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ON THE THEORY OF REFLECTION FROM A WIRE GRID PARALLEL TO AN INTERFACE BETWEEN HOMOGENEOUS MEDIA

By

James R. Wait

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ABSTRACT

The reflection from a wire grid parallel to a plane interface is considered. The respective media are homogeneous and either or both can be dissipative. The grid is composed of thin equi-spaced wires of finite conductivity. The plane wave solution for arbitrary incidence is then generalized for cylindrical-wave excitation. The energy absorbed from a magnetic line source by a grid situated on the surface of a dissipative half-space is treated in some detail. This latter problem is a two-dimensional analogy of a vertical antenna with a radial wire ground system.

Introduction

There have been many investigations of the electromagnetic properties of thin parallel wires composed of conductive material. The first quantitative study was made by Lamb[1] in 1898 who considered the plane wave incident normally on the grid. He showed that if the diameter, 2a, of the parallel wires was small, the reflection and transmission could be varied by changing the spacing. In 1914 von Ignatowsky[2] made a very exhaustive analysis of the scattering of incident plane waves by single metallic grids including the case where the wire spacing is comparable to the wavelength. His formulas have been reduced, extended and applied by other authors since that time.[3-11] A very illuminating treatment has been given by MacFarlane[5] who indicated that a single grid can be represented by an impedance shunted across an infinite transmission line whose characteristic impedance is proportional to the intrinsic impedance of the surrounding infinite medium. He showed that this shunt impedance was proportional to $\log(d/2\pi a)$ plus a correction factor which is a function of angle of incidence θ and the spacing d.

It is the purpose of the present paper to consider the effect of an interface on the equivalent shunt impedance of the wire grid. It can be expected that the evanescent (non-propagating) field of the grid will be modified by the interface. Since wire grids and meshes are often placed on the surface of the ground to improve the efficiency of antennas, it is desirable to consider the effect of the grid on the energy absorption in the lower medium for a source in the upper medium.

General Theory

The grid is illustrated in Fig. 1. The wires are of circular cross section with radius a and are composed of material having a conductivity σ, dielectric constant ϵ_1 and permeability μ_1. The wires are contained in the plane $x = h$ and are spaced a distance d between centers. It is assumed that $a << d$. The half space, defined by $x > 0$, surrounding the wires has a dielectric constant ϵ and a permeability μ. The half space $x < -h$ has a dielectric constant ϵ' and a permeability μ.

The electric field of the incident wave with the phase reference at the origin is taken for a time factor $e^{i\omega t}$ to be

$$\vec{E}^p = (E_x^p, 0, E_z^p)$$

$$= (E_o \sin\theta, 0, E_o \cos\theta) \exp\left[ik(x\cos\theta - z\sin\theta)\right] \qquad (1)$$

where $k = 2\pi/\text{wavelength}$, where E_o is the amplitude of the incident wave and where θ is the angle of incidence. The currents on the wires then have the form $Ie^{-ikz\sin\theta}$ where I is the unknown current at $z = 0$. The field $\vec{E}^{w}$ of the currents on the wires can be derived from an electric Hertz vector with only a z component Π_z^w since the induced currents are essentially in the z direction. Therefore

$$\vec{E}^{w} = k^2 \Pi_z^w \vec{i}_z + \text{grad} \frac{\partial \Pi_z^w}{\partial z} \tag{2}$$

where $\vec{i}_z$ is a unit vector in the z direction. The Hertz vector for the currents on a wire grid is easily obtained by adding the contributions from each of the wires, so that

$$k^2 \Pi_z^w = \frac{\mu\omega I}{4} \sum_{n=-\infty}^{+\infty} e^{-ikz\sin\theta} H_o^{(2)} [k \cos\theta \sqrt{(nd - y)^2 + (x - h)^2}\,] \tag{3}$$

where $H_o^{(2)}$ is the Hankel function of order zero of the second kind. The axial electric field of the currents on the wire grid is then given by

$$E_z^w = k^2 \Pi_z^w \cos^2\theta .$$

For the present application it is desirable to transform the Hankel function series to a simpler form. This is effected by using a transformation formula given previously[11], so that

$$\Pi_z^w = \frac{i\mu\omega I e^{-ikz\sin\theta}}{4\pi k^2} \sum_{m=-\infty}^{+\infty} e^{i2\pi my/d} \frac{\exp\left\{ - \frac{2\pi|x - h|}{d} \sqrt{m^2 - \left(\frac{d\cos\theta}{\lambda}\right)^2} \right\}}{\sqrt{m^2 - \left(\frac{d\cos\theta}{\lambda}\right)^2}} . \tag{4}$$

To satisfy boundary conditions at the interface, $x = 0$, it is now necessary to introduce secondary or scattered fields. For example in the absence of the grid the field could be derived from the following Hertz vectors:

$$\Pi_z^p + \Pi_z^r$$

$$= \frac{E_{oz}}{(k\cos\theta)^2}\exp(-ikz\sin\theta)[\exp(ikx\cos\theta) + R_o\exp(-ikx\cos\theta)] \tag{5a}$$

for $x > 0$ and

$$= \frac{E_{oz}T_o}{(k'\cos\theta')^2}\exp(-ikz\sin\theta)\exp(ik'x\cos\theta') \tag{5b}$$

for $x < 0$

where $k' = Nk$, or $N = (\epsilon'/\epsilon)^{1/2} = (\lambda/\lambda')$, $\sin\theta' = N\sin\theta$, and

$$R_o = T_o - 1 = \frac{\cos\theta' - N\cos\theta}{\cos\theta' + N\cos\theta} \quad . \tag{5c}$$

To account for the presence of the grid and its reaction on the interface at $x = 0$, it is necessary to consider the additional Hertz vectors:

$$\Pi_z^w + \Pi_z^{w'} = \frac{i\mu\omega I e^{-ikz\sin\theta}}{4\pi k^2} \cdot \sum_{m=-\infty}^{+\infty} e^{i2\pi my/d} \tag{6a}$$

$$\frac{e^{\frac{-2\pi|x-h|}{d}\sqrt{m^2-(d\cos\theta/\lambda)^2}} + R_m e^{\frac{-2\pi(x+h)}{d}\sqrt{m^2-(d\cos\theta/\lambda)^2}}}{\sqrt{m^2-(d\cos\theta/\lambda)^2}}$$

for $x > 0$ and

$$= \frac{i\mu\omega I e^{-ikz\sin\theta}}{4\pi(k')^2} \sum_{m=-\infty}^{+\infty} e^{i2\pi my/d}\, T_m \frac{e^{\frac{2\pi x}{d}\sqrt{m^2-(d\cos\theta'/\lambda')^2}}}{\sqrt{m^2-(d\cos\theta/\lambda)^2}}\, e^{\frac{-2\pi h}{d}\sqrt{m^2-(d\cos\theta/\lambda)^2}}\, \frac{\cos^2\theta}{\cos^2\theta'} \qquad (6c)$$

for $x < 0$. By invoking the condition that the tangential electric and magnetic fields are continuous at $x = 0$, the coefficients can be readily found, whence

$$R_m = T_m - 1$$

$$= \frac{\cos^2\theta'\sqrt{m^2-(d\cos\theta/\lambda)^2} - \cos^2\theta\sqrt{m^2-(d\cos\theta'/\lambda')^2}}{\cos^2\theta'\sqrt{m^2-(d\cos\theta/\lambda)^2} + \cos^2\theta\sqrt{m^2-(d\cos\theta'/\lambda')^2}} \,. \qquad (7)$$

It now remains to solve for the unknown current. This is accomplished by imposing the condition that the axial electric field at the wires is equal to $I\exp(-ikz\sin\theta)\, Z_i$ where Z_i is the internal impedance of the wires. Since $a << d$, it can be assumed that the field is uniform around the wires, and hence Z_i can be calculated by known methods[11] and is given by

$$Z_i = \frac{\eta_1 I_o(\gamma a)}{2\pi a\, I_1(\gamma a)} \qquad (8)$$

where

$$\eta_1 = [i\mu_1\omega/(\sigma_1 + i\omega\epsilon_1)]^{1/2} \text{ and}$$

$$\gamma = [i\mu_1\omega(\sigma_1 + i\omega\epsilon_1)]^{1/2} .$$

I_o and I_1 are modified Bessel functions of the first and second type of order zero. For metallic wires, the displacement currents are negligible since, even at microwave frequencies, $\omega\epsilon_1 << \sigma_1$. In addition, the frequency is usually sufficiently high that $|\gamma a| >> 1$ and hence

$$Z_i \simeq \left(\frac{\mu_1\omega}{2\sigma_1}\right)^{1/2} \left(\frac{1+i}{2\pi a}\right) \tag{9}$$

With the restriction $a << d$, it then follows that

$$-IZ_i = k^2 \cos^2\theta \, [\Pi_z^p + \Pi_z^r + \Pi_z^w + \Pi_z^{w'}]_{\substack{x=h+a\\ y=0}} = \frac{i\mu\omega I\cos^2\theta}{4\pi} \left\{ \sum_{m=-\infty}^{+\infty}{}' \frac{e^{\frac{-2\pi a|m|}{d}}}{|m|} + \frac{R_m^o \, e^{\frac{-2\pi(2h+a)|m|}{d}}}{|m|} \right.$$

$$\left. + \sum_{m=-\infty}^{+\infty}{}' \left[\frac{e^{\frac{-2\pi a}{d}\sqrt{m^2 - (d\cos\theta/\lambda)^2}} + R_m \, e^{\frac{-2\pi(2h+a)}{d}\sqrt{m^2-(d\cos\theta/\lambda)^2}}}{\sqrt{m^2 - (d\cos\theta/\lambda)^2}} \right. \right. \tag{10}$$

(cont.)

$$- \frac{e^{\frac{-2\pi a|m|}{d}} + R^{o} e^{\frac{-2\pi(2h+a)}{d}}}{|m|} \Bigg] \tag{10}$$

$$+ \frac{e^{\frac{-i2\pi a}{\lambda}\cos\theta} + R_o e^{\frac{-i2\pi(a+2h)}{\lambda}\cos\theta}}{i(d/\lambda)\cos\theta} \Bigg\}$$

$$+ E_o \cos\theta \left(e^{\frac{i2\pi(h+a)}{\lambda}\cos\theta} + R_o e^{\frac{-i2\pi(h+a)}{\lambda}\cos\theta} \right)$$

where the prime over the summation indicates that the term for $m = 0$ is to be omitted. In the above $R^o = [R_m]_{d=0}$ which is not to be confused with $R_o = [R_m]_{m=0}$.

Utilizing the identity[14]

$$\sum_{m=1}^{\infty} \frac{e^{-am}}{m} = -\log\left[1 - e^{-a}\right] \tag{11}$$

$$\simeq -\log a \text{ for } a << 1,$$

the determining equation for I can be transformed to

$$I = \frac{-E_o \cos\theta \, d \left\{ e^{\frac{i2\pi h}{\lambda}\cos\theta} + R_o e^{\frac{-i2\pi h}{\lambda}\cos\theta} \right\}}{\left(\frac{\eta\cos\theta}{2}\right)\left(1 + R_o e^{\frac{-4\pi i h}{\lambda}\cos\theta}\right) + \frac{i\mu\omega d}{2\pi}\cos^2\theta\left[\log\frac{d}{2\pi a} - R^o \log\left(1 - e^{\frac{-2\pi(2h+a)}{d}}\right) + \Delta\right] + dZ_i} \tag{12}$$

where

$$\Delta = \sum_{m=1}^{\infty}\left(\frac{1 + R_m e^{\frac{-4\pi h}{d}\sqrt{m^2 - (d\cos\theta/\lambda)^2}}}{\sqrt{m^2 - (d\cos\theta/\lambda)^2}}\right) - \left(\frac{1 + R^o e^{\frac{-4\pi h m}{d}}}{m}\right) \tag{13}$$

In the preceding equation a has been considered small compared to both d and λ. Δ can be regarded as a correction factor which becomes negligible for $d << \lambda$.

The Distant Field

Although the complete solution of the problem has not been obtained, it is very desirable to focus attention on the distant scattered field. For example, if $|x - h| >> \lambda$, it is evident that only the terms for $m = 0$ are significant if d/λ and $d/\lambda' < 1$. The higher values of m correspond to evanescent waves which are highly damped in the positive and negative x directions. For larger values of d, additional undamped waves can be scattered from the grid. The discussion will be limited here to the smaller grid spacings satisfying the above inequality. The distant magnetic field, which has only a y component, is given by

$$H_y = H_o \exp\left[i(2\pi/\lambda)(x \cos\theta - z \sin\theta)\right] - \left\{H_o R_o - \frac{I \cos\theta}{2d}\left[\exp(i\, 2\pi h \cos\theta/\lambda) + R_o \exp(-i 2\pi h \cos\theta/\lambda)\right]\right\} \exp\left[i(2\pi/\lambda)(-x\cos\theta - z \sin\theta)\right] \tag{14}$$

for large positive x, and

$$H_y = \left\{\left[H_o + \frac{I \cos\theta}{2d} \exp(-i\, 2\pi h \cos\theta/\lambda)\right] \cdot \exp\left[i(2\pi/\lambda')(x \cos\theta' + y \sin\theta')\right]\right\} \frac{T_o N \cos\theta}{\cos\theta'} \tag{15}$$

for large negative x,

where $R_o = T_o - 1 = (K' - K)/(K' + K)$

with $K = \eta \cos\theta,\ K' = \eta' \cos\theta',\ \eta = (\mu/\epsilon)^{1/2},\ \eta' = (\mu/\epsilon')^{1/2}$.

The current I can now be written

$$I = \frac{-E_o \cos\theta\, d \left\{\exp\left[i(2\pi/\lambda) h \cos\theta\right] + R_o \exp\left[-i(2\pi/\lambda) h \cos\theta\right]\right\}}{(\eta \cos\theta/2)(1 + R_o \exp\left[-i(4\pi/\lambda) h \cos\theta\right]) + Z_g} \tag{16}$$

where

$$Z_g = \frac{i\mu\omega d\cos^2\theta}{2\pi}\left[\log\frac{d}{2\pi a} - \frac{\cos^2\theta' - \cos^2\theta}{\cos^2\theta' + \cos^2\theta}\log\left(1 - e^{\frac{-2\pi(2h+a)}{d}}\right) + \Delta\right] + d\,Z_i \tag{17}$$

where

$$\Delta = \sum_{m=1}^{\infty}\left\{\frac{1}{\sqrt{m^2 - (d\cos\theta/\lambda)^2}}\left[1 + \frac{\dfrac{\cos^2\theta'}{\sqrt{m^2 - (d\cos\theta'/\lambda')^2}} - \dfrac{\cos^2\theta}{\sqrt{m^2 - (d\cos\theta/\lambda)^2}}}{\dfrac{\cos^2\theta'}{\sqrt{m^2 - (d\cos\theta'/\lambda')^2}} + \dfrac{\cos^2\theta}{\sqrt{m^2 - (d\cos\theta/\lambda)^2}}}\, e^{\frac{-4\pi h}{d}\sqrt{m^2 - (d\cos\theta/\lambda)^2}}\right]\right.$$

$$\left. - \frac{1}{m}\left[1 + \frac{\cos^2\theta' - \cos^2\theta}{\cos^2\theta' + \cos^2\theta}\, e^{\frac{-4\pi hm}{d}}\right]\right\} \tag{18}$$

The equivalent circuit which may be taken as the analogue of the wire grid is shown in Fig. 2. The space to the right of the interface, $x > 0$, is represented by a transmission line of characteristic impedance K and propagation constant Γ. The line constants for the space to the left, $x < 0$, are K' and Γ'. At $x = h$ the line is shunted by an impedance Z_g. The voltage V across the line can now be identified with the electric field E_z and the magnetic field H_y, respectively. The propagation constants are given by

$$\Gamma = i(2\pi/\lambda)\cos\theta \text{ and } \Gamma' = i(2\pi/\lambda')\cos\theta'.$$

At normal incidence, the constants of the equivalent circuit become

$$K = \eta, \quad K' = \eta' \tag{19}$$

$$Z_g = \frac{i\mu\omega d}{2\pi}\left[\log\frac{d}{2\pi a} + \Delta\right] + dZ_i$$

where now

$$\Delta = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m^2 - (d/\lambda)^2}}\left[1 + \frac{\sqrt{m^2 - (d/\lambda)^2} - \sqrt{m^2 - (d/\lambda')^2}}{\sqrt{m^2 - (d/\lambda)^2} + \sqrt{m^2 - (d/\lambda')^2}}\, e^{\frac{-4\pi h}{d}\sqrt{m - (d/\lambda)^2}}\right] - \frac{1}{m} \tag{20}$$

When the grid is located in the interface and returning to the case of oblique incidence, it follows that

$$Z_g = \frac{i\mu\omega d}{2\pi}\cos^2\theta\left[\frac{2\cos^2\theta'}{\cos^2\theta + \cos^2\theta'}\log\frac{d}{2\pi a} + \Delta\right] + dZ_i \tag{21}$$

where now

$$\Delta = \sum_{m=1}^{\infty} \frac{2\cos^2\theta'}{\cos^2\theta'\sqrt{m^2 - (d\cos\theta/\lambda)^2} + \cos^2\theta\sqrt{m^2 - (d\cos\theta'/\lambda)^2}} - \frac{2\cos^2\theta'}{(\cos^2\theta' + \cos^2\theta)m} \tag{22}$$

The special case of equation (22) for normal incidence which of course is the same as equation (20) with $h = 0$ is in agreement with a formula quoted to me by G. D. Monteath of the B. B. C.†

Then Z_g is given by equation (19) and

$$\Delta = \sum_{m=1}^{\infty} \frac{2}{\sqrt{m - (d/\lambda)^2} + \sqrt{m - (d/\lambda')^2}} - \frac{1}{m} \quad . \tag{23}$$

On previous occasions[12] it has been assumed that the equivalent shunt impedance Z_g for a wire grid is only dependent on the properties of the media in which it is immersed. In other words

$$Z_g = \frac{i\mu\omega\, d \cos^2\theta}{2\pi} \left[\log \frac{d}{2\pi a} + \Delta \right] + dZ_i \tag{24}$$

where

$$\Delta = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m^2 - (d \cos\theta/\lambda)^2}} - \frac{1}{m} \tag{25}$$

It can be seen from equation (17) that this is only justified if $h >> d$. In the case of large angles of incidence this condition becomes more stringent.

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† personal communication

Generalization to Arbitrary Incidence

In the preceding analysis it has been assumed that the electric vector is polarized in the plane of incidence (i.e., $E_y = 0$). It is of interest to consider the case where the incidence is arbitrary such that the electric field of the incident wave is given by

$$\vec{E}^p = \vec{A} \exp[ik(x \cos\phi \cos\theta + y \sin\phi \cos\theta - z \sin\theta)] \tag{26}$$

where $\vec{A}$ is the vector magnitude of the field. Under the assumption that the radius of the wires is small compared with the wavelength, only the z component of the electric field will excite currents in the grid. In the present instance, there will be a difference of phase of the incident field at adjacent wires of $kd \sin\phi \cos\theta$ radians together with a phase change of $k \sin\theta$ radians per unit length along each wire. The currents on the wires may then be represented by the expression

$$I e^{iknd \sin\phi \cos\theta} e^{-ikz \sin\theta} \quad (n = 0, \pm 1, \pm 2 \ldots)$$

where I is the current on the reference wire $(n = 0)$ at $x = h$, $y = 0$, $z = 0$.

The subsequent analysis for this problem is very similar to the special case $(\phi = 0)$ treated above. The algebra is, however, very cumbersome so further details will be omitted and the final result will be quoted directly in terms of the parameters of the equivalent circuit which has the same form as Fig. 2. The voltage on the line is now to be identified with E_z component and the current is to be identified with the H_y component.† The characteristic impedances and propagation constants are given by

$$K = \eta \cos\theta/\cos\phi,$$
$$K' = \eta' \cos\theta'/\cos\phi',$$
$$\Gamma = i(2\pi/\lambda)\cos\theta\cos\phi,$$
$$\Gamma' = i(2\pi/\lambda')\cos\theta'\cos\phi',$$

where $\eta'/\eta = \lambda'/\lambda = (\epsilon/\epsilon')^{\frac{1}{2}} = 1/N,$

$$\sin\theta' = (1/N)\sin\theta, \text{ and}$$
$$\sin\phi' = \frac{\sin\phi\cos\theta}{\sqrt{N^2 - \sin^2\theta}}.$$

The equivalent shunt impedance is now given by

$$Z_g = \frac{i\mu\omega\, d\cos^2\theta}{2\pi}\left[\log\frac{d}{2\pi a} + \frac{\cos^2\theta - \cos^2\theta'}{\cos^2\theta + \cos^2\theta'}\log\left(1 - e^{-\frac{2\pi(2h+a)}{d}}\right) + \Delta\right] + dZ_i \tag{27}$$

† The E_x and E_y components of the field are unaffected by the grid but, of course, they are modified by the dielectric interface.

where

$$\Delta \doteq \frac{1}{2} \sum_{m=1}^{\infty} \Bigg\{ \frac{1 + R_m e^{-\frac{4\pi h}{d}\sqrt{\left(m + \frac{d \cos\theta \sin\phi}{\lambda}\right)^2 - \left(\frac{d\cos\theta}{\lambda}\right)^2}}}{\sqrt{\left(m + \frac{d \cos\theta \sin\phi}{\lambda}\right)^2 - \left(\frac{d\cos\theta}{\lambda}\right)^2}}$$

$$+ \frac{1 + R_{-m} e^{-\frac{4\pi h}{d}\sqrt{\left(m - \frac{d \cos\theta \sin\phi}{\lambda}\right)^2 - \left(\frac{d\cos\theta}{\lambda}\right)^2}}}{\sqrt{\left(m - \frac{d \cos\theta \sin\phi}{\lambda}\right)^2 - \left(\frac{d\cos\theta}{\lambda}\right)^2}}$$

$$- \frac{2}{m}\left[1 - \frac{\cos^2\theta - \cos^2\theta'}{\cos^2\theta + \cos^2\theta'} e^{-\frac{4\pi h m}{d}} \right] \Bigg\} \qquad (28)$$

The equation for R_m is

$$R_m = \frac{\dfrac{\cos^2\theta'}{\sqrt{\left(m + d\cos\theta'\sin\phi'/\lambda'\right)^2 - \left(d\cos\theta'/\lambda'\right)^2}} - \dfrac{\cos^2\theta}{\sqrt{\left(m + d\cos\theta\sin\phi/\lambda\right)^2 - \left(d\cos\theta/\lambda\right)^2}}}{\dfrac{\cos^2\theta'}{\sqrt{\left(m + d\cos\theta'\sin\phi'/\lambda'\right)^2 - \left(d\cos\theta'/\lambda'\right)^2}} + \dfrac{\cos^2\theta}{\sqrt{\left(m + d\cos\theta\sin\phi/\lambda\right)^2 - \left(d\cos\theta/\lambda\right)^2}}} \tag{29}$$

and the corresponding equation for R_{-m} is obtained by replacing m with -m.

An important special case of (27) is when the electric vector is always parallel to the wires (i.e., $E_y = E_x = 0$) so that $\theta = \theta' = 0$ and then

$$Z_g = \frac{i\mu\omega d}{2\pi}\left[\log\frac{d}{2\pi a} + \Delta\right] + dZ_i \tag{30}$$

where

$$\Delta = \frac{1}{2} \sum_{m=1}^{\infty} \left\{ \frac{1 + R_m e^{-\frac{4\pi h}{d} \sqrt{\left(m + \frac{d \sin \phi}{\lambda}\right)^2 - \left(\frac{d}{\lambda}\right)^2}}}{\sqrt{\left(m + \frac{d \sin \phi}{\lambda}\right)^2 - \left(\frac{d}{\lambda}\right)^2}} \right.$$

$$\left. + \frac{1 + R_{-m} e^{-\frac{4\pi h}{d} \sqrt{\left(m - \frac{d \sin \phi}{\lambda}\right)^2 - \left(\frac{d}{\lambda}\right)^2}}}{\sqrt{\left(m - \frac{d \sin \phi}{\lambda}\right)^2 - \left(\frac{d}{\lambda}\right)^2}} - \frac{2}{m} \right\} \tag{31}$$

where

$$R_m = \frac{\sqrt{\left(m + d \sin \phi/\lambda\right)^2 - \left(d/\lambda\right)^2} - \sqrt{\left(m + d \sin \phi'/\lambda\right)^2 - \left(d/\lambda'\right)^2}}{\sqrt{\left(m + d \sin \phi/\lambda\right)^2 - \left(d/\lambda\right)^2} + \sqrt{\left(m + d \sin \phi'/\lambda'\right)^2 - \left(d/\lambda'\right)^2}} \tag{32}$$

Another limiting case is when $h >> d$, so that

$$Z_g = \frac{i\mu\omega d \cos^2\theta}{2\pi}\left[\log\frac{d}{2\pi a} + \Delta\right] + dZ_i \tag{33}$$

and

$$\Delta = \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{\sqrt{\left(m - \frac{d\cos\theta\sin\phi}{\lambda}\right)^2 - \left(\frac{d\cos\theta}{\lambda}\right)^2}} + \frac{1}{\sqrt{\left(m + \frac{d\cos\theta\sin\phi}{\lambda}\right)^2 - \left(\frac{d\cos\theta}{\lambda}\right)^2}} - \frac{2}{m} \tag{34}$$

In this case, the equivalent shunt impedance only depends on the properties of the media in which the grid is immersed. It is in agreement with a formula derived previously.

Line Source Excitation

To reduce the power absorbed in the ground from an antenna, a wire mesh is often placed on or just below the surface of the ground. For example, in broadcast antennas, this system takes the place of radial wires emanating from the base of the vertical mast. It is helpful to consider a two-dimensional analogue of this problem in order to further justify some of the approximation techniques previously employed in the three-dimensional counterpart.[13] Again the grid is assumed to consist of thin parallel wires and is located in the interface between the two half-spaces (i.e., $h = 0$) as indicated in Fig. 1. A line source is now located at $x = x_o$ carrying a magnetic current V (in volts) from $y = -\infty$ to $y = +\infty$. The primary field of this line source has only a y component and is given by

$$H_y^p = \frac{\epsilon\omega V}{4} H_o^{(2)}\left[k[(x - x_o)^2 + z^2]^{\frac{1}{2}}\right] \tag{35}$$

This can be rewritten in integral form as follows

$$H_y^p = \frac{i\epsilon\omega V}{4\pi} \int_{-\infty}^{+\infty} u^{-1} e^{-u|x - x_o|} e^{isz} ds \tag{36}$$

where $u = \sqrt{s^2 - k^2}$ with the contour being indented upward by a small semi-circle at $s = k$ and downward at $s = -k$.† The integration here can be regarded as a superposition over all plane waves whose angle of incidence θ, measured from the z axis, is related to s by

$$s = k \sin \theta$$

† Alternatively one may remove the indentations of the contour if k is considered to have a vanishing small negative imaginary part.

It is clear that the plane wave spectrum must include both real and complex angles of incidence.

Utilizing the results of the plane wave solution for the case of the H vector parallel to the wires, it readily follows that the resultant field is given by

$$H_y = \frac{i\epsilon\omega V}{4\pi} \int_{-\infty}^{+\infty} u^{-1} \left[e^{-u|x - x_o|} + R(s) e^{-u(x + x_o)} \right] e^{isz} \, ds \tag{37}$$

for $x > 0$ where

$$R(s) = \frac{K(s) - Z(s)}{K(s) + Z(s)} , \tag{38}$$

$$K(s) = \frac{\mu\omega}{ik^2} u = \frac{\mu\omega}{k} \sqrt{1 - (s/k)^2} , \tag{39}$$

$$Z(s) = \frac{K'(s)\, Z_g(s)}{K'(s) + Z_g(s)} , \quad K'(s) = \frac{\mu\omega}{k'} \sqrt{1 - (s/k')^2} , \tag{40}$$

$$Z_g(s) = \frac{i\mu\omega d}{2\pi} \left(1 - (s/k)^2\right) \left[\frac{2\left(1 - (s/k')^2\right)}{2 - (s/k)^2 - (s/k')^2} \log \frac{d}{2\pi a} + \Delta(s) \right] \tag{41}$$

and

$$\Delta(s) = \sum_{m=1}^{\infty} \frac{2\left(1 - (s/k')^2\right)}{\left(1 - (s/k')^2\right)\sqrt{m^2 + \left(s^2 - k^2\right)\left(d/2\pi\right)^2} + \left(1 - (s/k)^2\right)\sqrt{m^2 + \left(s^2 - (k')^2\right)\left(d/2\pi\right)^2}} - \frac{2\left(1 - (s/k')^2\right)}{\left[\left(1 - (s/k')^2\right) + \left(1 - (s/k)^2\right)\right] m} \tag{42}$$

When the field is observed at some large distance from the grid (say $x >> \lambda$) the integral in equation (37) can be evaluated by the principle of stationary phase. The saddle point is at $s = k \sin\bar{\theta}$ where $\bar{\theta} = \arctan\left(\frac{z}{x - x_o}\right)$. Since the function R(s) is slowly varying compared to the exponential factor it can be taken outside the integral and replaced by $R(k \sin\bar{\theta})$. The far field is then given by

$$H_y = \frac{\epsilon \omega V}{2\pi} \sqrt{\frac{2i}{\pi k r}}\, e^{-ikr} \left[1 + R(k \sin\bar{\theta}) e^{-i2x_o \cos\bar{\theta}}\right] \tag{43}$$

where $r = \sqrt{(x - x_o)^2 + z^2}$

It is of interest to consider the energy absorption, from a magnetic line source at $x = x_o$, in the lower half-space $(x < 0)$ of dielectric constant ϵ' which can have a finite negative imaginary part.† The power flow into the space $x > x_o$ is denoted by S^+ and that for $x < x_o$ is S^-. From Poynting's theorem it then follows that

$$S_\pm = \left[\int_{-\infty}^{+\infty} P_x \, dz \right]_{x = x_o \pm \delta} \tag{44}$$

with δ being a vanishing small positive quantity and

$$P_x = \frac{1}{2} \text{ Real Part of } E_z H_y^\dagger$$

where the footnote denotes a complex conjugate. The tangential fields are given by

$$H_y = \frac{i\epsilon\omega V}{4\pi} \int_{-\infty}^{+\infty} u^{-1} \left[e^{\mp(x - x_o)u} + R(s)\, e^{-(x + x_o)u} \right] e^{isz} \, ds \tag{45}$$

and

† The complex dielectric constant ϵ' in this case is replaced by $\epsilon' - i\,\sigma/\omega$ where now ϵ' is real and σ is the conductivity.

$$E_z = \frac{1}{i\epsilon\omega} \frac{\partial Hy}{\partial x}$$

$$= \frac{V}{4\pi} \int_{-\infty}^{+\infty} \left[\mp e^{\mp(x - x_o)u} - R(s)e^{-(x + x_o)u} \right] e^{isz} \, ds \qquad (46)$$

where the upper signs are taken for $x > x_o$ and the lower signs for $0 < x < x_o$. The expression for the power flow is then a threefold infinite integral

$$S_\pm = \frac{1}{2} \operatorname{Re}\left(\frac{V}{4\pi}\right)^2 \iiint_{-\infty}^{+\infty} f(s)\, g(s')e^{i(s - s')z} \, ds\, ds'\, dz \qquad (47)$$

with $f(s) = \mp e^{-u\delta} - R(s)e^{-2uh}$

and $g(s) = -\frac{i\epsilon\omega}{u^*}\left[e^{-u^*\delta} + R(s^*)e^{-2u^*h}\right]$

Noting that the unit impulse function $S(\alpha)$ at $\alpha = 0$ can be expressed, in the Lebesque sense, as

$$S(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda\alpha} \, d\lambda \qquad (48)$$

* The asterisk denotes a complex conjugate.

it then follows that

$$S_{\pm} = \mathrm{Re}\ \frac{V^2}{16\pi} \iint_{-\infty}^{+\infty} f(s)\ g(s')\ S(s - s')\ ds\,ds' \tag{49}$$

Now utilizing the sifting property of the impulse function, the integration with respect to s' can be carried out to yield

$$S_{\pm} = \mathrm{Re}\ \frac{V^2}{16\pi} \int_{-\infty}^{+\infty} f(s)\ g(s)\ ds \tag{50a}$$

$$= \mathrm{Im}\ \frac{V^2 \epsilon\omega}{16\pi} \int_{-\infty}^{+\infty} \frac{1}{u^*} \left[\mp e^{-u^* \delta} \mp R^*(s) e^{-u^* 2x_o} - R(s) e^{-u 2x_o} \right] d\lambda \tag{50b}$$

The preceding expression specifies the power flowing into the half-spaces above or below the line source. The total power which must be supplied by the line source is given by

$$S_- - S_+ = - \frac{\epsilon\omega V^2}{8\pi}\ \mathrm{Im} \int_{-\infty}^{+\infty} u^{-1} \left[e^{-u\delta} + R(s) e^{-u 2x_o} \right] ds \tag{51a}$$

$$S_- - S_+ = \frac{\epsilon\omega V^2}{8} \left[1 - \frac{1}{\pi} \operatorname{Im} \int_{-\infty}^{+\infty} R(s) e^{-2ux_o} u^{-1} ds \right] \tag{52b}$$

$$= \frac{\epsilon\omega V^2}{8} \left\{ \left[1 + J_o (2kx_o) \right] - \frac{1}{\pi} \operatorname{Im} \int_{-\infty}^{+\infty} [R(s) - 1] e^{-2ux_o} u^{-1} ds \right\} \tag{53b}$$

The integral term in the preceding equation can be regarded as a correction term which accounts for the imperfectly reflecting properties of the grid in the interface at x = 0. The first term is the power radiated from the line source when it is located a height x_o over a perfectly conducting flat surface. The additional power ΔS which the line source must be supplied to account for the losses in the lower half-space is therefore given by

$$\Delta S = -\operatorname{Im} \frac{\epsilon\omega V^2}{8\pi} \int_{-\infty}^{+\infty} [R(s) - 1] e^{-2ux_o} u^{-1} ds \tag{54}$$

In the preceding equation

$$R(s) - 1 = \frac{-2K' Z_g(s)}{KK' + Z_g(s)[K' + K]} \tag{55}$$

$$\text{where } K = \frac{\mu\omega}{k} \sqrt{1 - (s/k)^2} = \frac{\mu\omega}{ik^2} u = \frac{u}{i\epsilon\omega} \tag{56}$$

$$K' = \frac{\mu\omega}{(k')} \sqrt{1 - (s/k')^2} \tag{57}$$

and $Z_g(s)$ is given by equation (41). Since R(s) is slowly varying compared to the exponential factor in the integrand of equation (54) it can be replaced, subject to $Z_g << \eta$, by

$$R(s) - 1 \doteq - \frac{2\epsilon\omega}{u} \frac{\eta' Z_g}{\eta' + Z_g} \text{ with } \eta' = (\mu/\epsilon')^{\frac{1}{2}}, \tag{58}$$

$$Z_g = \frac{i\mu\omega d}{2\pi} \left[\log \frac{d}{2\pi a} + \Delta \right], \tag{59}$$

and

$$\Delta = \frac{1}{2} \sum_{m=1}^{\infty} \left[\frac{1}{\sqrt{m^2 - (d/\lambda)^2} + \sqrt{m^2 - (dN/\lambda)^2}} - \frac{2}{m} \right] \tag{60}$$

with $N = \sqrt{\epsilon'/\epsilon}$. Z_g is now the normal surface impedance of the grid when it is situated in the interface between media whose intrinsic impedances are η and η'. The integral for the energy absorption is then given by the approximate form

$$\Delta S = \frac{(\epsilon\omega V)^2}{4\pi} \mathrm{Re}\left(\frac{\eta' Z_g}{\eta' + Z_g}\right) \int_{-\infty}^{+\infty} \frac{e^{-2ux_o}}{u^2} ds \tag{61}$$

It is interesting to observe that this result is equivalent to the following

$$\Delta S = \mathrm{Re}\,\frac{1}{2} \left(\frac{\eta' Z_g}{\eta' + Z_g}\right) \int_{-\infty}^{+\infty} [H_y^{\infty}]^2 dz \tag{62}$$

where H_y^{∞} is the tangential magnetic field of the line source over the surface x = 0, assuming it to be a perfect conductor. To demonstrate this equivalence H_y^{∞} is expressed in integral form as

$$H_y^{\infty} = \frac{i\epsilon\omega V}{\pi} \int_{-\infty}^{+\infty} e^{-ux_o} e^{+isz} ds \tag{63}$$

and substituted into equation (62) forming a triple integral which can be readily reduced to equation (61) by making further use of sifting property of the impulse function. The factor $\eta' Z_g/(\eta' + Z_g)$ can be interpreted as the normal surface impedance at the interface x = 0 being composed on the surface impedance of the ground in parallel with that of the grid. Equation (62) can be derived directly by an application of the compensation theorem if one knows, a priori, the appropriate value to use for the composite surface impedance at the interface.[13] To express ΔS in terms of tabulated integrals it is desirable to start with the well-known result

$$\int_{-\infty}^{\infty} u^{-1} e^{-ua} \, ds = \frac{\pi}{i} \left[H_o^{(2)} \, (ka) \right] \tag{64}$$

and integrate both sides with respect to a from a to ∞ to give

$$\int_{-\infty}^{\infty} u^{-2} e^{-ua} \, ds = \frac{\pi}{i} \int_{a}^{+\infty} H_o^{(2)} \, (ka) \, da. \tag{65}$$

Therefore

$$\Delta S = \frac{(\epsilon\omega V)^2}{4} \, \mathrm{Im} \, \frac{\eta' Z_g}{\eta + Z_g} \int_{x_o}^{\infty} H_o^{(2)} \, (ka) \, da \tag{66}$$

where the integral is tabulated by Watson.[15]

Conclusion

A complete analysis has been given for the response of a wire grid in, or parallel to, an interface between homogeneous media. The results are valid for a plane wave with arbitrary polarization and angle of incidence. It is seen that the case for normal incidence or parallel polarization leads to considerably simpler formulae. However, subject to the smallness of the wire diameters, the equivalent circuit of the grid is always a pure shunt element and the respective media are homogeneous transmission lines.

The energy absorbed from a magnetic line source in the lower (dissipative) medium is shown to be appreciably modified by the presence of the wire grid in, or near, the interface. Some justification is given for the approximation techniques employed previously for the energy computations for monopole antennas with radial wire ground systems.

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N. B. In reference (11), $\cos\theta \cos\phi_o Z_i d/\eta_o$ should be replaced by $\cos^{-1}\theta \cos\phi_o Z_i d/\eta_o$ in equation (25).

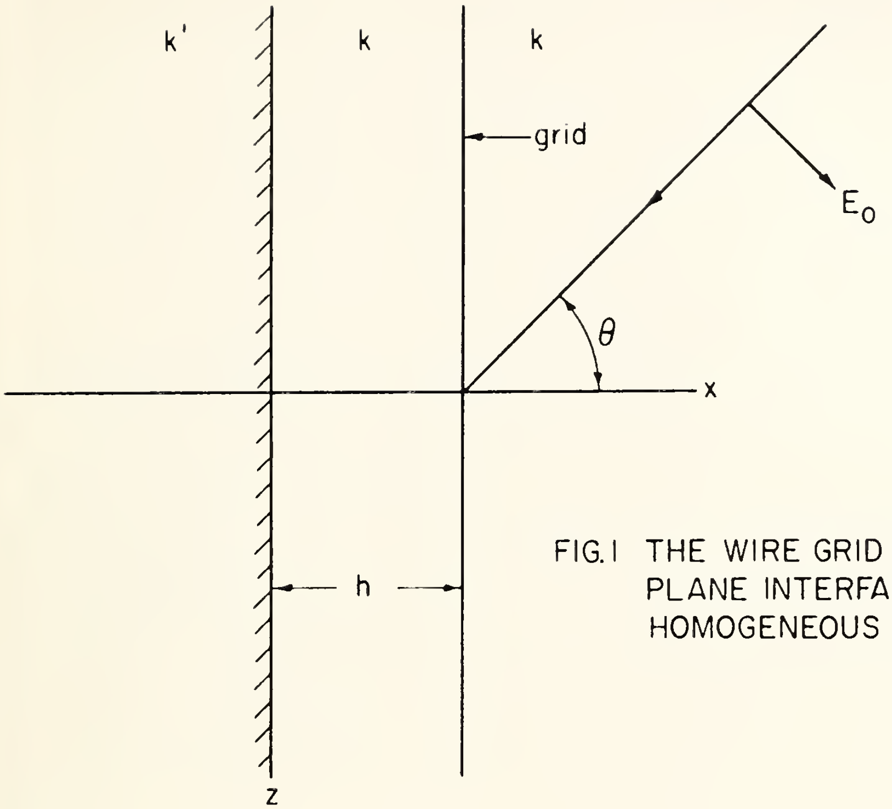

FIG. I THE WIRE GRID PARALLEL TO A PLANE INTERFACE BETWEEN TWO HOMOGENEOUS MEDIA.

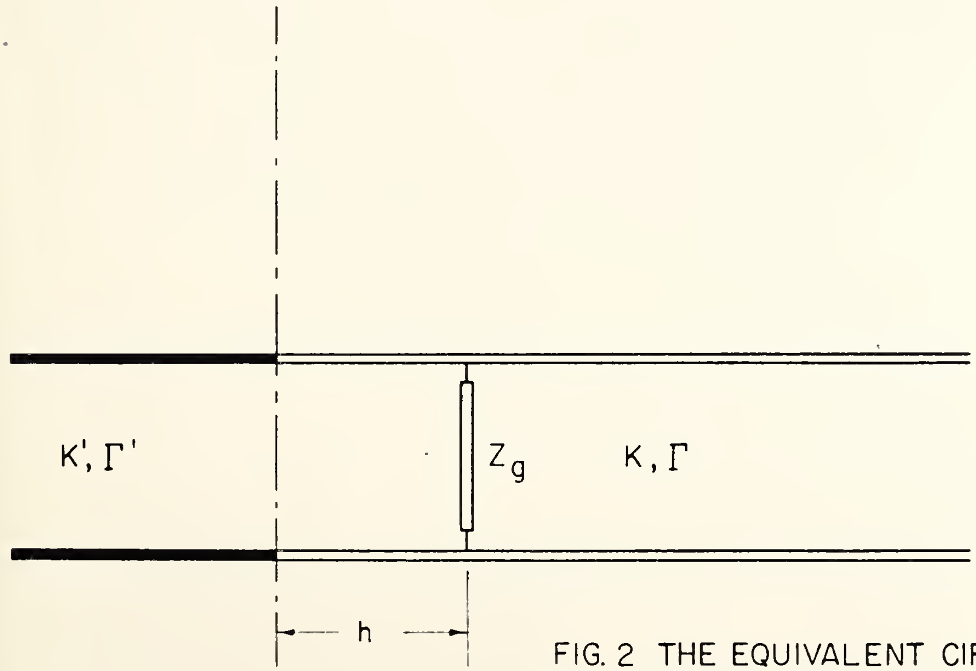

FIG. 2 THE EQUIVALENT CIRCUIT CONSISTING OF TWO SEMI-INFINITE TRANSMISSION LINES WITH A SHUNT ELEMENT ACROSS ONE.

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