NATIONAL BUREAU OF STANDARDS REPORT

4571

Probabilities of Non-Penetration and Targets Downed for Constant Characteristic Kill Probability

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NBS PROJECT **NBS REPORT**

1102-40-5126 4571

12 March 1956

Probabilities of Non-Penetration and Targets Downed for Constant Characteristic Kill Probability

Semi-Annual Progress Report on MIPR 55-2175-SC-91
July 1 to December 31, 1955

NBS

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

Probabilities of Non-Penetration and Targets Downed for Constant Characteristic Kill Probability

Semi-annual progress report on MIPR 55-2175-SC-91
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I. Introduction: This summary and the attached memoranda represent the analysis and computation done on the three systems proposed by Dr. E. Biser in his letter to the Bureau of Standards dated April 11, 1955.

In Section 2 the systems are defined in their general form. Their common property is the assumption of a constant kill probability without considerations of elapsed time and distance. They differ in the degree of coordination supposed in each case.

In Section 3 the results of the analysis are listed in closed form together with some observations on the behavior of these results. The two basic functions used to rate the systems are the number of targets downed (Tgt Dn) and its average (Avg Tgt Dn), and the probability of non-penetration (Pnp) and its average (Avg Pnp). One of the observations is that the behavior of the latter function is to a large extent determined by that of another function: the probability that all targets are fired at (P_F). The variances of these functions are also given.

In section 4 are listed further possibilities for analysis and computations suggested by the results already obtained and by Martin Orr of the Signal Corps in a recent conference. These

new investigations would be aimed at completing the overall view of the systems by introducing slight modifications in their definitions.

Section 5 is the compendium of memoranda sent to Dr. Biser between November 1955, and March 1956 with some revisions. They contain all the results listed in section 3 as well as the tables computed from these results. The memoranda containing tables are at the very end.

II. Definition of the Systems

1. General Assumptions

The number of batteries is B, the number of targets in a raid is T, and the total number of missiles available is m. The missiles are distributed equally among the batteries (m/B is an integer).

It is assumed that the single shot kill probability is a constant. This characteristic kill probability (CKP) is denoted by p.

2. Assignment Systems

An assignment system describes how batteries are assigned to targets in an engagement (including random assignment). The system describes, moreover, how many missiles are fired at each target in a given engagement (this may be a random variable).

The assignment system, together with the CKP p, determines the probability distribution of the random variable

N = number of targets down in an engagement.

That is, in hypothetically repeated engagements with the same B, T, m, and p, and with the same assignment system, N may take any integer value from zero to T. Roughly speaking, the probability distribution of N gives the frequencies of these values in the sequence of hypothetically repeated engagements.

In particular, the assignment system determines the average number of targets down (<u>Avg Tgt Dn</u>) in such a sequence of engagements. It also determines the average probability of non-penetration (<u>Avg Pnp</u>), which is the frequency of engagements in which all targets are downed, i.e., the probability that N equals T.

A. <u>Coordinated System</u>

The total number m of missiles is distributed as equally as possible among the targets. If m/T is an integer, each target is fired on by exactly m/T missiles and the precise scheme of assignment of batteries to targets is irrelevant.

(The case where m/T is not an integer is slightly more complicated. It is, however, a special case of the sector-coordinated system as described below).

B. Sector-Coordinated System

The set of B batteries is partitioned into S sectors, each containing the same number of batteries, and hence each having the same number m/S of missiles. A subset of the T targets is assigned to each sector (where these subsets may be overlapping). The batteries in a given sector distribute their m/S missiles as equally as possible among the targets assigned to them. If the m/S missiles cannot be distributed exactly equally among the targets, the "extras" are fired at a randomly chosen subset of the assigned targets.

An illustrative example is given of a sector-coordinated system involving 20 targets, 20 batteries, 5 sectors, and a particular assignment scheme. For this example, the following alternative method of distribution of fire is also considered: The m/S missiles fired by a sector are distributed as equally as possible among the targets assigned to that sector subject to the restriction that at least two missiles are fired at a target by a given sector if the sector fires at the target at all.

In sector-coordinated systems, the number of missiles fired at a target in an engagement depends on the number of sectors to which the target is assigned and on the distribution of fire in those sectors.

C. Battery-Wise Uncoordinated System

Each battery chooses one target at random (independent of choices made by other batteries) and fires all of its m/B missiles at that target. The number of missiles fired at a target in an engagement depends on the number of batteries which happen to choose that target.

III. Results. A list of the formulas for the various functions defined in section 2 is given below. Generally B indicates the number of batteries, T the number of targets, p the constant characteristic kill probability and m the number of missiles. Other symbols are defined as they are used. The values of these variables used in the computation of the tables to be found in the final memoranda are indicated at each step.

A. Coordinated system:

(A.1) $\text{Avg Tgt Dn} = T(1-b)$

(A.2) $\text{Avg Pnp} = (1-b)^T$

(A.3) $\text{Var Tgt Dn} = Tb\,(1-b)$

where $b = (1-p)^{m/T}$. Avg Tgt Dn and Avg Pnp are computed for $T = 20$, $m = 20(20)160$ and $p = 0.2, 0.3, 0.5, 0.7, 0.9$.

B. Sector-Coordinated System

Given T targets, m missiles, CKP p, and S sectors of batteries with m/S missiles per sector. Let $q = 1-p$. Suppose m/S is an integer.

Let $\delta_{ij} = \begin{array}{l} 1 \text{ if } i^{th} \text{ target assigned to } j^{th} \text{ sector} \\ 0 \text{ otherwise,} \end{array}$

$i = 1,\ldots,T; \quad j = 1,\ldots,S.$

Let $d_j = \sum_{i=1}^{T} \delta_{ij}, \; j = 1,\ldots,S.$

Let r_j, k_j, $(j = 1,\ldots,S)$ be integers defined by

$$m/S = k_j \, d_j + r_j, \quad 0 \leqslant r_j \leqslant d_j - 1.$$

$$\text{(B.1)} \quad \text{Avg Tgt Dn} = T - \sum_{i=1}^{T} \prod_{j=1}^{S} \left[\frac{r_j}{d_j} q^{\delta_{ij}(k_j + 1)} + \left(1 - \frac{r_j}{d_j}\right) q^{\delta_{ij} k_j} \right]$$

This formula applies when each sector distributes fire as equally as possible among the targets assigned to it.

Avg Tgt Dn, Special Case

Let $T = 20$, $S = 5$, and suppose targets one through ten are assigned to sectors two and four, targets eleven through twenty are assigned to sectors three and five, and targets eight through thirteen are assigned to sector one. The formula above specializes to the following.

$$\text{(B.2)} \quad \text{Avg Tgt Dn} = 20 - \frac{14}{100} q^{2k_1} (10 - r_1 + r_1 q)^2$$

$$- \frac{1}{100} q^{2k_1 + k_2} (10 - r_1 + r_1 q)^2 \, (6 - r_2 + r_2 q)$$

where k_1, k_2, r_1, r_2 are defined by

$$m/5 = 10k_1 + r_1 \ , \quad r_1 < 10$$

$$m/5 = 6k_2 + r_2, \quad r_2 < 6$$

for any m which is an integer multiple of 5.

If the equal distribution of fire is restricted by the requirement that at least two missiles be fired at a target if any are fired at it by a sector, then the formula above does not hold for m < 100. The correct formula for any m < 100 which is an integer multiple of 20 is the following.

(B.3) $$\text{Avg Tgt Dn} = 20 - \frac{14}{100}\left(10 - r_1' + r_1' q^2\right)^2$$

$$- \frac{1}{100}\, q^{2k_2'} \left(10 - r_1' + r_1' q^2\right)^2 \left(6 - r_2' + r_2' q^2\right)$$

where r_1', r_2', k_2' are integers defined by

$$m = 10\, r_1'$$

$$\frac{m}{10} = 6k_2' + r_2', \quad r_2' < 6.$$

Useful exact formulas for Avg Pnp can be given only in a few special cases (values of m) of the above particular sector-coordinated system. These, together with various approximations which were used for other values of m, are given below in memorandum 3.

A general formula for Avg Pnp which applies to all sector-coordinated systems with equal distribution of fire is also given below. Because it is not in very useful form, and because it requires a lengthy development of additional notation, it is omitted here.

Tables were computed for the particular sector-coordinated system described above, for CKP = 0.2, 0.3, 0.5, 0.7, 0.9, and m = 20(20)160.

C. <u>Uncoordinated System:</u>

(C.1) Avg Tgt Dn $= T\left[1-\left(1-\frac{1-a}{T}\right)^B\right]$

(C.2) Avg Pnp $= \sum_{i=0}^{T} (-1)^i \binom{T}{i}\left(1-\frac{1-a}{T}\, i\right)^B$

(C.3) $$= \frac{T!}{T^B} \sum_{i=0}^{B-T} \binom{B}{i} \mathcal{S}_{B-i}^{(T)} (aT)^i (1-a)^{B-i}$$

(C.4) $$P_F = \frac{T!}{T^B} \mathcal{S}_B^{(T)}$$

where $\mathcal{S}_B^{(T)}$ are the Stirling numbers of the second kind and $a = (1-p)^{m/B}$. The formulas (C.2) and (C.4) are special cases of the formulas for the probability that exactly n targets ($p_1(n)$) and the probability that exactly n targets are shot down ($p_2(n)$). Clearly $P_F = p_1(0)$ and Avg Pnp $= p_2(T)$.

(C.5) $$p_1(n) = \frac{T!}{n!\,T^B} \mathcal{S}_B^{(T-n)}$$

(C.6) $$p_2(n) = \binom{T}{n} \sum_{i=0}^{n} (-1)^i \binom{n}{i}\left(1-\frac{1-a}{T}\,(T-n+i)\right)^B$$

We observe that P_F is that largest value that can be taken by Avg Pnp with a fixed number of batteries and targets. It is also the approximate value taken by Avg Pnp when p is large or m/B is large, that is when a is small. Thus the order of magnitude of Avg Pnp for fixed number of batteries and targets is P_F.

This observation is formalized and generalized in the memorandum "Some Observations on Pnp". It is proved that Avg Pnp/P_F is a constant for all systems with similar engagement strategies. Thus strategies may be compared on the basis of their P_F.

Tables were computed for B = 10(10)40, T = 5(5)20, m/B = 1(1)9, 12, 15 and p = 0.2, 0.3, 0.5, 0.7, 0.9.

IV. Proposed Extensions. Graphs will be made to present the results in visual form. The nature of these graphs will be decided on in conference with Dr. Biser and Martin Orr.

As suggested by Mr. Orr we will compute two functions which will estimate the effect of the loss of information of the number of targets on the number of targets downed. These functions are $T[1-(\frac{1-a}{T})^B] - t[1-(\frac{1-a}{t})^B]$ and $t[1-(1-\frac{p}{t})^{\frac{Bt}{T}}]$ where t is the supposed number of targets.

Because the Stirling numbers of the second kind enter so frequently in our considerations of Pnp in the uncoordinated case, and especially in the formula for P_F, we propose extending

the known tables of these numbers to cover the cases for which they arise in this connection.

Because Avg Pnp/P_F is the basic function from which Avg Pnp can be computed for various equivalent strategies by multiplying by P_F for that strategy, we propose computing Avg Pnp/P_F in the uncoordinated system.

Because in actual practice the number of batteries is likely to be of the same order, and possibly less than, the number of targets in which case Avg Pnp would be too small for comfort, we propose computing Avg Pnp for a modified missile-uncoordinated system in which the number of batteries would be essentially extended and the number of missiles per battery would be essentailly reduced. This would have the effect of raising the value of P_F and thus of Avg Pnp.

To : Dr. E. Biser

From : K. Goldberg

Subject : Battery-Wise Uncoordinated System Based on Constant Kill Probability

After learning of your formula for Tgt Dn for the above case, we have derived additional formulas for this case. The results are as follows:

(1) $$\text{Avg Tg Dn} = T\left[1-\left(1-\frac{1-a}{T}\right)^B\right]$$

$$\text{Avg Pnp} = \sum_{i=0}^{T}(-1)^i\binom{T}{i}\left(1-\frac{1-a}{T}\,i\right)^B$$

where T is the number of targets, B the number of batteries, $a = (1-P)^{m/B}$ with p the constant kill probability and m/B the number of shots fired by each battery.

We also have

(2) $p_1(n)$ = prob. exactly n targets are not assigned = $\dfrac{T!}{n!\,T^B}\mathcal{S}_B^{(T-n)}$

$p_2(n)$ = avg. prob. exactly n targets shot down

$$= \binom{T}{n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left(1-\frac{1-a}{T}(T-n+i)\right)^B$$

where $\mathcal{S}_B^{(k)}$ is the Stirling number of the second kind defined

by

$$\sum_{s=0}^{\infty}\frac{k!\,\mathcal{S}_s^{(k)}}{s!}x^s = (e^x-1)^k \quad .$$

The formulas in (2) can be summed to give the probability for at least, or at most, n targets missing fire or not shot down. Note also that $p_2(T)$ is Avg. Pnp.

The derivations of these formulae follow from their basic definitions for a single engagement:

$$(3) \qquad \text{Tgt Dn} = \sum_{i=1}^{T} (1-a^{k_i})$$

$$\text{Pnp} = \prod_{i=1}^{T} (1-a^{k_i}) \qquad \sum_{i=1}^{T} k_i = B,\ k_i \geq 0 .$$

Prob. first n targets, and no others, shot down

$$= (1-a^{k_1}) \ldots (1-a^{k_n})\ a^{k_{n+1}} \ldots a^{k_T}$$

Number of unselected targets = number of k_i equal to 0.

In order to find Avg Tgt Dn, Avg Pnp and $p_2(n)$ we must sum the values for a single engagement over all possible distinct engagements and divide by the number of such engagements.

An engagement is defined by a vector $(k_1, k_2, \ldots k_T)$, distinct engagements having distinct vectors. Given any set $C = \{c_i\}$ of T non-negative integers there are

$$\binom{B}{C} = \frac{B!}{c_1!\ c_2! \ldots c_T!}$$

distinct vectors $(k_1, k_2, \ldots, k_T)$ with the set $K = \{k_i\}$ identically equal to the set $C = \{c_i\}$.

The symbol $\binom{B}{K}$ where $K = \{k_i\}$ can also be defined by a generating function which will prove useful:

$$\Sigma \binom{B}{K} x_1^{k_1} x_2^{k_2} \ldots x_T^{k_T} = (x_1 + x_2 + \ldots + x_T)^B .$$

The sum is taken over all partitions of B into T non-negative parts $k_1, k_2, \ldots, k_T$.

Thus the number of distinct engagements is

(4) $$\Sigma\binom{B}{K} = T^B .$$

The sum of Tgt Dn over all distinct engagements is

(5) $$\Sigma\binom{B}{K} \sum_{i=1}^{T} (1-a^{k_i}) = T^B - (a+T-1)^B$$

The sum of Pnp over all distinct engagements is

(6) $$\Sigma\binom{B}{K} \prod_{i=1}^{T} (1-a^{k_i}) = \sum_{i=0}^{T} (-1)^i \binom{T}{i} (ai+T-i)^B$$

The sum of the probabilities that only the first n planes are shot down over all distinct engagements is

(7) $$\Sigma\binom{B}{K} (1-a^{k_1}) \ldots (1-a^{k_n}) a^{k_{n+1}} \ldots a^{k_T} =$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (a(T-n+i)+n-i)^B$$

Therefore Avg Tgt Dn is the value of the formula in (5) divided by that in (4), Avg Pnp is the formula in (6) divided by that in (4), and $p_2(n)$ is the formula in (7) divided by that in (4) and multiplied by $\binom{T}{n}$ to get all the ways that exactly n, not just the first n, targets can be chosen.

To find $p_1(n)$ we first find $p_1(0)$. For each engagement the probability that no targets miss fire is 1 if all the k_i are positive and 0 otherwise. The sum of these probabilities over all distinct cases is $\Sigma\binom{B}{k}$ summed over all partitions of B into T positive integers $k_1, k_2, \ldots, k_T$. This is just B! times the coefficient of x^B in

$(e^x-1)^T$ or $T! \mathscr{S}_B^{(T)}$. Thus $p_1(0) = T! \mathscr{S}_B^{(T)} / T^B$.

Now suppose the first n targets miss fire and all the others receive fire. The probability that no targets among the last T-n miss fire from B batteries is $p_1(0)$ for T-n targets and B batteries or

$$(T-n)! \int_{B}^{(T-n)} / T^B .$$

To get $p_1(n)$, the probability that exactly n targets miss fire, we must multiply this by the number of ways of choosing the n lucky targets which is $\binom{T}{n}$.

Mrs. Joan Rosenblatt has derived all these formulas, except $p_1(n)$, by independent means and her results, which include a formula for the variance of Tgt Dn, will be transmitted to you in a separate memorandum.

Memorandum Number 2

Statistical Analysis of Uncoordinated System

Uncoordinated System Based on Constant Kill Probability

by

Joan Raup Rosenblatt
National Bureau of Standards

Given B batteries, T targets, m missiles, and constant kill probability p.

Let $a = (1 - p)^{m/B}$, where m/B is assumed to be an integer. The number a is the probability that a target will survive when fired on by one battery, since it is assumed that each battery will fire all of its m/B missiles at one target.

Under the uncoordinated system, each battery selects its target at random, independent of selections made by other batteries.

An engagement is characterized by the numbers k_1, k_2, ..., k_T, where k_i denotes the number of batteries which fire on target i. $B = k_1 + \dots + k_T$.

Quantities which may be used to describe the properties of this system include the following:

(1) Average number of targets down.

$$\text{Avg Tgt Dn} = T\left[1 - \left(1 - \frac{1-a}{T}\right)^B\right]$$

(2) Probability of non-penetration (i.e., probability that T targets are down).

$$\text{Pnp} = \sum_{i=0}^{T} (-1)^i \binom{T}{i} \left(1 - \frac{1-a}{T} i\right)^B$$

(3) Variance of the number of targets down.

$$\text{Var(Tgt Dn)} = T\left(1 - \frac{1-a}{T}\right)^B + T(T-1)\left(1 - 2\,\frac{1-a}{T}\right)^B - T^2\left(1 - \frac{1-a}{T}\right)^{2B}$$

(4) For each n $(n = 0, 1, \ldots, T)$, the probability p_n that exactly n targets are down.

$p_T = \text{Pnp}$ (see (2)),

$$p_{T-n} = \binom{T}{n} \sum_{i=0}^{T-n} (-1)^i \binom{T-n}{i} \left[1 - (n+i)\frac{1-a}{T}\right]^B, \quad n = 0, 1, \ldots, T.$$

In particular,

$$p_0 = a^B$$

$$p_1 = T\left(a + \frac{1-a}{T}\right)^B - Ta^B \quad .$$

These quantities are derived in the following.

<u>1. Distribution of Number of Targets Down.</u>

The random variable with which we are concerned is N, the number of targets down. Let the distribution of N be given by $(p_0, p_1, \ldots, p_T)$, where

$$p_n = \Pr(N = n) \quad , \quad n = 0, 1, \ldots, T.$$

Then, in particular,

$$Pnp = p_T .$$

From the distribution of N, we also obtain

$$\text{Avg Tgt Dn} = EN = \sum_{n=0}^{T} np_n ,$$

and

$$\text{Var(Tgt Dn)} = E(N - EN)^2 = \sum_{n=0}^{T} n^2 p_n - (EN)^2.$$

The quantities $p_0, \ldots, p_T$ are not convenient expressions, however, so that we do not evaluate EN and $\text{Var } N$ from the relations given above.

2. Evaluation of p_n.

Observe that there are T^B possible equally likely configurations of assignments of B batteries to T targets. Let $K = (k_1, \ldots, k_T)$ represent the configuration of assignments in one engagement where k_i batteries fire at the i*th* target. Corresponding to each K, there are $\binom{B}{K}$ configurations which differ only in that a different set of k_i batteries is assigned to the i*th* target.

Thus, we have

$$p_n = T^{-B} \sum_K \binom{B}{K} p_n^{(K)} \quad , \quad n = 0, 1, \ldots, T,$$

where $p_n^{(K)}$ denotes the conditional probability that exactly n targets are down in an engagement with configuration K.

Furthermore, for any function $f(n)$, we have

$$Ef(N) = \sum_{n=0}^{T} f(n)p_n = T^{-B} \sum_K \binom{B}{K} \left\{ \sum_{n=0}^{T} f(n)\, p_n^{(K)} \right\} .$$

We will write

$$E_K f(N) = \sum_{n=0}^{T} f(n)\, p_n^{(K)}$$

to denote the conditional expectation of $f(N)$ when K is fixed.

3. Evaluation of $p_n^{(K)}$.

Observe that N may be regarded as the sum of T random variables, $N = X_1 + \ldots + X_T$,

where X_i has the value one or zero according as the ith target is down or survives.

Now, when K is fixed, the X_i are mutually independent. The conditional distribution of X_i is given by

$$\Pr(X_i = 1 | K) = 1 - a^{k_i}$$

$$\Pr(X_i = 0 | K) = a^{k_i} \quad , \quad i = 1, \ldots, T.$$

We have, then,

$$p_T^{(K)} = \prod_{i=1}^{T} (1 - a^{k_i}) \quad ,$$

$$p_0^{(K)} = \prod_{i=1}^{T} a^{k_i} = a^B \quad ,$$

and for $n = 2, 3, \ldots, T - 1,$

$$p_n^{(K)} = \sum_{j_1 \neq j_2 \neq \ldots \neq j_n} \left(\frac{1 - a^{k_{j_1}}}{a^{k_{j_1}}} \right) \cdots \left(\frac{1 - a^{k_{j_n}}}{a^{k_{j_n}}} \right) a^B$$

Except for $p_0^{(K)}$, these are not convenient expressions to use in the relations given in the preceding section for calculating $Ef(n)$. Another method will be developed to carry out those calculations. We are, however, in a position to complete the evaluation of p_n $(n = 0, 1, \ldots, T)$.

4. Calculation of p_n.

a) Evidently, $p_0 = a^B$.

b) $$p_1 = T\left(a + \frac{1-a}{T}\right)^B - Ta^B .$$

$$p_1^{(K)} = \sum_{i=1}^{T}\left(\frac{1-a^{k_i}}{a^{k_i}}\right) a^B = \sum_{i=1}^{T} \prod_{j\neq i} a^{k_j} - Ta^B$$

$$p_1 = T^{-B} \sum_K \binom{B}{K} p_1^{(K)}$$

Now, in general,

$$T^{-B} \sum_K \binom{B}{K} a^{k_{i_1}} \cdots a^{k_{i_r}} = \left(1 - r\,\frac{1-a}{T}\right)^B .$$

Thus,

$$p_1 = T\left(1 - (T-1)\,\frac{1-a}{T}\right)^B - Ta^B .$$

c) The remaining expressions for p_n are calculated in similar fashion.

5. Calculation of Avg Tgt Dn.

$$EN = T^{-B} \sum_K \binom{B}{K} E_K N$$

$$E_K N = \sum_{i=1}^{T} E_K X_i = \sum_{i=1}^{T} (1 - a^{k_i})$$

The calculation of the expression given above for Avg Tgt Dn is straightforward.

6. Calculation of Var(Tgt Dn).

$$E(N - EN)^2 = E(N - E_K N)^2 + E(E_K N - EN)^2$$

a) $E(N - E_K N)^2 = T^{-B} \sum_K \binom{B}{K} E_K(N - E_K N)^2$

But $E_K(N - E_K N)^2$ is simply the variance of a sum of independent random variables, since $N = X_1 + \dots + X_T$ as noted above.

Now $\quad \text{Var } X_i = a^{k_i}(1 - a^{k_i})$, $i = 1, \dots, T$.

Hence

$$E(N - E_k N)^2 = T^{-B} \sum_K \binom{B}{K} \sum_{i=1}^{T} a^{k_i}(1 - a^{k_i}).$$

b) $E(E_K N - EN)^2 = T^{-B} \sum_K \binom{B}{K} (E_K N)^2 - (EN)^2$

Var N is obtained by evaluating the two terms.

Memorandum Number 3

Analysis of Coordinated and Sector Coordinated Systems

Coordinated and Sector-Coordinated Systems

Based on Constant Kill Probability

by

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The statistical method used to study the uncoordinated system is applied to the coordinated and sector-coordinated systems. The following is a summary of the investigations reported in these notes.

(1) Coordinated System. A formula is obtained for the variance of the number of targets down. For $n = 0, 1, \ldots, T$, the probability p_n that exactly n targets are down is given. These formulae as well as the formulae for Avg Tgt Dn and Pnp are also obtained for the case where m/T (number of missiles per target) is not an integer.

(2) Sector-Coordinated System. A formal method of solution is found. Avg Tgt Dn is obtained in reasonably computable form. Pnp would be extremely laborious to compute exactly; certain inequalities and an approximation are given. Var (Tgt Dn) could be computed exactly, but it would probably be sufficient to use an approximation for this quantity.

The device employed in obtaining a solution for the sector-coordinated system - an incidence matrix describing assignment of targets to sectors - could be used to analyze other possible forms of coordinated systems.

The system considered is different in one respect from the system described in the initial memorandum setting forth this problem. It was intended that the following rule should be invoked: a sector must fire at least two missiles at each target on which it fires. This rule was ignored.

(3) Distribution of Fire. For the coordinated system, equal distribution of fire is never worse than and can be better than distribution according to the "at least two" rule.

The two methods of distribution differ only when the number of missiles per target is less than two. The following comparisons hold when the two methods differ. Avg Tgt Dn is greater with equal distribution. Pnp is no less and sometimes greater with equal distribution.

The variance of the number of targets down, on the other hand, is smaller with the "at least two" rule.

(4) Examples: Sector-Coordinated System. For the assignment scheme outlined in the original memorandum on this problem (20 targets, 5 sectors), the exact formulas for Avg Tgt Dn have been worked out for m/S = 4(4)32. For m/S = 4(4)16, the equal distribution of fire and the "at least two" systems are different; Avg Tgt Dn formulas have been obtained for both cases. The equal distribution of fire system always gives a larger value for Avg Tgt Dn.

An approximation for Pnp is given for m/S = 4(4)32. The method of approximation does not distinguish between the two possible systems of distribution of fire for m/s = 4(4)16. Upper and lower

bounds for Pnp are given.

In two cases, the exact formula for Pnp is not too unwieldy. (i) Pnp is given exactly for m/S = 20. (ii) Pnp is given exactly for m/S = 12, under the "at least two" system.

1. Coordinated System.

Given T targets, m missiles, and a constant kill probability p.

Under this system, all engagements are alike. Each target receives m/T missiles. (If we suppose m/T is not an integer, the situation is slightly more complicated. The formulae for this case are given at the end of this section.)

Let $b = (1-p)^{m/T}$, the probability of survival of a target,

$$\text{Avg Tgt Dn} = T(1-b) \tag{1.1}$$

$$\text{Var (Tgt Dn)} = Tb(1-b) \tag{1.2}$$

$$\text{Pnp} = (1-b)^T \tag{1.3}$$

The derivation of these formulae follows.

Let X_i equal one or zero according as the i^{th} target is down or survives. $X_1, \ldots, X_T$ are independent and identically distributed with

$$\Pr(X_i = 1) = 1-b \quad , \quad i=1,\ldots,T \quad .$$

Now the number of targets down is $N = X_1 + \ldots + X_T$. It follows at once that

$$p_n = \Pr(N = n) = \binom{T}{n} b^{T-n}(1-b)^n \quad , \quad n=0,1,\ldots,T \quad ,$$

$$\text{Pnp} = p_T = (1-b)^T \quad ,$$

$$E\ N = T(1-b) \quad ,$$

$$\text{Var } N = Tb(1-b) \quad .$$

Now, suppose m/T is not an integer, say

$$\frac{m}{T} = k + \frac{s}{T} ,$$

where k is an integer and $1 \leq s \leq T - 1$. Using the argument developed in Dr. Alt's memorandum of 20 May, (T - s) targets will receive k missiles and s targets will receive (k + 1) missiles. Thus,

$$\Pr(X_i = 1) = \begin{cases} 1 - q^{k+1} & \text{if} \quad i=1,\ldots,s \\ 1 - q^k & \text{if} \quad i=s+1,\ldots,T \end{cases} ,$$

where $q = 1 - p$.

We obtain immediately:

$$\text{Pnp} = p_T = (1 - q^{k+1})^s (1 - q^k)^{T-s} , \qquad (1.4)$$

$$\text{Arg Tgt Dn} = E\,N = s(1 - q^{k+1}) + (T - s)(1 - q^k), \qquad (1.5)$$

$$\text{Var (Tgt Dn)} = \text{Var } N = sq^{k+1}(1 - q^{k+1}) + (T - s)q^k(1 - q^k) , \qquad (1.6)$$

and for $n = 0,1,\ldots,T$,

$$p_n = \sum_{i=0}^{n} \binom{s}{i} \binom{T-s}{n-i} (1 - q^{k+1})^i (q^{k+1})^{s-i} (1-q^k)^{n-i} (q^k)^{T-s-n+i} \qquad (1.7)$$

2. Sector-Coordinated System.

Given T targets, m missiles, CKP p, and S sectors of batteries with m/S missiles per sector. Let $q = 1 - p$. We suppose that m/S is an integer.

The assignment pattern for the sector-coordinated system is given by the T X S incidence matrix (δ_{ij}), where

$$\delta_{ij} = \begin{cases} 1 & \text{if the } i^{th} \text{ target is assigned to the } j^{th} \text{ sector} \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

$i = 1,\ldots,T$; $j = 1,\ldots,S$.

Let $d_j = \sum_{i=1}^{T} \delta_{ij}$ be the number of targets assigned to the j^{th} sector.

The following discussion does <u>not</u> include the requirement that a sector fire at least two missiles at a target if it fires on the target at all during an engagement. This modification could be made.

Let

$$m/S = k_j d_j + r_j \quad , \tag{2.2}$$

where $k_j = \left[m/S \cdot d_j \right]$ is the greatest integer contained in $m/S\ d_j$, and r_j $(0 \leq r_j \leq d_j - 1)$ is an integer.

Again using Dr. Alt's argument, we have in any engagement that $(d_j - r_j)$ of the targets assigned to the j^{th} sector receive k_j missiles; the remaining r_j targets receive $(k_j + 1)$ missiles.

Now there are

$$D = \prod_{j=1}^{S} \binom{d_j}{r_j} \tag{2.3}$$

equally likely configurations for particular engagements. A

configuration may be described by a matrix $L = (l_{ij})$ which is related to (δ_{ij}) in the following way: For $j = 1,\ldots,S$, $(d_j - r_j)$ of the d_j ones in the j^{th} column of (δ_{ij}) are replaced by zeros. That is, a matrix L has a one in its j^{th} column for every target which receives $(k_j + 1)$ missiles from the j^{th} sector in this engagement.

Now the number of targets down, N, may again be represented by the sum $N = X_1 + \ldots + X_T$. In any engagement, with L fixed, $X_1,\ldots,X_T$ are mutually independent. Thus, using the conditional-expectation notation developed in the discussion of the uncoordinated system, we have

$$\Pr(X_i = 1/L) = 1 - q^{\sum_{j=1}^{S} \delta_{ij}(k_j + l_{ij})}, \quad i=1,\ldots T, \tag{2.4}$$

where the exponent is the number of missiles fired at the i^{th} target in the engagement described by L.

From these quantities, we may obtain formally

$$E_L N = \sum_{i=1}^{T} \Pr(X_i = 1/L),$$

$$\text{Avg Tgt Dn} = E N = \frac{1}{D} \sum_{L} E_L N, \tag{2.5}$$

$$p_T^{(L)} = \prod_{i=1}^{T} Pr\ (X_i = 1/L) \quad ,$$

$$Pnp = p_T = \frac{1}{D} \sum_L p_T^{(L)} \quad , \tag{2.6}$$

where $\sum_L$ denotes summation over all possible configuration matrices L.

The variance of N and Pr (N = n) may also be obtained formally.

The computational difficulty arises from the fact that in order to evaluate these formulae, we must compute all the T X D values of the exponent

$$\sum_{j=1}^{S} \delta_{ij}\ (k_j + l_{ij}) \quad .$$

<u>Calculation of Avg Tgt Dn</u>

The following formula is derived, which does not depend on the individual configuration matrices L.

$$E\ N = T - \sum_{i=1}^{T} \prod_{j=1}^{S} \left[\frac{r_j}{d_j}\, q^{\delta_{ij}(k_j + 1)} + \left(1 - \frac{r_j}{d_j}\right) q^{\delta_{ij} k_j} \right] \tag{2.7}$$

The following derivation calls attention explicitly to certain assumptions which have been made implicitly above. Let

$$Y_{ij} = \begin{cases} 1 & \text{if the } j^{th} \text{ sector fails to kill the } i^{th} \text{ target} \\ 0 & \text{otherwise.} \end{cases}$$

In a given engagement, $(Y_{ij},\ldots,Y_{is})$ are mutually independent and have the distributions defined by

$$\Pr\,(Y_{ij}=1/L) \;=\; q^{\delta_{ij}(k_j+\,l_{ij})},\; i=1,\ldots,T,\, j=1,\ldots,S.$$

We write

$$E\,N \;=E\,(E_L\,N)\;\;,$$

where $E_L\,N$ denotes the conditional expectation of the number of targets down for fixed engagement pattern L, and $E(\cdot)$ denotes the average (i.e., expectation) over all possible equally likely engagements $E(\cdot)$ is equivalent to $D^{-1}\sum\limits_{L}(\cdot)$. Now

$$E_L\,N \;=\; \sum_{i=1}^{T}\Pr\,(X_i=1/L)\quad,$$

and, since $(Y_{i1},\ldots,Y_{is})$ are independent,

$$\Pr\,(X_i=1/L) \;=\; 1-\prod_{j=1}^{S}\Pr\,(Y_{ij}=1/L).$$

Hence

$$E\,N \;=\; T-\sum_{i=1}^{T}E\prod_{j=1}^{S}\Pr\,(Y_{ij}\;=1/L)\;.$$

Now, for fixed target (i), the random variables

$$\Pr (Y_{ij} = 1/L) \quad \text{and} \quad \Pr (Y_{ij'} = 1/L)$$

are independent ($j \neq j'$). This follows from the assumption that all engagement patterns L are equally likely, which implies that two sectors make independent random (equi-probable) selections of the targets which are to receive ($k_j + 1$) missiles. It follows that

$$E \prod_{j=1}^{S} \Pr (Y_{ij} = 1/L) = \prod_{j=1}^{S} E \Pr (Y_{ij} = 1/L) .$$

But

$$E \Pr (Y_{ij} = 1/L) = \frac{1}{D} \sum_{L} \Pr (Y_{ij} = 1/L)$$

$$= \frac{r_j}{d_j} q^{\delta_{ij}(k_j+1)} + (1 - \frac{r_j}{d_j}) q^{\delta_{ij}k_j} ,$$

since the proportion of engagements in which the j^{th} sector fires ($k_j + 1$) missiles at the i^{th} target is r_j/d_j. This completes the derivation of the expression for E N given above.

Calculation of Var (Tgt Dn).

A similar argument may be employed to derive

$$\text{Var } N = \sum_{i=1}^{T} \prod_{j=1}^{S} E\, Y_{ij} - \sum_{i=1}^{T} \prod_{j=1}^{S} (E\, Y_{ij})^2$$

$$+ 2 \sum_{1 \leq i_1 < i_2 \leq T} \prod_{j=1}^{S} \text{Cov} (Y_{i_1 j} , Y_{i_2 j}) \qquad (2.8)$$

where

$$E\,Y_{ij} = \frac{r_j}{d_j}\,q^{\delta_{ij}(k_j+1)} + \left(1 - \frac{r_j}{d_j}\right) q^{\delta_{ij}k_j} ,$$

and

$$\mathrm{Cor}\,(Y_{i_1j}\ ,\ Y_{i_2j})$$

$$= \frac{-r_j(d_j-r_j)}{d_j^{\,2}(d_j-1)}\,q^{(\delta_{i_1j}+\delta_{i_2j})k_j}\,(1-q^{\delta_{i_1j}})(1-q^{\delta_{i_2j}}) .$$

Calculation of Pnp.

To evaluate Pnp exactly would require enormous computations. The following inequalities may prove to be useful:

(1) $$\mathrm{Pnp} < \frac{1}{T}\ (\text{Avg Tgt Dn})$$

(2) $$\mathrm{Pnp} \geq \prod_{i=1}^{T}\left(1 - q^{\sum_{j=1}^{S}\delta_{ij}k_j}\right)$$

(3) $$\mathrm{Pnp} < \prod_{i=1}^{T}\left(1 - q^{\sum_{j=1}^{S}\delta_{ij}(k_j+1)}\right)$$

Equality will hold in (2) if and only if the numbers m/Sd_j are integers ($j=1,\ldots,S$). (It is assumed that the CKP $p=1-q$ is between zero and one. If $p=0$ or 1, then equality will hold in all three relations.)

The following formula is suggested as a possible approximation.

$$\text{Pnp} \doteq \prod_{i=1}^{T}\left[1 - q^{\sum_{j=1}^{S} \delta_{ij}(k_j + r_j/d_j)}\right] . \tag{2.9}$$

Here the exponent $\sum_{j=1}^{S} \delta_{ij}(k_j + \frac{r_j}{d_j})$ is the average number of missiles fired at the i^{th} target.

Approximation for Var (Tgt Dn).

The approximation given here is, like the approximation for Pnp, based on the average number of missiles fired at each target.

$$\text{Var } N \doteq \sum_{i=1}^{T} q^{a_i} (1 - q^{a_i}) , \tag{2.10}$$

where $a_i = \sum_{j=1}^{S} \delta_{ij} (k_j + \frac{r_j}{d_j})$.

3. Distribution of Fire.

A comparison is made between equal distribution of fire and distribution according to the "at least two" rule, for the coordinated system.

Given m missiles, T targets, and CKP $p = 1 - q$.

First, observe that if the number of missiles is at least 2T, then every target will receive at least two missiles by equal distribution of fire. Hence, the two methods of distribution will differ only if $m < 2T$.

Next, it is easy to see that if $m < 2T$, the "at least two" rule implies that at least one target will not be fired on. Hence $Pnp = 0$. For equal distribution of fire, Pnp is greater than zero for $T \leq m < 2T$ and $Pnp = 0$ only if $m < T$. In summary: with respect to Pnp, the two methods of distribution of fire are alike except when $T \leq m < 2T$; in this case, equal distribution is better.

Now, consider the comparison of the two methods with respect to Avg Tgt Dn and Var (Tgt Dn). Let $m = 2r + a$, $a = 0$ or 1.

<u>"At least two" rule.</u>

$$\text{Avg Tgt Dn} = (r - a)(1 - q^2) + a(1 - q^3)$$

$$\text{Var (Tgt Dn)} = (r - a)\, q^2\, (1 - q^2) + a\, q^3\, (1 - q^3)$$

<u>Equal distribution.</u>

$$\text{Avg Tgt Dn} = \begin{cases} (2r + a)(1 - q) & \text{if } 1 \leq m < T \\ (2r + a - T)(1 - q^2) + (2T - 2r - a)(1 - q) & \text{if } T \leq m < 2T \end{cases}$$

$$\text{Var (Tgt Dn)} = \begin{cases} (2r + a)\, q\, (1 - q) & \text{if } 1 \leq m < T \\ (2r + a - T)\, q^2\, (1 - q^2) + (2T - 2r - a)\, q\, (1 - q) & \text{if } T \leq m < 2T \end{cases}$$

It is readily verified that Avg Tgt Dn and Var (Tgt Dn) are always greater for the equal distribution case.

4. Example: Sector-Coordinated System.

Given $m/S = 4(4)32$, $T = 20$, $S = 5$, and the assignment scheme under which sector one fires on targets 8-13, sectors two and four fire on targets 1-10, sectors three and five fire on targets 11-20.

The general formula for Avg Tgt Dn may be specialized to the following.

$$\text{Avg Tgt Dn} = 20 - \frac{14}{100}\, q^{2k_1} (10 - r_1 + r_1 q)^2$$

$$- \frac{1}{100}\, q^{2k_1+k_2} (10 - r_1 + r_1 q)^2 (6 - r_2 + r_2 q)$$

where the integers k_1, k_2, r_1, r_2 are defined by

$$\frac{m}{5} = 10\, k_1 + r_1 \quad , \qquad 0 \leq r_1 < 10$$

$$\frac{m}{5} = 6\, k_2 + r_2 \quad , \qquad 0 \leq r_2 < 6.$$

For $m/5 \leq 16$, the above formula applies only if each sector distributes fire as equally as possible among the targets assigned to it. If the "at least two" rule is applied, then the following formula is correct.

$$\text{Avg Tgt Dn} = 20 - \frac{14}{100} (10 - r_1' + r_2' q^2)^2$$

$$- \frac{1}{100}\, q^{2k_2'} (10 - r_1' + r_1' q^2)\ (6 - r_2' + r_2' q^2)$$

where the integers k_2', r_1', r_2' are defined by

$$\frac{m}{10} = r_1'$$

$$\frac{m}{10} = 6\, k_2' + r_2' \quad , \quad 0 \leq r_2' < 6.$$

It is not always possible to obtain convenient exact expressions for Pnp. The following list gives, for each values of m/S, the exact expression for Pnp where possible, and otherwise an approximate expression. A discussion of the methods of approximation is added below.

<u>m/S = 4</u>

$Pnp = (1-q)^{20}/15(210)^2$ with equal distribution.

$Pnp = 0$ with "at least two" systems.

<u>m/S = 8</u>

$Pnp = (1-q^2)^{20}/15(210)^2$ with "at least two" systems.

Approximately, with equal distribution,

$$Pnp \doteq P_F(1-q^{8/5})^{14}\ (1-q^{44/15})^6$$

where $P_F = .533441$,

and $Pnp < P_F(1-q^2)^{20}$.

<u>m/S = 12</u>

$$Pnp = (1050)^{-2}\ (1-q^2)^{18}\ (1-q^4)^2 \left[\ 15(1+q^2+q^4)^2 + 110\ (1+q^2)^2\ (1+q^2+q^4) + 101\ (1+q^2)^4\right]^2$$

with "at least two" systems.

Approximately, with equal distribution,

$$Pnp \doteq (1-q^{12/5})^{14}\ (1-q^{22/5})^6,$$

and

$$(1-q^2)^{14}\ (1-q^4)^6 < Pnp < (1-q^3)^{20}.$$

m/S = 16

Approximately,

$$Pnp \doteq (1-q^{16/5})^{14}\ (1-q^{88/15})^{6} \quad \text{with equal distribution,}$$

$$Pnp \doteq P_F(1-q^{16/5})^{14}\ (1-q^{88/15})^{6} \quad \text{with "at least two" systems,}$$

where $P_F = .533441.$

With equal distribution,

$$(1-q^2)^{14}\ (1-q^4)^6 < Pnp < (1-q^4)^{20}.$$

m/S = 20

$$Pnp = (1-q^4)^{14}\ (1-q^7)^4\ (1-q^8)^2$$

(The equal distribution and "at least two" systems are the same for $m/S \geq 20$.)

m/S = 24

Approximately,

$$Pnp \doteq (1-q^{24/5})^{14}\ (1-q^{44/5})^{6}.$$

$$(1-q^4)^{14}\ (1-q^8)^6 < Pnp < (1-q^6)^{20}.$$

m/S = 28

Approximately,

$$Pnp \doteq (1-q^{28/5})^{14}\ (1-q^{154/15})^{6}.$$

$$(1-q^4)^{14}\ (1-q^8)^6 < Pnp < (1-q^6)^{14}\ (1-q^{11})^6.$$

m/S = 32

Approximately,

$$Pnp \doteq (1-q^{32/5})^{14}\ (1-q^{176/15})^{6}.$$

$$(1-q^6)^{14}\ (1-q^{11})^6 < Pnp < (1-q^8)^{20}.$$

The approximations are obtained by finding the average number of missiles fired at each target, as described in section two of this memorandum. For small values of m, however, these approximations have been improved by taking account of the fact that in some engagements some targets are not fired on at all. This is reflected in the use of the factor P_F which is the probability that all targets are fired on.

In some cases, it is possible to improve on the upper bounds for Pnp which were listed in section two. The following bound is used when it is better.

$$Pnp < (1-q^{m/T})^T.$$

Memorandum Number 4

Remarks on Avg Pnp

TO: Dr. E. Biser
Evans Signal Laboratories

February 8, 1956

FROM: K. Goldberg
National Bureau of Standards

SUBJECT: Some Remarks on Pnp

We have proved the following lemma which is relevant to our considerations of Avg Pnp;

Lemma: Let S be a strategy for firing missiles at targets. With S associate two functions: the average probability of non-penetration (Avg Pnp) and the probability that all targets are fired on (P_F). Also associate the set C of all possible different cases affected by S in which the probability of non-penetration (Pnp) is positive.

If S' is another strategy (with associated Avg Pnp', P_F' and C') such that Pnp' is equal to Pnp for each particular case, then C' = C implies

$$\text{(L)} \qquad P'_F \text{ (Avg Pnp)} = P_F \text{ (Avg Pnp')}$$

In other words if $P_F \neq 0$ then (Avg Pnp)/P_F is the same for all strategies in which the Pnp are equal for equal cases and C is a constant. If $P_F = 0$ then Avg Pnp = 0.

The implication of this lemma is that the order of magnitude of Pnp is determined primarily by P_F. Thus, it is reasonable to compare two strategies only if they have equal P_F, just as it is reasonable to compare two strategies only if they have equal CKP.

This has a direct bearing on our considerations of the uncoordinated system. This system represents a minimal strategy and should be useful in determining a minimum Avg Pnp for all strategies. However, the Avg Pnp in this system is actually too small, it provides an unreasonable minimum. The reason for this is that P_F is usually too small. As in our memorandum of November 1955:

$$P_F = p_1(0) = T! \mathcal{f}_B^{(T)} / T^B$$

Thus if B = T we have $P_F = T!/T^T$ or

$$P_F \sim e^{-T} (2\pi T)^{\frac{1}{2}}$$

which becomes very small very rapidly. On the other hand $(\text{Avg Pnp})/P_F = (1-a)^T$ when $B = T$ and this is of the proper order of magnitude for comparison purposes.

Thus we propose computing $(\text{Avg Pnp})/P_F$ for all cases in the uncoordinated system. This will provide a minimum for this function for all strategies with the same CKP and the same number of missiles and targets.

The proof of the lemma is straight-forward. Avg Pnp is the sum of the Pnp over all possible distinct cases ($\sum$ Pnp) divided by the total number of distinct cases (N). P_F is the number of distinct cases in which Pnp is positive (N_C) divided by N.

On the other hand $\sum$ Pnp is equal to the sum of the Pnp over those cases in which Pnp is positive ($\sum_C$ Pnp) since all other Pnp vanish. Thus if N_C does not vanish we have

$$(\text{Avg Pnp})/P_F = (\textstyle\sum_C \text{Pnp})/N_C.$$

Clearly this is a constant if C is constant and Pnp is derived in the same way for the same case. In fact both $\sum_C$ Pnp and N_C are constant under these conditions.

Memorandum Number 5

Tables of Avg Tgt Dn and Avg Pnp
in Coordinated System

COORDINATED SYSTEM

Pnp

m/20 \ P	0.2	0.3	0.5	0.7	0.9
1	.(13)105	.(10)349	.(6)954	.(3)798	.122
2	.(8) 134	.(5) 142	.(2)317	.152	.818
3	.(6) 587	.(3) 225	.(1)692	.578	.980
4	.(4) 265	.(2) 412	.275	.850	.998
5	.(3) 356	.0252	.530	.953	1.000
6	.(2) 229	.0818	.730	.986	1.000
7	.(2) 903	.179	.855	.996	1.000
8	.(1) 254	.305	.925	.999	1.000

Tgt Dn

	0.2	0.3	0.5	0.7	0.9
1	4.00	6.00	10.00	14.00	18.00
2	7.20	10.20	15.00	18.20	19.80
3	9.76	13.14	17.50	19.46	19.98
4	11.81	15.20	18.75	19.84	20.00
5	13.45	16.64	19.38	19.95	20.00
6	14.76	17.65	19.69	19.99	20.00
7	15.81	18.35	19.84	20.00	20.00
8	16.64	18.85	19.92	20.00	20.00

Memorandum Number 6

Tables of Avg Tgt Dn and Avg Pnp (approximate)
in Sector Coordinated System

Sector-Coordinated System

Tables of Avg Tgt Dn and Avg Pnp

The illustrative sector-coordinated system for which these computations were made is described in section four of the memorandum on sector-coordinated systems. There are twenty targets and five sectors of four batteries each. Each sector has the same number m/5 of missiles.

The tables were computed for m = 20(20)160, and for CKP p = .2, .3(.2).9.

Table I: Avg Tgt Dn is given for both the equal distribution of fire system and the "at least two" system. (There is no difference between the two systems for $m \geq 100$.)

Table II: Either Pnp or an approximate value for Pnp is given for both the equal distribution of fire system and the "at least two" system.

Table I: Avg Tgt Dn

m \ p	0.2 (a)	0.2 (b)	0.3 (a)	0.3 (b)	0.5 (a)	0.5 (b)	0.7 (a)	0.7 (b)	0.9 (a)	0.9 (b)
20	3.40,	3.75	4.69,	5.44	6.63,	8.48	7.84,	11.1	8.41,	13.3
40	6.46,	6.96	8.73,	9.73	11.9,	14.1	13.8,	17.0	14.7,	18.9
60	9.03,	9.48	11.8,	12.7	15.3,	16.9	17.0,	19.0	17.7,	19.9
80	11.2,	11.4	14.3,	14.6	17.6,	18.2	18.9,	19.6	19.4,	20.0
100	13.1		16.2		19.1		19.9		20.0	
120	14.3		17.1		19.4		19.9		20.0	
140	15.3		17.9		19.7		20.0		20.0	
160	16.2		18.5		19.8		20.0		20.0	

(a) For the "at least two" system.

(b) For equal distribution

Table II: Avg Pnp

m \ p	0.2 (a) (b)	0.3 (a) (b)	0.5 (a) (b)	0.7 (a) (b)	0.9 (a) (b)
20	0, .000	0, .000	0, .000	0, .000	0, .000
40	.000, .000A	.000, .000A	.000, .001A	.000, .049A	.000, .371A
60	.000, .000A	.000, .000A	.001, .039A	.015, .436A	.041, .945A
80	.000A, .000A	.001A, .002A	.096A, .180A	.393A, 737A	.529A, .991A
100	.000	.013	.390	.891	.999
120	.001A	.047A	.591A	.957A	1.000A
140	.005A	.111A	.743A	.984A	1.000A
160	.014A	.202A	.845A	.994A	1.000A

(a) For "at least two" system
(b) For equal distribution

"A" denotes approximate values. Others are exact to three places.
"O" denotes identically zero.

Memorandum Number 7

Tables of Avg Pnp in Uncoordinated System

MEMORANDUM TO: Dr. E. Biser

FROM : K. Goldberg

SUBJECT : Tables of Average P_{np} in Uncoordinated System

The average probability of non-penetration (Average P_{np}) has been computed for B = 10 (10) 40, T = 5 (5) 20, p = 0.2, 0.3 (0.2) 0.9 and m/B = 1 (1) 9, 12, 15. The tabulation of this computation is attached.

The formula used is

$$\text{Avg } P_{np} = \sum_{i=0}^{T} (-1)^i \binom{T}{i} \left(1 - \frac{1-a}{T}\right)^B \quad \text{where } a = (1-P)^{m/B}$$

The probability that all targets are fired on, P_F, is the number found under p = 0.9 and m/B = 15 for any particular B and T. This number determines both the maximum for Avg P_{np} and the order of magnitude for Avg P_{np} except for small CKP and small numbers of missiles. Because of the importance we tabulate it separately below:

P_F

T \ B	10	20	30	40
5	.522	.942	.993	.999
10	.000	.214	.629	.858
15	0	.001	.087	.338
20	0	.000	.001	.035

".000" indicates a value computed to three decimal places and is actually $T!/T^T$. "0" indicates absolute zero.

AVERAGE P_{np} IN BATTERYWISE UNCOORDINATED CASE

m/B \ p	0.2	0.3	0.5	0.7	0.9
			B = 10, T = 5		
1	.001	.010	.077	.226	.423
2	.022	.082	.273	.433	.512
3	.070	.188	.398	.496	.521
4	.135	.282	.461	.514	.522
5	.201	.354	.492	.520	.522
6	.261	.405	.507	.521	.522
7	.312	.441	.515	.522	.522
8	.355	.466	.518	.522	.522
9	.389	.483	.520	.522	.522
12	.455	.509	.522	.522	.522
15	.488	.518	.522	.522	.522
			B = 20, T = 5		
1	.042	.156	.499	.768	.906
2	.252	.516	.814	.911	.939
3	.479	.723	.895	.934	.942
4	.639	.822	.922	.940	.942
5	.740	.871	.933	.942	.942
6	.803	.898	.938	.942	.942
7	.844	.914	.940	.942	.942
8	.871	.924	.941	.942	.942
9	.890	.930	.942	.942	.942
12	.919	.938	.942	.942	.942
15	.931	.941	.942	.942	.942
			B = 30, T = 5		
1	.159	.510	.800	.946	.987
2	.556	.812	.962	.987	.993
3	.784	.928	.984	.992	.993
4	.887	.964	.990	.993	.993
5	.935	.978	.992	.993	.993
6	.958	.985	.993	.993	.993
7	.971	.988	.993	.993	.993
8	.978	.990	.993	.993	.993
9	.983	.991	.993	.993	.993
12	.989	.993	.993	.993	.993
15	.991	.993	.993	.993	.993

B = 40, T =5

m/B \ p	0.2	0.3	0.5	0.7	0.9
1	.323	.635	.927	.988	.998
2	.767	.933	.992	.998	.999
3	.919	.982	.997	.999	.999
4	.967	.993	.998	.999	.999
5	.984	.996	.999	.999	.999
6	.991	.997	.999	.999	.999
7	.994	.998	.999	.999	.999
8	.996	.998	.999	.999	.999
9	.997	.999	.999	.999	.999
12	.998	.999	.999	.999	.999
15	.999	.999	.999	.999	.999

B = 10, T = 10

Average P_{np} = 0 (to three places)

B = 20, T = 10

m/B \ p	0.2	0.3	0.5	0.7	0.9
1	.000	.000	.004	.038	.138
2	.000	.004	.056	.145	.206
3	.003	.026	.121	.192	.213
4	.013	.060	.165	.208	.214
5	.030	.095	.189	.212	.214
6	.051	.126	.201	.214	.214
7	.073	.150	.208	.214	.214
8	.095	.168	.211	.214	.214
9	.115	.182	.213	.214	.214
12	.160	.203	.214	.214	.214
15	.186	.210	.214	.214	.214

B = 30, T = 10

m/B \ p	0.2	0.3	0.5	0.7	0.9
1	.000	.003	.064	.262	.517
2	.010	.071	.326	.529	.618
3	.057	.210	.486	.600	.628
4	.138	.339	.561	.620	.629
5	.228	.432	.596	.626	.629
6	.310	.495	.612	.628	.629
7	.378	.538	.621	.628	.629
8	.432	.566	.625	.629	.629
9	.475	.586	.627	.629	.629
12	.554	.614	.628	.629	.629
15	.591	.624	.629	.629	.629

m/B \ p	0.2	0.3	0.5	0.7	0.9
			B = 40, T = 10		
1	.001	.021	.225	.550	.785
2	.056	.240	.621	.794	.851
3	.206	.483	.763	.841	.857
4	.373	.634	.815	.853	.858
5	.507	.719	.838	.856	.858
6	.604	.770	.848	.857	.858
7	.672	.800	.853	.857	.858
8	.720	.819	.855	.858	.858
9	.754	.832	.856	.858	.858
12	.811	.849	.857	.858	.858
15	.835	.855	.858	.858	.858
			B = 20, T = 15		
1	.000	.000	.000	.000	.000
2	.000	.000	.000	.000	.001
3	.000	.000	.000	.001	.001
4	.000	.000	.000	.001	.001
5	.000	.000	.001	.001	.001
6	.000	.000	.001	.001	.001
7	.000	.000	.001	.001	.001
8	.000	.000	.001	.001	.001
9	.000	.001	.001	.001	.001
12	.000	.001	.001	.001	.001
15	.001	.001	.001	.001	.001
			B = 30, T = 15		
1	.000	.000	.000	.006	.045
2	.000	.000	.011	.048	.082
3	.000	.003	.037	.074	.087
4	.001	.012	.058	.083	.087
5	.004	.025	.072	.086	.087
6	.010	.039	.079	.087	.087
7	.017	.051	.083	.087	.087
8	.025	.060	.085	.087	.087
9	.034	.068	.086	.087	.087
12	.056	.080	.087	.087	.087
15	.070	.085	.087	.087	.087

m/B \ p	0.2	0.3	0.5	0.7	0.9
			B = 40, T = 15		
1	.000	.000	.005	.063	.228
2	.000	.006	.094	.238	.326
3	.004	.042	.202	.307	.337
4	.020	.102	.268	.329	.338
5	.049	.161	.303	.335	.338
6	.086	.210	.320	.337	.338
7	.125	.247	.329	.337	.338
8	.161	.273	.333	.338	.338
9	.193	.292	.335	.338	.338
12	.261	.322	.337	.338	.338
15	.298	.332	.338	.338	.338

B = 20, T = 20

Average P_{np} = 0 (to three decimal places)

m/B \ p	0.2	0.3	0.5	0.7	0.9
			B = 30, T = 20		
1	.000	.000	.000	.000	.000
2	.000	.000	.000	.000	.001
3	.000	.000	.000	.000	.001
4	.000	.000	.000	.001	.001
5	.000	.000	.000	.001	.001
6	.000	.000	.001	.001	.001
7	.000	.000	.001	.001	.001
8	.000	.000	.001	.001	.001
9	.000	.001	.001	.001	.001
12	.000	.001	.001	.001	.001
15	.000	.001	.001	.001	.001
			B = 40, T = 20		
1	.000	.000	.000	.001	.014
2	.000	.000	.002	.016	.033
3	.000	.000	.011	.028	.035
4	.000	.002	.021	.033	.035
5	.000	.006	.027	.035	.035
6	.001	.012	.031	.035	.035
7	.004	.017	.033	.035	.035
8	.006	.021	.034	.035	.035
9	.010	.025	.035	.035	.035
12	.019	.032	.035	.035	.035
15	.026	.034	.035	.035	.035

<u>Memorandum Number 8</u>

Tables of Avg Tgt Dn in Uncoordinated System

TO : Dr. E. Biser February 16, 1956

FROM : K. Goldberg

SUBJECT : Avg Tgt Dn in the Uncoordinated System

The table of values of the average number of targets downed is attached. It was computed from your formula

$$\text{Avg Tgt Dn} = T\left\{1-\left(1-\frac{1-a}{T}\right)^B\right\}, \quad a = (1-p)^{m/B}$$

You will note that for large p or large $^m/B$ the values tend to a limit. In these cases $a \sim 0$ so that

$$(\text{Avg Tgt Dn})/T \lesssim 1-\left(1-\frac{1}{T}\right)^B \sim 1 - e^{-\frac{B}{T}}$$

and in such cases Avg Tgt Dn may be approximated from this formula.

We tabulate the limiting values for Avg Tgt Dn below:

T \ B	10	20	30	40
5	4.5	4.9	5.0	5.0
10	6.5	8.8	9.6	9.9
15	7.5	11.2	13.1	14.1
20	8.0	12.8	15.7	17.4

Avg Tgt Dn in the Uncoordinated System

T = 5

B = 10

m/B	0.2	0.3	0.5	0.7	0.9
1	1.7	2.3	3.3	3.9	4.3
2	2.6	3.3	4.0	4.3	4.4
3	3.2	3.8	4.3	4.4	4.5
4	3.6	4.0	4.4	4.5	4.5
5	3.8	4.2	4.4	4.5	4.5
6	4.0	4.3	4.4	4.5	4.5
7	4.1	4.3	4.5	4.5	4.5
8	4.2	4.4	4.5	4.5	4.5
9	4.3	4.4	4.5	4.5	4.5
12	4.4	4.4	4.5	4.5	4.5
15	4.4	4.5	4.5	4.5	4.5

B = 20

m/B	0.2	0.3	0.5	0.7	0.9
1	2.8	3.5	4.4	4.8	4.9
2	3.9	4.4	4.8	4.9	4.9
3	4.4	4.7	4.9	4.9	4.9
4	4.6	4.8	4.9	4.9	4.9
5	4.7	4.9	4.9	4.9	4.9
6	4.8	4.9	4.9	4.9	4.9
7	4.8	4.9	4.9	4.9	4.9
8	4.9	4.9	4.9	4.9	4.9
9	4.9	4.9	4.9	4.9	4.9
12	4.9	4.9	4.9	4.9	4.9
15	4.9	4.9	4.9	4.9	4.9

B = 30

m/B	0.2	0.3	0.5	0.7	0.9
1	3.5	4.2	4.8	4.9	5.0
2	4.5	4.8	5.0	5.0	5.0
3	4.8	4.9	5.0	5.0	5.0
4	4.9	5.0	5.0	5.0	5.0
5	4.9	5.0	5.0	5.0	5.0
6	5.0	5.0	5.0	5.0	5.0
7	5.0	5.0	5.0	5.0	5.0
8	5.0	5.0	5.0	5.0	5.0
9	5.0	5.0	5.0	5.0	5.0
12	5.0	5.0	5.0	5.0	5.0
15	5.0	5.0	5.0	5.0	5.0

B = 40

m/B	0.2	0.3	0.5	0.7	0.9
1	4.0	4.6	4.9	5.0	5.0
2	4.7	4.9	5.0	5.0	5.0
3	4.9	5.0	5.0	5.0	5.0
4	5.0	5.0	5.0	5.0	5.0
5	5.0	5.0	5.0	5.0	5.0
6	5.0	5.0	5.0	5.0	5.0
7	5.0	5.0	5.0	5.0	5.0
8	5.0	5.0	5.0	5.0	5.0
9	5.0	5.0	5.0	5.0	5.0
12	5.0	5.0	5.0	5.0	5.0
15	5.0	5.0	5.0	5.0	5.0

T = 10

B = 10

m/B	0.2	0.3	0.5	0.7	0.9
1	1.8	2.6	4.0	5.2	6.1
2	3.1	4.1	5.4	6.1	6.5
3	3.9	4.9	6.0	6.4	6.5
4	4.6	5.5	6.3	6.5	6.5
5	5.0	5.8	6.4	6.5	6.5
6	5.4	6.0	6.5	6.5	6.5
7	5.6	6.2	6.5	6.5	6.5
8	5.8	6.3	6.5	6.5	6.5
9	6.0	6.4	6.5	6.5	6.5
12	6.2	6.5	6.5	6.5	6.5
15	6.4	6.5	6.5	6.5	6.5

B = 20

m/B	0.2	0.3	0.5	0.7	0.9
1	3.3	4.6	6.4	7.7	8.5
2	5.2	6.5	7.9	8.5	8.8
3	6.3	7.4	8.4	8.7	8.8
4	7.0	7.9	8.6	8.8	8.8
5	7.5	8.2	8.7	8.8	8.8
6	7.8	8.4	8.7	8.8	8.8
7	8.1	8.5	8.8	8.8	8.8
8	8.2	8.6	8.8	8.8	8.8
9	8.4	8.7	8.8	8.8	8.8
12	8.6	8.7	8.8	8.8	8.8
15	8.7	8.8	8.8	8.8	8.8

B = 30

m/B	0.2	0.3	0.5	0.7	0.9
1	4.5	6.0	7.9	8.9	9.4
2	6.7	7.9	9.0	9.4	9.6
3	7.8	8.7	9.4	9.5	9.6
4	8.4	9.1	9.5	9.6	9.6
5	8.8	9.3	9.5	9.6	9.6
6	9.0	9.4	9.6	9.6	9.6
7	9.2	9.4	9.6	9.6	9.6
8	9.3	9.5	9.6	9.6	9.6
9	9.3	9.5	9.6	9.6	9.6
12	9.5	9.6	9.6	9.6	9.6
15	9.5	9.6	9.6	9.6	9.6

B = 40

m/B	0.2	0.3	0.5	0.7	0.9
1	5.5	7.0	8.7	9.5	9.8
2	7.7	8.8	9.6	9.8	9.8
3	8.6	9.3	9.7	9.8	9.9
4	9.1	9.6	9.8	9.8	9.9
5	9.4	9.7	9.8	9.9	9.9
6	9.5	9.8	9.8	9.9	9.9
7	9.6	9.8	9.8	9.9	9.9
8	9.7	9.8	9.8	9.9	9.9
9	9.7	9.8	9.9	9.9	9.9
12	9.8	9.8	9.9	9.9	9.9
15	9.8	9.8	9.9	9.9	9.9

T = 15

B = 10

m/B	0.2	0.3	0.5	0.7	0.9
1	1.9	2.7	4.3	5.7	6.9
2	3.2	4.4	6.0	7.0	7.4
3	4.2	5.4	6.8	7.3	7.5
4	5.0	6.9	7.1	7.4	7.5
5	5.5	6.5	7.3	7.5	7.5
6	5.9	6.8	7.4	7.5	7.5
7	6.3	7.0	7.4	7.5	7.5
8	6.5	7.2	7.5	7.5	7.5
9	6.7	7.3	7.5	7.5	7.5
12	7.1	7.4	7.5	7.5	7.5
15	7.3	7.5	7.5	7.5	7.5

B = 20

m/B	0.2	0.3	0.5	0.7	0.9
1	3.5	5.0	7.4	9.2	10.6
2	5.8	7.5	9.6	10.7	11.2
3	7.3	8.9	10.5	11.1	11.2
4	8.3	9.7	10.9	11.2	11.2
5	9.0	10.2	11.1	11.2	11.2
6	9.5	10.5	11.1	11.2	11.2
7	9.9	10.8	11.2	11.2	11.2
8	10.2	10.9	11.2	11.2	11.2
9	10.4	11.0	11.2	11.2	11.2
12	10.8	11.2	11.2	11.2	11.2
15	11.0	11.2	11.2	11.2	11.2

B = 30

m/B	0.2	0.3	0.5	0.7	0.9
1	5.0	6.8	9.6	11.2	12.7
2	7.8	9.7	11.8	12.7	13.1
3	9.4	11.1	12.5	13.0	13.1
4	10.5	11.8	12.8	13.1	13.1
5	11.2	12.3	13.0	13.1	13.1
6	11.7	12.6	13.0	13.1	13.1
7	12.0	12.7	13.1	13.1	13.1
8	12.3	12.9	13.1	13.1	13.1
9	12.5	12.9	13.1	13.1	13.1
12	12.8	13.0	13.1	13.1	13.1
15	13.0	13.1	13.1	13.1	13.1

B = 40

m/B	0.2	0.3	0.5	0.7	0.9
1	6.2	8.3	11.1	12.8	13.7
2	9.3	11.2	13.1	13.8	14.0
3	11.0	12.5	13.6	14.0	14.0
4	12.0	13.1	13.9	14.0	14.1
5	12.6	13.5	14.0	14.0	14.1
6	13.0	13.7	14.0	14.0	14.1
7	13.3	13.8	14.0	14.0	14.1
8	13.5	13.9	14.0	14.1	14.1
9	13.6	13.9	14.0	14.1	14.1
12	13.8	14.0	14.0	14.1	14.1
15	14.0	14.0	14.1	14.1	14.1

m/B	0.2	0.3	0.5	0.7	0.9	0.2	0.3	0.5	0.7	0.9
			B = 10					B = 20		
1	1.9	2.8	4.5	6.0	7.4	3.6	5.2	7.9	10.2	12.0
2	3.3	4.6	6.4	7.4	8.0	6.1	8.1	10.7	12.1	12.8
3	4.4	5.7	7.2	7.9	8.0	7.8	9.7	11.8	12.6	12.8
4	5.2	6.4	7.6	8.0	8.0	9.0	10.8	12.3	12.8	12.8
5	5.8	6.9	7.8	8.0	8.0	9.9	11.4	12.6	12.8	12.8
6	6.3	7.3	7.9	8.0	8.0	10.6	11.9	12.7	12.8	12.8
7	6.6	7.5	8.0	8.0	8.0	11.1	12.1	12.8	12.8	12.8
8	6.9	7.7	8.0	8.0	8.0	11.5	12.4	12.8	12.8	12.8
9	7.2	7.8	8.0	8.0	8.0	11.7	12.5	12.8	12.8	12.8
12	7.6	7.9	8.0	8.0	8.0	12.3	12.7	12.8	12.8	12.8
15	7.8	8.0	8.0	8.0	8.0	12.6	12.8	12.8	12.8	12.8
			B = 30					B = 40		
1	5.2	7.3	10.6	13.1	15.0	6.6	9.1	12.7	15.2	16.8
2	8.4	10.8	13.6	15.1	15.6	10.3	12.9	15.7	16.9	17.4
3	10.5	12.7	14.8	15.5	15.7	12.6	14.7	16.7	17.3	17.4
4	11.9	13.7	15.3	15.7	15.7	14.0	15.8	17.1	17.4	17.4
5	12.8	14.4	15.5	15.7	15.7	14.9	16.3	17.3	17.4	17.4
6	13.5	14.8	15.6	15.7	15.7	15.6	16.7	17.3	17.4	17.4
7	14.0	15.1	15.7	15.7	15.7	16.0	16.9	17.4	17.4	17.4
8	14.4	15.3	15.7	15.7	15.7	16.3	17.1	17.4	17.4	17.4
9	14.7	15.4	15.7	15.7	15.7	16.6	17.2	17.4	17.4	17.4
12	15.2	15.6	15.7	15.7	15.7	17.0	17.4	17.4	17.4	17.4
15	15.5	15.7	15.7	15.7	15.7	17.2	17.4	17.4	17.4	17.4

Memorandum No. 2
of NBS Report No. 4571

Uncoordinated System Based on
Constant Kill Probability

by

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Given B batteries, T targets, m missiles, and constant kill probability p.

Let $a = (1 - p)^{m/B}$, where m/B is assumed to be an integer. The number a is the probability that a target will survive when fired on by one battery, since it is assumed that each battery will fire all of its m/B missiles at one target.

Under the uncoordinated system, each battery selects its target at random, independent of selections made by other batteries.

An engagement is characterized by the numbers k_1, k_2, ..., k_T, where k_i denotes the number of batteries which fire on target i. $B = k_1 + \dots + k_T$.

Quantities which may be used to describe the properties of this system include the following:

(1) Average number of targets down.

$$\text{Avg Tgt Dn} = T\left[1 - \left(1 - \frac{1-a}{T}\right)^B\right]$$

(2) Probability of non-penetration (i.e., probability that T targets are down).

$$\text{Pnp} = \sum_{i=0}^{T} (-1)^i \binom{T}{i}\left(1 - \frac{1-a}{T}\, i\right)^B$$

(3) Variance of the number of targets down.

$$\text{Var(Tgt Dn)} = T\left(1 - \frac{1-a}{T}\right)^B + T(T-1)\left(1 - 2\,\frac{1-a}{T}\right)^B - T^2\left(1 - \frac{1-a}{T}\right)^{2B}$$

(4) For each n $(n = 0, 1, \ldots, T)$, the probability p_n that exactly n targets are down.

$p_T = \text{Pnp}$ (see (2)),

$$p_{T-n} = \binom{T}{n} \sum_{i=0}^{T-n} (-1)^i \binom{T-n}{i} \left[1 - (n+i)\frac{1-a}{T}\right]^B, \quad n = 0, 1, \ldots, T.$$

In particular,

$$p_0 = a^B$$

$$p_1 = T\left(a + \frac{1-a}{T}\right)^B - Ta^B \quad .$$

These quantities are derived in the following.

<u>1. Distribution of Number of Targets Down.</u>

The random variable with which we are concerned is N, the number of targets down. Let the distribution of N be given by $(p_0, p_1, \ldots, p_T)$, where

$$p_n = \Pr(N = n) \quad , \quad n = 0, 1, \ldots, T.$$

Then, in particular,

$$Pnp = p_T .$$

From the distribution of N, we also obtain

$$\text{Avg Tgt Dn} = EN = \sum_{n=0}^{T} np_n ,$$

and

$$\text{Var(Tgt Dn)} = E(N - EN)^2 = \sum_{n=0}^{T} n^2 p_n - (EN)^2.$$

The quantities $p_0, \ldots, p_T$ are not convenient expressions, however, so that we do not evaluate EN and Var N from the relations given above.

2. Evaluation of p_n .

Observe that there are T^B possible equally likely configurations of assignments of B batteries to T targets. Let $K = (k_1, \ldots, k_T)$ represent the configuration of assignments in one engagement where k_i batteries fire at the ith target. Corresponding to each K, there are $\binom{B}{K}$ configurations which differ only in that a different set of k_i batteries is assigned to the ith target.

Thus, we have

$$p_n = T^{-B} \sum_K \binom{B}{K} p_n^{(K)} , \quad n = 0, 1, \ldots, T,$$

where $p_n^{(K)}$ denotes the conditional probability that exactly n targets are down in an engagement with configuration K.

Furthermore, for any function $f(n)$, we have

$$Ef(N) = \sum_{n=0}^{T} f(n) p_n = T^{-B} \sum_K \binom{B}{K} \left\{ \sum_{n=0}^{T} f(n)\, p_n^{(K)} \right\} .$$

We will write

$$E_K f(N) = \sum_{n=0}^{T} f(n)\, p_n^{(K)}$$

to denote the conditional expectation of $f(N)$ when K is fixed.

3. Evaluation of $p_n^{(K)}$.

Observe that N may be regarded as the sum of T random variables,

$$N = X_1 + \ldots + X_T ,$$

where X_i has the value one or zero according as the ith target is down or survives.

Now, when K is fixed, the X_i are mutually independent. The conditional distribution of X_i is given by

$$\Pr\,(X_i = 1 | K) = 1 - a^{k_i}$$

$$\Pr\,(X_i = 0 | K) = a^{k_i} \quad , \quad i = 1, \ldots, T.$$

We have, then,

$$p_T^{(K)} = \prod_{i=1}^{T} (1 - a^{k_i}) \quad ,$$

$$p_0^{(K)} = \prod_{i=1}^{T} a^{k_i} = a^B \quad ,$$

and for $n = 2, 3, \ldots, T - 1,$

$$p_n^{(K)} = \sum_{j_1 \neq j_2 \neq \ldots \neq j_n} \left(\frac{1 - a^{k_{j_1}}}{a^{k_{j_1}}} \right) \cdots \left(\frac{1 - a^{k_{j_n}}}{a^{k_{j_n}}} \right) a^B$$

Except for $p_0^{(K)}$, these are not convenient expressions to use in the relations given in the preceding section for calculating $Ef(n)$. Another method will be developed to carry out those calculations. We are, however, in a position to complete the evaluation of p_n $(n = 0, 1, \ldots, T)$.

4. Calculation of p_n.

a) Evidently, $p_0 = a^B$.

b) $$p_1 = T\left(a + \frac{1-a}{T}\right)^B - Ta^B .$$

$$p_1^{(K)} = \sum_{i=1}^{T}\left(\frac{1-a^{k_i}}{a^{k_i}}\right) a^B = \sum_{i=1}^{T} \prod_{j \neq i} a^{k_j} - Ta^B$$

$$p_1 = T^{-B} \sum_K \binom{B}{K} p_1^{(K)}$$

Now, in general,

$$T^{-B} \sum_K \binom{B}{K} a^{k_{i_1}} \cdots a^{k_{i_r}} = \left(1 - r\,\frac{1-a}{T}\right)^B .$$

Thus,

$$p_1 = T\left(1 - (T-1)\,\frac{1-a}{T}\right)^B - Ta^B .$$

c) The remaining expressions for p_n are calculated in similar fashion.

5. Calculation of Avg Tgt Dn.

$$EN = T^{-B} \sum_K \binom{B}{K} E_K N$$

$$E_K N = \sum_{i=1}^{T} E_K X_i = \sum_{i=1}^{T} (1 - a^{k_i})$$

The calculation of the expression given above for Avg Tgt Dn is straightforward.

6. Calculation of Var(Tgt Dn).

$$E(N - EN)^2 = E(N - E_K N)^2 + E(E_K N - EN)^2$$

a) $$E(N - E_K N)^2 = T^{-B} \sum_K \binom{B}{K} E_K(N - E_K N)^2$$

But $E_K(N - E_K N)^2$ is simply the variance of a sum of independent random variables, since $N = X_1 + \ldots + X_T$ as noted above.

Now $$\operatorname{Var} X_i = a^{k_i} (1 - a^{k_i}) , \quad i = 1, \ldots, T .$$

Hence

$$E(N - E_k N)^2 = T^{-B} \sum_K \binom{B}{K} \sum_{i=1}^{T} a^{k_i} (1 - a^{k_i}).$$

b) $$E(E_K N - EN)^2 = T^{-B} \sum_K \binom{B}{K} (E_K N)^2 - (EN)^2$$

Var N is obtained by evaluating the two terms.

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