F$  with  $i$  levels, we introduce  $i-1$  parameters, which may be referred to as the linear effect, the quadratic effect, the cubic effect, etc. The elements in the vector of coefficients (a column of  $E$ ) of each of these effects are the values of the corresponding orthogonal polynomial [2].

For example, if a factor has three levels, 0, 1 and 2, then there are two parameters: the linear effect and the quadratic effect. The coefficient of the linear effect is -1, 0, or 1 and of the quadratic effect is 1, -2 or 1, depending on whether the factor is at level 0, 1 or 2.

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Other parameters are associated with several factors simultaneously, and may be called interaction effects. The vector of coefficients of such an effect (a column of E) is obtained by taking the element by element products of the several corresponding vectors. For example, consider factors  $F_1$ ,  $F_2$  and  $F_3$  which have 2, 2 and 3 levels, respectively.

The vector of coefficients of the linear by linear by quadratic effect is obtained as indicated below:

Level			Effect			
$F_1$	$F_2$	$F_3$	Linear, $F_1$	Linear, $F_2$	Quad., $F_3$	$L \times L \times Q$
0	0	0	-1	-1	1	1
0	1	0	-1	1	1	-1
1	0	0	1	-1	1	-1
1	1	0	1	1	1	1
0	0	1	-1	-1	-2	-2
0	1	1	-1	1	-2	2
1	0	1	1	-1	-2	2
1	1	1	1	1	-2	-2
0	0	2	-1	-1	1	1
0	1	2	-1	1	1	-1
1	0	2	1	-1	1	-1
1	1	2	1	1	1	1

We have defined E, and now shall examine the definitions of the parameters. We may write

$$E(dI) = C$$

where  $(dI)$  is a diagonal matrix and C is an orthogonal matrix. The element  $d_{jj}$  of  $(dI)$  is  $(\sum_i e_{ij}^2)^{-1/2}$ , where  $e_{ij}$  is the element in the  $i$ th row and  $j$ th column of E.

The first part of the paper is devoted to a general discussion of the problem. It is shown that the problem is well-posed and that the solution exists and is unique. The second part of the paper is devoted to the construction of the solution. It is shown that the solution can be constructed by the method of successive approximations. The third part of the paper is devoted to the numerical solution of the problem. It is shown that the numerical solution can be obtained by the method of finite differences.

TABLE I		TABLE II		TABLE III	
1	2	3	4	5	6
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9
10	10	10	10	10	10

The results of the numerical solution are shown in Table I. It is seen that the numerical solution is in good agreement with the analytical solution. The error is of the order of  $10^{-4}$ . The results of the numerical solution are also shown in Table II. It is seen that the numerical solution is in good agreement with the analytical solution. The error is of the order of  $10^{-4}$ . The results of the numerical solution are also shown in Table III. It is seen that the numerical solution is in good agreement with the analytical solution. The error is of the order of  $10^{-4}$ .

Because  $E(dI)$  is orthogonal,

$$[E(dI)]^{-1} = [E(dI)]^T,$$

whence

$$E^{-1} = (dI)^2 E^T$$

Therefore,

$$\underline{p}(CF) = (dI)^2 E^T \underline{\eta}(CF).$$

This equation defines the effects in terms of the expected responses.

100

100

100

100

100

100

100

A one-half replicate of the  $2^3 3^2$ .

The following set of tables, Tables 1, ..., 8 traces the application of the method to a particular 1/2 replicate of the  $2^3 3^2$  complete factorial.

Table 1 contains the 36 treatment combinations which comprise the design, together with data for an illustrative example. There are three factors, A, B, and C which have two levels, and two factors,  $\alpha$  and  $\beta$ , which have three levels. The collection  $C_1$  contains A, B, C and the collection  $C_2$  contains  $\alpha$  and  $\beta$ . The sets of  $C_1$  are

	<u>A</u> <u>B</u> <u>C</u>			<u>A</u> <u>B</u> <u>C</u>
	0 0 0			1 0 0
$m_1$ :	1 1 0	and	$m_2$ :	0 1 0
	1 0 1			0 0 1
	0 1 1			1 1 1

and the sets of  $C_2$  are

	<u><math>\alpha</math></u> <u><math>\beta</math></u>			<u><math>\alpha</math></u> <u><math>\beta</math></u>
	0 0			0 1
	0 2			1 0
$n_1$ :	1 1	and	$n_2$ :	1 2
	2 0			2 1
	2 2			

The treatment combinations of  $m_1$  are adjoined to those of  $n_1$ , and those of  $m_2$  to those of  $n_2$  to form the  $4 \cdot 5 + 4 \cdot 4 = 36$  treatment combinations in Table 1.

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In SENATE,  
January 10, 1891.

REPORT  
OF THE  
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PASSED BY THE SENATE  
MAY 10, 1890.

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TABLE 1

Treatment Combinations\*  
(Observed values in parentheses)

A	0	1	1	0	1	0	0	1	0	1	1	0	
B	0	1	0	1	0	1	0	1	0	1	0	1	
C	0	0	1	1	0	0	1	1	0	0	1	1	
$\alpha$	0	0	0	0	0	0	0	0	0	0	0	0	
$\beta$	0	0	0	0	1	1	1	1	2	2	2	2	
	(	85.9)	(	115.5)	(	42.0)	(	78.4)	(	164.8)	(	190.3)	
		(	119.8)	(	88.9)	(	203.9)	(	123.2)				
		(	99.3)		(	142.0)							
A	1	0	0	1	0	1	1	0	1	0	0	1	
B	0	1	0	1	0	1	0	1	0	1	0	1	
C	0	0	1	1	0	0	1	1	0	0	1	1	
$\alpha$	1	1	1	1	1	1	1	1	1	1	1	1	
$\beta$	0	0	0	0	1	1	1	1	2	2	2	2	
	(	92.8)	(	110.4)	(	54.8)	(	128.2)	(	110.2)	(	99.3)	
		(	94.9)	(	144.6)		(	104.4)		(	104.4)	(	205.3)
		(	167.2)		(	144.4)							
A	0	1	1	0	1	0	0	1	0	1	1	0	
B	0	1	0	1	0	1	0	1	0	1	0	1	
C	0	0	1	1	0	0	1	1	0	0	1	1	
$\alpha$	2	2	2	2	2	2	2	2	2	2	2	2	
$\beta$	0	0	0	0	1	1	1	1	2	2	2	2	
	(	80.6)	(	110.2)	(	121.6)	(	127.7)	(	178.4)	(	168.0)	
		(	178.5)	(	141.0)		(	141.0)		(	197.5)	(	172.1)
		(	145.9)		(	189.2)							

\* The entries are read vertically, in groups of five. For example, the first entry indicates that all of the factors are at their zero level.

# TABLE

OF THE

11	12	13	14
15	16	17	18
19	20	21	22
23	24	25	26
27	28	29	30
31	32	33	34
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43	44	45	46
47	48	49	50
51	52	53	54
55	56	57	58
59	60	61	62
63	64	65	66
67	68	69	70
71	72	73	74
75	76	77	78
79	80	81	82
83	84	85	86
87	88	89	90
91	92	93	94
95	96	97	98
99	100	101	102
103	104	105	106
107	108	109	110
111	112	113	114
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235	236	237	238
239	240	241	242
243	244	245	246
247	248	249	250
251	252	253	254
255	256	257	258
259	260	261	262
263	264	265	266
267	268	269	270
271	272	273	274
275	276	277	278
279	280	281	282
283	284	285	286
287	288	289	290
291	292	293	294
295	296	297	298
299	300	301	302
303	304	305	306
307	308	309	310
311	312	313	314
315	316	317	318
319	320	321	322
323	324	325	326
327	328	329	330
331	332	333	334
335	336	337	338
339	340	341	342
343	344	345	346
347	348	349	350
351	352	353	354
355	356	357	358
359	360	361	362
363	364	365	366
367	368	369	370
371	372	373	374
375	376	377	378
379	380	381	382
383	384	385	386
387	388	389	390
391	392	393	394
395	396	397	398
399	400	401	402
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407	408	409	410
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415	416	417	418
419	420	421	422
423	424	425	426
427	428	429	430
431	432	433	434
435	436	437	438
439	440	441	442
443	444	445	446
447	448	449	450
451	452	453	454
455	456	457	458
459	460	461	462
463	464	465	466
467	468	469	470
471	472	473	474
475	476	477	478
479	480	481	482
483	484	485	486
487	488	489	490
491	492	493	494
495	496	497	498
499	500	501	502
503	504	505	506
507	508	509	510
511	512	513	514
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519	520	521	522
523	524	525	526
527	528	529	530
531	532	533	534
535	536	537	538
539	540	541	542
543	544	545	546
547	548	549	550
551	552	553	554
555	556	557	558
559	560	561	562
563	564	565	566
567	568	569	570
571	572	573	574
575	576	577	578
579	580	581	582
583	584	585	586
587	588	589	590
591	592	593	594
595	596	597	598
599	600	601	602
603	604	605	606
607	608	609	610
611	612	613	614
615	616	617	618
619	620	621	622
623	624	625	626
627	628	629	630
631	632	633	634
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643	644	645	646
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683	684	685	686
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691	692	693	694
695	696	697	698
699	700	701	702
703	704	705	706
707	708	709	710
711	712	713	714
715	716	717	718
719	720	721	722
723	724	725	726
727	728	729	730
731	732	733	734
735	736	737	738
739	740	741	742
743	744	745	746
747	748	749	750
751	752	753	754
755	756	757	758
759	760	761	762
763	764	765	766
767	768	769	770
771	772	773	774
775	776	777	778
779	780	781	782
783	784	785	786
787	788	789	790
791	792	793	794
795	796	797	798
799	800	801	802
803	804	805	806
807	808	809	810
811	812	813	814
815	816	817	818
819	820	821	822
823	824	825	826
827	828	829	830
831	832	833	834
835	836	837	838
839	840	841	842
843	844	845	846
847	848	849	850
851	852	853	854
855	856	857	858
859	860	861	862
863	864	865	866
867	868	869	870
871	872	873	874
875	876	877	878
879	880	881	882
883	884	885	886
887	888	889	890
891	892	893	894
895	896	897	898
899	900	901	902
903	904	905	906
907	908	909	910
911	912	913	914
915	916	917	918
919	920	921	922
923	924	925	926
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931	932	933	934
935	936	937	938
939	940	941	942
943	944	945	946
947	948	949	950
951	952	953	954
955	956	957	958
959	960	961	962
963	964	965	966
967	968	969	970
971	972	973	974
975	976	977	978
979	980	981	982
983	984	985	986
987	988	989	990
991	992	993	994
995	996	997	998
999	1000	1001	1002

THESE ARE THE RESULTS OF THE CALCULATIONS MADE BY THE

TABLE 2

Expected Values of the Responses  
in a  $2^3$  Factorial

<u>Response at Combination</u>	<u>Effect</u>							
	<u>E(000,00)</u>	<u>E(100,00)</u>	<u>E(010,00)</u>	<u>E(001,00)</u>	<u>E(110,00)</u>	<u>E(101,00)</u>	<u>E(011,00)</u>	<u>E(111,00)</u>
	$M_1$							
000	[	+	-	-	-	+	+	+
110		+	+	+	-	+	-	-
101		+	+	-	+	-	+	-
011		+	-	+	+	-	-	+
	$M_2$							
100	[	+	+	-	-	-	+	+
010		+	-	+	-	+	-	+
001		+	-	-	+	+	-	+
111		+	+	+	+	+	+	+

# TABLE 1

Summary of the results of the analysis of variance for the effect of the treatment on the response of the subjects to the treatment

TABLE 1

								Total	
								Sum of Squares	
								df	
								Mean Square	
								F	
								p	
								Value	
								Significance	
								Level	
								of Error	
								Total	
								Sum of Squares	
								df	
								Mean Square	
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								Significance	
								Level	
								of Error	
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								p	
								Value	
								Significance	
								Level	
								of Error	
								Total	

TABLE 3

Expected Values of the Responses  
in a  $3^2$  Factorial

<u>Response at Combination</u>	<u>Effect</u>								
	<u>E(000,00)</u>	<u>E(000,10)</u>	<u>E(000,20)</u>	<u>E(000,01)</u>	<u>E(000,02)</u>	<u>E(000,11)</u>	<u>E(000,12)</u>	<u>E(000,21)</u>	<u>E(000,22)</u>
	$N_1$								
00	1	-1	1	-1	1	1	-1	-1	1
02	1	-1	1	1	1	-1	-1	1	1
11	1	0	-2	0	-2	0	0	0	4
20	1	1	1	-1	1	-1	1	-1	1
22	1	1	1	1	1	1	1	1	1
	$N_2$								
01	1	-1	1	0	-2	0	2	0	-2
10	1	0	-2	-1	1	0	0	2	-2
12	1	0	-2	1	1	0	0	-2	-2
21	1	1	1	0	-2	0	-2	0	-2



TABLE 4

The Matrices  $M_1^T M_1$  and  $M_2^T M_2$  $M_1^T M_1$ 

-

4	0	0	0	0	0	0	-4
	4	0	0	0	0	-4	0
		4	0	0	-4	0	0
			4	-4	0	0	0
				4	0	0	0
					4	0	0
						4	0
							4

Symmetric

 $M_2^T M_2$ 

-

4	0	0	0	0	0	0	4
	4	0	0	0	0	4	0
		4	0	0	4	0	0
			4	4	0	0	0
				4	0	0	0
					4	0	0
						4	0
							4

Symmetric

THEORY

The following table shows the results of the experiment.

Time (s)	Distance (m)	Velocity (m/s)	Acceleration (m/s <sup>2</sup> )
0.0	0.0	0.0	0.0
0.2	0.1	0.5	2.5
0.4	0.4	1.0	2.5
0.6	0.9	1.5	2.5
0.8	1.6	2.0	2.5
1.0	2.5	2.5	2.5

Time (s)	Distance (m)	Velocity (m/s)	Acceleration (m/s <sup>2</sup> )
0.0	0.0	0.0	0.0
0.2	0.1	0.5	2.5
0.4	0.4	1.0	2.5
0.6	0.9	1.5	2.5
0.8	1.6	2.0	2.5
1.0	2.5	2.5	2.5



# Table

of the ...

1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8

...

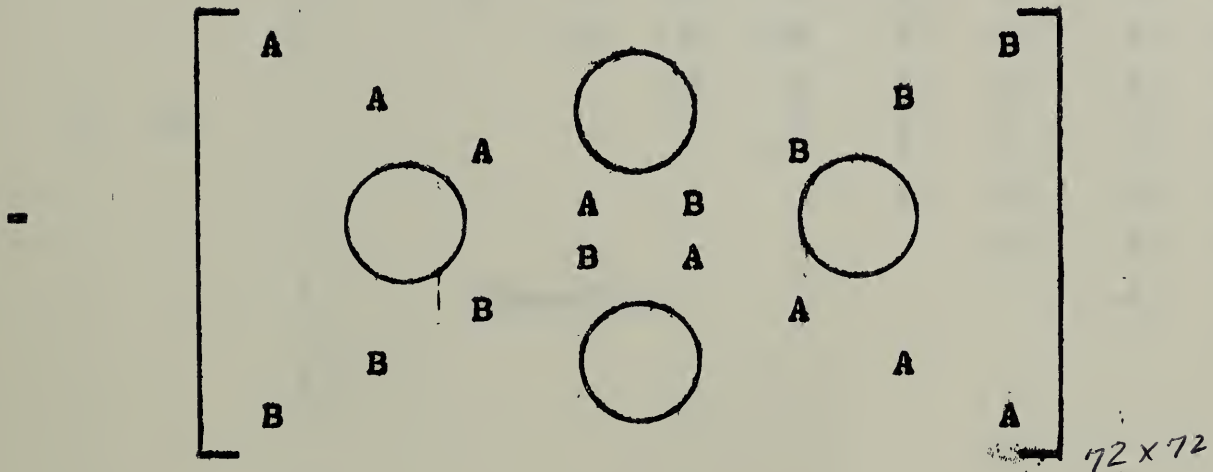
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8

...

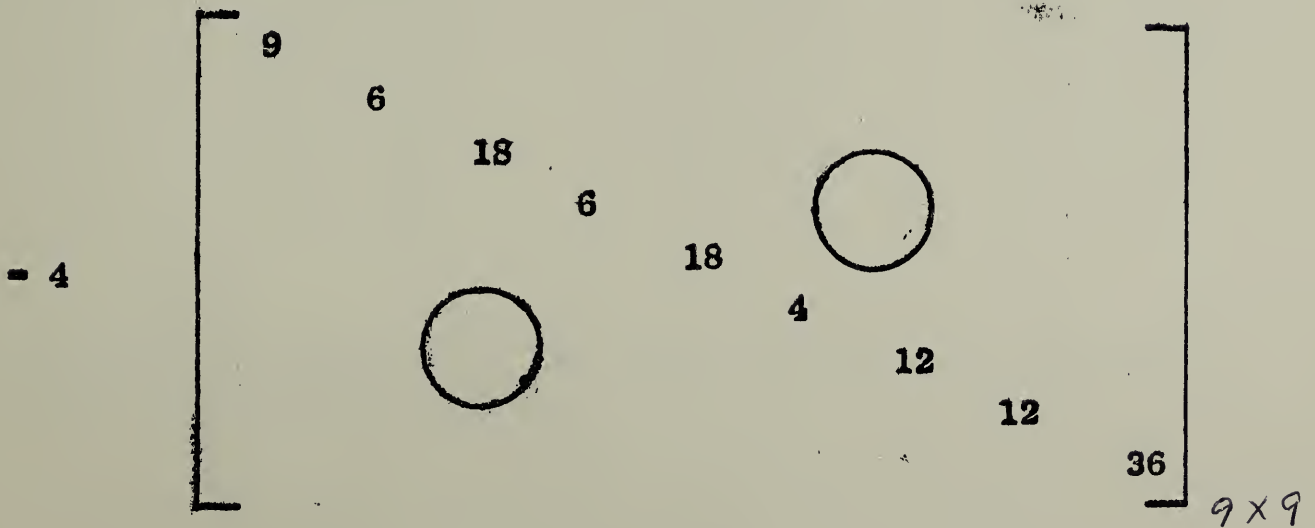
TABLE 6

The Coefficients in the Normal Equations

$$(M_1^T M_1 \otimes N_1^T N_1) + (M_2^T M_2 \otimes N_2^T N_2)$$



where  $A = 4(N_1^T N_1 + N_2^T N_2)$





and  $B = -4(N_1^T N_1 - N_2^T N_2)$

$-4$

1	0	4	0	4	0	0	0	0	16
	2	0	0	0	0	8	0	0	0
		-2	0	16	0	0	0	0	-8
			2	0	0	0	8	0	0
				-2	0	0	0	0	-8
					4	0	0	0	0
						-4	0	0	0
							-4	0	0
								4	4

Symmetric

9x9



TABLE 7

Effects Which Occur in Each Subset of the Normal Equations

<u>Subset</u>			
<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>
E(000,00)*	E(100,00)*	E(010,00)*	E(001,00)*
E(000,10)*	E(100,10)*	E(010,10)*	E(001,10)*
E(000,20)*	E(100,20)*	E(010,20)*	E(001,20)*
E(000,01)*	E(100,01)*	E(010,01)*	E(001,01)*
E(000,02)*	E(100,02)*	E(010,02)*	E(001,02)*
E(000,11)*	E(100,11)	E(010,11)	E(001,11)
E(000,12)*	E(100,12)	E(010,12)	E(001,12)
E(000,21)*	E(100,21)	E(010,21)	E(001,21)
E(000,22)*	E(100,22)	E(010,22)	E(001,22)
E(111,00)	E(011,00)*	E(101,00)*	E(110,00)*
E(111,10)	E(011,10)	E(101,10)	E(110,10)
E(111,20)	E(011,20)	E(101,20)	E(110,20)
E(111,01)	E(011,01)	E(101,01)	E(110,01)
E(111,02)	E(011,02)	E(101,02)	E(110,02)
E(111,11)	E(011,11)	E(101,11)	E(110,11)
E(111,12)	E(011,12)	E(101,12)	E(110,12)
E(111,21)	E(011,21)	E(101,21)	E(110,21)
E(111,22)	E(011,22)	E(101,22)	E(110,22)

---

\*) Indicates the grand mean, the main effects, and the two-factor interaction effects.



TABLE 8

Estimation of the Main and Two-Factor Interaction Effects in Subset 2

$$4 \begin{bmatrix} 9 & & & & & & \\ & 6 & & & & & \\ & & 18 & & & & \\ & & & 6 & & & \\ & & & & 18 & & \\ & & & & & 6 & \\ & & & & & & 9 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ -4 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} \hat{E}(100,00) \\ \hat{E}(100,10) \\ \hat{E}(100,20) \\ \hat{E}(100,01) \\ \hat{E}(100,02) \\ \hat{E}(011,00) \end{bmatrix} = \begin{bmatrix} Y(100,00) \\ Y(100,10) \\ Y(100,20) \\ Y(100,01) \\ Y(100,02) \\ Y(011,00) \end{bmatrix}$$

$$= \frac{1}{2^8 3^2} \begin{bmatrix} \hat{E}(100,00) \\ \hat{E}(100,10) \\ \hat{E}(100,20) \\ \hat{E}(100,01) \\ \hat{E}(100,02) \\ \hat{E}(011,00) \end{bmatrix} = \frac{1}{2^8 3^2} \begin{bmatrix} 65 & 0 & 2 & 0 & 2 & +9 \\ & 96 & 0 & 0 & 0 & 0 \\ & & 36 & 0 & 4 & +18 \\ & & & 96 & 0 & 0 \\ & & & & 36 & +18 \\ & & & & & 81 \end{bmatrix} \begin{bmatrix} Y(100,00) \\ Y(100,10) \\ Y(100,20) \\ Y(100,01) \\ Y(100,02) \\ Y(011,00) \end{bmatrix}$$

Symmetric



Tables 2 and 3 contain sketches of the equations of expectation. At the left are the expected responses  $\eta(000)$ ,  $\eta(110)$ , etc., with  $\eta(\quad)$  omitted. The equality signs have been omitted and the parameters (effects) have been indicated at the top, instead of as column vectors. In Table 2 it is understood that the + and - are to be read as +1 and -1. The matrices  $M_1$ ,  $M_2$ ,  $N_1$  and  $N_2$  have been indicated.

The notation for the effects is as follows: The symbol E denotes "effect", and the five positions correspond, in order, to A, B, C,  $\alpha$  and  $\beta$ . The comma separates A, B, and C from  $\alpha$  and  $\beta$ . Zero (0) in a position indicates that no effect of the corresponding factor is involved in the definition of E( ). One (1) in a position indicates that the linear effect of the corresponding factor is involved, and two (2) that the quadratic effect is involved.

The following typical examples should suffice to explain the notation:

Grand average	E(000,00)
Main effect of A	E(100,00)
Interaction, A with B	E(110,00)
Linear effect of $\alpha$	E(000,10)
Quadratic effect of $\alpha$	E(000,20)
Linear $\alpha$ by linear $\beta$	E(000,11)
Linear $\alpha$ by quadratic $\beta$	E(000,12)
Linear A by linear $\alpha$	E(100,10)
Linear A by quadratic $\alpha$	E(100,20)



Tables 4 and 5 exhibit the matrices  $M_1^T M_1$ ,  $M_2^T M_2$ ,  $N_1^T N_1$ , and  $N_2^T N_2$ ; and Table 6 displays the matrix of coefficients of the normal equations. This matrix is of order 72, and decomposes into four identical submatrices of order 18. These submatrices are of rank 9, which implies that it is impossible to uniquely estimate the 72 effects.

At this point it is reasonable to equate certain effects to zero, and to solve for the remaining effects. A selection of effects to be estimated is indicated by asterisks in Table 7. (The unstarred effects are put equal to zero.) The effects to be estimated are all of the main effects and two-factor interactions.

The subsets in Table 7 correspond to the submatrices of Table 6. Solutions for the effects in subset 1 are particularly easy, because the corresponding matrix of coefficients is a diagonal matrix.

Table 8 displays the solution for the effects in subsets 2, 3 and 4. The symbol  $Y(i_1 i_2 i_3, j_1 j_2)$  - where  $i_1$ ,  $i_2$ , and  $i_3$  take on the values 0 and 1; and  $j_1$  and  $j_2$  take on the values 0, 1 and 2 - denotes a linear combination of the observed responses. It occurs in the normal equation which corresponds to  $E(i_1 i_2 i_3, j_1 j_2)$ ; and is the inner product of the vector of responses with the vector of coefficients of  $E(i_1 i_2 i_3, j_1 j_2)$  in the equations of expectation.



From the inverse matrix, one may read the variances and covariances of the estimates. The maximum correlation between estimates is  $18/[(36)(81)]^{1/2} = 1/3$ .

There is very little correlation among the estimates. In fact, of the  $27(26)/2 = 351$  pairs of estimates, there is correlation between the estimates in only 18 pairs.

If, instead of using the present design, one used the design\*which is formed by adjoining the treatments of  $m_1$  to those of  $n_1$  and  $n_2$ , only 21 of the chosen effects would be estimable (the effects  $E(100,00)$ ,  $E(011,00)$ ,  $E(010,00)$ ,  $E(101,00)$ ,  $E(001,00)$ , and  $E(110,00)$  would not be estimable).

It is interesting to compare the variances of the estimates obtained from the two designs. They are identical, except in six instances. For  $\hat{E}(100,20)$ ,  $\hat{E}(100,02)$ ,  $\hat{E}(010,20)$ ,  $\hat{E}(010,02)$ ,  $\hat{E}(001,20)$ , and  $\hat{E}(001,02)$  the variance for the alternative design is  $8/9$  as large as for the present design.

\* As suggested in [1].



An illustrative example

The following example is taken from a paper by Youden and Zimmerman [3]. Youden was concerned with comparing various methods of producing tomato plant seedlings prior to transplanting in the field. Comparison was made by planting in the field and then weighing the ripe produce. Thus, the observations were pounds of tomatoes.

Although Youden used five methods of production, we shall select only three: flats, fibre pots, and fibre pots soaked in one percent sodium nitrate solution. Other factors considered were different soil conditions, different sizes of pots, different varieties of tomato, and different locations on the field. The factors and their levels are recorded below:

<u>Factor</u>	<u>Levels</u>	
Soil condition, A	Field Soil	0
	Plus fertilizer	1
Size of pot, B	Three-inch	0
	Four-inch	1
Variety of tomato, C	Bonny Best	0
	Marglobe	1
Method of production, $\alpha$	Flat	0
	Fibre	1
	Fibre + NO <sub>3</sub>	2
Location on field, $\beta$		0,1,2



The object of the experiment was to evaluate the effects of these factors on the yield of tomatoes. The observations have been recorded, along with the experiment design, in Table 1.

The first stage in the analysis is the same as for a complete factorial. We form a number of summary tables, which are given below:

TABLE 9  
Summary Tables

		Soil Condition (A)			
		<u>0</u>	<u>1</u>	<u>Total</u>	
Size of Pot (B)	0:	993.7	1210.9	2204.6	
	1:	1100.7	1415.9	2516.6	
	Total:	2094.4	2626.8	4721.2	

		Soil Condition (A)			
		<u>0</u>	<u>1</u>	<u>Total</u>	
Variety (C)	0:	980.3	1078.8	2059.1	
	1:	1114.1	1548.0	2662.1	
	Total:	2094.4	2626.8	4721.2	

		Size of Pot (B)			
		<u>0</u>	<u>1</u>	<u>Total</u>	
Variety (C)	0:	931.1	1128.0	2059.1	
	1:	1273.5	1388.6	2662.1	
	Total:	2204.6	2516.6	4721.2	

		Method of Production (α)				
		<u>0</u>	<u>1</u>	<u>2</u>	<u>Total</u>	
Soil Cond. (A)	0:	640.5	608.2	845.7	2094.4	
	1:	813.5	848.3	965.0	2626.8	
	Total:	1454.0	1456.5	1810.7	4721.2	



Table 9 (Continued)

		Method of Production ( $\alpha$ )			
		<u>0</u>	<u>1</u>	<u>2</u>	<u>Total</u>
Size of Pot (B)	0:	705.3	601.7	897.6	2204.6
	1:	748.7	854.8	913.1	2516.6
	Total:	1454.0	1456.5	1810.7	4721.2

		Method of Production ( $\alpha$ )			
		<u>0</u>	<u>1</u>	<u>2</u>	<u>Total</u>
Variety (C)	0:	676.9	595.7	786.5	2059.1
	1:	777.1	860.8	1024.2	2662.1
	Total:	1454.0	1456.5	1810.7	4721.2

		Location ( $\beta$ )			
		<u>0</u>	<u>1</u>	<u>2</u>	<u>Total</u>
Soil Cond. (A)	0:	617.0	635.2	842.2	2094.4
	1:	784.0	767.6	1075.2	2626.8
	Total:	1401.0	1402.8	1917.4	4721.2

		Location ( $\beta$ )			
		<u>0</u>	<u>1</u>	<u>2</u>	<u>Total</u>
Size of Pot (B)	0:	652.5	592.9	959.2	2204.6
	1:	748.5	809.9	958.2	2516.6
	Total:	1401.0	1402.8	1917.4	4721.2

		Location ( $\beta$ )			
		<u>0</u>	<u>1</u>	<u>2</u>	<u>Total</u>
Variety (C)	0:	595.4	552.7	911.0	2059.1
	1:	805.6	850.1	1006.4	2662.1
	Total:	1401.0	1402.8	1917.4	4721.2

		Method of Production ( $\alpha$ )			
		<u>0</u>	<u>1</u>	<u>2</u>	<u>Total</u>
Location ( $\beta$ )	0:	420.5	465.3	515.2	1401.0
	1:	351.3	472.0	579.5	1402.8
	2:	682.2	519.2	716.0	1917.4
	Total:	1454.0	1456.5	1810.7	4721.2



The entry in any cell is the sum of the observations which have the row and column designations associated with the cell. For example, in the last table, the first entry is the sum of the observations in Table 1 which have  $\alpha = 0$  and  $\beta = 0$ , i.e., the first four observations.

We calculate the Y's from the summary tables. For example, from the first summary table we compute

$$Y(100,00) = -993.7 - 1100.7 + 1210.9 + 1415.9 = 532.4$$

$$Y(010,00) = -993.7 + 1100.7 - 1210.9 + 1415.9 = 312.0$$

$$\text{and } Y(110,00) = 993.7 - 1100.7 - 1210.9 + 1415.9 = 98.0$$

We observe that there is some repetition of Y's from the various tables. For instance,  $Y(100,00)$  is obtained from all four tables which have A as a factor. This repetition is unnecessary and in practice would be avoided. Having already calculated a Y from a previous table, we would not calculate it again.

The complete list of 27 distinct Y's is given in Table 10.



TABLE 10  
Values of Y's

$Y_1 = Y(000,00) = 4721.2$	$Y_{15} = Y(001,20) = -192.3$
$Y_2 = Y(100,00) = 532.4$	$Y_{16} = Y(000,01) = 516.4$
$Y_3 = Y(010,00) = 312.0$	$Y_{17} = Y(000,02) = 512.8$
$Y_4 = Y(110,00) = 98.0$	$Y_{18} = Y(100,01) = 66.0$
$Y_5 = Y(001,00) = 603.0$	$Y_{19} = Y(100,02) = 135.2$
$Y_6 = Y(101,00) = 335.4$	$Y_{20} = Y(010,01) = -97.0$
$Y_7 = Y(011,00) = -81.8$	$Y_{21} = Y(010,02) = -339.0$
$Y_8 = Y(000,10) = 356.7$	$Y_{22} = Y(001,01) = -114.8$
$Y_9 = Y(000,20) = 351.7$	$Y_{23} = Y(001,02) = -289.2$
$Y_{10} = Y(100,10) = -53.7$	$Y_{24} = Y(000,11) = -60.9$
$Y_{11} = Y(100,20) = -187.9$	$Y_{25} = Y(000,21) = 354.7$
$Y_{12} = Y(010,10) = -27.9$	$Y_{26} = Y(000,12) = -327.9$
$Y_{13} = Y(010,20) = -447.3$	$Y_{27} = Y(000,22) = 391.3$
$Y_{14} = Y(001,10) = 137.5$	

The formulae and estimates of the main and interaction effects are given in Table 11.



TABLE 11  
Estimated Effects

$$\begin{aligned}\hat{E}(000,00) &= Y_1/36 = 131.1 & \hat{E}(100,10) &= Y_{10}/12 = -4.5 \\ \hat{E}(000,10) &= Y_8/12 = 29.7 & \hat{E}(100,01) &= Y_{18}/12 = 5.5 \\ \hat{E}(000,01) &= Y_{16}/12 = 43.0 & \hat{E}(010,10) &= Y_{12}/12 = -2.3 \\ \hat{E}(000,20) &= Y_9/24 = 14.7 & \hat{E}(010,01) &= Y_{20}/12 = -8.1 \\ \hat{E}(000,02) &= Y_{17}/24 = 21.4 & \hat{E}(001,10) &= Y_{14}/12 = 11.5 \\ \hat{E}(000,11) &= Y_{24}/8 = -7.6 & \hat{E}(001,01) &= Y_{22}/12 = -9.6 \\ \hat{E}(000,21) &= Y_{25}/16 = 22.2 \\ \hat{E}(000,12) &= Y_{26}/16 = -20.5 \\ \hat{E}(000,22) &= Y_{27}/32 = 12.2\end{aligned}$$

$$\begin{aligned}\hat{E}(100,00) &= (65Y_2 + 2Y_{11} + 2Y_{19} + 9Y_7)/1152 = 29.3 \\ \hat{E}(100,20) &= (Y_2 + 18Y_{11} + 2Y_{19} + 9Y_7)/384 = -8.6 \\ \hat{E}(100,02) &= (Y_2 + 2Y_{11} + 18Y_{19} + 9Y_7)/384 = 4.8 \\ \hat{E}(011,00) &= (Y_2 + 2Y_{11} + 2Y_{19} + 9Y_7)/128 = -2.4\end{aligned}$$

$$\begin{aligned}\hat{E}(010,00) &= (65Y_3 + 2Y_{13} + 2Y_{21} + 9Y_6)/1152 = 18.9 \\ \hat{E}(010,20) &= (Y_3 + 18Y_{13} + 2Y_{21} + 9Y_6)/384 = -14.1 \\ \hat{E}(010,02) &= (Y_3 + 2Y_{13} + 18Y_{21} + 9Y_6)/384 = -9.5 \\ \hat{E}(101,00) &= (Y_3 + 2Y_{13} + 2Y_{21} + 9Y_6)/128 = +13.7\end{aligned}$$

$$\begin{aligned}\hat{E}(001,00) &= (65Y_5 + 2Y_{15} + 2Y_{23} + 9Y_4)/1152 = 34.0 \\ \hat{E}(001,20) &= (Y_5 + 18Y_{15} + 2Y_{23} + 9Y_4)/384 = -6.7 \\ \hat{E}(001,02) &= (Y_5 + 2Y_{15} + 18Y_{23} + 9Y_4)/384 = -10.7 \\ \hat{E}(110,00) &= (Y_5 + 2Y_{15} + 2Y_{23} + 9Y_4)/128 = 4.1\end{aligned}$$



These formulae have been arranged into four sets. All of the estimated effects in the first set are uncorrelated. In addition the estimated effects within any set are uncorrelated with the estimated effects in any other set.

The analysis of variance is given in Table 12.

Table 12  
Analysis of Variance

<u>Source of Variation</u>	<u>Degrees of Freedom</u>	<u>Sum of Squares</u>	<u>Mean Square</u>
Effects	26	58975.51	2268
Residual	<u>9</u>	<u>4150.08</u>	<u>461</u>
Total	35	63125.59	

In order to evaluate the effects, t-tests were carried out. It was found that  $\hat{E}(100,00)$ ,  $\hat{E}(001,00)$ ,  $\hat{E}(000,10)$  and  $\hat{E}(000,01)$  are significant at the .01 level of significance. There were no other significant effects.



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