NATIONAL BUREAU OF STANDARDS REPORT

3625

A PROBLEM IN SELF-HEATING
OF A
SPHERICAL BODY

by

Samuel M. Genensky

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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A Problem in Self-Heating of a Spherical Body

by

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Abstract

An analytic steady-state solution is developed for a spherical body in which heat is generated according to a first order unimolecular reaction law, and lost at the surface in accordance with Newton's law of cooling. The temperature within the sphere depends chiefly on the radial distance from the center, but also on the ambient temperature, the surface heat loss coefficient, and the material properties which are assumed constant.

Introduction

The analysis described in this paper was pursued in connection with an investigation into the self-heating of fibrous materials carried on at the National Bureau of Standards.

Comparison of the results of this analysis with the experimental results indicated that the assumption of a single first order unimolecular reaction was an over-simplification for the material investigated experimentally. However, the analysis serves to indicate the relative importance of the various parameters entering into the problem, and may be helpful in indicating the kind of data that would be of greatest use in further investigations in which the assumptions may be applicable.

A homogeneous sphere of radius B is generating heat, under steady-state conditions, in accordance with a first order unimolecular reaction law $Ae^{-\frac{E}{RT}}$ and losing heat from its surface

to an atmosphere at constant temperature T_a according to Newton's law of cooling. The temperature T(r) of the sphere is independent of time and depends upon the radial distance r from the sphere's center. Within the sphere transfer of heat by means other than conduction is considered negligible. The thermal conductivity k, gas constant R, rate of reaction-frequency product A and activation energy E of the material of the sphere are assumed to be known constants as are the temperature T_o and gradient $\frac{dT}{dr} = 0$ at the center of the sphere. The problem is to find the surface temperature of the sphere T_B at r = B and the heat transfer coefficient h between the surface of the sphere and a surrounding atmosphere.

Analysis

Mathematically the problem becomes:

$$k\left[\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr}\right] + Ae^{-\frac{E}{RT}} = 0 \qquad (0 < r < B) \qquad (1)$$

$$T = T_o \quad \text{at } r = 0 \qquad (2)$$

$$\frac{dT}{dr} = 0 \quad \text{at } r = 0 \qquad (3)$$

$$k\left(\frac{dT}{dr}\right)_{r=B} = -h\left[T_B - T_a\right] \qquad (4)$$

Let

$$\eta = \frac{r}{B}, \quad V(\eta) = \frac{R}{E}T(\eta) \text{ and } C = \frac{ARB^2}{Ek}$$

Then equations (1)-(4) become:

$$\frac{d^2V}{d\eta^2} + \frac{2}{\eta}\frac{dV}{d\eta} + Ce^{-\frac{1}{V}} = 0 \qquad (0 < \eta < 1) \qquad (5)$$

$$V = V_o = \frac{RT_o}{E} \text{ at } \eta = 0 \tag{6}$$

$$\frac{dV}{d\eta} = 0 \quad \text{at} \quad \eta = 0 \tag{7}$$

$$k\left(\frac{dV}{d\eta}\right)_{\eta=1} = -\frac{BhR}{E}\left[T_B - T_a\right] \tag{8}$$

It is further assumed that (a) the temperature gradient within the sphere exists and is continuous throughout the interval $0 \leq \eta \leq 1$ and (b) that the absolute value of the difference between $T(\eta)$ $0 \leq \eta \leq 1$ and T_o, is small in comparison to T_o. Under these restrictions a technique presented by Chambré[2] proves useful to observe that:

$$\frac{1}{V} = \frac{1}{V_o - \Delta V} = \frac{1}{V_o}\left[1/\left(1-\frac{\Delta V}{V_o}\right)\right] = \frac{1}{V_o}\sum_{i=0}^{\infty}\left(\frac{\Delta V}{V_o}\right)^i \tag{9}$$

where $\Delta V = V_o - V$

Now since $\left|\frac{\Delta V}{V_o}\right| < 1$ the series coverges and for a sufficiently large positive integer N, $\frac{1}{V_o}\sum_{i=0}^{N}\left(\frac{\Delta V}{V_o}\right)^i$ is a very good approximation to $\frac{1}{V}$.

Therefore (5) may be written approximately as

$$\frac{d^2V}{d\eta^2} + \frac{2}{\eta}\frac{dV}{d\eta} + Ce^{-\frac{1}{V_o}\sum_{i=0}^{N}\left(\frac{\Delta V}{V_o}\right)^i} = 0 \qquad (0 < \eta < 1) \tag{10}$$

Now $$\sum_{i=0}^{N}\left(\frac{\Delta V}{V_o}\right)^i = \sum_{i=0}^{N}\left(1-\frac{V}{V_o}\right)^i = (N+1) - B_1V + B_2V^2 - \dots + (-1)^N B_N V^N$$

where

$$B_k = \sum_{n=0}^{N}\frac{n!}{(n-k)!\;k!\;V_o^k} \qquad k = 1,2,\dots,N$$

and in the special case $V = V_o$

$$(N+1)-B_1V + B_2V^2- \dots +(-1)^N B_N V^N = 1 \quad N = 0,1,2,\dots$$

Therefore (10) may be written

$$\frac{d^2V}{d\eta^2} + \frac{2}{\eta}\frac{dV}{d\eta} + Ce^{-\frac{N+1}{V_o}} e^{+\frac{1}{V_o}\left[B_1V-B_2V^2 + \dots -(-1)^N B_N V^N\right]} = 0 \tag{11}$$

Since (11) is analytic and regular except at $\eta = 0$ and further since $\frac{dV}{d\eta} = 0$ at $\eta = 0$ the solution of (11) may be written in the form:

$$V = V(\eta) = \sum_{i=0}^{\infty} a_i\eta^i \qquad (0 \leq \eta \leq 1) \tag{12}$$

Recalling (6), equation (12) at $\eta = 0$ yields:

$$V_o = V(0) = \frac{T_oR}{E} = a_o \tag{13}$$

and further differentiating (12) once and considering (7) it is found that

$$a_1 = 0 \tag{14}$$

Differentiating (12) twice, substituting these first two derivatives into equation (11) and rearranging terms:

$$\sum_{i=2}^{\infty}(i^2+i)a_i\eta^{i-2} = -De^{+\frac{1}{V_o}\left[B_1V - B_2V^2 + \dots -(-1)^N B_N V^N\right]} \tag{15}$$

where $D = Ce^{-\frac{N+1}{V_o}}$

and the a_i are given by

$$a_i = -\frac{1}{(i^2+i)}\,\frac{d^{i-2}\left[De^{+\frac{1}{V_o}\left[B_1V-B_2V^2 + \dots -(-1)^N B_N V^N\right]}\right]}{(i-2)!\;d\eta^{i-2}}\Bigg|_{\eta=0} \tag{16}$$

$i=2,3,4\dots$

Using the coefficients obtained from (13), (14), and (16) and substituting them into equation (12), V can be computed for $0 \leq \eta \leq 1$.

In particular for $\eta = 1$ equation (12) gives the dimensionless surface temperature V(1) and thus recalling that $T(\eta) = \frac{E}{R}V(\eta)$ the surface temperature T(1) which equals T_B is easily found. Differentiating equation (12) once and evaluating this derivative at $\eta = 1$, h can then be found by solving equation (8), for all the other factors involved in this equation are either given or have been computed.

The first six non-zero coefficients have been evaluated and are:

$$a_0 = V_o$$

$$a_2 = -\frac{C}{6}\, e^{-\frac{1}{V_o}}$$

$$a_4 = \frac{C^2}{120}\, \frac{e^{-\frac{2}{V_o}}}{V_o^2}$$

$$a_6 = \frac{C^3}{1008\, V_o^3}\left[\frac{8}{15}\,\frac{C}{V_o} - \frac{2}{3}\right] e^{-\frac{3}{V_o}}$$

$$a_8 = \frac{C^4}{51840\, V_o^4}\left[\frac{122C^2}{63V_o^2} - \frac{122C}{21\, V_o} + \frac{10}{3}\right] e^{-\frac{4}{V_o}}$$

$$a_{10} = \frac{C^5}{4435200\, V_o^5}\left[\frac{5032\, C^3}{405\, V_o^3} - \frac{8428\, C^2}{135\, V_o^2} + \frac{1486\, C}{27\, V_o} - \frac{560}{9}\right] e^{-\frac{5}{V_o}}$$

Observe that a_{2i-1} $(i = 1,2,3,\ldots)$ are zero. This follows from $a_1 = 0$ and that spherical symmetry of the problem.

Figure 1 is a plot of V(1), as a function of C and $1/V_o$, where C was allowed to vary over the range $10^{-1} \leq C \leq 10^{19}$ and $1/V_o$ took on selected values in the range $5 \leq 1/V_o \leq 100$.

The points were obtained using

$$V(\eta) = \sum_{i=0}^{10} a_i \eta^i \tag{17}$$

over its range of applicability for the values of C and $1/V_o$ within the above limits.

In the foregoing, it has been assumed that the temperature at the center of the sphere T_o is known, and also that the physical condition indicated by equation (8) is satisfied.

The more common steady state problem of finding the temperature at the center of a sphere which generates heat according to a first order unimolecular reaction law and loses heat according to Newton's law of cooling may be solved using the analysis described above. In this case E, R, k, A, h, T_a, T_B and B are assumed known.

If the sphere being considered really satisfies the assumptions of the problem then its temperature must satisfy equation (1). Since equation (1) is of the second order only two boundary conditions can be preassigned. Further since the temperature at the center of the sphere must remain finite, one of these preassigned boundary conditions must be $\frac{dT}{dr} = 0$ at $r = 0$. This leaves but one free boundary condition. However at the surface of the sphere both

$$\left(\frac{dT}{dr}\right)_{r=B} = \frac{h}{k}\left[T_B - T_a\right] \tag{18}$$

and

$$T = T_B \quad \text{at } r = B \tag{19}$$

must be satisfied; and since only one of these two boundary conditions may be formally assigned, it must be assumed that the other is consistent with the problem. Therefore Fig. 1, which represents equation (17), may be used to solve this problem. Since R, E, k, A and B are known, both C and V(1) (recall that $T_B = \frac{E}{R} V(1)$) can be computed. Thus using Fig. 1, $1/V_o$ may be found by interpolation provided that the point (C, V(1)) under consideration lies within the region of applicability of equation (17) as indicated on Page 6. Since the center temperature of the sphere is T_o, that is, $\frac{E}{R} V_o$, the problem is solved.

Here no use was made of condition (18) and as mentioned earlier this condition must be satisfied independently, if the assumptions made are fulfilled. Thus the degree of consistency between conditions (18) and (19) serves as an indication of the applicability of the assumptions made in this analysis.

References

1. Mitchell, N. D. "New Light on Self Ignition" National Fire Protection Association Quarterly October, 1951.

2. Chambre, P. L. "On the Solution on the Poisson-Boltzman Equation with Application to the Theory of Thermal Explosions" J. Chem. Phys. Vol. 20, No. 11, pp. 1795-1797, 1952.

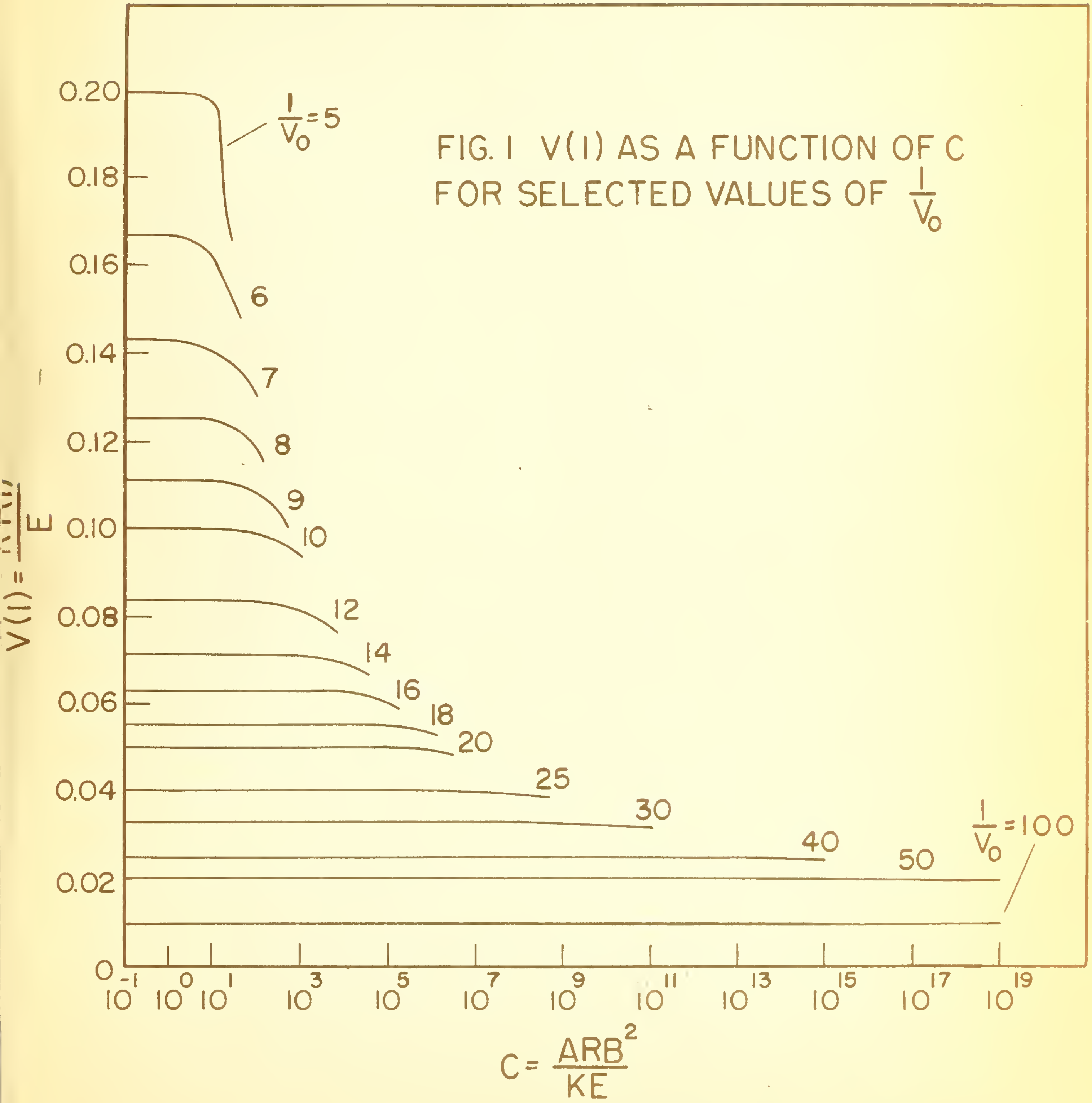

FIG. 1 V(1) AS A FUNCTION OF C FOR SELECTED VALUES OF $\frac{1}{V_0}$
$V(1) = \frac{K(1)}{E}$
$\frac{1}{V_0} = 5$
6
7
8
9
10
12
14
16
18
20
25
30
40
50
$\frac{1}{V_0} = 100$
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
10^{-1} 10^{0} 10^{1} 10^{3} 10^{5} 10^{7} 10^{9} 10^{11} 10^{13} 10^{15} 10^{17} 10^{19}
$C = \frac{ARB^2}{KE}$

THE NATIONAL BUREAU OF STANDARDS

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