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SOME FORMULAS FOR COMPUTING PROBABILITIES FOR COMMON STATISTICS

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This note gives formulas for computing the probability that a random variable will exceed a specified value in the case of the F, t, X^2, and normal distribution. These formulas were taken or adapted from standard statistical text books and from the volumes of the Bateman Manuscript Project referenced herein as follows:

[1] Higher Transcendental Functions, Vol. 1

[2] Higher Transcendental Functions, Vol. 2.

Most of those from the latter source are new or not well-known to statisticians. As many such formulas as conveniently available were compiled for the purpose of developing computational techniques suitable to electronic or punch card machines. It was considered desirable to preserve them even though they are not in regular publishable form. For convenience, some of the well-known classical relationships are also given.

I The F distribution

$$P_r\{F > F_0\} = \int_{F_0}^{\infty} \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \frac{F^{\frac{n_1-2}{2}}\, dF}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$$= I_{x_0}\left(\frac{n_2}{2}, \frac{n_1}{2}\right) \quad \text{where } x_0 = \frac{n_2}{n_2+n_1F} \qquad 0 < x_0 < 1$$

Also we have, [1] p87,

$$P_r\{F > F_0\} = 1 - I_{1-x_0}\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$$

$$= \frac{2}{n_2} x_0^{\frac{n_2}{2}} F\left(\frac{n_2}{2}, -\frac{n_1}{2}+1, \frac{n_2}{2}+1; x_0\right)$$

$$= 1 - \frac{2}{n_1}(1-x_0)^{\frac{n_1}{2}} F\left(\frac{n_1}{2}, -\frac{n_2}{2}+1, \frac{n_1}{2}+1; 1-x_0\right)$$

where the hypergeometric function $F(\quad)$ is given by

$$F(a, b, c; x) = F(b, a, c; x) = (1-x)^{c-a-b} F(c-a, c-b, c; x)$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n, \qquad |x| < 1$$

with $(u)_0 = 1$, $(u)_n = u(u+1) \cdots (u+n-1)$

Using this definition we may write

$$P_r\{F > F_0\} = 2 x_0^{\frac{n_2}{2}} \sum_{n=0}^{\infty} \frac{\left(-\frac{n_1}{2}+1\right)_n}{(2n+n_2)\, n!} x_0^n$$

(which has only a finite number of terms for n_1 even)

$$= \frac{2}{n_2} x_0^{\frac{n_2}{2}} (1-x_0)^{\frac{n_1}{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{n_1+n_2}{2}\right)_n}{\left(\frac{n_2}{2}+1\right)_n} x_0^n$$

The following formulas are useful when x_0 is near 1 (F_0 small)

$$P_r\{F>F_0\} = x_0^{\frac{n_2}{2}}\left[B\left(\frac{n_1}{2},\frac{n_2}{2}\right)\sum_{n=0}^{\infty}\frac{\left(\frac{n_2}{2}\right)_n}{n!}(1-x_0)^n - \frac{2}{n_1}\sum_{n=0}^{\infty}\frac{\left(\frac{n_1+n_2}{2}\right)_n}{\left(\frac{n_1}{2}+1\right)_n}(1-x_0)^n\right]$$

[1] p108(1)

$$P_r\{F>F_0\} = B\left(\frac{n_1}{2},\frac{n_2}{2}\right) + x_0^{\frac{n_2}{2}-1}\left(1-\frac{1}{x_0}\right)^{\frac{n_2}{2}}\sum_{n=0}^{\infty}\frac{\left(-\frac{n_2}{2}\right)_{n+1}}{\left(\frac{n_1}{2}\right)_{n+1}}\left(1-\frac{1}{x_0}\right)^n$$

[1] p109 (4)

This last formula has only a finite number of terms if n_2 is even

II The t distribution

$$Pr\{|t| > t_0\} = 2 \int_{t_0}^{\infty} \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \frac{dt}{(1+\frac{t^2}{n})^{\frac{n+1}{2}}}$$

for $n=1$ this reduces to $\left(1 - \frac{2}{\pi} \tan^{-1} t_0\right)$.

For small n and large values of t this probability is approximately $2 c_n t^{-n}$ where c_n has the values given in the table:

n	1	2	3	4	5	6	7
c_n	0.317	0.488	1.04	2.7	8.2	26	84

From $P_r\{|t| > t_0 / df = n\} = P_r\{F > t_0^2 / df = 1, n\}$ we have

$$P_r\{|t| > t_0\} = I_{x_0}\left(\tfrac{n}{2}, \tfrac{1}{2}\right) \qquad \text{where } x_0 = \frac{n}{n+t_0^2}$$

$$= 1 - I_{x_0}\left(\tfrac{1}{2}, \tfrac{n}{2}\right)$$

$$= \frac{2}{n} x_0^{n/2} F\left(\tfrac{n}{2}, \tfrac{1}{2}, \tfrac{n}{2}+1 ; x_0\right)$$

$$= 1 - 2(1-x_0)^{1/2} F\left(\tfrac{1}{2}, -\tfrac{n}{2}+1, \tfrac{3}{2} ; 1-x_0\right)$$

$$= 2 x_0^{n/2} \sum_{r=0}^{\infty} \frac{(\frac{3}{2})_r}{(2r+n)\, r!} x_0^r$$

$$= \frac{2}{n} x_0^{n/2} (1-x_0)^{\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(\frac{n+1}{2})_r}{(\frac{n}{2}+1)_r} x_0^r$$

III The χ^2 distribution

$$P = Pr\{\chi^2 > \chi_0^2\} = \int_{\chi_0^2}^{\infty} \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n}{2}-1} d\chi^2$$

for even degrees of freedom

$$P = e^{-\frac{1}{2}\chi_0^2}\left\{1 + \frac{\chi_0^2}{2} + \cdots + \frac{\chi_0^{n-2}}{2\cdot 4\cdot 6 \cdots (n-2)}\right\}$$

$$= \sum_{k=0}^{c-1} \frac{e^{-m} m^k}{k!} \qquad \text{where } c = \frac{n}{2}, \quad m = \frac{\chi_0^2}{2}$$

$$Pr\{\chi^2 > \chi_0^2\} = \frac{\Gamma(\frac{n}{2}, \frac{\chi_0^2}{2})}{\Gamma(\frac{n}{2})},$$

where $\Gamma(a,x)$ is the incomplete gamma function [2] p 134 (1)

$$\Gamma(a,x) = \int_x^{\infty} t^{a-1} e^{-t} dt$$

$$= \Gamma(a) - \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{x^{a+r}}{a+r}$$

Hence, [2] p 135 (4)

$$Pr\{\chi^2 > \chi_0^2\} = 1 - \frac{1}{\Gamma(\frac{n}{2})} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{(\frac{1}{2}\chi_0^2)^{\frac{n}{2}+r}}{\frac{n}{2}+r}$$

$$= 1 - \frac{e^{-\frac{1}{2}\chi_0^2}}{\Gamma(\frac{n}{2})} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}\chi_0^2)^{r+\frac{n}{2}}}{(\frac{n}{2})_{r+1}}$$

An asymptotic expression for large χ_0^2 is [2] p 135 (6)

$$P_r\{\chi^2 > \chi_0^2\} = \frac{\left(\frac{1}{2}\chi_0^2\right)^{\frac{n}{2}-1} e^{-\frac{1}{2}\chi_0^2}}{\Gamma\left(\frac{n}{2}\right)} \left[\sum_{r=0}^{M-1} \frac{\left(1-\frac{n}{2}\right)_r (-1)^r}{\left(\frac{1}{2}\chi_0^2\right)^r} + O\left\{\left(\tfrac{1}{2}\chi_0^2\right)^{-M}\right\} \right]$$

As a continued fraction [2] p 136 (13)

$$P_r\{\chi^2 > \chi_0^2\} = \frac{\left(\frac{1}{2}\chi_0^2\right)^{\frac{n}{2}} e^{-\frac{1}{2}\chi_0^2}}{\Gamma\left(\frac{n}{2}\right)} \left[\frac{1}{\frac{1}{2}\chi_0^2 +} \; \frac{1-\frac{n}{2}}{1 +} \; \frac{1}{\frac{1}{2}\chi_0^2 +} \; \frac{2-\frac{n}{2}}{1 +} \cdots \right]$$

From Nielsen's expansion we have [2] p 139 (2)

$$P_r\{\chi^2 > \chi_0^2 + h\} = P_r\{\chi^2 > \chi_0^2\} - \frac{\left(\frac{1}{2}\chi_0^2\right)^{\frac{n}{2}-1} e^{-\frac{1}{2}\chi_0^2}}{\Gamma\left(\frac{n}{2}\right)} \sum_{r=0}^{\infty} \left(1-\tfrac{n}{2}\right)_r \left(-\tfrac{1}{2}\chi_0^2\right)^{-r} \left[1 - e^{-h} e_r(h)\right]$$

where $|h| < \chi_0^2$

[2] p 136 (16) and $e_r(h) = \sum_{m=0}^{r} \frac{h^m}{m!}$, the Truncated exponential series

Also, [2] p 139 (3)

$$P_r\{\chi^2 > \chi_0^2\} = 1 - e^{-\frac{1}{2}\chi_0^2} \left(\tfrac{1}{2}\chi_0^2\right)^{\frac{n}{4}} \sum_{r=0}^{\infty} e_r(-1) \left(\tfrac{1}{2}\chi_0^2\right)^{\frac{r}{2}} I_{r+\frac{n}{2}}\left(2\sqrt{\tfrac{1}{2}\chi_0^2}\right)$$

is rapidly convergent for all $\chi_0^2 > 0$, where [2] p 5 (12)

$$I_p(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{2m+p}}{m!\,\Gamma(m+p+1)}$$

$$\left\{ = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\chi_0^2\right)^{m+\frac{r}{2}+\frac{n}{4}}}{m!\,\Gamma\left(m+r+\frac{n}{2}+1\right)} \right.$$

is the modified Bessel function of the first kind.

There is also an asymptotic formula for χ_0^2 large compared to n:

$$P_r\{\chi^2 > \chi_0^2\} = \frac{\left(\frac{1}{2}\chi_0^2\right)^{\frac{n}{2}} e^{-\frac{1}{2}\chi_0^2}}{\frac{1}{2}\chi_0^2 - \frac{n}{2} + 1}\left[1 - \frac{\frac{n}{2} - 1}{\left(\frac{1}{2}\chi_0^2 - \frac{n}{2} + 1\right)^2} + \frac{n-2}{\left(\frac{1}{2}\chi_0^2 - \frac{n}{2} + 1\right)^3} + O\left\{\left(\frac{n}{2} - 1\right)^2\left(\frac{1}{2}\chi_0^2 - \frac{n}{2} + 1\right)^{-4}\right\}\right]$$

[2] p 140 (4)

IV The normal distribution

$$Pr\{x > x_0\} = \frac{1}{\sqrt{2\pi}} \int_{x_0}^{\infty} e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{x_0\sqrt{2\pi}} e^{-\frac{x_0^2}{2}} \left[1 - \frac{1}{x_0^2} + \frac{1\cdot 3}{x_0^4} - \frac{1\cdot 3\cdot 5}{x_0^6} + \cdots\right]$$

$$= \frac{1}{x_0\sqrt{2\pi}} e^{-\frac{x_0^2}{2}} \left[1 - \frac{1}{x_0^2+3} - \frac{6}{x_0^6+13x_0^4+25x_0^2+145} - \cdots\right]$$

$$Pr\{x > x_0\} = \frac{1}{\sqrt{\pi}} \operatorname{Erfc} \frac{x_0}{\sqrt{2}} = \frac{1}{2\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}, x_0^2\right) = \tfrac{1}{2} Pr\{\chi^2 > x_0^2 / n=1\}$$

where $\operatorname{Erfc} x = \int_x^{\infty} e^{-t^2} dt = \frac{1}{2}\Gamma\left(\frac{1}{2}, x^2\right) = \frac{1}{2}\sqrt{\pi} - \operatorname{Erf} x$

The formulas for χ^2 may thus be used. In addition [2] p147 (6)

$$Pr\{x > x_0\} = \tfrac{1}{2} - \tfrac{1}{\sqrt{\pi}} \operatorname{Erf} \frac{x_0}{\sqrt{2}}$$

$$= \tfrac{1}{2} - \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}x_0^2\right)^{n+\frac{1}{2}}}{n!\,(2n+1)} = \tfrac{1}{2} - \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x_0^2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}x_0^2\right)^{n+\frac{1}{2}}}{\left(\frac{3}{2}\right)_n}$$

which becomes [2] p147 (8)

$$= \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{2}x_0^2} \left[\sum_{m=0}^{M} \frac{(-1)^m \left(\frac{1}{2}\right)_m}{\left(\frac{1}{2}x_0^2\right)^{m+\frac{1}{2}}} + O\left\{\left(\tfrac{1}{2}x_0^2\right)^{-M-\frac{1}{2}}\right\}\right]$$

asymptotically as $x_0 \to \infty$.

Expansion in series of modified Bessel functions of the first kind gives [2] p148 (19) & (20):

$$Pr\{x > x_0\} = \tfrac{1}{2} - \tfrac{1}{2}\left(\frac{x_0^2}{\sqrt{2}}\right)^{\frac{1}{2}} e^{-\frac{1}{2}x_0^2} \sum_{n=0}^{\infty} e_n(-1)\left(\tfrac{1}{2}x_0^2\right)^{\frac{n}{2}} I_{n+\frac{1}{2}}\left(\sqrt{2}\,x_0\right)$$

$$= \tfrac{1}{2} - \tfrac{1}{\sqrt{2}} \sum_{n=0}^{\infty} (-1)^{\left[\frac{n}{2}\right]} I_{n-\frac{1}{2}}\left(\tfrac{1}{2}x_0^2\right)$$