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THE COMPONENTS OF POWER APPEARING IN THE HARMONIC ANALYSIS OF A STATIONARY PROCESS

by

M. M. Siddiqui

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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS
BOULDER LABORATORIES
Boulder, Colorado

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1. INTRODUCTION

Let $Y(t)$ be a real stationary process with mean zero and auto-covariance function $C(t) = EY(s)Y(s+t)$. Here Ex denotes the expected value of the random variable x and we assume $C(t)$ to be finite for all values of t. Suppose we have a continuous record of $Y(t)$ over the time interval $(0, T)$ measured in seconds and we take $2n+1$ equally spaced readings $Y_o, Y_1, \ldots, Y_{2n}$ at times $t=0, \delta, 2\delta, \ldots, 2n\delta = T$ seconds. It is customary to represent the readings by the Fourier series

$$(1.1) \quad Y_i = \alpha_o' + \sum_{j=1}^{n} \left(\alpha_j' \cos \frac{2\pi ji}{2n} + \beta_j' \sin \frac{2\pi ji}{n} \right), \quad i = 0, 1, \ldots, 2n,$$

to estimate the power spectrum of the process $Y(t)$.

Assuming that the true representation of $Y(t)$ is given by

$$(1.2) \quad Y(t) = \sum_{j=1}^{\infty} \left(\alpha^*(j) \cos \frac{2\pi jt}{T} + \beta^*(j) \sin \frac{2\pi jt}{T} \right)$$

$$+ \sum_{m=2}^{\infty} \left(\alpha^*(\tfrac{1}{m}) \cos \frac{2\pi t}{mT} + \beta^*(\tfrac{1}{m}) \sin \frac{2\pi t}{mT} \right),$$

it will be shown that the power ascribed to the frequency j/T cycles per second (cps) in the analysis of (1.1) consists of three components:

(1) the true power in the frequency j/T cps;

(2) the powers in the frequencies $(2kn \pm j)/T$ cps, $k = 1, 2, \ldots$;

(3) a part of the powers in the frequencies $1/(mT)$ cps, $m = 2, 3, \ldots$.

The frequencies mentioned in (2) are called the 'aliases' to the frequency j/T cps and the transfer of their powers to the frequency j/T cps is well known. The study of the effect of the presence of frequencies lower than $1/T$ cps in the process on the harmonic analysis, however, seems to have been somewhat neglected. Spetner [1] considers the error in power spectra due to finite sample from a continuous Gaussian process, and to the author's knowledge no other references are found in the literature. The following results do not suppose the process to be Gaussian and the only restrictions placed on the process are given by equations (2.2) and (2.3).

2. LOW FREQUENCIES IN THE PROCESS

The record of $Y(t)$ over the interval of time $(0, T)$ can be exactly represented by

$$(2.1) \quad Y(t) = a_o + \sum_{j=1}^{\infty} \left(a_j \cos \frac{2\pi jt}{T} + \beta_j \sin \frac{2\pi jt}{T} \right) .$$

However, let the true representation of the process Y(t) be

$$(2.2) \quad Y(t) = \sum_{j=1}^{\infty} \left(\alpha^*(j) \cos \frac{2\pi jt}{T} + \beta^*(j) \sin \frac{2\pi jt}{T} \right)$$

$$+ \sum_{m=2}^{\infty} \left(\alpha^*(\tfrac{1}{m}) \cos \frac{2\pi t}{mT} + \beta^*(\tfrac{1}{m}) \sin \frac{2\pi t}{mT} \right) ,$$

where α^* and β^* are random variables satisfying the following conditions:

$$(2.3) \quad E\alpha^*(s) = E\beta^*(s) = 0 \quad \text{for all index values of } s ,$$

$$E\alpha^{*2}(s) = E\beta^{*2}(s) = \sigma^{*2}(s) ,$$

$$E\alpha^*(s)\alpha^*(r) = E\beta^*(s)\beta^*(r) = 0 \ \text{ if } s \neq r ,$$

$$E\alpha^*(s)\beta^*(r) = 0 \ \text{ for all } s \text{ and } r ,$$

$$\sum_s \sigma^{*2}(s) = \sigma_Y^2 < \infty .$$

The power in the frequency s/T cps is then defined as $\sigma^{*2}(s)$.

The conditions (2.3) are satisfied, for example, if the typical term of (2.2) is represented by $\gamma_s \cos\left(\frac{2\pi st}{T} + \epsilon_s\right)$ and we assume γ_s and ϵ_s to be independent variates with $E\gamma_s = 0$, $E\gamma_s^2 = 2\sigma^{*2}(s)$ and ϵ_s to be distributed uniformly over the interval $(0, 2\pi)$.

Now, for $0 < t < T$

$$(2.4) \quad \cos \frac{2\pi t}{mT} = \frac{m}{2\pi} \sin \frac{2\pi}{m} - \frac{m}{\pi} \sin \frac{2\pi}{m} \sum_{j=1}^{\infty} (j^2m^2-1)^{-1} \cos \frac{2\pi jt}{T}$$

$$+ \frac{m^2}{\pi} (1 - \cos \frac{2\pi}{m}) \sum_{j=1}^{\infty} j(j^2m^2-1)^{-1} \sin \frac{2\pi jt}{T} ,$$

$$\sin \frac{2\pi t}{mT} = \frac{m}{2\pi} (1 - \cos \frac{2\pi}{m}) - \frac{m}{\pi} (1 - \cos \frac{2\pi}{m}) \sum_{j=1}^{\infty} (j^2m^2-1)^{-1} \cos \frac{2\pi jt}{T}$$

$$- \frac{m^2}{\pi} \sin \frac{2\pi}{m} \sum_{j=1}^{\infty} j(j^2m^2-1)^{-1} \sin \frac{2\pi jt}{T} .$$

Since (2.1) and (2.2) are identical over the interval (0, T), we obtain

$$(2.5) \quad a_o = \sum_{m=2}^{\infty} \left[\alpha^*(\frac{1}{m}) \frac{m}{2\pi} \sin \frac{2\pi}{m} + \beta^*(\frac{1}{m}) \frac{m}{2\pi} (1 - \cos \frac{2\pi}{m}) \right] ,$$

and for $j > 0$

$$(2.6) \quad a_j = \alpha^*(j) - \sum_{m=2}^{\infty} \left[\alpha^*(\frac{1}{m})(j^2m^2-1)^{-1} \frac{m}{\pi} \sin \frac{2\pi}{m} + \beta^*(\frac{1}{m})(j^2m^2-1)^{-1} \frac{m}{\pi} (1 - \cos \frac{2\pi}{m}) \right] ,$$

$$\beta_j = \beta^*(j) + \sum_{m=2}^{\infty} \left[\alpha^*(\frac{1}{m}) j(j^2m^2-1)^{-1} \frac{m^2}{\pi}(1 - \cos \frac{2\pi}{m}) - \beta^*(\frac{1}{m}) j(j^2m^2-1)^{-1} \frac{m^2}{\pi} \sin \frac{2\pi}{m} \right] .$$

We, therefore, have

$$(2.7) \quad E\alpha_j = E\beta_j = 0 \quad \text{for } j = 0, 1, \ldots ,$$

$$(2.8)\quad E\alpha_o^2 = \sum_{m=2}^{\infty} \sigma^{*2}\left(\frac{1}{m}\right) \frac{m^2}{\pi^2} \sin^2 \frac{\pi}{m} ,$$

$$E\alpha_j^2 = \sigma^{*2}(j) + 4 \sum_{m=2}^{\infty} \sigma^{*2}\left(\frac{1}{m}\right) \frac{m^2 \sin^2 \frac{\pi}{m}}{\pi^2(j^2m^2-1)^2} \quad \text{for } j > 0 ,$$

$$E\beta_j^2 = \sigma^{*2}(j) + 4 \sum_{m=2}^{\infty} \sigma^{*2}\left(\frac{1}{m}\right) \frac{m^4 j^2 \sin^2 \frac{\pi}{m}}{\pi^2 (j^2m^2-1)^2} \quad \text{for } j > 0 ,$$

$$E\alpha_o\alpha_j = -2 \sum_{m=2}^{\infty} \sigma^{*2}\left(\frac{1}{m}\right) \frac{m^2 \sin^2 \frac{\pi}{m}}{\pi^2 (j^2m^2-1)} \quad \text{for } j > 0 ,$$

$$E\alpha_i\alpha_j = 4 \sum_{m=2}^{\infty} \sigma^{*2}\left(\frac{1}{m}\right) \frac{m^2 \sin^2 \frac{\pi}{m}}{\pi^2(j^2m^2-1)(i^2m^2-1)} \quad \text{for } i \neq j,\ i,j \neq 0,$$

$$E\beta_i\beta_j = 4 \sum_{m=2}^{\infty} \sigma^{*2}\left(\frac{1}{m}\right) \frac{m^4\, ij \sin^2 \frac{\pi}{m}}{\pi^2(j^2m^2-1)(i^2m^2-1)} \quad \text{for } i \neq j ,$$

$$E\alpha_i\beta_j = 0 \qquad \text{for all } i \text{ and } j .$$

Writing

$$\sigma_o^2 = E\alpha_o^2,\ \sigma_j^2 = \tfrac{1}{2} E(\alpha_j^2 + \beta_j^2) \qquad \text{for } j > 0 ,$$

we find

$$(2.9)\quad \sigma_o^2 = \sum_{m=2}^{\infty} \left(\frac{m}{\pi} \sin \frac{\pi}{m}\right)^2 \sigma^{*2}\left(\frac{1}{m}\right) ,$$

$$\sigma_j^2 = \sigma^{*2}(j) + 2 \sum_{m=2}^{\infty} \frac{j^2m^2+1}{(j^2m^2-1)^2} \left(\frac{m}{\pi} \sin \frac{\pi}{m}\right)^2 \sigma^{*2}\left(\frac{1}{m}\right), \text{ for } j > 0.$$

Performing the calculations so as to account for at least 90% of the power in each frequency, we have

$$(2.10)\quad \sigma_o^2 = 0.405\,\sigma^{*2}(\tfrac{1}{2}) + 0.684\,\sigma^{*2}(\tfrac{1}{3}) + 0.811\,\sigma^{*2}(\tfrac{1}{4}) + 0.874\,\sigma^{*2}(\tfrac{1}{5}) + 0.912\,\sigma^{*2}(\tfrac{1}{6}) + 0.933\,\sigma^{*2}(\tfrac{1}{7}) + 0.950\,\sigma^{*2}(\tfrac{1}{8}) + \theta \sum_{m=9}^{\infty} \sigma^{*2}(\tfrac{1}{m})$$

where $0.96 < \theta < 1$,

$$\sigma_1^2 = \sigma^{*2}(1) + 0.450\,\sigma^{*2}(\tfrac{1}{2}) + 0.214\,\sigma^{*2}(\tfrac{1}{3}) + 0.123\,\sigma^{*2}(\tfrac{1}{4}) + 0.079\,\sigma^{*2}(\tfrac{1}{5}) + \ldots,$$

$$\sigma_2^2 = \sigma^{*2}(2) + 0.061\,\sigma^{*2}(\tfrac{1}{2}) + 0.041\,\sigma^{*2}(\tfrac{1}{3}) + \ldots,$$

$$\sigma_3^2 = \sigma^{*2}(3) + 0.024\,\sigma^{*2}(\tfrac{1}{2}) + 0.018\,\sigma^{*2}(\tfrac{1}{3}) + \ldots,$$

$$\sigma_j^2 \cong \sigma^{*2}(j) \qquad \text{for } j \geq 4 .$$

We observe that more than 95% of the power in the frequencies lower than $1/(7T)$ cps is transferred to the zero frequency. Since the mean of the sample is usually subtracted from each observation before further analysis, the power σ_o^2 will be subtracted from the spectrum. Furthermore, more than 90% of the power in frequencies ranging from $\frac{1}{2T}$ cps to zero cps is transferred to the zero and the first two lowest frequencies analysed.

REFERENCE

[1] Lee M. Spetner, "Errors in power spectra due to finite sample," Journal of Applied Physics 25 (1954), pp. 653-659.

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