NBS REPORT

5595

EVALUATION OF THE FIRST AND SECOND MOMENT INTEGRALS OF A CERTAIN PROBABILITY DENSITY FUNCTION BY AN APPLICATION OF THE THEORY OF GAUSSIAN QUADRATURE

by

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and

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U. S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

BOULDER LABORATORIES

Boulder, Colorado

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NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT | NBS REPORT
8330-11-8332 | August 25, 1958 | 5595

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ABSTRACT

The first moment integral, $|\bar{\Omega}|$, and the second moment integral, σ_Ω^2, of the probability density function, $p(\Omega)$,

$$|\bar{\Omega}| = 2\int_0^\pi \Omega\, p(\Omega)\, d\Omega,$$

and

$$\sigma_\Omega^2 = 2\int_0^\pi \Omega^2\, p(\Omega)\, d\Omega,$$

are evaluated by an application of the theory of Gaussian quadrature. The mean absolute value, $|\bar{\Omega}|$, and the standard deviation, σ_Ω, are tabulated as a function of the random Raleigh distributed component of relative intensity, k^2.

Introduction to the Problem

The investigations of the time-harmonic Fourier integral by the authors[1] led to the conclusion that high efficiency could be obtained in the numerical evaluation of such integrals by employing the theory of Gaussian quadrature[2]. It seemed quite likely that many of the more simple integrals which appear as a result of the theoretical study of radio wave propagation could be evaluated to a high degree of precision by an application of this method. The first and second moment probability density function integrals evaluated in this paper illustrate the power of this method of analysis. These integrals are also important[3] in the analysis of the effects of a rough ionosphere on the propagation of radio waves.

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[1] J. R. Johler and L. C. Walters, "Transmission of a Ground Wave Pulse Around a Finitely Conducting Spherical Earth," National Bureau of Standards Report No. 5566, April 17, 1958 (to be published).

[2] Z. Kopal, "Numerical Analysis," John Wiley and Sons, Inc., New York, N. Y., (1955), p. 367. See also p. 348.

[3] K. A. Norton, "Transmission Loss in Radio Propagation, II," National Bureau of Standards Report No. 5092, July 25, 1957. Appendix I, pp 1-11. See especially eqs. I-35 and I-36.

Theory

The evaluation of the mean absolute value, $|\bar{\Omega}|$, of the vector argument or "phase," Ω, and the corresponding variance, σ_{Ω}^2, integrals[3],

$$|\bar{\Omega}| = 2 \int_0^{\pi} \Omega \, p(\Omega) \, d\Omega, \tag{1}$$

and

$$\sigma_{\Omega}^2 = 2 \int_0^{\pi} \Omega^2 \, p(\Omega) \, d\Omega \tag{2}$$

as a function of the modulus squared or "intensity" variational component, k^2 (determined relative to the steady component) is the prime task of this paper.

The probability distribution, $p(\Omega)$, for this problem is[3],

$$2\pi p(\Omega) = \left\{ 1 + \sqrt{\pi} \, z \exp(z^2) \left[1 + \mathrm{erf}(z) \right] \right\} \exp(-1/k^2), \tag{3}$$

where $z = \frac{\cos \Omega}{k}$ (4)

and, $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} \, du.$ (5)

The error function, erf(z), can be evaluated by the following convergent series:

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \left[z - \frac{z^3}{1!3} + \frac{z^5}{2!5} - \frac{z^7}{3!7} + \ldots \right] \tag{6}$$

It is frequently more efficient for large values of z, (z >1) to use the asymptotic expansion for real values of z,

$$1 - \mathrm{erf}(z) = \frac{1}{z\sqrt{\pi}} \exp(-z^2)\left[1 - \frac{1}{2z^2} + \frac{1\cdot 3}{(2z^2)^2} - \frac{1\cdot 3\cdot 5}{(2z^2)^3} + \frac{1\cdot 3\cdot 5\cdot 7}{(2z^2)^4} - \dots\right] \qquad (7)$$

The mean absolute value, $|\bar{\Omega}|$, and the variance, σ_Ω^2, integrals can be represented as a finite sum,

$$|\bar{\Omega}| = \frac{1}{\pi} \exp(-1/k^2) \int_0^\pi \Omega p_o(\Omega)\, d\Omega = \frac{1}{\pi} \exp(-1/k^2) \sum_{m=1}^{M} W_m \Omega_m p_o(\Omega_m) + \epsilon_M , \qquad (8)$$

$$\sigma_\Omega^2 = \frac{1}{\pi} \exp(-1/k^2) \int_0^\pi \Omega^2 p_o(\Omega)\, d\Omega = \frac{1}{\pi} \exp(-1/k^2) \sum_{m=1}^{M} W_m \Omega^2_m p_o(\Omega_m) + \epsilon'_M . \qquad (9)$$

ϵ_M and ϵ'_M are error terms which can in general be made arbitrarily small by increasing M.

$$p_o(\Omega) = 2\pi \exp(1/k^2)\, p(\Omega) \qquad (10)$$

$$m = 1, 2, 3 \dots M \qquad (11)$$

$$\Omega_m = \frac{1}{2}\left[\Omega_b - \Omega_a\right] x_m + \frac{1}{2}\left[\Omega_b + \Omega_a\right] \qquad (12)$$

$\Omega_b = \pi$, and $\Omega_a = 0$, are the upper and lower limits of integration, respectively. The weight functions can be determined from the limits of integration,

$$W_m = \frac{1}{2}\left[\Omega_b - \Omega_a\right] H_m . \qquad (13)$$

Thus, integer M and the limits of integration, Ω_a, Ω_b, determine the particular values of the integrand to be calculated in the quadrature. The "universal" constants of the theory of Gaussian quadrature, H_m, x_m, Table II[4], can be determined for various M,

$$\int_{-1}^{1} f(x)dx = \sum_{m=1}^{M} H_m \, f(x_m). \tag{14}$$

The abscissas, x_m, are the roots of the Legendre polynomials defined by,

$$\frac{d^m}{dx^m} (x^2 - 1)^m - 2^m \, m! \, P_m(x) = 0, \tag{15}$$

$$\begin{aligned} P_o(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{3}{2} x^2 - \frac{1}{2} \\ P_3(x) &= \frac{5}{2} x^2 - \frac{3}{2} x \\ P_4(x) &= \frac{35}{8} x^4 - \frac{15}{4} x^2 - \frac{3}{8} \end{aligned} \tag{16}$$

Polynomials of higher degree are determined by use of the recursion formula,

$$(m + 1) P_{m+1}(x) + m P_{m-1}(x) - (2m + 1) x P_m(x) = 0. \tag{17}$$

---

4 P. Davis, P. Rabinowitz, "Abscissas and Weights for Gaussian Quadrature of High Order," National Bureau of Standards Journal of Research, Vol. 56, Research Paper 2645, 1956, p. 35-37.

The weight coefficients, H_m, can be determined from the roots, x_m,

$$H_m = \frac{2}{(1 - x_m^2)\left[P'_m(x_m)\right]^2} \tag{18}$$

Since the available weights, H_m, and abscissas, x_m, are limited, M = 48, Table II, it is quite possible for very precise work to split each integral somewhat arbitrarily but consistent with efficiency,

$$\int_{\Omega_a}^{\Omega_b} f(\Omega)\, d\Omega = \int_{\Omega_a}^{\Omega_1} f(\Omega)\, d\Omega + \int_{\Omega_1}^{\Omega_2} f(\Omega)\, d\Omega + \int_{\Omega_2}^{\Omega_3} f(\Omega)\, d\Omega + \ldots + \int_{\Omega_n}^{\Omega_b} f(\Omega)\, d\Omega, \tag{19}$$

in which a specified number of intervals, n = 1, 2, 3 . . . , has been selected with limits of integration, Ω_a, Ω_1, Ω_2, Ω_3, . . . Ω_b. Each integral is evaluated by the previously described quadrature (8, 9) with the maximum available abscissas and weights (M = 48).

Computation

The first and second moment probability density function integrals, $|\overline{\Omega}|$, and σ_Ω^2, were evaluated between the limits, $\Omega_a = 0$, $\Omega_b = \pi$, employing the maximum available (M = 48) Gaussian abscissas, x_m, and weight, H_m, for values of the random Raleigh distributed component of relative intensity, k^2, between 0.01 and 1000. According to the theory of Gaussian quadrature, this integration is equivalent to fitting a 95th degree polynomial (2m - 1) degree at 48 points, to the integrand, which points are weighted according to previously described rules (13) at the particular values of phase, $\Omega = \Omega_m$ (12).

The integration was checked at values, $k^2 = 0.01, 0.1, 1, 10, 100$ and 1000, by splitting the integral into two integrals, $\Omega_a = 0$, $\Omega_1 = \frac{\pi}{2}$, $\Omega_b = \pi$, $n = 1$, (19). A maximum precision variation of ± 1 was noted in the eighth significant figure. This indicated that the computation precision for this range of k^2 values was governed by the digital capacity of the electronic computer eight significant floating point system[5] rather than the quadrature. The results of the computation are illustrated, Fig 1., and are presented for eight significant figures in Table I.[6]

Discussion and Conclusions

The results of the computation of the first and second moment probability density function integrals demonstrate the high computation precision which can be achieved by an application of the theory of Gaussian quadrature to integrals with finite limits. The method is, of course, designed about digital e. d. p.[7] and would indeed prove quite tedious when applied to m. d. p. However, this

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5 I. B. M. 650 -407 - E. D. P. machine.

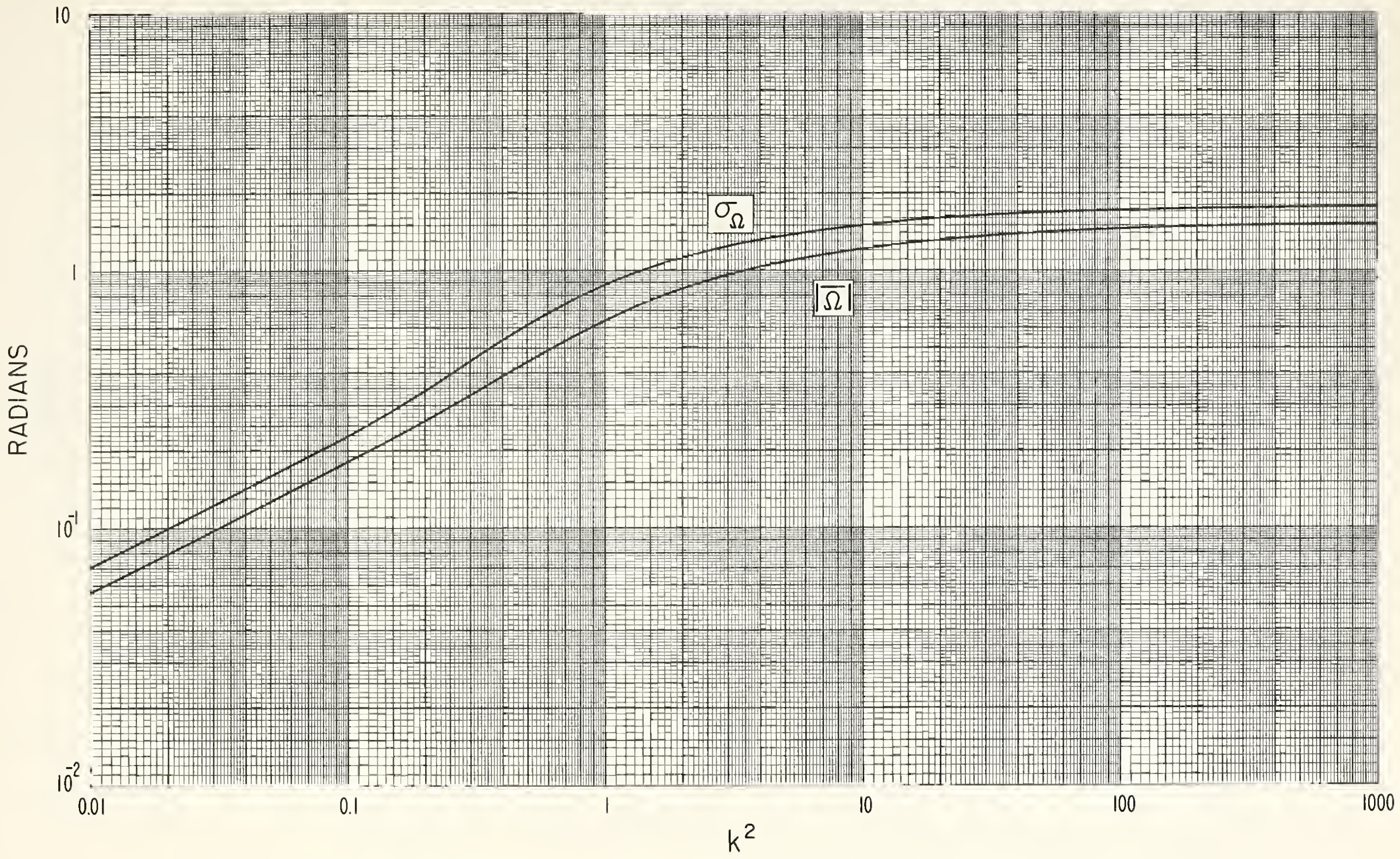
6 The integer to the right of each table entry, if present, indicates the power of the factor ten, (10) by which the number is multiplied, thus positioning the decimal point. For example, 5. 6513905 - 2 = 0. 056513905. Note that: $k^2 = \infty$, $\sigma_\Omega = \pi/\sqrt{3}$ and $|\bar{\Omega}| = \pi/2$.

7 m. d. p. = manual data processing.
e. d. p. = electronic data processing.

type of problem would probably not be attempted, at least to a high degree of precision, by m. d. p. The only limitation to e. d. p. seems to be the digit capacity of the machine. The efficiency of the computation could be improved by a reduction in the number, M, to 40, 32, 24, 20, 16, 8 or 4, Table II, provided the required computation precision is not degraded.

Acknowledgements

This work was suggested by K. A. Norton, of the National Bureau of Standards, in connection with theoretical sky-wave studies of NBS Project 8330-11-8332. The careful electronic computer work was developed by L. C. Walters (co-author) and C. M. Lilley was responsible for printing the tables. The manuscript was typed by Eileen Brackett and the drawing was prepared by John Harman.



<u>Fig. 1.</u> Mean absolute value, $|\overline{\Omega}|$, and standard deviation, σ_Ω, as a function of the random Raleigh distributed component relative intensity, k^2.

TABLE I.

k^2	$\lvert\bar{\Omega}\rvert$ radians	σ_Ω radians
0.010	5.6513905 - 2	7.0889665 - 2
0.011	5.9281802 - 2	7.4368405 - 2
0.012	6.1929049 - 2	7.7695630 - 2
0.015	6.9273763 - 2	8.6933250 - 2
0.020	8.0059192 - 2	1.0051253 - 1
0.021	8.2050463 - 2	1.0302171 - 1
0.022	8.3996053 - 2	1.0547390 - 1
0.025	8.9586354 - 2	1.1252438 - 1
0.030	9.8223130 - 2	1.2342886 - 1
0.031	9.9864166 - 2	1.2550275 - 1
0.032	1.0148000 - 1	1.2754525 - 1
0.035	1.0618661 - 1	1.3349834 - 1
0.040	1.1361951 - 1	1.4291103 - 1
0.041	1.1505149 - 1	1.4472622 - 1
0.042	1.1646690 - 1	1.4652096 - 1
0.045	1.2061991 - 1	1.5179011 - 1
0.050	1.2726013 - 1	1.6022582 - 1
0.051	1.2854991 - 1	1.6186600 - 1
0.052	1.2982785 - 1	1.6349168 - 1
0.055	1.3359414 - 1	1.6828606 - 1
0.060	1.3966386 - 1	1.7602361 - 1
0.061	1.4084930 - 1	1.7753642 - 1
0.062	1.4202581 - 1	1.7903826 - 1
0.065	1.4550327 - 1	1.8348099 - 1
0.070	1.5113924 - 1	1.9069203 - 1
0.071	1.5224405 - 1	1.9210734 - 1
0.072	1.5334169 - 1	1.9351423 - 1
0.075	1.5659419 - 1	1.9768590 - 1
0.080	1.6188698 - 1	2.0448601 - 1
0.081	1.6292752 - 1	2.0582473 - 1
0.082	1.6396238 - 1	2.0715680 - 1
0.085	1.6703356 - 1	2.1111367 - 1
0.090	1.7204794 - 1	2.1758636 - 1
0.091	1.7303579 - 1	2.1886366 - 1
0.092	1.7401916 - 1	2.2013540 - 1
0.095	1.7694134 - 1	2.2391945 - 1

k^2	$\|\Omega\|$ radians	σ_{Ω} radians
0.10	1.8172472 - 1	2.3012770 - 1
0.11	1.9099601 - 1	2.4221559 - 1
0.12	1.9992282 - 1	2.5393404 - 1
0.15	2.2508374 - 1	2.8749828 - 1
0.20	2.6329686 - 1	3.4032320 - 1
0.21	2.7053872 - 1	3.5059207 - 1
0.22	2.7767192 - 1	3.6077869 - 1
0.25	2.9848460 - 1	3.9085106 - 1
0.30	3.3147043 - 1	4.3923906 - 1
0.31	3.3783684 - 1	4.4863226 - 1
0.32	3.4413045 - 1	4.5792448 - 1
0.35	3.6258665 - 1	4.8517326 - 1
0.40	3.9199002 - 1	5.2841280 - 1
0.41	3.9767434 - 1	5.3672815 - 1
0.42	4.0329502 - 1	5.4493255 - 1
0.45	4.1978140 - 1	5.6888385 - 1
0.50	4.4604899 - 1	6.0664350 - 1
0.51	4.5112729 - 1	6.1388105 - 1
0.52	4.5614885 - 1	6.2101680 - 1
0.55	4.7088003 - 1	6.4182710 - 1
0.60	4.9436251 - 1	6.7460915 - 1
0.61	4.9890499 - 1	6.8089385 - 1
0.62	5.0339766 - 1	6.8709150 - 1
0.65	5.1658413 - 1	7.0517715 - 1
0.70	5.3763008 - 1	7.3371715 - 1
0.71	5.4170535 - 1	7.3919745 - 1
0.72	5.4573767 - 1	7.4460535 - 1
0.75	5.5758218 - 1	7.6040620 - 1
0.80	5.7651767 - 1	7.8540860 - 1
0.81	5.8018927 - 1	7.9022010 - 1
0.82	5.8382359 - 1	7.9497130 - 1
0.85	5.9450880 - 1	8.0887390 - 1
0.90	6.1162274 - 1	8.3093795 - 1
0.91	6.1494579 - 1	8.3519350 - 1
0.92	6.1823668 - 1	8.3939890 - 1
0.95	6.2792124 - 1	8.5172250 - 1

k^2	$\lvert\bar{\Omega}\rvert$ radians	σ_Ω radians
1.0	6.4346186 - 1	8.7133750 - 1
1.1	6.7247370 - 1	9.0744135 - 1
1.2	6.9902834 - 1	9.3992080 - 1
1.5	7.6682649 - 1	1.0205470
2.0	8.5195625 - 1	1.1175455
2.1	8.6599590 - 1	1.1331300
2.2	8.7925345 - 1	1.1477455
2.5	9.1498336 - 1	1.1866651
3.0	9.6399295 - 1	1.2389914
3.1	9.7254727 - 1	1.2480052
3.2	9.8075157 - 1	1.2566181
3.5	1.0034854	1.2803230
4.0	1.0361770	1.3140093
4.1	1.0420645	1.3200272
4.2	1.0477624	1.3258374
4.5	1.0638134	1.3421326
5.0	1.0875730	1.3660625
5.1	1.0919370	1.3704336
5.2	1.0961852	1.3746814
5.5	1.1082832	1.3867404
6.0	1.1265439	1.4048374
6.1	1.1299436	1.4081929
6.2	1.1332664	1.4114684
6.5	1.1428021	1.4208462
7.0	1.1573986	1.4351373
7.1	1.1601429	1.4378157
7.2	1.1628333	1.4404390
7.5	1.1705980	1.4479959
8.0	1.1826098	1.4596450
8.1	1.1848849	1.4618459
8.2	1.1871207	1.4640069
8.5	1.1936018	1.4702620
9.0	1.2037103	1.4799901
9.1	1.2056364	1.4818398
9.2	1.2075323	1.4836595
9.5	1.2130479	1.4889462

k^2	$\lvert\overline{\Omega}\rvert$ radians	σ_Ω radians
10	1.2217077	1.4972268
11	1.2372944	1.5120705
12	1.2509646	1.5250259
15	1.2836835	1.5558026
20	1.3212337	1.5907359
21	1.3271198	1.5961754
22	1.3326079	1.6012386
25	1.3470845	1.6145548
30	1.3662719	1.6321174
31	1.3695492	1.6351074
32	1.3726740	1.6379555
35	1.3812409	1.6457515
40	1.3933403	1.6567297
41	1.3954936	1.6586796
42	1.3975702	1.6605589
45	1.4033835	1.6658143
50	1.4118941	1.6734927
51	1.4134456	1.6748905
52	1.4149523	1.6762476
55	1.4192256	1.6800932
60	1.4256277	1.6858461
61	1.4268134	1.6869104
62	1.4279704	1.6879488
65	1.4312811	1.6909181
70	1.4363217	1.6954337
71	1.4372656	1.6962787
72	1.4381902	1.6971060
75	1.4408523	1.6994873
80	1.4449535	1.7031527
81	1.4457283	1.7038446
82	1.4464888	1.7045237
85	1.4486897	1.7064882
90	1.4521112	1.7095402
91	1.4527618	1.7101201
92	1.4534017	1.7106904
95	1.4552605	1.7123466

k^2	$\|\Omega\|$ radians	σ_Ω radians
100	1.4581716	1.7149389
110	1.4633887	1.7195798
120	1.4679420	1.7236250
150	1.4787613	1.7332182
200	1.4910552	1.7440872
210	1.4929717	1.7457784
220	1.4947562	1.7473525
250	1.4994527	1.7514919
300	1.5056546	1.7569509
310	1.5067115	1.7578805
320	1.5077186	1.7587659
350	1.5104767	1.7611897
400	1.5143646	1.7646036
410	1.5150558	1.7652101
420	1.5157220	1.7657947
450	1.5175857	1.7674297
500	1.5203113	1.7698191
510	1.5208079	1.7702541
520	1.5212896	1.7706766
550	1.5226565	1.7718738
600	1.5247021	1.7736652
610	1.5250810	1.7739967
620	1.5254503	1.7743202
650	1.5265072	1.7752451
700	1.5281154	1.7766522
710	1.5284164	1.7769155
720	1.5287113	1.7771734
750	1.5295601	1.7779157
800	1.5308671	1.7790585
810	1.5311138	1.7792741
820	1.5313563	1.7794859
850	1.5320572	1.7800987
900	1.5331466	1.7810506
910	1.5333538	1.7812315
920	1.5335573	1.7814094
950	1.5341489	1.7819263
1000	1.5350752	1.7827352

TABLE II.

GAUSSIAN QUADRATURE ABSCISSAS AND WEIGHT COEFFICIENTS

Abscissas, x_m		Weights, H_m	
M = 2			
0.5773502691	89625764509	1.0000000000	0000000000
M = 4			
0.8611363115	94052575224	0.3478548451	3745385737
0.3399810435	84856264803	0.6521451548	6254614262
M = 8			
0.9602898564	97536231684	0.1012285362	9037625915
0.7966664774	13626739592	0.2223810344	5337447054
0.5255324099	16328985818	0.3137066458	7788728733
0.1834346424	95649804939	0.3626837833	7836198296
M = 16			
0.9894009349	91649932596	0.0271524594	1175409485
0.9445750230	73232576078	0.0622535239	3864789286
0.8656312023	87831743880	0.0951585116	8249278481
0.7554044083	55003033895	0.1246289712	5553387205
0.6178762444	02643748447	0.1495959888	1657673208
0.4580167776	57227386342	0.1691565193	9500253818
0.2816035507	79258913230	0.1826034150	4492358886
0.0950125098	37637440185	0.1894506104	5506849628
M = 20			
0.9931285991	85094924786	0.0176140071	3915211831
0.9639719272	77913791268	0.0406014298	0038694133
0.9122344282	51325905868	0.0626720483	3410906357
0.8391169718	22218823395	0.0832767415	7670474872
0.7463319064	60150792614	0.1019301198	1724043503
0.6360536807	26515025453	0.1181945319	6151841731
0.5108670019	50827098004	0.1316886384	4917662689
0.3737060887	15419560673	0.1420961093	1838205132
0.2277858511	41645078080	0.1491729864	7260374678
0.0765265211	33497333755	0.1527533871	3072585069
M = 24			
0.9951872199	97021360180	0.0123412297	9998719954
0.9747285559	71309498198	0.0285313886	2893366318
0.9382745520	02732758524	0.0442774388	1741980616
0.8864155270	04401034213	0.0592985849	1543678074
0.8200019859	73902921954	0.0733464814	1108030573
0.7401241915	78554364244	0.0861901615	3195327591
0.6480936519	36975569252	0.0976186521	0411388827
0.5454214713	88839535658	0.1074442701	1596563478
0.4337935076	26045138487	0.1155056680	5372560135
0.3150426796	96163374387	0.1216704729	2780339120
0.1911188674	73616309159	0.1258374563	4682829612
0.0640568928	62605626085	0.1279381953	4675215697
M = 32			
0.9972638618	49481563545	0.0070186100	0947009660
0.9856115115	45268335400	0.0162743947	3090567060
0.9647622555	87506430774	0.0253920653	0926205945
0.9349060759	37739689171	0.0342738629	1302143310
0.8963211557	66052123965	0.0428358980	2222668065
0.8493676137	32569970134	0.0509980592	6237617619
0.7944837959	67942406963	0.0586840934	7853554714
0.7321821187	40289680387	0.0658222227	7636184683
0.6630442669	30215200975	0.0723457941	0884850622
0.5877157572	40762329041	0.0781938957	8707030647
0.5068999089	32229390024	0.0833119242	2694675522
0.4213512761	30635345364	0.0876520930	0440381114
0.3318686022	82127649780	0.0911738786	9576388471
0.2392873622	52137074545	0.0938443990	8080456563
0.1444719615	82796493485	0.0956387200	7927485941
0.0483076656	87738316235	0.0965400885	1472780056
M = 40			
0.9982377097	10559200350	0.0045212770	9853319125
0.9907262386	99457006453	0.0104982845	3115281361
0.9772599499	83774262663	0.0164210583	8190788871
0.9579168192	13791655805	0.0222458491	9416695726
0.9328128082	78676533361	0.0279370069	8002340109
0.9020988069	68874296728	0.0334601952	8254784739
0.8659595032	12259503821	0.0387821679	7447201764
0.8246122308	33311663196	0.0438709081	8567327199
0.7783056514	26519387695	0.0486958076	3507223206
0.7273182551	89927103281	0.0532278469	8393682435
0.6719566846	14179548379	0.0574397690	9939155136
0.6125538896	67980237953	0.0613062424	9292893916
0.5494671250	95128202076	0.0648040134	5660103807
0.4830758016	86178712909	0.0679120458	1523390382
0.4137792043	71605001525	0.0706116473	9128677969
0.3419940908	25758473007	0.0728865823	9580405906
0.2681521850	07253681141	0.0747231690	5796826420
0.1926975807	01371099716	0.0761103619	0062624237
0.1160840706	75255208483	0.0770398181	6424796558
0.0387724175	06050821933	0.0775059479	7842481126
M = 48			
0.9987710072	52426118601	0.0031533460	5230583863
0.9935301722	66350757548	0.0073275539	0127626210
0.9841245837	22826857745	0.0114772345	7923453949
0.9705915925	46247250461	0.0155793157	2294384872
0.9529877031	60430860721	0.0196161604	5735552781
0.9313866907	06554333114	0.0235707608	3932437914
0.9058791367	15569672822	0.0274265097	0835694820
0.8765720202	74247885906	0.0311672278	3279808890
0.8435882616	24393530711	0.0347772225	6477043889
0.8070662040	29442627083	0.0382413510	6583070631
0.7671590325	15740339254	0.0415450829	4346474921
0.7240341309	23814654674	0.0446745608	5669428041
0.6778723796	32663905212	0.0476166584	9249047482
0.6288673967	76513623995	0.0503590355	5385447495
0.5772247260	83972703818	0.0528901894	8519366709
0.5231609747	22233033678	0.0551995036	9998416286
0.4669029047	50958404545	0.0572772921	0040321570
0.4086864819	90716729916	0.0591148396	9839563574
0.3487558862	92160738160	0.0607044391	6589388005
0.2873624873	55455576736	0.0620394231	5989266390
0.2247637903	94689061225	0.0631141922	8625402565
0.1612223560	68891718056	0.0639242385	8464818662
0.0970046992	09462698930	0.0644661644	3595008220
0.0323801709	62869362033	0.0647376968	1268392250

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- Office of Basic Instrumentation
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