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Motivation for working in numerical analysis

by

John Todd

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Motivation for working in numerical analysis

The profession of numerical analysis is not yet so desirable that it is taken up by choice; indeed, although it is one of the oldest professions, it is only now becoming respectable. Most of those who are now working in this field have been, more or less, drafted into it, either in World War I or in World War II, or more recently. The question at issue is why have we stayed in this field and not returned to our earlier interests.

Our answer is that numerical analysis is an attractive subject where mathematics of practically all sorts can be used significantly, and from which, on the other hand, many of its branches can benefit. We call attention here to the applications of functional analysis by the Russian school led by Kantorovitch [1] (A survey of this has been given recently by Collatz [1a]). In the other direction we would recall the developments in analytic number theory by Lehmer and Rademacher which followed MacMahon's computations of p(n) for Hardy and Ramanujan (see [2]).

Before proceeding to a discussion of some individual topics in numerical analysis, some general remarks are in order.

We have, on various occasions, distinguished between classical and modern numerical analysis, the latter being material required in connection with the exploitation of high speed automatic digital computing machines. It now seems desirable to recognize ultra-modern numerical analysis, which may be specified as adventures with high speed automatic digital computing machines[See

50, 51]. There are, of course, no sharp boundaries between these parts of the subject, and, as we shall see, there is room for development in the classical phases as well as in the newer.

In distinction from the deliberate explorations contemplated in ultra-modern numerical analysis, there is much routine work in numerical analysis which must necessarily be of an experimental or empirical nature. It is just not feasible to carry out rigorous error estimates for all problems of significant complication: it is necessary to place considerable reliance, on the one hand, on the experience of those familiar with similar problems and, on the other, on the good judgment of the setter of the problem. To justify this remark we consider three examples. The solution of systems of twenty or more first order differential equations is being handled regularly. To see the complication of theoretical error estimates (in which the fact that all numbers handled are finite (binary) decimals is disregarded) we refer to Bieberbach [3]. The complication of a stability analysis in a system of fourteen equations is evident from a study carried out by F. J. Murray [4]. Again, the extent of a complete error estimate for the problem of matrix inversion is familiar from the work of von Neumann and Goldstine [5, 5a] and Turing [6]. Finally, there is the analysis of the triple diagonal method for determining the characteristic roots of a symmetric matrix by Givens [7, 7a, 7b].

What the numerical analyst has to do is to be aware of the

precision of results obtained from, for instance, the conformal mapping of an ellipse on a circle by a certain process, and from this to extrapolate to cases of regions of comparable shape. On the one hand, he has to examine general error analyses for their realism by comparison with cases where the explicit exact results are known. On the other hand, he must devote time to the construction and study of bad examples so as to counteract any tendency to too much extrapolation.

The main part of this paper is devoted to a discussion of some topics in numerical analysis which appear attractive. These have been chosen among those with which we are familiar, to point out some of the techniques of the subject and to indicate some of those who have made distinguished contributions in the field. In addition, our choice has been controlled by our opinion that separation between theoretical and practical numerical analysis is undesirable. As an illustration of the practicality of some of the techniques used, we refer to computations of the radiation from a simple source which is reflected from a Lambert plane recently carried out by P. Henrici [8] where the ideas of §2 and §5a below were used.

1. Evaluation of polynomials

What is the best way of computing polynomials, for instance,

$$f(x) = a_o x^n + \ldots + a_{n-1} x + a_n$$

for a series of values of x, not equally spaced? (In the case where the values of f(x) for a series of equally spaced values of x are required, building up f(x) from its differences might be the most convenient.) The usual answer is to suggest the recurrence scheme;

$$f_o = a_o$$
$$f_{r+1} = xf_r + a_{r+1} \qquad r = 0,1,\ldots,n-1.$$

which was known to Newton but is usually ascribed to Horner [9]. In this way we get f(x) by n additions and n multiplications. Is this the best possible algorithm? Consider an alternative, in the case of

$$f(x) = 1 + 2x + 3x^2.$$

If we proceed as follows:

$$2x,\ x^2,\ 3x^2,\ 1 + 2x + 3x2$$

we need 3 multiplications and 2 additions compared with the 2 multiplications and 2 additions needed in applying the above algorithm; thus

$$3x,\ 3x+2,\ x(3x+2),\ x(3x+2) + 1.$$

This problem was formulated as one in abstract algebra by Ostrowski and he showed [9] that the above algorithm was indeed the best for polynomials of degree not exceeding 4. It was

recently taken up again by Motzkin [10] who showed that the result is not true in practice for larger n. We give a simple example in the case n = 6. Consider the evaluation of

$$P = x^6 + Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F.$$

Introduce the following polynomials

$$P_1 = x + ax$$

$$P_2 = (P_1 + x + b)\,(P_1 + c)$$

$$P_3 = (P_2 + d)\,(P_1 + e)$$

and determine a,b,c,d,e and f by identifying P and P_3+f. This can be done by the solution of linear equations and a single quadratic. This evaluation is done once for all and then P can be evaluated at the expense of three multiplications only, with a significant economy over the other process if we have to evaluate P for a sufficiently large number of values of x. Whether it is possible to improve on the Newton-Horner scheme by a purely rational process does not seem to have been decided.

The details of the evaluations are as follows. The result of equating coefficients in P and P_3+f is:

(1.1) $3a + 1 = A$

(1.2) $3a^2 + 2a + b + c + e = B$

(1.3) $a^2 + a^2 + 2ab + 2ac + 2ae + c + e = C$

(1.4) $a^2 b + a^2 c + a^2 e + ac + ae + bc + be + ce + d = D$

(1.5) $abc + abe + ace + ad + ce = E$

(1.6) $ebc + de + f = F$

From (1.1) we find a. Hence we can rewrite (1.2) and (1.3) in the form

(1.2') $b + c + e = B'$

(1.3') $2a(b + c + e) + c + e = C'$

(We use primed capitals to indicate new known constants). Using (1.2') in (1.3') we get

(1.3") $c + e = C''$

which with (1.2') gives us b explicitly. Using a,b, c+e, we can write (1.4) as

(1.4") $d + ce = D'$

Using a,b, c+e, d+ce in (1.5) we find

(1.5') $ce = E'$

which gives d from (1.4'). From (1.3") and (1.5') we can find c,e by solving a quadratic equation and then from (1.6) we can find f.

2. Increasing the speed of convergence of sequences

The construction of processes which increase the speed of convergence of sequences and series has been a favorite topic for many numerical analysts. For instance, there is the h^2 extrapolation process of L. F. Richardson [11], the converging-factor method of J. R. Airey [12], [12a], the Euler summation process [13] and a whole subject associated with the name of

Chebyshev [14]. We shall discuss the δ^2-process which has been popularized in numerical analysis by A. C. Aitken [15,e.g]; it dates back at least to Kummer [16].

If

$$x_n \to x$$

and

$$x_n - x \doteq A\,\lambda^n, \qquad |\lambda| < 1 \tag{2.1}$$

then

$$\frac{x_{n+2} - x}{x_{n+1} - x} \doteq \frac{x_{n+1} - x}{x_n - x} \tag{2.2}$$

From (2.2) we find

$$x \doteq x_{n+2} - \frac{(x_{n+2} - x_{n+1})^2}{x_{n+2} - 2x_{n+1} + x_n}$$

This suggests that the sequence $\{\bar{x}_{n+2}\}$ defined by

$$\bar{x}_{n+2} = x_{n+2} - \frac{(x_{n+2} - x_{n+1})^2}{x_{n+2} - 2x_{n+1} + x_n}, \qquad n=0,1,2,\ldots$$

converges more rapidly to x than the original sequence. This is indeed the case for if

$$x_n - x = A\,\lambda^n + o(\lambda^n), \qquad |\lambda| < 1$$

then it follows that

$$\bar{x}_n - x_n = o(\lambda^n).$$

Several remarks are in order. First, this process can be iterated to remove successively components in the remainder of the form

$$A\lambda^n,\ B\mu^n,\ C\nu^n,\ldots$$

where $1 > |\lambda| > |\mu| > |\nu| > \ldots$. The cases when there are equalities such as $|\lambda| = |\mu|$ can be handled by simple modifications. Second, it is important to note that this process can make things worse if the convergence is not geometric as required by (2.1). Here is a simple example involving two of the standard iterative processes for determining the reciprocal of a number N. Consider the sequences

$$y_{n+1} = (1 - N)\, y_n + 1; \quad z_{n+1} = z_n\,(2 - Nz_n).$$

In the case $N = \frac{1}{2}$ with $y_0 = 1$, $z_0 = 1$, we obtain the following table:

y_n			$\bar{y}_n$	z_n			$\bar{z}_n$
1			–	1			–
1.5	.5	-.125	–	1.5	.5	-.125	
1.75	.25	-.125	2	1.875	.375	-.2578	3.0000
1.875	.125		2	1.9922	.1172		2.0455

The sequence $\{\bar{y}_n\}$ appears to converge more rapidly than $\{y_n\}$ while the sequence $\{\bar{z}_n\}$ appears to converge less rapidly than $\{z_n\}$. These results can be easily established. First of all, each

sequence converges to N^{-1} [illegible] if $0 < N < 1$ for

$$y_n - N^{-1} = (1 - N^{-1})^N (y_o - N^{-1})$$

and

$$z_n - N^{-1} = - N^{2^n - 1} (z_o - N^{-1})^{2^n} .$$

Thus $\{y_n\}$ satisfies the condition (2.1) while $\{z_n\}$ does not, converging too rapidly. In the present case we have $y_n \equiv N^{-1}$. On the other hand it can be shown that

$$\frac{\bar{z}_n - N^{-1}}{z_n - N^{-1}} \rightarrow \infty .$$

Note, however, that to justify the application of this process it is sufficient to show the existence of an expansion of the form (2.1) with $|\lambda| < 1$.

Extensive use of this process was made in experiments in conformal mapping by G. Blanch and L. K. Jackson [17] and by John Todd and S. E. Warshawski [18]. For instance, in the latter the mapping of an ellipse (of axis ratio 5:1) on a circle, it was found that while about 50 iterations, each requiring about 30 minutes of computing on SEAC, were required to secure directly about nine correct decimals in the value of the boundary function. It was however, possible to obtain the same accuracy by a double use of the Aitken process on the first 14 iterants -- the extra time required for this being negligible.

3. Modified Differences

We shall show here how the use of quadratic interpolation enables the table-maker to cut down on the size of a table at the expense of some work by the table-user. We shall then show how a further saving in space can be accomplished, at no further expense to the user but at some to the table-maker, by the use of modified differences. For simplicity and definiteness we consider the construction of a table of sin x, to four places of decimals, for x in the range $(0, \frac{1}{2}\pi)$.

a) Linear Interpolation.

The error involved in linear interpolation, i.e., the assumption that

$$f(a+ph) = f(a) + p\{f(a+h)-f(a)\}, \quad 0 < p < 1,$$

can be estimated as

$$h^2 \left|\left(\frac{p}{2}\right)\right| \max \left|f''(x)\right|.$$

If this is to be less than $\frac{1}{2}x10^{-4}$ an appropriate choice for h is .02. This requires a table of some 80 entries, part of which is shown below.

x	sin x
0.00	0.0000
0.02	0.0200
0.04	0.0400
0.06	0.0600
0.08	0.0799
⋮	⋮
1.20	0.9320
1.22	0.9391
1.24	0.9458
1.26	0.9521
1.28	0.9580
⋮	⋮

Interpolation, say for x=1.234, in this table is carried out as follows:

$$\sin 1.234 = .9391 + \frac{14}{20}(.9458 - .9391) = .9438$$

b) Quadratic interpolation

We now consider using the Everett interpolation formula

$$(3.1)\quad f_p = qf_0 + pf_1 + E_0^{2}\delta^2 f_0 + E_1^{2}\delta^2 f_1 + E_0^{4}\delta^4 f_0 + E_1^{4}\delta^4 f_1 + \ldots$$

where $q = 1-p, E_0^{2} = q(q^2-1)/6, E_1^{2} = p(p^2-1)/6, \ldots$

If we retain the first four terms the truncation error can be estimated as

$$h^4 \left|\binom{p+1}{4}\right| \max \left|f^4(x)\right|$$

$$< h^4 \times .024 \times 1.$$

For this to be less than $\frac{1}{2} \times 10^{-4}$ we can conveniently take h=.2.

The corresponding complete table is given below.

x	sin x	δ^2
0.0	0.0000	0
0.2	0.1987	- 80
0.4	0.3894	-155
0.6	0.5646	-224
0.8	0.7174	-287
1.0	0.8415	-336
1.2	0.9320	-371
1.4	0.9854	-392
1.6	0.9996	-400
1.8	0.9738	-387

For interpolation we now have either to compute the Everett coefficients or to obtain them from a table; we find

$$p = .17,\ E_0^{2} = -.0430,\ E_1^{2} = -.0275.$$

We then have

$$\sin 1.234 = .9320 + \frac{34}{200}(.9854 - .9320) + (.0430)(.0371)$$

$$+ (.0275)(.0392) = .9438$$

c) Comrie's throwback

This device, introduced by Comrie [19] depends essentially on the fact that the ratio

$$k(p) = \frac{E_o^4}{E_0^2} = \frac{(p+1)(p-3)}{20}$$

is approximately constant for $0 < p < 1$. Various ways of choosing a mean value for this have been discussed [20,]. The preferred value is $k = -.18393$. With this value of k we rewrite the first four terms of (3.1) as:

$$(3.2) \qquad f_p \doteq qf_o + pf_1 +$$

$$E_o^2(\delta^2 f_o + k\delta^4 f_o) + E_1^2(\delta^2 f_1 + k\delta^4 f_1).$$

Therefore if we define

$$\delta_m^2 f = \delta^2 f + k\,\delta^4 f$$

and use these modified second differences in exactly the same way as we used the ordinary second differences in the preceding subsection, we can obtain the desired accuracy of interpolation with a much larger interval. In fact the error in (3.2) is made up of the truncation now bounded by

$$(3.3) \qquad h^6 \left|\binom{p+3}{6}\right| \max \left|f^{(6)}(x)\right| < h^6 x\ .0049$$

together with the error caused by the modification. It can be shown that the latter is less than half a unit in the last place if the fourth differences are less than 1000 units (and the fifth less than 70 units).

The condition (3.3) gives $h < .46$, which suggests that $h = .5$ might be acceptable. For this value of h, a bound for the fourth difference is

$$(.5)^4 \times |f^4(x)| < .0625$$

which is acceptable, although the bound for the fifth is not. Nevertheless we shall use $h = .5$ without carrying out a more precise estimate.

The complete table is given below:

x	sin x	$\delta^2 m$
0.0	0.0000	0
0.5	0.4794	-1225
1.0	0.8415	-2154
1.5	0.9975	-2552
2.0	0.9093	-2326

For interpolation we first find the Everett coefficients

$p = .468$: $E_0^2 = .0636$, $E_1^2 = -.0609$.

We then have

$$\sin 1.234 = .8415 + \frac{234}{500}(.9975 - .8415) + (.0636 \times .2154),$$

$$+ (.0609 \times .2552) = .9145 + .0137 + .0155 = .9437$$

The discrepancy between the results can be explained either by

the marginal choice of h, or by rounding errors.

More elaborate methods of throw-backs e.g. of the sixth difference as well as the fourth into the second have also been given by Comrie [21]. Discussions of the minor disadvantages of the modified differences are also available [19].

4. Characteristic roots of finite matrices

Considerable effort has been expended in problems of numerical analysis involving matrices [see e.g., 72, 73, and 22, 23]. The two main problems are the inversion of matrices and the determination of their characteristic values. In both problems the practical determination of bounds for characteristic roots is important. In this connection we call attention to the following lemma of Gerschgorin [24] which has many applications.

All the characteristic roots of $A = (a_{ij})$ lie in the union of the circular regions

$$|a_{ij} - z| \leq \sum_{i \neq j} |a_{ij}|, \quad i = 1,2,\ldots,n.$$

This is proved by use of the fact that a determinant with dominant main diagonal does not vanish. This last result has been generalized by many writers: for an account of some of this work (up to 1947) see O. Taussky [25].

One of the preferred and practical methods of getting all the characteristic roots of a symmetric matrix depends on the reduction of the matrix to pure diagonal forms (Goldstine, von Neumann and Murray [26], Gregory [27]) by superposing orthogonal trans-

formations involving two variables at a time. Theoretically we obtain

$$TAT' = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

and then $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the exact characteristic values. In practice we find

$$TAT' = (\varepsilon_{ij})$$

where the ε_{ij} are small for $i \neq j$. We then ask how near are the ε_{ii} to the λ_i? If we disregard the question of the transformation not being truly orthogonal -- and therefore the characteristic roots of (ε_{ij}) not being identical with those of (a_{ij}) - the answer comes at once from the lemma. If the ε_{ij}, $i \neq j$ are sufficiently small then

$$|\lambda_i - \varepsilon_{ii}| < \sum_{i \neq j} |\varepsilon_{ij}|, \quad i = 1,2,\ldots,n.$$

Allowance can easily be made in this inequality for round-off error in the product TAT'.

5. Quadrature, integral equations

a) We begin with an example to show that there is still scope for new ideas in classical numerical analysis. A typical quadrature formula is

$$\int_a^b f(x)\,dx \doteq \sum p_i\, f(x_i)$$

and the error

$$E = \left| \int_a^b f(x)dx - \sum p_i\, f(x_i) \right|$$

is estimated as a multiple of a (high) derivative $f^n(x)$ of $f(x)$ at a point in (a,b). In many cases it is far from convenient to obtain bounds on $f^n(x)$, or to estimate these by computing the corresponding differences manually. Recently P. Davis and P. Rabinowitz [28], [28a] reconsidered this problem in the case when $f(x)$ is analytic in a region including the segment (a,b). W. F. Eberlein [29] has also contributed to this problem. The case of ellipses $\mathcal{E}$ with foci at the end-points, which we normalize to $(1,0)$ $(-1,0)$, can be handled elegantly, in terms of the Chebyshev polynomials.

$$(1 - z^2)^{-\frac{1}{2}} \sin((n+1)\arccos z)$$

which are orthogonal over the area of such an ellipse. It can be shown that

$$E < \sigma_{\mathcal{E}} \|f\|$$

where $\sigma_{\mathcal{E}}$ is a constant depending only on the ellipse $\mathcal{E}$ and where

$$\|f\| = \iint_{\mathcal{E}} |f(z)|^2 \, dx\, dy$$

(Note that $\|f\|$ increases as $\mathcal{E}$ expands; however, $\sigma_{\mathcal{E}}$ then decreases and there is a problem of optimal choice of $\mathcal{E}$). The $\sigma_{\mathcal{E}}$ can be tabulated once for all, and $\|f\|$ can be estimated in terms e.g., of $\max|f|$.

As an example consider the evaluation of

$$\int_3^4 \Gamma(x)dx$$

using a 7-point Gaussian rule. To evaluate and bound the fourteenth derivative of $P(z)$ seems rather out of the question. Simple estimates can be used in the method just described to find

$$E < 2.04 \times 10^{-12}.$$

A comparison of this estimate with the usual one [30, 31]:

$$\frac{f^{(2n)}(\xi)}{(2n!)^3}\frac{(n!)^4}{(2n+1)}, \qquad 3 \leqslant \xi \leqslant 4,$$

where the derivative is now estimated by the use of Cauchy's formula, shows that the new one is somewhat better.

b) Among the basic problems in the numerical analysis of integral equations is the relation between the eigenvalues of a (symmetric) kernel and those of an approximating matrix. A satisfactory account of this was given recently by Wielandt [32] in support of some experiments on conformal mapping [18] which were being carried out

on SEAC, the National Bureau of Standards Eastern Automatic Computer. The continuous problem is the solution of

$$\int_0^1 K(x,\xi)\, y(\xi)\, d\xi = ky(x)$$

We make this discrete by introducing a quadrature

$$\int_0^1 f(\xi)\, d\xi \doteq \sum p_\nu\, f(\xi_\nu) \tag{5.1}$$

and are therefore led to consider the matrix problem

$$\sum_\nu K(\xi_\mu, \xi_\nu)\, p_\nu\, y_\nu = \bar{k}\, y_\mu$$

What are the relations between the finite number of $\bar{k}$ and the infinity of the k? We quote a typical result. If we take for (5.1) the trapezoidal quadrature

$$\int_0^1 f(\xi)\, d\xi = \frac{1}{n}\left\{\tfrac{1}{2}f_0 + f_1 + \ldots + f_{n-1} + \tfrac{1}{2}f_n\right\}, \quad f_i = f\left(\frac{i-1}{n}\right)$$

then, provided K satisfies

$$|K(x,y) - K(\alpha,\beta)| < L\,(|x-\alpha| + |y-\beta|)$$

where α, β run through the points $(i/n, j/n)$ and where $|x-\alpha| < \frac{1}{2}n^{-1}$, $|y-\beta| < \frac{1}{2}n^{-1}$, we have

$$k - \bar{k} < \frac{CL}{n-1}$$

where the constant $C = \frac{1}{4} + \sqrt{1/12}$ is best possible.

6. Game theory and related developments

There are here problems in which the intuition of a geometer can play an essential part; for instance, the theory of polyhedra and convex bodies and fixed point theory are all highly relevant. The foundations of a theory of games were laid down by von Neumann [33], [39, 74, 75] beginning in 1928. The theory of two-person zero sum games is well developed but the practical problem of finding the value of such a game and the optimal strategies is difficult and solutions available so far are not entirely satisfactory. (Among related and essentially equivalent problems are the solution of systems of linear inequalities and the solution of linear programs in the sense of G.B.Dantzig.)

Among the methods of attacking these problems are the Simplex Method [34], the Relaxation Method [35], the Double Description Method [36]. We shall, however, discuss a very simple example by a natural approach due to G. W. Brown [37], the validity of which was established by J. Robinson [38], A related continuous solution of this discrete problem has been given by Brown and von Neumann [37a]; we take up this idea of continuous approach to discrete problems again §8c..

Consider the following game played between two players R and C each of whom has two strategies, which we may interpret as the

choice of a row or a column in the pay-off matrix $P = (p_{ij})$:

$$P = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} .$$

If R chooses the ith row and C chooses the jth column then R gets p_{ij} from C. This is manifestly an unfair game, and R should pay to play it.

The value of this game is 2.5 and the optimal strategies are the following: R should choose 1 and 2 each with probability $\frac{1}{2}$; C should choose 1 with probability $\frac{1}{4}$ and 2 with probability 3/4. The significance of these statements are the following: if R plays in this way his expected gain is not less than 2.5 while if C plays this way his expected loss if not greater than 2.5.

To prove this statement is simple. Let R play 1 with probability $r \geq 0$ and 2 with probability $1-r \geq 0$; let C play 1 with probability $c \geq 0$ and 2 with probability $1-c \geq 0$. Then the expectation of R is:

$$E = 1 \times rc + 4(1-r)c + 3r(1-c) + 2\,(1-r)(1-c).$$

We have:

$$E = -4\,(r-\tfrac{1}{2})\,(c-\tfrac{1}{4}) + 5/2.$$

This shows that when $r = \frac{1}{2}$ then $E = 5/2$ for any c, and that if $r \neq \frac{1}{2}$ then c can be chosen to make $E < 5/2$; similarly, if $c = \frac{1}{4}$ then $E = 5/2$ and if $c \neq \frac{1}{4}$ then r can be chosen to make $E > 5/2$. (cf. McKinsey [39]).

How can we arrive at these results, or approximations to them? Suppose we consider a sequence of plays of the game in which each player is trying his best in the following sense. After n plays suppose C has chosen 1 in $c^{(n)}$ x n plays and 2 in the remaining $c_2^{(n)}$x n plays. If C continues this pattern, the expectation of R in the next play is

$$e_1 = c_1 < 3c_2 \quad \text{if he chooses 1,}$$
$$e_2 = 4c_1 + 2c_2 \quad \text{if he chooses 2.}$$

R therefore chooses 1 or 2 according to whether $e_1 \geq e_2$ or $e_1 < e_2$. Similarly, if R has chosen 1 in $r_1^{(n)}$x n plays and 2 in the remaining $r_2^{(n)}$x n plays, then C chooses 1 or 2 according to the expected size of his loss which is

$$f_1 = r_1 + 4r_2 \quad \text{if he chooses 1,}$$
$$f_2 = 3r_1 + 2r_2 \quad \text{if he chooses 2.}$$

Specifically C chooses 1 if $f_1 \leq f_2$ and 2 if $f_1 > f_2$.

This sequence of plays is determined after we make an (arbitrary) assignment of the initial play e.g., that each player chooses 1. The resultant sequence of plays is:

(1,1), (2,1), (2,2), (2,2), (2,2), (2,2), (2,2),
(2,2), (1,2), (1,2), (1,2), (1,2), (1,2), (1,1), (1,1),
(2,1), (2,2), (2,2), (2,2), (2,2), (1,2), (1,2), (1,2),
(1,1), (1,1), (2,1), (1,1), (2,1), (2,2), (2,2), (2,2), (2,2),
(2,2), (2,2), (2,2), (2,2), (2,2), (2,2), (2,2), (2,2), (1,2),
(1,2), (1,2), (1,2), (1,2), (1,2), (1,2), (1,2), (1,2), (1,2),...

It has been shown [38], that as $n \to \infty$, the sequences $\{c_1^{(n)}\}$, $\{r_1^{(n)}\}$ converge to the optimal strategy i.e., $c_1^{(n)} \to \frac{1}{4}$, $r_1^{(n)} \to \frac{1}{2}$, and that the average pay-off $p^{(n)}$ converges to the value of the game. In our case

$$r_1^{(50)} = .48, \quad c_1^{(50)} = .2, \quad p^{(50)} = 2.4,$$

The structure of the sequence above, consisting of blocks of identical elements, is typical; this can obviously be used to speed the computations. For some practical experiments in this field see [40].

We shall now discuss an application of the theory of games to the so-called assignment problem. This problem is to assign n square pegs to n round holes in such a way as to maximize the total goodness of fit. In other words, (a_{ij}) being given, we have to choose a permutation P_i of $(1,2,\ldots,n)$ so as to maximize

$$\sum a_{iP_i} \tag{6.1}$$

This is trivial theoretically: we have only to find the largest of the n! sums of the form (6.1). In practice, however, this may be out of the question and so we may have to settle for some approximation to the maximum. One way of doing this (suggested by von Neumann [41]), is to set up an equivalent game theory problem --- it turns out to be a sort of hide and seek --- and solve this approximately by the method just discussed. The first

player chooses a pair of indices (i,j) $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n)$; he has n^2 strategies. The second then elects first to guess the first or second of these two indices, and then guesses it by choosing $k(1 \leqslant k \leqslant n)$; he has $2n$ strategies. In the first case if $k=i$, and in the second case if $k=j$, the first player pays the second $(a_{ij})^{-1}$, otherwise there is no pay-off.

Assignment problems for n=12 have been handled by this method.

An up-to-date account of this problem and its generalizations has been given by Motzkin [42]. Among these are the transportation problem, the caterer problem, the problem of contract awards, the traveling salesman problem [43, 44]. Solutions to problems of this type are now obtained on a routine basis, on high speed computers, as an aid to management decisions in industrial and military situations [43]. Among other problems of this general character, which are in the research stage, are those concerned with organization theory which have been studied by Tompkins and Marshak [46].

7. Monte Carlo

This is a subject with large areas unsoiled by theorems, as can be seen by reference to the reports on various symposia held on the subject [67, 68]. For instance, during the last four years we have been generating millions of pseudo-random numbers on SEAC using such relations as

$r_n = 2^{-42}x_n$, $x_{n+1} = \rho x_n \pmod{2^{42}}$, $x_0 = 1$, $\rho = 5^{17}$

or

$r_n = 2^{-44}\, x_{2n}$, $x_{n+1} = x_n + x_{n-1} \pmod{2^{44}}$, $x_0 = 0$, $x_1 = 1$.

The r_n behave as if they came from a uniform distribution in the interval 0.1. The results we obtained were satisfactory in all cases where we had independent checks. We have, however, no theorems at all about the "randomness" of these sequences or about the distributions of blocks of the size used in our calculations.

We mention here also the quasi-Monte Carlo processes studied by Peck and Richtmyer [47,48]. Here high-power algebraic number theory is used to evaluate the error committed by replacing integrals by sums of the integrands at points determined by certain algebraic numbers.

8. Recent activity in numerical analysis

We mention here a few areas, with which we are familiar, and which we have found interesting. This personal selection omits reference to many areas in which there have been important advances (e.g. meteorology) and to areas which have been discussed elsewhere in this volume, or its predecessor.

a) Ultramodern numerical analysis

One class of experiments may be described as follows.

It has been usual in discussing properties of matter to regard the medium as continuous, set up differential equations, look at them for a while, give up and replace them by difference equations. These difference equations were then solved and no attention was paid for their physical significance, if any. An alternative approach is to handle the problem discretely from the beginning, lumping the "molecules" together in groups as small as the computing equipment can handle.

Among those who have handled problems in this general way are Seeger, von Neumann and Polachek (cf. Seeger [49]), who were concerned with shock wave phenomena. Recently Ulam [50] has studied the mixing of fluids and the motions of star clusters in this way. Metropolis and Fermi [51] have investigated the equations of state of individually interacting particles forming an idealized liquid.

b) Biological applications

There has been pioneering work by Turing [52] on the problem of morphogenesis. Turing constructs a mathematical model of a growing embryo and shows how well-known physical laws are sufficient to account for many of the facts about the development of its anatomical structure.

Another application has been the study of the reaction of nerve-fibers to electrical stimuli. This phenomenon is governed by a system of four non-linear ordinary differential equations

(Hodgkin-Huxley). The system has been studied on SEAC (by K. S. Cole, H. A. Antosiewicz, and P. Rabinowitz [53], and in particular the threshold value of the input current has been determined. The agreement with the results of millions of experiments indicates the reliability of the model and encourages further investigation.

c) Combinatorial analysis

This is an obvious source of problems. There have been recent reports on this topic by S. S. Cairns [54] and C. B. Tompkins [55]. The numerical analyst, however, soon finds himself out of his depth if he uses straight-forward approaches.

One new idea which was tried is the following: that of a continuous approach to discrete problems, in particular to the search for perfect difference sets. . . . . A perfect difference set is a set of n+1 integers whose n(n+1) differences take on all non-zero values mod $n^2 + n+1$. For example, the differences of 1,2,4, are ± 1, ± 2, ± 3 i.e., all non-zero values mod 7, and so 1,2,4 form a perfect difference set mod 7.

A perfect difference set S can be specified by $N=n^2+n+1$ constants x_r where $x_r = 1$ if $r \in S$, $x_r = 0$ otherwise. In this case we have

$$y_0 = \sum_r x_r^2 = (n+1)$$

$$y_s = \sum_r x_r x_{r+s} = 1, \quad s = 1,2,\ldots,N-1$$

(The subscript $r+s$ is to be understood mod N); hence

(8.1) $$\sum_r y_r = (n+1)^2 .$$

It follows, therefore, that such a set x_r minimizes

$$J = (n+1) \sum_s y_s^2 - n\, y_o^2$$

for, in view of (8.1), J differs by a constant from

$$(y_o - (n+1))^2 + (n+1) \sum_{s>o} (y_s - 1)^2 .$$

This suggests an attempt to obtain a set of x_r by minimizing J, now regarded as a function of the N continuous real variables x, subject to (8.1), (and perhaps to other relations such as $0 \leq x_r \leq 1$).

Such an attempt was made on SWAC, by a steepest descent process. Although admissible values of y were obtained rapidly, the corresponding values of x were not integers.

d) Number theory and algebra

These subjects are natural sources of problems and there has been many applications of high speed computers, particularly in elementary, algebraic and analytic number theory and some in algebra proper [55a, 56].

Recent work on SWAC (The National Bureau of Standards Western Automatic Computer), mainly on elementary number theory by D. H. and E. Lehmer and their collaborators, has been discussed by E. Lehmer [57].

Among other work has been a study of the divisibility of $((p-1)! + 1)/p$ by p. This was known to be the case for p=5, p=13; K. Goldberg [58] found that it was also the case for p=563 and for no other $p < 10{,}000$.

Problems in algebraic number theory are more complicated to handle. A survey of computational problems in this field has been given by O. Taussky [59]. Since then there has been work by H. Cohn and S. Gorn [60] on units in cubic fields.

There have been various attempts to study the zeros of the Riemann-zeta function: among those is that of Turing [61].

e. Topology

It is clear that approximate computations of quantities known to be integers serve to define them if the absolute value of the error is known to be less than $\frac{1}{2}$. This idea is used in the work on p(n) mentioned earlier. Ulam [50] has suggested that further applications can be made in essentially topological problems, e.g. the structure of the lines of force caused by current in two infinite straight wires which are skew.

A simple application of this is to the location of the zeros of a polynomial P(x). We use the fact that

$$n = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{P'(z)}{P(z)}\, dz$$

where n is the number of zeros inside the simple closed rectifiable curve C. It is possible to choose C to be a square so large as to contain all the zeros of P(z) and then, by process of quadrisection, to locate the zeros approximately. The quadrature must be accomplished with absolute error less than $\frac{1}{2}$; if this proves difficult due to vanishing or near vanishing of P(z) on the boundary, then we know that we are in the neighborhood of a zero and can act accordingly. A constructive proof of the fundamental theorem of algebra along these lines had been given by Rosenbloom [62].

9. Theory of machines or automata

Among those who have contributed to basic research in this direction have been Turing [63], Shannon [64] and von Neumann [65]. There has been some efforts of a character of supporting research: the use of machines to design circuits for better machines, the design of self-correcting codes, and improvements in the use of machines, e.g. more automatic coding. Much of this belongs more to the domain of logicians than to that of the numerical analyst.

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