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STATISTICAL ANALYSIS OF A STATIONARY PROCESS

by

M. M. Siddiqui

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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS
BOULDER LABORATORIES
Boulder, Colorado

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ABSTRACT

The techniques developed in recent years to deal with stationary processes are presented in a manner suited to the purposes of those who use Fourier analysis or its variations for the analysis of their data. Necessary probability background is provided and the spectral analysis of discrete-time and continuous-time processes is carried out in detail with a number of examples to illustrate the applications of the theory.

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1. INTRODUCTION AND SUMMARY

In recent years the approach to time series has undergone some important changes, and the methods of analysis have been modified accordingly.

The classical approach to time series may be summarized as follows. We have observations, $x_1, \ldots, x_n$, on a time variable and we represent

$$x_t = m_t + \xi_t, \qquad t = 1, 2, \ldots, n$$

where m_t is the deterministic and ξ_t the random component of x_t at time t. ξ_t, $t = 1, \ldots, n$, are independent and identically distributed random variables, each with mean zero and variance σ^2. The distribution of ξ_t is usually assumed to be normal and the variance σ^2 to be unknown. m_t is represented either as a polynomial in t

$$p(t) = c_o + c_1 t + \ldots + c_k t^k,$$

where the degree of the polynomial, k, and the coefficients, $c_o, \ldots, c_k$, are assumed to be unknown constants; or m_t is represented as the sum of trigonometric terms

$$s(t) = a_o + a_1 \cos \lambda_1 t + b_1 \sin \lambda_1 t + \ldots + a_p \cos \lambda_p t + b_p \sin \lambda_p t,$$

where p, a's, b's and λ's are assumed to be unknown constants; or, more generally, m_t is assumed to be the sum of p(t) and s(t). The method of least-squares is then employed to estimate the coefficients in any of these representations and the periodogram, i.e. $(a_j^2 + b_j^2)$ multiplied by a constant, is used to test the significance of periods $2\pi/\lambda_j$.

For a summary treatment of these methods the reader may refer to [10, Chapters 29, 30].

In more recent years, the concept of stationarity of time series has been playing an important role. We may still try to decompose a time series into a deterministic component, m_t, and a random component, ξ_t, if there are reasons to believe that some of the factors underlying our observations have been changing deterministically during the course of our observation. However, there are reasons to believe that in many physical experiments these factors are in statistical equilibrium, i.e. changing randomly with a fixed distribution function, during the course of our observation, or, even if they change more violently, this change is random rather than deterministic. In any case the deterministic component, m_t, is dropped, and a completely stochastic model takes its place. Several stochastic models have been developed to deal with non-stationary processes which represent certain phenomena, e.g. population growth, Brownian motion, epidemics, traffic on telephones and highways, servicing etc., under specified conditions. We will not describe them here.

In this report we will confine attention to the techniques developed in recent years to deal with stationary processes, i.e. time series under statistical equilibrium. These techniques are specially suited to analyse radio propagation data, taken over a short interval of time under more or less constant conditions. The material presented herein, excepting one or two original results, is explicitly or implicitly available elsewhere. However, it seemed desirable to present it here in a manner suited to the purposes of those who use Fourier analysis or its variations in dealing with time series data. It is interesting to note that the classical representation, s(t), of a time series, still occupies a prominent place in the analysis, although the coefficients, a's and b's, are not considered constants any more but random variables.

In section 2 necessary probability background will be provided. In sections 3 and 4 the spectral analysis of discrete-time and continuous-time processes, respectively, will be dealt with. In section 5 the case of a discrete sample from a continuous-time process will be considered. Linear filters, the estimation of spectral densities, and some examples of stationary processes will be presented in sections 6, 7 and 8, respectively. The references at the end of the report include only such publications which are directly connected with the subject matter of this report. For a more complete bibliography the reader may consult the Bibliography at the end of [9].

2. SOME CONCEPTS IN PROBABILITY THEORY

The basic idea in statistical analysis is that of a sample space. A sample space, Ω, may be defined as the set of all possible distinct observations, ω, on a physical experiment, actual or hypothetical, repeated under essentially similar conditions. Any subset, S, of Ω is called the event S. An additively closed family, B, of subsets of Ω is constructed and a probability measure, P, is defined on B. These concepts are introduced as a theoretical foundation of probability theory. From a practical point of view, the usual frequency concept of the probability of an event will be sufficient for proceeding further.

It may be that the observation on an experiment is a real number or a set of real numbers, e.g. observing the pressure and temperature of a given volume of a gas, in which case Ω may be taken an Euclidean space of proper dimensions, each characteristic being a coordinate. In certain other experiments the observation may not be expressed as a set of real numbers, e.g. observing the sex and color of hair of an individual, in which case we may associate real numbers to these characteristics under some conventional system, e.g. 0 for female and 1 for male. A random variable, $X(\omega)$, is, then, defined as a real-valued function of ω, i.e., a function which associates a real number to every element ω of Ω. A complex-valued function of ω will be called a complex-valued random variable. The subsets of Ω are associated through $X(\omega)$ with the subsets of the real line, R_1, and hence a probability measure is induced on R_1. If S is a subset of R_1, we may speak of the probability that $X(\omega)$ belongs to S and write $\Pr(X(\omega) \in S)$. In particular

$$P(y) = \Pr(X(\omega) \leq y),$$

where y is a real number, is called the distribution function of $X(\omega)$. $P(y)$ is a non-decreasing function of its argument with $P(-\infty) = 0$ and $P(\infty) = 1$. It can be a step function or an absolutely continuous function or a mixture of both. In case $P(y)$ is absolutely continuous its derivative $P'(y) = p(y)$ is called the probability density function or the frequency function of the random variable $X(\omega)$. Obviously

$$p(y) \geq 0, \quad \Pr(X(\omega) \leq y) = P(y) = \int_{-\infty}^{y} p(x)dx, \quad \int_{-\infty}^{\infty} p(y)dy = 1.$$

The idea of a vector random variable presents no conceptual difficulty. In case of more than one random variable we speak of the joint distribution and the joint probability density functions.

A random process, $\{x(t,\omega)\}$, besides being a function of ω is also a function of t. We will speak of t as a time variable though it may be an arbitrary variable. For every value of t, $x(t,\omega)$ is a random variable. Thus a random process is a collection of random variables. For every fixed ω, $x(t,\omega)$ is a sample function or realization of the process. Thus

$$x(t,\omega) = A(\omega) \cos\left(\lambda(\omega)t + \phi(\omega)\right)$$

where (A, λ, ϕ) is a vector random variable, is a random process.

A random process may be visualized as a large number of simultaneous independent measurements of the same experiment over a long period of time. Theoretically the number of measurements is to be indefinitely large and the time of observation infinite in both directions. In practice we shall be dealing with a single realization of the process over a finite interval of time.

The time variable t may vary continuously or may take only discrete values for a given random process. In the former case we shall call the process a continuous-time process and in the latter case, a discrete-time process. A continuous-time process may be made discrete by considering it at discrete time points.

There is no conceptual difficulty in talking about a vector random process nor in considering t as a vector variable. In the following discussions, however, we will confine ourselves to a one-dimensional random process and t will always represent time. Furthermore, the letter ω will not occur explicitly and we will write $\{x(t)\}$ for a continuous-time process with $x(t)$ as a sample function and $\{x_t\}$ for a discrete-time process with x_t as a sample function. If u is a random variable, Eu will be used to denote its mean value over the sample space Ω.

A real random process, $\{x(t)\}$, will be called weakly stationary or stationary if, for all t and s,

(i) $Ex^2(t) < \infty$

(ii) $Ex(t) = Ex(0)$

(iii) $Ex(t)x(t+s) = Ex(0)x(s)$.

It will be called strictly stationary if the joint distribution of $x(t_1+h), \ldots, x(t_n+h)$ is identical with the joint distribution of $x(t_1), \ldots, x(t_n)$ for every n, $t_1, \ldots, t_n$ and every h.

A process is called Gaussian if $x(t_1), \ldots, x(t_n)$ have a joint normal distribution for every $n, t_1, \ldots, t_n$. A Gaussian process is completely specified by $Ex(t)$ and $Ex(t)x(s)$ given for all t and s. Hence a weakly stationary Gaussian process is strictly stationary.

3. FOURIER ANALYSIS OF A DISCRETE-TIME PROCESS

Let $\{x_t\}$ be a stationary discrete-time process with $Ex_t = m$ and $E(x_t-m)(x_{t+s}-m) = C_s$. C_s will be called the <u>autocovariance function</u> and $\rho_s = C_s/C_o$, the <u>autocorrelation function</u> of the process. Obviously, $C_{-s} = C_s$, $\rho_{-s} = \rho_s$. If $sC_s \to 0$ as $s \to \infty$, then, the series

$$\sum_{s=-\infty}^{\infty} C_s e^{-2\pi i s\lambda} = C_o + 2\sum_{s=1}^{\infty} C_s \cos 2\pi s\lambda$$

is a Fourier series of its sum $f(\lambda)$, so that

$$f(\lambda) = \sum_{s=-\infty}^{\infty} C_s e^{-2\pi i s\lambda} = C_o + 2\sum_{s=1}^{\infty} C_s \cos 2\pi s\lambda \tag{1}$$

$$C_s = \int_{-1/2}^{1/2} f(\lambda) e^{2\pi i s\lambda}\, d\lambda = \int_{-1/2}^{1/2} f(\lambda)\cos 2\pi s\lambda\, d\lambda. \tag{2}$$

We readily see from (1) that $f(-\lambda) = f(\lambda)$ and from (2) that $\int_{-1/2}^{1/2} f(\lambda)\, d\lambda = C_o$. $f(\lambda)$ is called the <u>power spectral density function</u> of the process $\{x_t\}$.

Consider the <u>periodogram</u>

$$I_n(\mu) \equiv \frac{1}{n}\left|\sum_{t=1}^{n} (x_t-m)e^{2\pi i t\mu}\right|^2 = \frac{1}{n}\sum_{k=1}^{n}\sum_{t=1}^{n} (x_t-m)(x_k-m)\, e^{2\pi i(t-k)\mu} . \tag{3}$$

We have $I_n(-\mu) = I_n(\mu)$ and

$$(4)\ EI_n(\mu) = \frac{1}{n} \sum_{k=1}^{n} \sum_{t=1}^{n} C_{k-t}\ e^{2\pi i(t-k)\mu} = \sum_{s=-(n-1)}^{n-1} \left(1 - \frac{|s|}{n}\right) C_s\, e^{-i2\pi s\mu} .$$

If $\sum_{1}^{\infty} sC_s < \infty$, we have

$$(5) \qquad \lim_{n\to\infty} EI_n(\mu) = \sum_{-\infty}^{\infty} C_s\, e^{-i2\pi s\mu} = f(\mu).$$

Since $I_n(\mu) \geq 0$, $f(\mu) \geq 0$.

Thus we may summarize the properties of $f(\lambda)$ in the following manner:

(i) $f(-\lambda) = f(\lambda)$ i.e. $f(\lambda)$ is symmetric,

(ii) $f(\lambda) \geq 0$ i.e. $f(\lambda)$ is non-negative,

(iii) $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, d\lambda = C_0$;

so that $f(\lambda)/C_0$ is a symmetrical frequency function over $(-\frac{1}{2}, \frac{1}{2})$. We will set $f(\lambda) = 0$ outside the interval $(-\frac{1}{2}, \frac{1}{2})$.

Writing

$$(6) \qquad F(\lambda) = \int_{-\frac{1}{2}}^{\lambda} f(\mu)\, d\mu, \qquad -\tfrac{1}{2} \leq \lambda \leq \tfrac{1}{2},$$

we will call $F(\lambda)$ the (power) spectral distribution function of $\{x_t\}$. $f^*(\lambda) = f(\lambda)/C_o$ and $F^*(\lambda) = f(\lambda)/C_o$ will be called the normalized spectral density and the normalized spectral distribution functions, respectively. Since $f(\lambda)$ is the Fourier series transformation of C_s, $f^*(\lambda)$ is the Fourier series transformation of ρ_s. Thus we have proved the following theorem.

Theorem 1. If the Fourier series transformation, $f^*(\lambda)$, of the autocorrelation function, ρ_s, of a discrete-time real stationary process exists, then it is a symmetrical frequency function over the interval $(-\frac{1}{2}, \frac{1}{2})$.

It can also be shown that given a symmetrical frequency function, $f^*(\lambda)$, over $(-\frac{1}{2}, \frac{1}{2})$, we can find a real discrete-time stationary process, $\{x_t\}$, such that the Fourier coefficients of $f^*(\lambda)$ are the autocorrelations of the process $\{x_t\}$.

Returning to the equation (5) we observe that if we have a sample $x_1, \ldots, x_n$ then the periodogram $I_n(\lambda)$ is an asymptotically unbiased estimator of $f(\lambda)$. However, if we insert the value of C_s from (2) into (4), we obtain

$$f_n(\mu) \equiv EI_n(\mu) = \sum_{s=-(n-1)}^{n-1} \left(1 - \frac{|s|}{n}\right) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i s(\lambda-\mu)} f(\lambda)\, d\lambda$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda) \left[\sum_{s=-(n-1)}^{n-1} \left(1 - \frac{|s|}{n}\right) e^{2\pi i s(\lambda-\mu)}\right] d\lambda .$$

Now, using the identities

$$\sum_{s=1}^{p} z^s = \frac{z-z^{p+1}}{1-z} ; \qquad \sum_{s=1}^{p} s z^s = \frac{z-(p+1)\, z^{p+1} + p z^{p+2}}{(1-z)^2} ,$$

with $p = n-1$, and $z = e^{\pm 2\pi i(\lambda-\mu)}$, we can sum the series in the brackets and, using certain trigonometrical identities, we finally obtain

$$(7)\quad f_n(\mu) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda) \frac{\sin^2 \pi n(\lambda-\mu)}{n \sin^2 \pi(\lambda-\mu)}\, d\lambda = \frac{1}{n^2} \int_{n(\mu-\frac{1}{2})}^{n(\mu+\frac{1}{2})} f(\mu - \frac{\nu}{n}) \frac{\sin^2 \pi\nu}{\sin^2(\pi\nu/n)}\, d\nu .$$

Hence $I_n(\mu)$ is an unbiased estimate of a 'filtered' spectral density $f_n(\mu)$ for finite n. Only when $n \to \infty$, $f_n(\mu) \to f(\mu)$.

For convenience we suppose that $n = 2N+1$ and represent the sample $x_1, \ldots, x_n$ by the trigonometric series

$$(8)\qquad x_t = a_o + \sum_{j=1}^{N} \left(a_j \cos \frac{2\pi jt}{n} + b_j \sin \frac{2\pi jt}{n} \right), \quad t = 1, 2, \ldots, n.$$

The coefficients a's and b's are determined by the usual method of harmonic analysis, i. e., by the relations

$$(9)\qquad \begin{aligned} a_o &= \frac{1}{n} \Sigma\, x_t \equiv \bar{x} , \\ a_j &= \frac{2}{n} \Sigma\, x_t \cos \frac{2\pi jt}{n} = \frac{2}{n} \Sigma\, (x_t - c) \cos \frac{2\pi jt}{n}, \quad j = 1, \ldots, N , \\ b_j &= \frac{2}{n} \Sigma\, x_t \sin \frac{2\pi jt}{n} = \frac{2}{n} \Sigma\, (x_t - c) \sin \frac{2\pi jt}{n}, \quad j = 1, \ldots, N , \end{aligned}$$

where the summations are with respect to t from 1 to n and where c is an arbitrary constant. Thus, setting $c = m$, we have

$$\text{(10)}\quad \frac{n}{4}\left(a_j^2 + b_j^2\right) = \frac{1}{n}\left(\Sigma(x_t - m)\cos\frac{2\pi jt}{n}\right)^2 + \frac{1}{n}\left(\Sigma(x_t - m)\sin\frac{2\pi jt}{n}\right)^2$$

$$= \frac{1}{n}\left|\Sigma(x_t - m)\, e^{\frac{i\,2\pi jt}{n}}\right|^2 = I_n\left(\frac{j}{n}\right), \quad j = 1, \ldots, N.$$

However, $I_n(0) = \frac{1}{n}\,|\Sigma(x_t - m)|^2 \equiv n(a_o - m)^2$, and depends on the knowledge of the process mean m. If we replace m by $\bar{x}$ everywhere, $I_n(\frac{j}{n})$ will not change for $j = 1, \ldots, N$, but $I_n(0) \equiv 0$ and no estimate at zero frequency will be available. It seems desirable, therefore, to estimate the process mean, m, by a much longer sample preferably from the observations which do not include $x_1, \ldots, x_n$.

Thus the representation (8) provides the periodogram estimate of the spectral density function at the frequencies j/n, $j = 1, \ldots, N$, in terms of the coefficients a_j and b_j.

We have

$$\text{(11)}\quad Ea_o = m,$$

$$\operatorname{var} a_o = E(a_o - m)^2 = \frac{1}{n^2}\, E\,|\Sigma(x_t - m)|^2 = \frac{1}{n}\, EI_n(0)$$

$$= \frac{1}{n}\, f_n(0).$$

Also, for $j \neq 0$,

$$E\,a_j = Eb_j = 0$$

$$\operatorname{var} a_j = Ea_j^2 = \frac{4}{n^2} E \sum_{t=1}^{n} \sum_{s=1}^{n} (x_t - m)(x_s - m) \cos \frac{2\pi jt}{n} \cos \frac{2\pi js}{n}$$

$$= \frac{2}{n^2} \sum_{t=1}^{n} \sum_{s=1}^{n} C_{t-s} \left[\cos \frac{2\pi j(t-s)}{n} + \cos \frac{2\pi j(t+s)}{n} \right]$$

$$= \frac{2}{n} \sum_{r=-(n-1)}^{n-1} \left(1 - \frac{|r|}{n}\right) C_r \cos \frac{2\pi jr}{n} + \frac{2}{n^2} C_0 \sum_{s=1}^{n} \cos \frac{4\pi js}{n}$$

$$+ \frac{4}{n^2} \sum_{r=1}^{n-1} \sum_{s=1}^{n-r} C_r \cos \frac{2\pi j(2s+r)}{n} .$$

The first summation is seen to be equal to $2/n\ EI_n(j/n) = 2/n\ f_n(j/n)$.

Using the trigonometrical identity

$$\sum_{s=1}^{p} \cos(\alpha + 2s\beta) = \operatorname{cosec} \beta \sin(p\beta) \cos\{\alpha + (p+1)\beta\}$$

with $\beta = 2\pi j/n$, $\alpha = 2\pi jr/n$, we find

$$\sum_{s=1}^{n-r} \cos \frac{2\pi j(2s+r)}{n} = -\cot \frac{2\pi j}{n} \sin \frac{2\pi jr}{n} .$$

Hence for $j \neq 0$,

$$\text{(12)} \qquad \operatorname{var} a_j = \frac{2}{n} f_n\left(\frac{j}{n}\right) - \frac{4}{n^2} \cot \frac{2\pi j}{n} \sum_{r=1}^{n-1} C_r \sin \frac{2\pi jr}{n}$$

$$= \frac{2}{n} f_n\left(\frac{j}{n}\right) + O\left(\frac{1}{n^2}\right), \text{ since } \sum_{r=1}^{\infty} C_r \sin \frac{2\pi jr}{n} < \infty .$$

It can be shown similarly that

$$\text{(12a)} \qquad \operatorname{var} b_j = \frac{2}{n} f_n\left(\frac{j}{n}\right) + \frac{4}{n^2} \cot \frac{2\pi j}{n} \sum_{r=1}^{n-1} C_r \sin \frac{2\pi jr}{n}$$

$$= \frac{2}{n} f_n\left(\frac{j}{n}\right) + O\left(\frac{1}{n^2}\right),$$

$$E\, a_i a_j = O\left(\frac{1}{n^2}\right), \quad E\, b_i b_j = O\left(\frac{1}{n^2}\right), \quad i \neq j,$$

$$E\, a_i b_j = O\left(\frac{1}{n^2}\right), \quad \text{all } i, j.$$

Thus the correlation between any two coefficients is of order $1/n$.

Now, if $\{x_t\}$ is a Gaussian process, a_j and b_j will be normally distributed and since different coefficients are approximately uncorrelated, they may be assumed to be independent of each other. Thus the $I_n(j/n)$ are also approximately independent of each other and

$$2\, I_n(j/n) \,/\, f_n(j/n)$$

is approximately a χ^2 variate with 2 degrees of freedom. We, therefore, have

$$\text{(13)} \qquad \operatorname{var} I_n(j/n) \cong f_n^{\,2}(j/n), \quad j = 1, \ldots, N.$$

Even if the process is not Gaussian, the result (13) can be proved to be valid. (13) shows that the periodogram, $I_n(\lambda)$, has standard error equal to (in fact a little greater than) $EI_n(\lambda)$, which makes it an undesirable estimate.

We will discuss some proposed modifications of $I_n(\lambda)$ in section 7.

4. FOURIER ANALYSIS OF A CONTINUOUS-TIME PROCESS

Let $\{x(t)\}$ be a continuous-time real stationary process with $Ex(t) = m$ and $E[x(t)-m][x(t+s)-m] = C(s)$, $\rho(s) = C(s)/C(0)$. If $\rho(s)$ is continuous at $s = 0$ then it can be shown that it is continuous for all s and also that

$$\lim_{s \to 0} E\,|x(t+s) - x(t)|^2 = 0 .$$

Consider the periodogram

$$(14)\quad I_T(\lambda) \equiv \frac{1}{T}\left|\int_0^T [x(t)-m]e^{2\pi i t\lambda}dt\right|^2 = \frac{1}{T}\int_0^T\int_0^T [x(t)-m][x(k)-m]e^{i2\pi(t-k)\lambda}dk\,dt.$$

We have

$$(15)\quad g_T(\lambda) \equiv EI_T(\lambda) = \frac{1}{T}\int_0^T\int_0^T C(k-t)e^{i2\pi(t-k)\lambda}dk\,dt = \int_{-T}^{T}\left(1 - \frac{|s|}{T}\right) C(s)e^{-i2\pi s\lambda}ds.$$

Since $I_T(\lambda) \geq 0$, it follows that

$$(16)\quad \int_0^T\int_0^T \rho(k-t)e^{i2\pi(t-k)\lambda}\,dk\,dt \geq 0 .$$

Since $|\rho(s)| \leq 1$, $\rho(0) = 1$, $\rho(s)$ is continuous and symmetric and the integral in (16) is non-negative, therefore [5, p. 91] we have the following theorem.

Theorem 2. If the autocorrelation function $\rho(s)$ of a continuous-time real stationary process is continuous at $s = 0$, it is the characteristic function of a symmetric distribution, i. e.

$$\rho(s) = \int_{-\infty}^{\infty} e^{2\pi i s\lambda} dG^*(\lambda) \tag{17}$$

where $G^*(\lambda)$ is a distribution function of a symmetric distribution over $(-\infty, \infty)$.

Conversely, it can be shown that for a symmetric distribution $dG^*(\lambda)$ there exists a real continuous-time stationary process such that the characteristic function of $dG^*(\lambda)$ is identical with the autocorrelation function of the process.

If $G^*(\lambda)$ is absolutely continuous and has a frequency function $g^*(\lambda) = dG^*/d\lambda$, then it follows that

$$g^*(\lambda) = \int_{-\infty}^{\infty} \rho(s) e^{-2\pi i s\lambda} ds , \tag{18}$$

and that $g^*(\lambda)$ is continuous. Writing $g(\lambda) = C(0) g^*(\lambda)$, we have the transform pair

$$C(s) = \int_{-\infty}^{\infty} e^{2\pi i s\lambda} g(\lambda)\, d\lambda \tag{19}$$

$$g(\lambda) = \int_{-\infty}^{\infty} e^{-2\pi i s\lambda} C(s)\, ds . \tag{20}$$

Since $C(-s) = C(s)$, we have $g(-\lambda) = g(\lambda)$; also $g(\lambda) \geq 0$ as $g^*(\lambda)$ is a frequency function, and

$$\int_{-\infty}^{\infty} g(\lambda)\, d\lambda = C(0).$$

$g(\lambda)$ is called the power spectral density function of the process $\{x(t)\}$.

If $\int_{0}^{\infty} sC(s)ds$ exists, then from (15) it follows that

$$\lim_{T\to\infty} g_T(\lambda) = g(\lambda).$$

However, if we insert the value of $C(s)$ from (19) into (15), we obtain

(21) $$g_T(\mu) = \int_{-T}^{T}\int_{-\infty}^{\infty}\left(1 - \frac{|s|}{T}\right) g(\lambda)\, e^{i2\pi(\lambda-\mu)s}\, d\lambda\, ds$$

$$= \int_{-\infty}^{\infty} g(\lambda)d\lambda \int_{-T}^{T}\left(1 - \frac{|s|}{T}\right) e^{i2\pi(\lambda-\mu)s}\, ds$$

$$= \int_{-\infty}^{\infty} g(\lambda)\, \frac{\sin^2 \pi T(\lambda-\mu)}{\pi^2 T(\lambda-\mu)^2}\, d\lambda$$

$$= \int_{0}^{\infty} \left[g\left(\mu + \frac{\nu}{T}\right) + g\left(\mu - \frac{\nu}{T}\right)\right] \frac{\sin^2 \pi\nu}{\pi^2 \nu^2}\, d\nu .$$

Thus, if we have a continuous sample over $(0, T)$, then the periodogram, $I_T(\mu)$, is an unbiased estimate of $g_T(\mu)$ which is a 'filtered' spectral density.

Now, the sample function over $(0, T)$ can be represented by a Fourier series, i.e.,

$$(22) \qquad x(t) = a_o + \sum_{j=1}^{\infty} \left(a_j \cos \frac{2\pi jt}{T} + b_j \sin \frac{2\pi jt}{T} \right), \quad 0 < t < T .$$

We then have

$$(23) \qquad a_o = \frac{1}{T} \int_o^T (x(t)\, dt = \bar{x} ,$$

$$a_j = \frac{2}{T} \int_o^T x(t) \cos \frac{2\pi jt}{T}\, dt = \frac{2}{T} \int_o^T [x(t)-c] \cos \frac{2\pi jt}{T}\, dt, \quad j \neq 0,$$

$$b_j = \frac{2}{T} \int_o^T x(t) \sin \frac{2\pi jt}{T}\, dt = \frac{2}{T} \int_o^T [x(t)-c] \sin \frac{2\pi jt}{T}\, dt,$$

where c is an arbitrary constant. Setting $c = m$, we have

$$(24) \qquad \frac{T}{4}\left(a_j^2 + b_j^2\right) = \frac{1}{T} \left| \int_o^T [x(t)-m]\, e^{\frac{i2\pi jt}{T}}\, dt \right|^2 = I_T(\tfrac{j}{T}), \quad j \neq 0.$$

Thus the representation (22) provides the periodogram estimate of $g_T(\lambda)$ at the frequencies $\lambda = j/T$, $j = 1, 2, \ldots,$ in terms of the Fourier coefficients a_j and b_j and from (23) we see that these estimates are independent of the process mean m. However,

$$I_T(0) = \frac{1}{T} \left| \int_o^T [x(t)-m]\, dt \right|^2 = T(a_o-m)^2 ,$$

and depends on the knowledge of m. If we replace m by $\bar{x}$ then $I_T(0) = 0$; hence, for an estimate at $\lambda = 0$, we require the knowledge of m independent of the sample mean.

With calculations similar to those carried out for the discrete-time case [see also 12 pp. 157-160] we can verify that

$$
\begin{aligned}
(25) \qquad & Ea_o = m, \quad Ea_j = Eb_j = 0, \quad j \neq 0, \\
& \text{var } a_o = \frac{1}{T} g_T(0), \\
& \text{var } a_j = \frac{2}{T} g_T(\frac{j}{T}) + O(\frac{1}{T^2}) \\
& \text{var } b_j = \frac{2}{T} g_T(\frac{j}{T}) + O(\frac{1}{T^2}) \\
& Ea_i a_j = O(\frac{1}{T^2}), \quad Eb_i b_j = O(\frac{1}{T^2}), \quad i \neq j, \\
& Ea_i b_j = O(\frac{1}{T^2}), \quad \text{all } i, j.
\end{aligned}
$$

Also

$$(26) \qquad \text{var } I_T(\lambda) \cong g_T^2(\lambda), \quad \lambda = j/T, \quad j \neq 0.$$

5. FOURIER ANALYSIS OF A CONTINUOUS-TIME PROCESS OBSERVED AT DISCRETE TIMES

The most interesting case is one in which we have a continuous record over $(0, T)$ as in section 4 but select a systematic sample $x(\alpha)$, $x(\alpha+\delta)$, $x(\alpha+2\delta)$, ..., $x(\alpha+\overline{n-1}\delta)$, where $0 \leq \alpha < \alpha+\overline{n-1}\,\delta \leq T$.

We may then consider two processes $\{x(t)\}$ and $\{x_t\}$ where $x_t = x(\alpha + \overline{t-1}\,\delta)$, $t = 0, \pm 1, \pm 2, \ldots.$ We therefore have $C_t = C(t\delta)$. If $f(\lambda)$ and $g(\lambda)$ are the spectral densities of $\{x_t\}$ and $\{x(t)\}$ respectively, we have

$$\int_{\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, e^{i2\pi t\lambda}\, d\lambda = \int_{-\infty}^{\infty} g(\lambda)\, e^{i2\pi t\delta\lambda}\, d\lambda$$

$$= \frac{1}{\delta} \sum_{r=-\infty}^{\infty} \int_{r-\frac{1}{2}}^{r+\frac{1}{2}} g(\lambda'/\delta)\, e^{i2\pi t\lambda'}\, d\lambda'$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi t\lambda} \left[\frac{1}{\delta} \sum_{r=-\infty}^{\infty} g\left(\frac{\lambda+r}{\delta}\right) \right] d\lambda$$

for every integer t. From the uniqueness of Fourier coefficients it follows that

$$f(\lambda) = \frac{1}{\delta} \sum_{r=-\infty}^{\infty} g\left(\frac{\lambda+r}{\delta}\right), \quad -\tfrac{1}{2} \le \lambda \le \tfrac{1}{2}. \tag{27}$$

The frequencies

$$\frac{\lambda \pm 1}{\delta}, \quad \frac{\lambda \pm 2}{\delta}, \ldots$$

are called aliases to λ/δ as they become indistinguishable from λ/δ when we consider the discrete-time process $\{x_t\}$ instead of the continuous time process $\{x(t)\}$.

If we set $\alpha = \delta$ and $n\delta = T$, we have

$$(28) \quad \frac{1}{n} f\left(\frac{j}{n}\right) = \frac{1}{T} \sum_{r=-\infty}^{\infty} g\left(\frac{j+nr}{T}\right)$$

$$= \frac{1}{T} g\left(\frac{j}{T}\right) + \frac{1}{T} \sum_{r=1}^{\infty} \left[g\left(\frac{j+nr}{T}\right) + g\left(\frac{j-nr}{T}\right) \right], -\frac{n}{2} \leq j \leq \frac{n}{2} .$$

Since $\int_0^{\infty} g(x+\mu)\, d\mu$ exists for every x, we have

$$K_n = \int_{n/T}^{\infty} g(x+\mu)\, d\mu \to 0 \qquad \text{as } n \to \infty .$$

Now $1/T \sum_{k=o}^{\infty} g\left(x + \frac{n+k}{T}\right)$ is an approximating Riemann sum to the integral K_n with $\Delta\mu = 1/T$ and hence tends to zero as $n \to \infty$. Therefore, if n is large, we have for small values of j

$$(28a) \qquad \frac{1}{n} f\left(\frac{j}{n}\right) \cong \frac{1}{T} g\left(\frac{j}{T}\right) .$$

In practice, when we are analysing a record with superposed undesired 'noise' such as due to recording or reading instrument we will select δ or n in such a way that $g\left(\frac{n-1}{2T}\right)$ is dominated by the noise spectral density at the frequency $(n-1)/2T$. Here, we are assuming that the noise spectral density is negligible as compared to the spectral density of our data in low frequency ranges. We are then not interested in the value of $g(\lambda)$ for $\lambda > (n-1)/2T$, or, perhaps, the values of $g(\lambda)$ for $\lambda > (n-1)/2T$ are considered negligible. In this situation the effect of aliasing will be negligible except at the point $\lambda = (n-1)/2T$, where it will make our estimate twice as large as it actually should be.

6. LINEAR FILTERS

If we write $a^*(\lambda)$ for the Fourier transform of a function $a(t)$ and if

$$a(t) = \int_{-\infty}^{\infty} b(t-u)\, c(u)\, du\,,$$

then it is easy to show that

$$a^*(\lambda) = b^*(\lambda)\, c^*(\lambda)\,.$$

$a(t)$ is called the convolution of $b(t)$ and $c(t)$.

Let $\{x(t)\}$ and $\{z(t)\}$ be two continuous-time real stationary processes such that

(29) $$x(t) = \int_{-\infty}^{\infty} W(t-u)\, z(u)\, du = \int_{-\infty}^{\infty} W(u)\, z(t-u)\, du\,,$$

where $W(t)$ is a real function integrable over $(-\infty, \infty)$. $W(t)$ is called the linear filter, $z(t)$ the input and $x(t)$ the output of the filter $W(t)$. if $C_x(t)$ and $C_z(t)$ denote the autocovariance and $g_x(\lambda)$ and $g_z(\lambda)$ the spectral densities of the processes $\{x(t)\}$ and $\{z(t)\}$

respectively, and $W^*(\lambda)$ is the Fourier transform of $W(t)$, then we have

$$C_x(s) = Ex(t)x(t+s) = E\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(u)W(v)z(t-u)z(t+s-v)dudv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(u)W(v)\, C_z\,(s+u-v)dudv\ .$$

$$g_x(\lambda) = \int_{-\infty}^{\infty} e^{-2\pi i s\lambda} C_x(s)ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^{-2\pi i s\lambda} W(u)W(v)C_z(s-u-v)dudvds.$$

Making the transformation $s' = s+u-v$, $u=u$, $v=v$, we have

$$(3)\quad g_x(\lambda) = \int_{-\infty}^{\infty} d^{-2\pi i s'\lambda}\, C_z(s')ds' \int_{-\infty}^{\infty} e^{i2\pi u\lambda}\, W(u)du \int_{-\infty}^{\infty} e^{-i2\pi v\lambda}\, W(v)dv$$

$$= g_z(\lambda)\ \ |W^*(\lambda)|^2\ , \qquad -\infty \leq \lambda \leq \infty\ .$$

Now, if the filter is known and the spectral density of one of the processes is known, then the relation (30) gives the spectral density of the other process. It is sometimes possible to adjust the filter in such a way that the output $x(t)$ is approximately a white noise, i.e., has a constant spectral density over a wide frequency range. The use of such a filter on $z(t)$ is called the <u>prewhitening</u> of $z(t)$ and is a powerful method in obtaining a reliable estimate of its spectral density.

If in (29) we replace $x(t)$ by x_t etc. and integrals by summations, we obtain

$$f_x(\lambda) = |W^*(\lambda)|^2 f_z(\lambda), \quad -\tfrac{1}{2} \le \lambda \le \tfrac{1}{2},$$

where, now,

$$W^*(\lambda) = \sum_{k=-\infty}^{\infty} W_k e^{-i2\pi k\lambda}.$$

For further study on this subject see [12, Ch. 5].

7. ESTIMATION OF SPECTRAL DENSITY

The purpose of Fourier analysis of a stationary process is to construct an estimator of the true spectral density function which has at least two desirable properties of being unbiased and consistent. It turns out that no estimator of $g(\lambda)$ or $f(\lambda)$ exists which is unbiased for finite sample size, although asymptotically unbiased estimators are available. The periodogram is one such estimator. Unfortunately, the variance of the periodogram remains bounded from below no matter how large a sample we take, i.e., it is an inconsistent estimator of $g(\lambda)$ or $f(\lambda)$. We therefore try to construct some other estimators which, besides being asymptotically unbiased, are consistent. In fact, several such estimators have been proposed and a general method of 'filtering' the periodogram is available which produces such estimators. We will only mention two such modified periodogram estimators, one suggested by Bartlett and the other by Tukey.

Assuming $Ex_t = 0$ and writing

$$(32) \qquad c_k \equiv \frac{\sum_{r=1}^{n-k} x_r x_{r+k}}{n-k}, \quad k = 0, 1, \ldots, n-1, c_{-k} = c_k,$$

we observe that

$$(33) \quad I_n(\lambda) = \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n}\right) c_k \cos 2\pi k\lambda, \quad \lambda = \frac{j}{n}, \quad j = 0, 1, \ldots, \frac{n-1}{2}.$$

Here, $I_n(0)$ can only be obtained when the process mean is known to be zero. Otherwise, $I_n(0)$ will remain arbitrary depending on our estimate of the process mean.

Bartlett suggests [1, also 9, p. 146]

$$(34) \quad I_n^{(1)}(\lambda) = \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m}\right) c_k \cos 2\pi k\lambda, \quad \lambda = \frac{j}{m}, \quad j = 0, 1, \ldots, m-1,$$

where $m = 0(n)$ but $m \to \infty$ as $n \to \infty$. Here, the sample covariances of large order have been avoided as they are subject to great sampling fluctuations. It is easy to see that $\lim_{n \to \infty} EI_n^{(1)}(\lambda) = f(\lambda)$.

The variance is given by

$$(35) \qquad \operatorname{var} I_n^{(1)}(\lambda) \cong \begin{cases} \frac{2m}{3n} f^2(\lambda), & \lambda \neq 0 \\ \frac{4m}{3n} f^2(0), & \lambda = 0. \end{cases}$$

Tukey has proposed [9, p.149; also 2]

$$(36) \qquad I_n^{(2)}(\lambda) = c_o + 2 \sum_{k=1}^{m-1} \left(.46 \cos \frac{\pi k}{m} + .54\right) c_k \cos \pi k\lambda$$

$$+ (.46 \cos \pi + .54)\, c_m \cos \pi m\lambda,$$

$$\lambda = \frac{j}{m}, \quad j = 0, 1, \ldots, m-1,$$

as an estimate of $f(\lambda)$, where $m = 0(n)$ but $m \to \infty$ as $n \to \infty$. $I_n^{(2)}(\lambda)$ is also an asymptotically unbiased estimator of $f(\lambda)$, and its variance is

$$(37) \qquad \operatorname{var} I_n^{(2)}(\lambda) \cong \begin{cases} \frac{m}{n} f^2(\lambda)[(.46)^2 + 2(.54)^2], & \lambda \neq 0 \\ \frac{2m}{n} f^2(0)[(.46)^2 + 2(.54)^2], & \lambda = 0. \end{cases}$$

Tukey has also suggested a method of setting confidence bands on $f(\lambda)$, by showing that $I_n^{(2)}(\lambda)$ is approximately a χ^2 distribution with approximately $2n/m$ degrees of freedom, [2]. However, $I_n^{(2)}(\lambda_1)$ and $I_n^{(2)}(\lambda_2)$ are not independent if $|\lambda_1 - \lambda_2| \leq 3/m$, so that the total degrees of freedom do not exceed the available degrees of freedom n.

If the sample mean is subtracted from data before any further analysis, then as we have seen $I_n(0) \equiv 0$. It is difficult to give any meaning to $I_n^{(1)}(0)$ and $I_n^{(2)}(0)$ in such a situation.

8. EXAMPLES

(i) White noise. Let $\{\xi_t\}$ be a stationary process such that $E\xi_t\xi_{t+s} \equiv C_s = 0$ if $s \neq 0$. Then, from (1), we have

$$f(\lambda) = C_o \equiv \sigma_\xi^2, \quad -\tfrac{1}{2} \leq \lambda \leq \tfrac{1}{2}.$$

A process which has a constant spectral density is called a white noise. Conversely, if $f(\lambda) = C$, $-\tfrac{1}{2} \leq \lambda \leq \tfrac{1}{2}$, we have from (2), $C_o = C$, $C_s = 0$, $s \neq 0$.

If, besides being uncorrelated, ξ_t and ξ_{t+s} are independent for all $s \neq 0$, the discrete time process $\{\xi_t\}$ is called a pure white noise [9, p. 42].

From equation (19) we observe that we cannot set $g(\lambda)$ = constant in a continuous time process as this would imply that $C(0)$ was unbounded. However, by allowing the existence of Dirac's delta function and its Fourier transform, some authors have extended the concept of white noise to continuous time processes. We may, however, set $g(\lambda) \cong$ constant over a wide frequency band in some cases without making $C(0)$ infinite.

(ii) Moving average and autoregressive processes. Consider

$$x_t = a_o y_t + a_1 y_{t-1} + \dots + a_p y_{t-p} = \sum_{k=o}^{p} a_k y_{t-k}, \tag{38}$$

where x_t and y_t are stationary processes and a_k are real constants. Since x_t is the convolution of W_t and y_t, where

$$W_t = a_t \text{ if } 0 \leq t \leq p$$
$$= 0 \text{ otherwise,}$$

we obtain

$$f_x(\lambda) = \left| \sum_{k=o}^{p} a_k e^{-i2\pi k\lambda} \right|^2 f_y(\lambda). \tag{39}$$

If y_t is a white noise, x_t is called a moving average process. On the other hand if x_t is a white noise and $\sum_{k=o}^{p} a_k z^{p-k} = 0$ has all the roots within the unit circle $|z| = 1$ in complex plane, y_t is called an autoregressive process of order p. In this case x_t is independent of y_{t-1}, y_{t-2}, ... [9, p. 38].

Special cases. (a) Simple moving average. If we set $a_k = 1/(p+1)$, $k = 0, 1, \ldots, p$, in (38) we obtain

$$x_t = \frac{1}{p+1} \sum_{j=o}^{p} y_{t-j}. \tag{40}$$

If y_t is a white noise, the above scheme is called a simple moving average scheme. Since

$$\left| \sum_{i=o}^{p} e^{-i\omega} \right| = \left| \frac{1 - e^{-i(p+1)\omega}}{1-e^{-i\omega}} \right| = \left| \frac{e^{-i(p+1)/2}}{e^{-i\omega/2}} \frac{\sin(p+1)\omega/2}{\sin \omega/2} \right| = \left| \frac{\sin(p+1)\omega/2}{\sin \omega/2} \right|,$$

we have from (39)

$$f_x(\lambda) = \frac{1}{(p+1)^2} \frac{\sin^2 (p+1)\pi\lambda}{\sin^2 \pi\lambda} f_y(\lambda).$$

However, since y_t is a white noise, $f_y(\lambda) = \sigma_y^2$ and from (40)

$$f_y(\lambda) = \sigma_y^2 = (p+1)\sigma_x^2, \quad -½ \leq \lambda \leq ½,$$

and hence

(41) $$f_x(\lambda) = \frac{\sin^2 (p+1)\pi\lambda}{(p+1)\sin^2 \pi\lambda} \sigma_x^2, \quad -½ \leq \lambda \leq ½.$$

In estimating $f_x(\lambda)$, then, we only estimate σ_x^2 by the usual method of estimating a variance.

(b) Markov scheme. We consider the first-order autoregressive scheme (Markov scheme) in the form

(42) $$y_t = \rho y_{t-1} + x_t,$$

where $|\rho| < 1$, and x_t is a white noise. From (39)

$$f_y(\lambda) = f_x(\lambda)/|1 - \rho e^{-i2\pi\lambda}|^2 = \sigma_x^2 (1 - 2\rho \cos 2\pi\lambda + \rho^2)^{-1}, \quad -½ \leq \lambda \leq ½,$$

since $f_x(\lambda) = \sigma_x^2$. Since x_t is independent of y_{t-1}, we obtain from (42)

$$\sigma_y^2 = \rho^2 \sigma_y^2 + \sigma_x^2$$

or

$$\sigma_x^2 = \sigma_y^2 (1 - \rho^2).$$

Thus

$$(43) \qquad f_y(\lambda) = \sigma_y^2 (1-\rho^2)(1-2\rho\cos 2\pi\lambda + \rho^2)^{-1}, \quad -\tfrac{1}{2} \le \lambda \le \tfrac{1}{2}.$$

σ_y^2 and ρ are estimated by the sample variance and first serial correlation coefficient respectively.

(c) Yule scheme. Consider the second-order autoregressive scheme (Yule scheme) in the form

$$(44) \qquad y_t = \alpha y_{t-1} - \beta y_{t-2} + x_t ,$$

where x_t is a white noise and the roots of $z^2 - \alpha z + \beta = 0$ lie within the unit circle $|z| = 1$ in the complex plane. From (39) we have

$$f_y(\lambda) = \frac{f_x(\lambda)}{1 + \alpha^2 + \beta^2 - 2\alpha(1+\beta)\cos 2\pi\lambda + 2\beta\cos 4\pi\lambda}, \quad -\tfrac{1}{2} \le \lambda \le \tfrac{1}{2}.$$

Now, since x_t is independent of y_{t-1} and y_{t-2}, multiplying the equation (44) by y_{t-1} and y_{t-2} respectively and taking expectations, we obtain

$$C_1 = \alpha C_o - \beta C_1, \quad C_2 = \alpha C_1 - \beta C_o ,$$

or

$$\rho_1 = \alpha - \beta\rho_1, \quad \rho_2 = \alpha\rho_1 - \beta ,$$

so that

$$(45) \qquad \alpha = \frac{\rho_1(1-\rho_2)}{1-\rho_1^2}, \quad \beta = \frac{\rho_1^2 - \rho_2}{1-\rho_1^2} .$$

If ρ denotes the multiple correlation coefficient between y_t and (y_{t-1}, y_{t-2}), it is given by

$$(46) \qquad (1-\rho^2) = (1-\rho_1^2)^{-1} \begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix} .$$

With simple evaluation we then obtain

$$\sigma_x^2 = \sigma_y^2 (1-\rho^2) .$$

Since $f_x(\lambda) = \sigma_x^2$, $-\frac{1}{2} \le \lambda \le \frac{1}{2}$, we finally have

$$(47) \quad f_y(\lambda) = \frac{\sigma_y^2 (1-\rho^2)}{1+\alpha^2+\beta^2 - 2\alpha(1+\beta)\cos 2\pi\lambda + 2\beta \cos 4\pi\lambda} , \quad -\tfrac{1}{2} \le \lambda \le \tfrac{1}{2} .$$

To estimate ρ, α, and β, we need estimates of σ_y, ρ_1 and ρ_2. These estimates are provided by the sample variance and the first and second serial correlation coefficients respectively.

(iii) Processes related through a differential equation. For continuous-time processes we may consider the differential equation

$$(48) \qquad a_p \frac{d^p y}{dt^p} + a_{p-1} \frac{d^{p-1} y}{dt^{p-1}} + \dots + a_o y = x(t) .$$

To obtain a relationship between the spectral densities $g_x(\lambda)$ and $g_y(\lambda)$ in this and similar cases, we outline a technique suggested by Cramér [3, 4] of representing the process as a Fourier transform of an orthogonal set function.

If I_1 and I_2 are two disjoint sets on a real line, a set function $z(I)$ is called <u>orthogonal</u> if

$$Ez(I_1)\overline{z(I_2)} = 0 .$$

Here $\overline{z}$ denotes the complex conjugate of z. Writing

$$dz(\mu) = z(d\mu) ,$$

Cramér shows that every stationary continuous-time process, $\{x(t)\}$, may be represented as

$$x(t) = \int_{-\infty}^{\infty} e^{i2\pi t\lambda} dz_x(\lambda) \tag{49}$$

where $z_x(\lambda)$ is an orthogonal set function with $E|dz_x(\lambda)|^2 = dG_x(\lambda)$, and $G_x(\lambda)$ is the spectral distribution function. If $x(t)$ is real, then, also

$$x(t) = \int_{-\infty}^{\infty} e^{-i2\pi t\mu} \overline{dz_x(\mu)} .$$

Thus

$$C_x(s) = Ex(t)x(t+s) = E\int\int e^{-i2\pi t\lambda} e^{+i2\pi(t+s)\mu} \overline{dz_x(\lambda)} dz_x(\mu)$$

$$= \int_{-\infty}^{\infty} e^{+i2\pi s\lambda} dG_x(\lambda) ,$$

which is the same as (19) in the case $G_x(\lambda)$ is absolutely continuous, Returning to (48) let us represent

$$y(t) = \int_{-\infty}^{\infty} e^{i2\pi t\lambda}\, dz_y(\lambda)$$

$$x(t) = \int_{-\infty}^{\infty} e^{i2\pi t\lambda}\, dz_x(\lambda)\ .$$

Then, from (48)

$$\int_{-\infty}^{\infty} e^{i2\pi t\lambda}\left[\sum_{k=o}^{p} a_k(2\pi i\lambda)^k\right] dz_y(\lambda) = \int_{-\infty}^{\infty} e^{i2\pi t\lambda}\, dz_x(\lambda)\ .$$

From the uniqueness of the Fourier transform, we have

$$\left[\sum_{k=o}^{p} a_k(2\pi i\lambda)^k\right] dz_y(\lambda) = dz_x(\lambda)\ .$$

Multiplying by the complex conjugates and taking expectations, we obtain

$$\left|\sum_{k=o}^{p} a_k\,(2\pi i\lambda)^k\right|^2 dG_y(\lambda) = dG_x(\lambda)$$

or

$$g_x(\lambda) = \left|\sum_{k=o}^{p} a_k\,(2\pi i\lambda)^k\right|^2 g_y(\lambda)\ . \tag{50}$$

A representation similar to (49) for a discrete-time process, $\{x_t\}$, is

(51) $$x_t = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i t\lambda}\, dz_x(\lambda),$$

where, now, $E|dz_x(\lambda)|^2 = dF_x(\lambda)$, $Edz_x(\lambda)\overline{dz_x(\mu)} = 0$, $\lambda \neq \mu$, and $F_x(\lambda)$ is the spectral distribution function of $\{x_t\}$.

(iv) Shot noise. Let us consider instants, t_k, distributed at random on the time axis according to a Poisson process of density $\beta > 0$. Let $n(t_2' - t_1')$ be the number of these instants belonging to the interval (t_1', t_2'). Writing $dn(t) = n(dt)$, it is then known that

$$Edn(t) = \beta dt, \quad E[dn(t)]^2 = \beta dt, \quad E[dn(t)dn(s)] = 0, \quad t \neq s,$$

so that $n(\cdot)$ is an orthogonal set function.

Consider the process

(52) $$x(t) = \sum_{tj \leq t} a(t-t_j),$$

where $a(t)$ is a real function with $a(t) = 0$ if $t < 0$, and $a(t)$ is integrable over $(0, \infty)$.

Equation (52) may also be written

$$x(t) = \int_{-\infty}^{\infty} a(t-s)\, dn(s), \quad dn(s) = n(ds).$$

We have

$$Ex(t) = \beta \int_{-\infty}^{\infty} a(t-s)\,ds = \beta \int_{0}^{\infty} a(s)\,ds \equiv m, \text{ say.}$$

Also,

$$C(u) = E[x(t)-m][x(t+u)-m] = \int_{-\infty}^{\infty} a(t-s)\,a(t+u-s) \operatorname{var} dn(s)$$

$$= \beta \int_{-\infty}^{\infty} a(t')\,a(t'+u)\,dt' .$$

The right hand side, except for the factor β, is the convolution of $a(u)$ and $a(-u)$. Hence, if $a^*(\lambda)$ is the Fourier transform of $a(t)$, we obtain

(43) $$g_x(\lambda) = \beta\,|a^*(\lambda)|^2 .$$

Thus $x(t)$ is composed of a 'direct current' $\beta \int_0^\infty a(t)dt$ and a 'shot noise' with continuous spectral density $\beta|a^*(\lambda)|^2$.

<u>Special case.</u> $$a(t) = \begin{cases} \frac{1}{T} e^{-t/T} & t \geq 0 \\ 0 & t < 0 . \end{cases}$$

$$|a^*(\lambda)|^2 = \frac{1}{1+4\pi^2T^2\lambda^2} ,$$

$$g_x(\lambda) = \frac{\beta}{1+4\pi^2T^2\lambda^2} ; \quad Ex = \beta, \ \operatorname{var} x = \frac{\beta}{2T} .$$

For further examples see [12, chs. 3,4,5].

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