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NATIONAL BUREAU OF STANDARDS REPORT

3262

CONTRIBUTIONS TO THE THEORY OF
RANK ORDER STATISTICS

by

I. Richard Savage

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NBS PROJECT | **NBS REPORT**

1103-11-1107 | 3 May 1954 | 3262

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Approved for public release by the Director of the National Institute of Standards and Technology (NIST) on October 9, 2015

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ABSTRACT

The problem of the construction of optimum nonparametric tests and the evaluation of their power functions is considered. All of the situations treated involve mutually independent random variables $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ where the X's have a continuous cumulative distribution function F(x) and the Y's have a continuous cumulative distribution function G(x). When the observed values of the random variables are arranged in an ascending sequence the rank order is defined as $z = (z_1, \ldots, z_i, \ldots, z_{m+n})$ where $z_i = 0(1)$ if the i-th observation in the ascending sequence is from the F(G) distribution. A rank order test is formed by assigning a probability a(z) to each rank order and if a rank order z occurs H_0: F = G is rejected with probability a(z).

If for the rank orders z and z' we have $P(Z=z) > P(Z=z')$ for all alternatives under consideration then the criterion of admissability requires that a = 1 whenever a' > 0. When enough relationships of the above type fail to exist the construction of best tests becomes extremely complicated. It is shown that for such alternatives as H_S: $F(x) \geq G(x)$, where the inequality holds for some x, there does not exist an adequate number of the desired relationships. For the alternative H_{KD}: F and G have density functions $f(x) = A(x)B(\theta_1) \exp[C(x)D(\theta_1)]$ and $g(x) = A(x)B(\theta_2) \exp[C(x)D(\theta_2)]$ where C(x) and D(θ) are increasing

functions, it is shown that if two rank orders z and z' are identical except for the i and j ($i<j$) elements which are (0,1) for z and (1,0) for z' then $P(Z=z) > P(Z=z')$. It is then shown that the Wilcoxon statistic and the c_1 statistic of Terry satisfy the above criterion of admissability but that the one-sided Smirnov test is inadmissable.

Next the hypothesis H_L: $G(x) = [F(x)]^\delta$ where $\delta \geq 1$ is considered. H_L is a special case of H_{KD} when F(x) has a density function, but even if there is no density function the previously mentioned result regarding the ordering of the probabilities of the rank orders is applicable. However, in this case more can be done towards the construction of optimum tests. It is shown that

$$P(Z=z) = m!n!\delta^n / \prod_{i=1}^{m+n} [\sum_{j=1}^{i} (1-z_j+\delta z_j)] \quad .$$

Thus for a particular value of δ under H_L it is possible to compute the probabilities of the rank orders and to construct the most powerful rank order test. For small sample sizes it is possible to examine the expressions given for the probabilities of the rank orders and to find other pairs z and z' than those given above such that $P(Z=z) > P(Z=z')$. For very small sample sizes enough ordering relationships are obtained for constructing

uniformly most powerful rank order tests. For slightly larger sample sizes where, for some levels of significance, uniformly most powerful rank order tests fail to exist, it is possible using the above expression for the probability of a rank order to construct most stringent rank order tests. Under H_L it is shown that tests based on small values of

$$T = \sum_{i=1}^{m+n} z_i D_{Ni} \text{ , where } D_{Ni} = \sum_{j=i}^{m+n} j^{-1}$$

are locally most powerful and that T is approximately normally distributed for large samples. It is indicated that decisions resulting from the T and the Wilcoxon tests will frequently be the same; but the decisions of the Wilcoxon and Terry tests will be in agreement even more often.

Various tables are given of (1) probabilities of the rank orders under H_L, (2) the distribution of T, and (3) the power functions of proposed tests. These tables enable one to conduct tests of H_L against H_0 and estimate their power for all combinations of sample sizes. An important empirical conclusion which can be drawn from these tables is that for all practical purposes the test based on T will give results almost as powerful as the best test even for very small sample sizes.

1. Introduction. The idea of a statistical test of a hypothesis and the other concepts introduced by Neyman and Pearson have served as a model for much of modern statistics. In nonparametric work it is seldom possible to apply all of these concepts. This results from the fact that for most of the alternatives that have been considered there do not exist optimum critical regions or analytic tools for finding power functions. The sign test gives an illustration where it is possible to find the exact power function; on the other hand this procedure is seldom optimum. The c_1 test [Terry, 1952]* has optimum limiting properties but little is known about its power function for small samples. The Kolmogorov and Smirnov tests [Kolmogorov, 1941] have a certain intuitive appeal but their only justification is consistency. The Wilcoxon test [Mann and Whitney, 1947] is justified on the basis that it is analogous to a good parametric procedure but has little direct justification.

* References in brackets will be found in "Bibliography of nonparametric statistics and related topics", I. Richard Savage, Journal of the American Statistical Association, 48 (1953), pp. 844-906. References in parentheses will be found at the end of this report.

In the course of this paper we will consider several non-parametric hypotheses that have been treated previously. In section 5 it will be indicated that for the two-sample problem with such alternatives as slippage, there do not exist optimum nonparametric tests. In particular we show that the class of admissable tests is too large to be of use. In section 6 alternatives are considered which admit a sufficient statistic, and a simple admissability criterion is given. In particular two normal populations differing only in mean value are considered. It is shown that several of the previously proposed tests of this hypothesis satisfy this criterion. Section 7 deals with a special subclass of the alternatives used in section 6. Members of this sublcass are the extreme-value distribution and the exponential distribution. For these alternatives we not only have the results of the previous section on the construction of admissable tests, but also are able to carry out the construction of optimum nonparametric tests for small samples and to evaluate the operating characteristics of these tests. These small-sample tests are uniformly most powerful rank order tests and most stringent rank order tests. Also the limiting optimum test is given. The successful application of the theory is demonstrated by the fact that sufficient tables are supplied so that optimum tests for all sample sizes and estimates of the power are available.

2. <u>Notation</u>. The main concern in the following will be the situation where there are random variables $X_1, \ldots, X_m$ independently distributed each with continuous distribution function $F(x)$, and random variables $Y_1, \ldots, Y_n$ which are independent of the X's and are independently distributed each with continuous distribution $G(x)$, i.e., two independent samples.

The observed values $x_1, \ldots, x_m$ of the random variables $X_1, \ldots, X_m$ will be called the first sample and the observed values $y_1, \ldots, y_n$ of the random variables $Y_1, \ldots, Y_n$ will be called the second sample. When all of the observed values are ordered from smallest to largest they form a sequence which will be denoted by $w_1, \ldots, w_{m+n}$. A new sequence $z_1, \ldots, z_{m+n}$ can be formed from the w sequence by letting $z_i = 0$ if w_i comes from the first sample, and $z_i = 1$ if w_i comes from the second sample ($i=1, \ldots, m+n$). From the z sequence two other sequences are defined by the following formulas:

$$(2.1) \qquad \begin{cases} v_i = \sum_{j=1}^{i} z_j \\ \\ u_i = i - v_i \end{cases} \qquad (i=1, \ldots, m+n) \quad .$$

The rank of x_i, denoted by r_i, is the number of observations less than or equal to x_i; the rank of y_i, denoted by s_i, is

the number of observations less than or equal to y_i. Corresponding to the observed values w_i, z_i, u_i, v_i, r_i, and s_i are the random variables W_i, Z_i, U_i, V_i, R_i, and S_i. An entire sequence such as $u_1, \dots, u_{m+n}$ will be denoted by the corresponding letter u without a subscript. It should be noted that any one of z, u, v, r, or s determines the others and in general these sequences will be referred to as rank orders. All of the above quantities are uniquely defined with probability one as a result of the assumption of continuity of the original distribution functions.

The following symbols will be used to denoted special rank orders:

I < II, (sample I is less than sample II) all of the x's are less than all of the y's.

I a< II, (sample I is almost less than sample II) all of the x's are less than all of the y's except that there is one x larger than one y.

The symbols II < I and II a< I are defined analogously. Thus, when m = n = 3 there is the following dual representation for some of the rank orders:

I < II	000111
I a< II	001011
II a< I	110100
II < I	111000

When a distribution function F has a density function it will be denoted by the corresponding lower case letter f. Integrals involving variables $x_1, x_2, \cdots, x_N$ will be taken over regions of the form

$$- \infty < x_1 < x_2 < \cdots < x_N < \infty$$

unless specifically noted.

3. Hypotheses. For all testing situations considered the following basic assumption will be assumed valid.

Basic Asumption. The random variables $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ are mutually independent. The X's have a common continuous cumulative distribution function F(x), and the Y's have a common continuous cumulative distribution function G(x).

The null hypothesis will be

$$H_0: \ F(x) \equiv G(x) \quad .$$

The following alternatives will be treated.

H_S (Slippage): $F(x) \geq G(x)$, where the inequality holds for some x.

H_T (Translation): $G(x) \equiv F(x-\theta)$ where $\theta > 0$.

H_{TS} (Translation and Symmetry): $G(x) \equiv F(x-\theta)$ where $\theta > 0$ and $F(x) + F(-x) \equiv 1$.

H_{TSU} (Translation, Symmetry, and Unimodal): $G(x) \equiv F(x-\theta)$; where $\theta>0$, $F(x)+F(-x) \equiv 1$, and where $b>a>0$ and $c>0$ implies $F(a+c)-F(a) \geq F(b+c)-F(b)$.

H_{KD} (Koopman-Darmois): $F(x)$ and $G(x)$ have density functions $f(x)$ and $g(x)$, where $f(x)=h(x,\theta_1)$ and $g(x) = h(x,\theta_2)$, $\theta_2 > \theta_1$, and $h(x,\theta)$ is a probability density function of the form

$$A(x)B(\theta) \exp[C(x)D(\theta)] \quad .$$

$C(x)$ and $D(\theta)$ are increasing functions.

H_L (Lehmann): $F(x) = [H(x)]^{\Delta_1}$ and $G(x) = [H(x)]^{\Delta_2}$, where $\Delta_2>\Delta_1>0$ and $H(x)$ is a continuous cumulative distribution function.

H_E (Exponential): $F(x)=\Theta(x,\Delta_1)$ and $G(x)=\Theta(x,\Delta_2)$ where $\Delta_2 > \Delta_1 > 0$ and

$$\Theta(x,\Delta) = \begin{cases} e^{\Delta x} & \text{if } x < 0 \\ 1 & \phantom{\text{if }} x \geq 0 \end{cases} \quad .$$

H_{EV} (Extreme Value): $F(x) = \Omega(x,\Delta_1)$ and $G(x) = \Omega(x,\Delta_2)$ where $\Delta_2 > \Delta_1$ and

$$\Omega(x,\Delta) = e^{-e^{-(x-\Delta)}} .$$

H_N (Normal): $F(x)$ and $G(x)$ have the density functions $f(x) = N(x,\theta_1)$ and $g(x) = N(x,\theta_2)$ where $\theta_2 > \theta_1$ and

$$N(x,\theta) = \frac{1}{\sqrt{2\pi}} \exp[-(x-\theta)^2/2] .$$

The basic assumption of continuity of the distribution functions implies that the occurrence of equal observations is an event with zero probability. In practice, ties will occur and the methods of this paper will need to be modified to accomodate this situation. The choice of the constants in the alternative hypotheses is made so that we need only consider one-sided tests. However, the methods of this paper can be adapted to consider the two-sided case.

The distribution of rank orders under H_0 is not affected by the underlying distribution function. Therefore, as far as rank order tests are concerned H_0 may be considered a simple hypothesis. The alternative hypotheses can be thought of as

either simple or composite. The interpretation used will be clear from the text. Thus in the alternative H_T we have a simple hypothesis if F(x) and θ are held fixed; a composite hypothesis if F(x) is held fixed and all $\theta > 0$ are considered; a composite hypothesis if we consider arbitrary F(x) and all $\theta > 0$.

The alternative hypotheses are related in the following ways:

1. All of the alternatives are special cases of H_S.
2. H_{TSU} is a special case of H_{TS} which is a special case of H_T.
3. When H(x) in H_L has a density function, H_L is a special case of H_{KD}.
4. H_E and H_{EV} are special cases of H_L and of H_{KD}.
5. H_N is a special case of H_{KD}.

Of the above results only (3) needs a word of proof. Under H_L

$$F(x) = [H(x)]^{\Delta_1}$$

$$= \exp\,[\Delta_1 \ln H(x)]$$

and thus

$$f(x) = \Delta_1 \frac{d[\ln H(x)]}{dx} \exp\,[\Delta_1 \ln H(x)] \,,$$

which is the form appearing in H_{KD}.

Nonparametric tests of H_L against H_0 will be introduced in section 7. As a basis for determining the effectiveness of these procedures, their operating characteristics will be compared to those of the best parametric test of H_E against H_0. Since H_L is a nonparametric alternative there is no best parametric procedure. However if $H(x)$ is known and an observation x is replaced by $\ln H(x)$, the testing situation becomes the parametric one described above. Thus the parametric situation serves as a basis of comparison.

The following theorem summarizes the details of the parametric procedure:

<u>Theorem</u> 3.1. Under H_E

1. $\sum_{i=1}^{m} X_i$ and $\sum_{i=1}^{n} Y_i$ are the sufficient statistics.

2. The ratio

$$F = (n \sum_{i=1}^{m} X_i) \Big/ (\delta m \sum_{i=1}^{n} Y_i) ,$$

where $\delta = \Delta_2 / \Delta_1$, has an F-distribution with 2m and 2n degrees of freedom.

3. The best test with a similar region of the null hypothesis $\Delta_1 = \Delta_2$ against the one-sided alternative that $\Delta_2 > \Delta_1$ is based on large values of δF.

[illegible]

[illegible]

[illegible]

[illegible]

4. If a test at the α level of significance with power at least $1 - \beta$ is desired, then δ must be at least as large as

$$F^{\alpha}_{2m,2n} \; F^{\beta}_{2n,2m}$$

where $F^{\varepsilon}_{g,h}$ is the upper ε percentage point of the F-distribution with g and h degrees of freedom.

Eisenhart (Eisenhart, Hastay, and Wallis, 1947, chapter 8, sections 4 and 6.2 and tables 8.3 and 8.4) treats a similar situation to that of theorem 3.1.

For large samples we have

<u>Theorem</u> 3.2. Under H_E if m and n tend to infinity in a fixed ratio, the quantity δF has a limiting normal distribution with expected value δ and variance $\delta^2(m^{-1}+n^{-1})$. If in this limiting process

$$\delta = 1 + c/(m+n)^{\frac{1}{2}} \text{ , c constant,}$$

then the limiting power of the test based on large values of δF at the α level of significance is

$$\Phi(\lambda_{1-\beta})$$

where

$$\lambda_{1-\beta} = \lambda_\alpha - c\sqrt{mn} / (m+n)$$

and

$$\Phi(\lambda_\varepsilon) = \int_{\lambda_\varepsilon}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-x^2/2]dx = \varepsilon \quad .$$

If a test at the α level of significance is desired with power $1 - \beta$, then

$$c = (\lambda_\alpha - \lambda_{1-\beta})(m+n) / \sqrt{mn} \quad .$$

Theorem 3.2 follows from Cramér [1946, page 366] and Pitman [1948, section 18].

Table I gives values of $F^{\alpha}_{2m,2n}$ $F^{\beta}_{2n,2m}$ and $(\lambda_\alpha - \lambda_{1-\beta})\sqrt{m+n} / \sqrt{mn}$.

4. <u>Construction of rank order tests for small samples.</u> The stochastic model for rank order tests can be described in the following manner. There is a random variable Z which can take on values, z^i, in a finite space. The points in the finite space can be identified with the rank orders. Under the null hypothesis each point has equal probability. Let the number of points be J. For the two sample problem, $J = \binom{m+n}{n}$. A test at the K/J level of significance consists of choosing probabilities $a_1, \cdots, a_J$ such that

$$K = \sum_{i=1}^{J} a_i \tag{4.1}$$

and the null hypothesis is rejected with probability a_i if the i-th point is observed. The proper choice of the a_i's depends on the alternative hypothesis.

If under the alternative hypothesis the points z^i can be arranged so that

$$P(Z=z^{i_1}) \geq P(Z=z^{i_2}) \geq \cdots \geq P(Z=z^{i_{J-1}}) \geq P(Z=z^{i_J}) \quad , \tag{4.2}$$

where $i_1, i_2, \cdots, i_{J-1}, i_J$ is a permutation of the first J integers, then an optimum choice of the a_i's is

$$\begin{cases} a_{i_1} = \cdots = a_{i_{[K]}} = 1 \\ a_{i_{[K]+1}} = K - [K] \\ a_{i_{[K]+2}} = \cdots = a_{i_J} = 0 \end{cases} \tag{4.3}$$

where [K] is the largest integer $\leq$ K. If the alternative hypothesis is composite this procedure will lead to a uniformly most powerful test. Specifically for the nonparametric situation

this will be a uniformly most powerful rank order test. When the alternative is simple, (4.2) is always possible and the resulting procedure is a most powerful test.

The above situation is ideal and seldom will occur except for very small J. The least favorable situation for constructing optimum procedures occurs when it is impossible to find any ordering relationships between the probabilities of the points of the finite space that will hold for all of the alternatives under consideration. Such a situation will be described in section 5. The following is more common and examples of it will be given in sections 6 and 7.

The points z^i can be divided into three disjoint sets B_i (which depend on K) such that

1. z^i is in B_1 if there are at least J - [K] other points always less probable than z^i.
2. z^i is in B_3 if there are at least [K] + 1 (K if K is an integer) other points always more probable than z^i.
3. z^i is in B_2 if it is not in B_1 or B_3.

It is clear that the following condition must hold for an admissable test. If for all alternative hypotheses under consideration

$$P(Z=z^i) > P(Z=z^j) \tag{4.4}$$

then $a_j > 0$ implies $a_i = 1$. Making use of this we have

1. $a_i = 1$ if z^i is in B_1. If z^i is in B_1 then the total of the probabilities assigned to points other than the points always less probable than z^i is at most $J-(J-[K])=[K]$. So that if in particular $a_i<1$ it would be necessary for a point always less probable than z^i to have an $a > 0$ which would not be compatible with admissability.

2. $a_i = 0$ if z^i is in B_3. If z^i is in B_3 and $a_i > 0$ a more powerful test can be obtained by distributing the probability assigned to z^i to some of the points which are always more probable. This can be done for each of the points in B_3 so that finally if z^i is in B_3, admissability requires $a_i = 0$.

3. Admissability will not generally determine the a's for points in B_2. However, the assignment of the a's in B_2 must be consistent with the admissability criterion arising from (4.4). When B_2 contains exactly one point, probability of $K-[K]$ should be assigned to it and the resulting test will be uniformly most powerful. When B_2 is empty the resulting test is uniformly most powerful.

Consideration of order relationships between the probabilities of the points has given criteria for admissable tests and under special circumstances uniformly most powerful tests. In order to construct tests with further optimum properties, it is necessary to consider the actual dependence of the probabilities on the alternatives. In section 7 we will treat a case where the functional form of the probabilities of the points (rank orders) can be given explicitly. For that situation, depending on the sample sizes, it will be possible to construct uniformly most powerful, most stringent, or locally most powerful rank order tests.

5. <u>Slippage alternatives</u>. In this section we consider the alternatives H_S, H_T, H_{TS}, and H_{TSU} introduced in section 3. Admissable and other optimum tests will not be constructed. Instead several examples will be given indicating that the class of admissable tests is so large it is unlikely that uniformly most powerful or related optimum tests exist. This does not mean that there do not exist tests of these hypotheses with some optimum properties. For instance, there exist unbiased tests of these hypotheses [Lehmann, 1951]. However, there is no evidence that Lehmann's procedure is the best unbiased test.

A reasonable conjecture appears to be that I < II (the first sample is less than the second) is the most probable rank

order under H_S: $F(x) \geq G(x)$. This is the case in the extreme situation where there exists an x_0 such that $F(x_0)[1-G(x_0)] = 1$. In section 6, it will be shown that I < II is the most probable rank order when two samples are taken from normal populations which are the same except that the mean of the second is larger than that of the first. Other statistically important examples will be given showing that H_S is compatible with I < II being the most probable rank order.

However, H_S is not always sufficient to insure that I < II is the most probable rank order. In fact it will be shown by example 1 that even under H_{TSU}, I < II need not be the most probable rank order. Here it should be recalled that H_{TSU} is $G(x) \equiv F(x-\theta)$, where $\theta > 0$, $F(x) + F(-x) \equiv 1$, and $F(a+c) - F(a) \geq F(b+c) - F(b)$ where $b > a > 0$ and $c > 0$.

Example 1. Let

$$(5.1) \qquad f(x) = \begin{cases} 0 & x < -5/2 \\ \theta/2 & -5/2 \leq x < -1/2 \\ 1-2\theta & -1/2 \leq x < 1/2 \\ \theta/2 & 1/2 \leq x < 5/2 \\ 0 & 5/2 \leq x \end{cases}, \qquad (\theta < 2/5)$$

and $g(x) \equiv f(x-1)$.

Let A be the rank order in which all of the observations from the first sample are less than all of the observations from

the second sample except that there is one observation from the first sample larger than all of the other observations. Thus in the case that $m = 4$, $n = 2$, A is the rank order 000110. The result will be proved by showing that for some θ, m, and n

$$P(A) > P(I < II) \quad . \tag{5.2}$$

Let B be the event that all of the observations from the second sample are in the interval (1/2, 3/2), and $\overline{B}$ the complement of this event. Let C_i be the event that m-i observations from the first sample are less than 1/2 and the remaining i observations from the first sample are in the interval (1/2, 3/2). Let D_i be the event that m-i observations from the first sample are less than 1/2, that i-1 observations from the first sample are in the interval (1/2, 3/2) and that one observation from the first sample is in the interval (3/2, 5/2).

With this notation it follows that

$$P(A) - P(I < II) = P(B)[P(A|B) - P(I < II|B)] + P(\overline{B})[P(A|\overline{B}) - P(I < II|\overline{B})]$$

$$= P(B)\left\{\sum_{i=1}^{m}[P(AC_i|B) + P(AD_i|B)] - \sum_{i=0}^{m} P(I<II \cdot C_i|B)\right\} + P(\overline{B})[P(A|\overline{B}) - P(I < II|\overline{B})] \quad . \tag{5.3}$$

It is clear that

a) $P(B) = (1-2\theta)^n$

b) $P(\overline{B}) = 1 - (1-2\theta)^n$

c) $P(AC_i|B) = P(I < II \cdot C_i|B) \quad i=1, \cdots, m$

d) $P(I < II \cdot C_0|B) = (1-\theta)^m$

e) $P(AD_1|B) = m\theta(1-\theta)^{m-1}/2$

f) $P(AD_i|B) > 0 \quad i=2, \cdots, m$.

Hence

$$P(A) - P(I < II) > (1-2\theta)^n \left[\frac{m\theta}{2}(1-\theta)^{m-1} - (1-\theta)^m\right] + [1-(1-2\theta)^n][P(A|\overline{B}) - P(I < II|\overline{B})] . \tag{5.4}$$

Now let $m \to \infty$, $\theta = k/m$ and n remains fixed, then

a) $(1-2\theta)^n \to 1$

b) $1 - (1-2\theta)^n \to 0$

c) $(1-\theta)^m \to (1-\theta)^{m-1} \to e^{-k}$

and thus

$$P(A) - P(I < II) > e^{-k}\left(\frac{k}{2} - 1\right) . \tag{5.5}$$

Hence for sufficiently large m and k > 2 the desired result is obtained.

While the above example covers all of the slippage alternatives it is only for large m. A counter example against H_S which holds for small m and n is

Example 2. Let

$$(5.6) \qquad f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x < \varepsilon \\ 0 & \varepsilon \leq x < 2\varepsilon \\ 1 & 2\varepsilon \leq x < 1 + \varepsilon \\ 0 & 1 + \varepsilon \leq x \end{cases}$$

and

$$(5.7) \qquad g(x) = \begin{cases} 0 & x < \varepsilon \\ 1 & \varepsilon \leq x < 1 + \varepsilon \\ 0 & 1 + \varepsilon \leq x \end{cases} .$$

Then

$$(5.8) \qquad P(I \; a< II) - P(I < II) = \varepsilon^{m}(mn[1-\varepsilon]^{n}-1) > 0 ,$$

so long as $\varepsilon < 1 - (mn)^{-1/n}$. When n = 1, this difference is maximized if $\varepsilon = \frac{m-1}{m+1}$ in which case

$$(5.9) \qquad P(Ia<II) - P(I<II) = \left(\frac{m-1}{m+1}\right)^{m+1} .$$

[Note: Theorem 5.1 shows that this last result is actually the best possible.]

Using the same distributions and letting $m = n = 2$, the following is obtained

$$
(5.10) \quad \begin{cases}
P(I<II) = P(0011) \\
\qquad = \varepsilon^4 + 2\varepsilon^3(1-\varepsilon) + \varepsilon^2(1-\varepsilon)^2 + R_\varepsilon \\
P(Ia<II) = P(0101) \\
\qquad = 2\varepsilon^2(1-\varepsilon)^2 + R_\varepsilon \\
P(0110) = 2\varepsilon^3(1-\varepsilon) + 2\varepsilon^2(1-\varepsilon)^2 + R_\varepsilon \\
P(1001) = R_\varepsilon \\
P(IIa<I) = P(1010) = R_\varepsilon \\
P(II<I) = P(1100) = \varepsilon^2(1-\varepsilon)^2 + R_\varepsilon
\end{cases}
$$

where $R_\varepsilon = 2\varepsilon(1-\varepsilon)^3/3 + (1-\varepsilon)^4/4$.

Now then for all ε the intuitively least probable rank order II < I has a greater probability than the rank orders 1001 and 1010. However each rank order beginning with 1 is less probable than any rank order beginning with 0. Also

P(0110) > P(0101) for all ε. Finally

$$(5.11)\quad \begin{cases} P(0110) - P(0011) = \varepsilon^2(1-2\varepsilon) \\ \\ P(0101) - P(0011) = \varepsilon^2(2\varepsilon^2 - 4\varepsilon + 1)\ . \end{cases}$$

The first of these differences is greater than 0 provided $\varepsilon < \frac{1}{2}$ and the second difference is greater than 0 provided $\varepsilon < 1 - 1/\sqrt{2}$.

<u>Theorem</u> 5.1. If n = 1, F(x) ≥ G(x), then

$$P(I<II) - P(Ia<II) \geq -\left(\frac{m-1}{m+1}\right)^{m+1} .$$

<u>Proof</u>. Let Δ = P(I<II) - P(Ia<II). Then

$$(5.12)\qquad \Delta = \int [(m+1)F^m - mF^{m-1}]dG \quad ,$$

and integration by parts yields

$$(5.13)\qquad \Delta = \Big[(m+1)F^m G - mF^{m-1}G\Big]_{-\infty}^{\infty} - \int SdF = 1 - \int SdF \quad ,$$

where

$$(5.14)\qquad S = mGF^{m-2}[(m+1)F - (m-1)] \quad .$$

Dividing the range of integration into the parts where S is positive and negative and making the change of variables F = F(x), we obtain

$$\Delta = 1 - \int_0^{\frac{m-1}{m+1}} SdF - \int_{\frac{m-1}{m+1}}^1 SdF \geq 1 - \int_{\frac{m-1}{m+1}}^1 SdF$$

(5.15)

$$\geq 1 - \int_{\frac{m-1}{m+1}}^1 mF^{m-1}[(m+1)F - (m-1)]dF \quad ,$$

since $F \geq G$. Hence ,

(5.16) $$\Delta \geq - \left(\frac{m-1}{m+1}\right)^{m+1} \quad .$$

The next three theorems show that under the slippage alternatives there exists an ordering of the probabilities of rank orders for very special combinations of sample sizes. As would be expected it is possible to prove such an ordering for a larger range of sample sizes for the hypotheses with more restrictive assumptions.

<u>Theorem</u> 5.2. Under H_S, i.e., $F(x) \geq G(x)$, where the inequality holds for some x, and $m = n = 1$,

$$P(I<II) > \tfrac{1}{2} > P(II<I) \quad .$$

<u>Proof</u>.

(5.17)
$$\begin{aligned} P(I<II) &= \int FdG \\ &> \int GdG \\ &= \tfrac{1}{2} \quad . \end{aligned}$$

Theorem 5.3. Under H_{TS}, i.e., $G(x) \equiv F(x-\theta)$; where $\theta > 0$, $F(x) + F(-x) \equiv 1$, and $m = 2$, $n = 1$,

$$P(I<II) > P(Ia<II) > P(II<I)$$

Proof.

$$
\begin{aligned}
P(I<II) &= \int F^2 \, dG \\
&= \int F^2(x)\, dF(x-\theta) \\
&= \int F^2(\theta-x)\, dF(-x) \\
&= \int [1-F(x-\theta)]^2 dF(x) \qquad (5.18) \\
&= 2\int F(x)[1-F(x-\theta)]dF(x-\theta) \\
&> 2\int F(1-F)\, dG \\
&= P(I\ a<\ II)
\end{aligned}
$$

and $P(I\ a< II) > P(II < I)$ is proved using a similar transformation and integration by parts.

Theorem 5.4. Under H_{TSU}, i.e., $G(x) \equiv F(x-\theta)$, $\theta>0$; $F(x) + F(-x) \equiv 1$; $F(a+c)-F(a) \geq F(b+c)-F(b)$, where $b>a>0$ and $c>0$, and $m=3$, $n=1$,

$$P(0001) > P(0010) > P(0100) > P(1000) \quad .$$

Proof.

$$\text{Let } \Delta = P(0001) - P(0010)$$

then

$$\Delta = \int F^3 \, dG - 3\int F^2 \, (1-F)\, dG$$

(5.19)
$$= \int (4F^3 - 3F^2)\, dG \quad .$$

Integrating by parts we obtain

$$\Delta = \Big[(4F-3)F^2G\Big]_{-\infty}^{\infty} - 6\int GF(2F-1)\,dF$$

(5.20) $$= 1 - 6\int GF(2F-1)\,dF \quad .$$

then using

(5.21) $$1 = 6\int F^2(2F-1)\,dF \quad ,$$

we have that

(5.22) $$\Delta = 6\int (F-G)F(2F-1)\,dF \quad .$$

To show that $\Delta > 0$ it is sufficient to note the following

(a) $dF(x) = dF(-x)$

(b) $[2F(x)-1] = -[2F(-x)-1] > 0, \; x > 0$

(c) $F(x) > F(-x), \; x > 0$

(d) $F(x) - G(x) > F(-x) - G(-x), \; x > 0$.

The second inequality is proved as follows:

$$P(0010) - P(0100) = 3\int [F^2(1-F) - F(1-F)^2]\,dG$$

(5.23) $$= 3\int F(1-F)(2F-1)\,dG$$

$$> 0$$

since $F(1-F)(2F-1)$ is an odd function ≥ 0 when $x > 0$ and

$dG(x) - dG(-x) > 0$ when $x > 0$. Finally, $P(0100) > P(1000)$ is proved like the first inequality.

Although the assumptions of the slippage hypotheses are not sufficient to show that I < II is the most probable rank order, the assumptions of H_S and hence all of the others do show that I < II is more probable after slippage than before. This is given formally in

<u>Theorem</u> 5.5. If $F(x) \geq G(x)$ where the inequality holds for some x, then

$$P(I<II) > \binom{m+n}{n}^{-1}$$

<u>Proof</u>.

$$P(I<II) = \int nF^m (1-G)^{n-1} dG$$

$$> n\int G^m (1-G)^{n-1} dG \tag{5.24}$$

$$= \binom{m+n}{n}^{-1}.$$

Example 3 below illustrates a situation where H_S implies an ordering of the probabilities of the rank orders such as that of (4.2) and thus the construction of optimum rank order tests is possible for all combinations of sample sizes.

Example 3. Let $X_1, \dots, X_m$ be a sample from the rectangular distribution with range from 0 to 1, and let $Y_1, \dots, Y_n$ be an

independent sample from a rectangular distribution with range from 0 to L (where $L > 1$) then

1. The probability of a rank order depends on the length of the last run of 1's only.

2. The longer the last run of 1's the more probable is the rank order.

Let A stand for a specific rank order. Then

$$P(A) = \sum_{i=0}^{n} P(A | i \text{ of the } Y\text{'s} > 1)\, P(i \text{ of the } Y\text{'s} > 1)$$

$$(5.25) \qquad = \sum_{i=0}^{n} \binom{n}{i} L^{-n} (L-1)^{i} \binom{m+n-i}{m}^{-1} \Gamma(A) \quad ,$$

where $\Gamma(A) = 1$ if A can occur when there are as many as i of the Y's > than all of the X's, and otherwise $\Gamma(A) = 0$. From this the results are immediate.

Example 2 with theorem 4.2 shows that H_S is sufficient for I < II to be the most probable rank order only when $m = n = 1$. Also in this example when $m = n = 2$ we have further evidence that for these alternatives the methods of section 4 for constructing optimum tests are inadequate.

Example 1 with theorem 5.4 shows that H_{TSU} implies that I < II only for certain m and n. The example could also be used for showing that there are rank orders, other than the one treated,

that are sometimes more probable than I < II. Thus even for this more restrictive alternative it does not appear possible to apply the methods of section 4.

Example 3 is a situation under slippage where it is actually possible to construct the best test. The more common statistical situations will be discussed in the next two sections. For these cases it will turn out that the hypotheses induce a partial ordering of the probabilities of rank orders which are intermediate between the orderings given by the examples of this section. For the alternatives discussed in these latter sections the partial ordering will be adequate to give a useful criterion for the construction of admissable tests. Finally, in section 7 a case is treated where not only is it possible to construct various types of best tests but also their operating characteristics are given.

6. Koopman-Darmois alternatives. The density functions introduced in H_{KD}, i.e.,

$$A(x)B(\theta)\exp[C(x)D(\theta)] \tag{6.1}$$

were used by Koopman (1936) and Darmois (1935) in the study of sufficient statistics. These density functions have been used in nonparametric work by Lehmann and Stein [1949]. That these

density functions are important to statistics, results from the fact that they include as special cases the normal, chi-square, exponential, and extreme-value distributions. In this section it is shown that for this type of alternative it is possible to give an easily applied criterion for the admissability of rank order tests.

Theorem 6.1. If the random variables $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ are mutually independent and the X's have the density function

$$A(x)B(\theta_1) \exp[C(x)D(\theta_1)]$$

and the Y's have the density function

$$A(x)B(\theta_2) \exp[C(x)D(\theta_2)]$$

where $\theta_2 > \theta_1$ and $C(x)$ and $D(\theta)$ are increasing functions, then the rank order z is more probable than z' when

(1) $z_i = 1 - z'_i = 1 - z_j = z'_j = 0$ for $i < j$ and $z_k = z'_k$ for $i \neq k \neq j$, or equivalently

(2) the two rank orders are identical except for their i-th and j-th elements $(i<j)$, which are (0,1) for z and (1,0) for z'.

Proof. Define D as

(6.2) $$D = P(Z=z) - P(Z=z').$$

Then

(6.3) $$D = \int \cdots \int \Lambda \Gamma(x_i, x_j; \theta_1, \theta_2)\, dx_1 \cdots dx_{m+n}$$

where

(6.4) $$\Lambda = m!\,n!\,[B(\theta_1)]^m [B(\theta_2)]^n \left[\prod_{k=1}^{m+n} A(x_k)\right] \exp\left[\sum_{\substack{k=1 \\ i\neq k\neq j}}^{m+n} C(x_k) D(\theta_{1+z_k})\right]$$

and

(6.5) $$\Gamma(x_i, x_j; \theta_1, \theta_2) = \exp[C(x_i)D(\theta_1) + C(x_j)D(\theta_2)] - \exp[C(x_i)D(\theta_2) + C(x_j)D(\theta_1)].$$

If the integrand $\Lambda\Gamma$ is always non-negative the theorem will be proved. It is clear that $\Lambda \geq 0$ and so all that remains is to examine Γ. However,

(6.6) $$\Gamma(x_i, x_j; \theta_1, \theta_2) > 0$$

is equivalent to

(6.7) $$[D(\theta_2) - D(\theta_1)][C(x_j) - C(x_i)] > 0 .$$

But (6.7) follows from the fact that $C(x)$ and $D(\theta)$ are increasing and $x_j > x_i$ as well as $\theta_2 > \theta_1$.

The following corollary is a restatement of theorem 6.1, giving, an easily applied necessary criterion for admissability of rank order tests.

> Corollary 6.1. Admissable rank order tests of H_0 against H_{KD} must satisfy the following condition. If rank orders z and z' are related as in theorem 6.1 then, if the null hypothesis is rejected with non-zero probability when z' occurs, it must be rejected with probability one when z occurs.

Thus when m = n = 2 the rank order z = (0,1,0,1) must be put into the critical region with probability one before the rank order z' = (1,0,0,1) is put into the critical region with non-zero probability. In the equal sample case, the one-sided Smirnov test [Kolmogorov, 1941] is based on large values of the statistic

$$\text{(6.8)} \qquad \underset{1 \leq i \leq m+n}{\text{Maximum}} (i - 2v_i)$$

where it should be recalled that

$$\text{(6.9)} \qquad v_i = \sum_{j=1}^{i} z_j$$

or v_i is the number of observations from the second sample amongst the i smallest observations of the combined sample. However,

for the two rank orders just mentioned the Smirnov statistic has the same value, i.e., 1. Thus, the Smirnov procedure could lead to the use of inadmissable tests of H_0 against H_{KD}.

Many procedures proposed for testing H_0 against H_{KD} are based on statistics of the form

$$\sum_{i=1}^{m+n} c_i z_i \tag{6.10}$$

where the c_i's are an increasing (decreasing) sequence and large (small) values of (6.10) are critical. Some typical examples of this are:

1. The Wilcoxon statistic [Mann and Whitney, 1947] where $c_i = i$ is an increasing sequence.
2. The c_1 statistic [Terry, 1952] where the coefficients c_i = the expected value of the i-th order statistic in a sample of m+n observations from the standardized normal distribution is an increasing sequence.
3. The T statistic (introduced in section 7) where $c_i = \sum_{j=i}^{m+n} 1/j$ is a decreasing sequence.

Statistics of the form (6.10) satisfy the admissability criterion of corollary 6.1 for if rank orders z and z' are in the desired relationship, the difference in the corresponding values

of the statistic will be $c_j - c_i$ which is positive (negative) when large (small) values are critical.

In the next section we study a subclass of the alternatives H_{KD}. For this subclass we will find that it is possible to construct tests with further optimum properties than admissability.

7. <u>Lehmann Alternatives</u>. Alternatives of the form H_L were introduced by Lehmann [1953] in order to study nonparametric procedures when the alternatives themselves are given in a non-parametric form. In this section we continue the study of these alternatives and show that for them it is possible to construct optimum critical regions of various types. The H_L alternatives are of statistical interest since they include the extreme-value and exponential distributions as was pointed out in section 3.

7.a. <u>General Formulas</u>. One of the reasons why the non-parametric treatment of the H_L alternatives can be so complete from the Neyman-Pearson point of view is that it is possible to give in explicit form the probabilities of the rank orders. This will be done in Corollary 7.a.1.

<u>Theorem</u> 7.a.1. If the random variables $X_1, \ldots, X_N$ are mutually independent and X_i has the cumulative distribution function $[H(x)]^{\Delta_i}$ where $\Delta_i > 0$ and $H(x)$ is a continuous

distribution function then

$$P(X_1<X_2<\cdots<X_{N-1}<X_N) = (\prod_{i=1}^{N} \Delta_i)/\prod_{i=1}^{N} (\sum_{j=1}^{i} \Delta_j) .$$

By a proper numbering of the X's the probability of any ordering can be found.

Proof. Let

(7.a.1) $$P = P(X_1<X_2< \cdots < X_{N-1}<X_N).$$

Then

(7.a.2) $$P = \int \cdots \int \prod_{i=1}^{N} d[H(x_i)]^{\Delta_i} .$$

Making the change of variables

(7.a.3) $$y_i = H(x_i) \qquad (i=1, \cdots, N) ,$$

we have

(7.a.4) $$P = \int_{0\leq y_1\leq\cdots\leq y_N\leq 1} \cdots \int \prod_{i=1}^{N} d(y_i)^{\Delta_i}$$

$$= (\prod_{i=1}^{N} \Delta_i) \int_{0\leq y_1\leq\cdots\leq y_N\leq 1} \cdots \int \prod_{i=1}^{N} (y_i^{\Delta_i-1} dy_i)$$

$$= (\prod_{i=1}^{N} \Delta_i)/\prod_{i=1}^{N} (\sum_{j=1}^{i} \Delta_j) .$$

The following corollary is equivalent to equation 4.5 of Lehmann [1953].

Corollary 7.a.1. Under H_L the probability of a rank order z is

$$m!n!\, \Delta_1^m \Delta_2^n / \prod_{i=1}^{m+n} \left(\sum_{j=1}^{i} [(1-z_j)\Delta_1 + z_j\Delta_2] \right)$$

or

$$m!n!\, \delta^n / \prod_{i=1}^{m+n} (u_i + v_i\delta)$$

where $\delta = \Delta_2/\Delta_1$.

The quantity $\prod_{i=1}^{m+n} (u_i+\delta v_i)$ occurring in corollary 7.a.1 is a polynomial in δ whose coefficients depend on the rank order z. For convenience denote this polynomial by $f_z(\delta)$. The non-zero coefficients of $f_z(\delta)$ are positive integers. Using $u_i + v_i = i$ and setting $\delta = 1$, the sum of the coefficients is $(m+n)!$ If $\underline{r} = \min(r_1, \ldots, r_m)$, i.e. $\underline{r}$ is the rank of the smallest observation from the first sample, then the smallest power of δ with a non-zero coefficient is $\underline{r}-1$. In particular if $z_1 = 0$ the polynomial has a constant term. If $\underline{s} = \min(s_1, \ldots, s_n)$, i.e. $\underline{s}$ is the rank of the smallest observation from the second sample, then the largest power of δ with a non-zero coefficient is $m+n-\underline{s}+1$. In particular if $z_1 = 1$ the polynomial is of degree m+n. If $I = \max(\underline{r}, \underline{s})$, the coefficient of $\delta^{\underline{r}-1}$ is $(I-1)! \prod_{i=I}^{m+n} u_i$ and the coefficient

of $\delta^{m+n-\underline{s}+1}$ is $(I-1)! \prod_{i=I}^{m+n} v_i$. All of the non-zero coefficients of $f_z(\delta)$ are $\geq m!$.

Let z be a rank order for sample sizes m and n and $z^0(z^1)$ be a rank order for sample sizes m+1 and n(m and n+1) such that the first m+n elements of $z^0(z^1)$ are the same as the elements of z and the (m+n+1) element of $z^0(z^1)$ is a 0(1) then

$$(7.a.5) \quad \begin{cases} f_{z^0}(\delta) = [(m+1) + n\delta]\, f_z(\delta) \\ \\ (f_{z^1}(\delta) = [m+(n+1)\delta] f_z(\delta)) \end{cases} .$$

When two rank orders z and z' are identical except in their k-th and k+1-st elements which are (0,1) for z and (1,0) for z', then we have the following relationship between their probabilities

$$(7.a.6) \qquad P(Z=z) = \frac{(u_i+\delta v_i+\delta-1)}{(u_i+\delta v_i)} P(Z=z') ,$$

where u_i and v_i are computed for z. The probability of I < II, all of the first sample less than the second, is

$$(7.a.7) \qquad P(I<II) = n!\delta^n \Big/ \prod_{i=1}^{n} (m+i\delta)$$

and the probability of II < I, all of the second sample less than the first, is

(7.a.8) $$P(II<I) = m!/\prod_{i=1}^{m}(i+n\delta) .$$

7.b. <u>Simple Alternatives</u>. If one wishes to test H_0 against H_L for a specific value of $\delta = \Delta_2/\Delta_1$, then the procedure is straight forward. The probabilities of the rank orders are computed. Then the rank orders are arranged so that (4.2) is satisfied. The test is constructed in the manner indicated by (4.3).

Table IIa gives values of the probabilities of some of the rank orders for all combinations of sample sizes such that max $(m,n) \leq 5$ for certain specific alternatives, i.e., $\delta = \Delta_2/\Delta_1$. The values of δ used were taken from table Ia which gives the alternatives required to obtain desired power at specified levels of significance using the best parametric test of H_0 against H_E. The powers of the resulting best tests are given in table IVa. For the values computed the power of the nonparametric procedures is substantially below that of the corresponding best parametric procedures. This is explained by the following:

1. Unless a nonparametric test is equivalent to the best parametric test it must be less powerful. They are not equivalent in this case.

2. When the sample sizes are small there are only a few rank orders, and if the level of significance is small, the test will consist in rejecting H_0 with some small probability ε if a particular (possibly several) rank order(s) is observed. Under these conditions no rank order test has power greater than ε.

3. The best parametric procedure is based on an F-statistic with 2m and 2n degrees of freedom. This is the same statistic that would be used for testing the equality of variances for samples of size 2m and 2n from normal distributions with known mean. But if there were 2m and 2n observations, the corresponding nonparametric test would not be the same and would be more powerful.

Although this section was entitled "simple alternatives" it should be noted that in fact the alternatives considered were composite since H(x) was not specified and the parameters Δ_1 and Δ_2 were only specified by their ratio. The distinction between "simple" as used in this section and "composite" as used in the next section lies in whether δ is specified or not.

Finally it should be noted that table IIa was prepared by first computing the probability of the rank order $I < II$ by use of (7.a.7) and the remaining values were obtained by the use of (7.a.6). The tables were checked by recomputing $P(I<II)$ and by computing directly some of the probabilities of the other rank orders making use of corollary 7.a.1. In those cases where the probabilities were computed for all of the rank orders an additional check was obtained by seeing if the probabilities summed to one.

7.c. Composite alternatives. In this section optimum tests of H_0 against H_L where Δ_1 and Δ_2 are restricted only by $\delta = \Delta_2/\Delta_1 > 1$ are considered. Corollary 6.1 gives an easily applied criteria for admissability. It will be possible in this section to go further and find more details about the structure of the sets B_i introduced in section 4. The following two theorems give simple criteria which when applicable determine when one rank order is sometimes more probable than another. The first of these theorems considers small values of δ and the second theorem considers large values of δ. If for a pair of rank orders one of these theorems shows that for some δ the first rank order is more probable than the second, and the other theorem shows that for some other δ the second is

more probable than the first, then it is clear that the choosing of probabilities for putting these rank orders into the critical region must be done carefully unless some other criteria has already determined these probabilities.

The statistic T(z) or simply T defined as

$$(7.c.1) \qquad T(z) = \sum_{i=1}^{m+n} v_i/i$$

will be used in the next theorem. T(z) will be the center of discussion of the remaining subsections.

<u>Theorem</u> 7.c.1. Under H_L, if $T(z) < T(z')$, then there exists a δ say δ^*, such that $\delta^* > 1$ and for δ in the interval $(1,\delta^*)$ the probability of z is greater than the probability of z'.

<u>Proof</u>. Let

$$(7.c.2) \qquad D = P(Z=z) - P(Z=z') .$$

Then making use of corollary 7.a.1 we have

$$(7.c.3) \qquad D = m!n!\delta^n \left\{ [\prod_{i=1}^{m+n} (u_i+\delta v_i)]^{-1} - [\prod_{i=1}^{m+n} (u_i'+\delta v_i')]^{-1} \right\} .$$

Now using $u_i + v_i = u_i' + v_i' = i$ and rearranging (7.c.3), we have

$$(7.c.4) \qquad D = \frac{m!n!\delta^n}{(m+n)!} \left\{ \left[\prod_{i=1}^{m+n} \left(1 + \frac{(\delta-1)v_i}{i}\right) \right]^{-1} - \left[\prod_{i=1}^{m+n} \left(1 + \frac{(\delta-1)v_i'}{i}\right) \right]^{-1} \right\} .$$

The braces in (7.c.4) can be expanded in a Taylor series about $\delta = 1$ with remainder if $1 \leq \delta < 2$ and we obtain

$$(7.c.5) \qquad D = \frac{m!n!\delta^n}{(m+n)!} (\delta-1)[T(z') - T(z)] + O[(\delta-1)^2] .$$

The theorem then follows from (7.c.5) since there must exist a δ^* which will make the remainder term smaller than the principal term.

In subsection 7.e. there will be a discussion of the size of δ^*-1 when m and n are large. It will be indicated that this difference is of the order of magnitude $(m+n)^{-\frac{1}{2}}$ which is the same as that introduced in theorem 3.2.

Theorem 7.c.2. Under H_L the rank order z will be more probable than the rank order z' for sufficiently large δ if $\underline{s} > \underline{s}'$ or if $\underline{s} = \underline{s}'$ and

$$(I-1)! \prod_{i=I}^{m+n} v_i < (I'-1)! \prod_{i=I'}^{m+n} v_i' .$$

[Note: For the meaning of the above symbols see the discussion following corollary 7.a.1.]

Proof. The conclusion follows immediately from the discussion after corollary 7.a.1 since the coefficient of the term of highest degree of a polynomial dominates its behavior for large values of the argument.

Thus in order for the rank order z to always be more probable than the rank order z' under H_L, it is necessary that $\underline{s}$ and $\underline{s}'$ satisfy the conditions of theorem 7.c.2. When this is the case the necessary and sufficient condition for z to be more probable than z' is that the polynomial

$$f_{z,z'}(\delta) = f_{z'}(\delta) - f_z(\delta) \tag{7.c.6}$$

has no (real) roots larger than 1. This results from the fact that the condition on (7.c.6) is equivalent to the denominator of the formula for the probability of z being less than the denominator of the formula for the probability of z', where these formulas are given in corollary 7.a.1.

Several simple criteria were applied to determine whether (7.c.6) had no roots larger than 1. (1) If the sequence of coefficients of $f_{z,z'}(\delta)$ has only one change in sign it follows that the polynomial has only one root larger than 0. However, it has a root at 1 since $f_z(1) = f_{z'}(1) = (m+n)!$ and hence no roots larger than 1. (2) If the sequence of coefficients of the polynomial has two changes of sign and the polynomial has a

positive derivative at 1, it can have no roots larger than 1. It has at most two roots greater than 0. One of these roots is at 1 and the function is positive for large values of δ. Thus if it had any roots between 1 and ∞, it would have at least two which would be a contradiction. (3) If the partial sums of the sequence of coefficients of $f_{z,z'}(\delta)$ are all non-positive, then it has no roots larger than 1 (see Conkwright, 1941, pages 40-41).

The following figure gives relationships between probabilities of rank orders. The numbers in the figure are the numbers assigned to the rank orders in table IIIa. If for $i < j$ it is possible to connect i and j by a sequence of ascending segments, i.e., segments connecting a smaller to a larger number, then the rank order with number i is always (under H_L) more probable than the rank order with the number j. If this is not possible, rank order i is more probable than j for some δ's, and rank order i is less probable than j for other δ's.

The diagrams were drawn using the criteria given by corollary 6.1 and (7.c.6). If a line connecting i and j ($i<j$) is unmarked then it was obtained by making use of corollary 6.1 or the criteria (1) and (3) given above. When this is so it is possible to state that rank order j is always more probable than rank order i when $\delta < 1$. This information is of use if we are to

consider as alternatives $\delta < 1$ (which would occur in the two-sided testing situation) or in the construction of the corresponding diagrams for $m > n$. The lines carrying check marks were obtained making use of (2) and do not necessarily have this reversible property.

When the diagram corresponding to a particular combination of sample sizes is in the form of a simple chain, the situation is that of (4.2) and it is possible to construct a uniformly most powerful rank order test for every level of significance. When $m = 1$, (4.2) is easily verified through the use of (7.a.6). The same situation holds for $m = 2$ and $n = 2,3,4$, or 5 so that for these combinations of sample sizes uniformly most powerful rank order tests of H_0 against H_L can be formed for every level of significance. In the case that $m = 2$ and $n = 5$ the checked line between rank orders 6 and 7 is reversible, i.e., the probability of rank order 7 is greater than the probability of rank order 6 when $\delta < 1$. However, the checked lines between 11 and 12, and between 15 and 16 are not reversible. Thus rank order 15 is more probable than rank order 16 if $0 < \delta < \frac{1}{2}$ and less probable if $\frac{1}{2} < \delta < 1$. Not all cases with $m = 2$ give a simple ordering; for instance $m = 2$, $n = 6$ (diagram not given).

Figure

Diagrams of partial orderings of probabilities of rank orders

m = 1   n = 1

1 —— 2

m = 1   n = 2

1 —— 2 —— 3

m = 1   n = 3

1 —— 2 —— 3 —— 4

m = 2   n = 2

1 —— 2 —— 3 —— 4 —— 5 —— 6

m = 1   n = 4

1 —— 2 —— 3 —— 4 —— 5

Figure , continued

m = 2   n = 3

1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 10

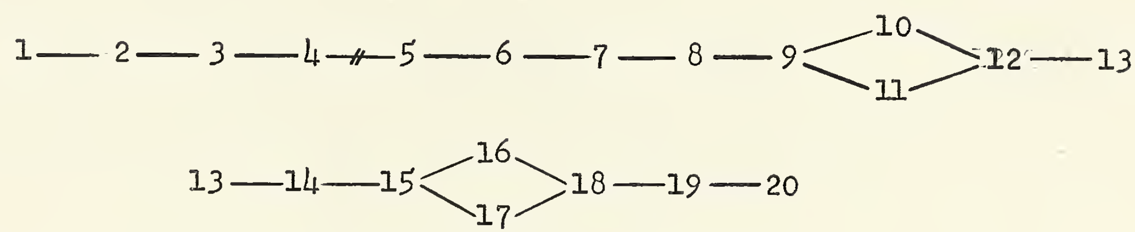
m = 1   n = 5

1 — 2 — 3 — 4 — 5 — 6

m = 2   n = 4

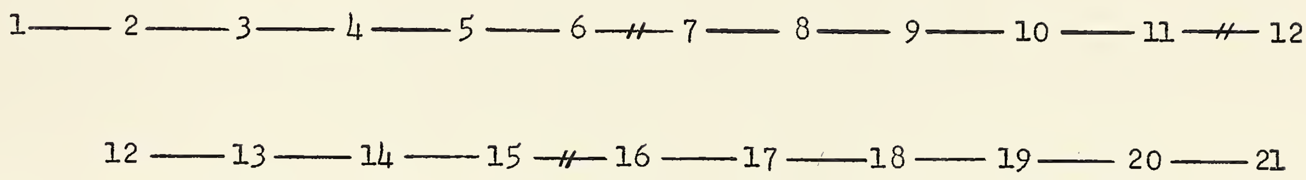
1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 10 — 11 — 12 — 13 — 14 — 15

m = 3   n = 3

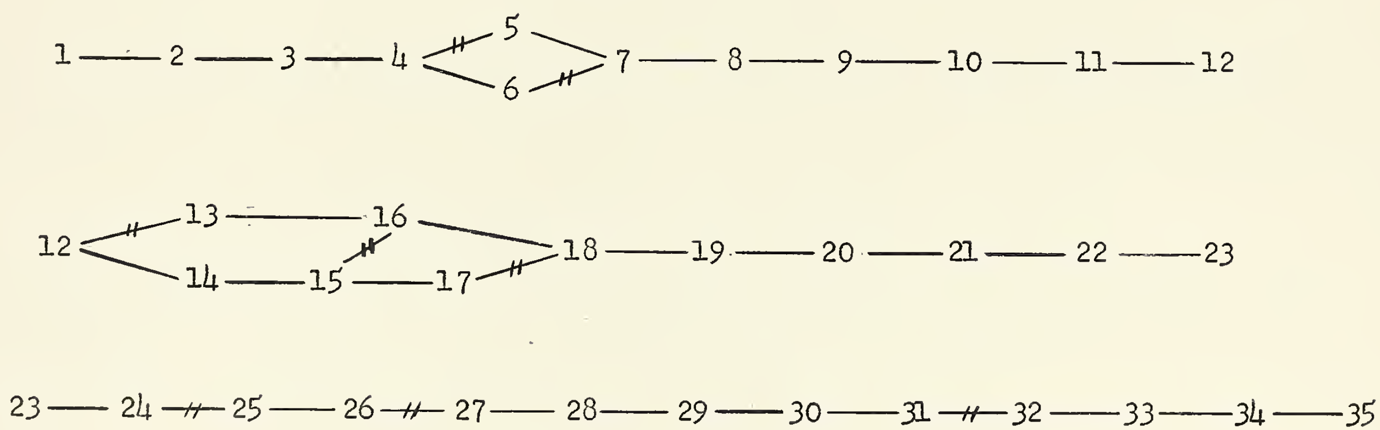


Figure, continued

m = 2 n = 5



m = 3 n = 4



Figure, continued

m = 3   n = 5

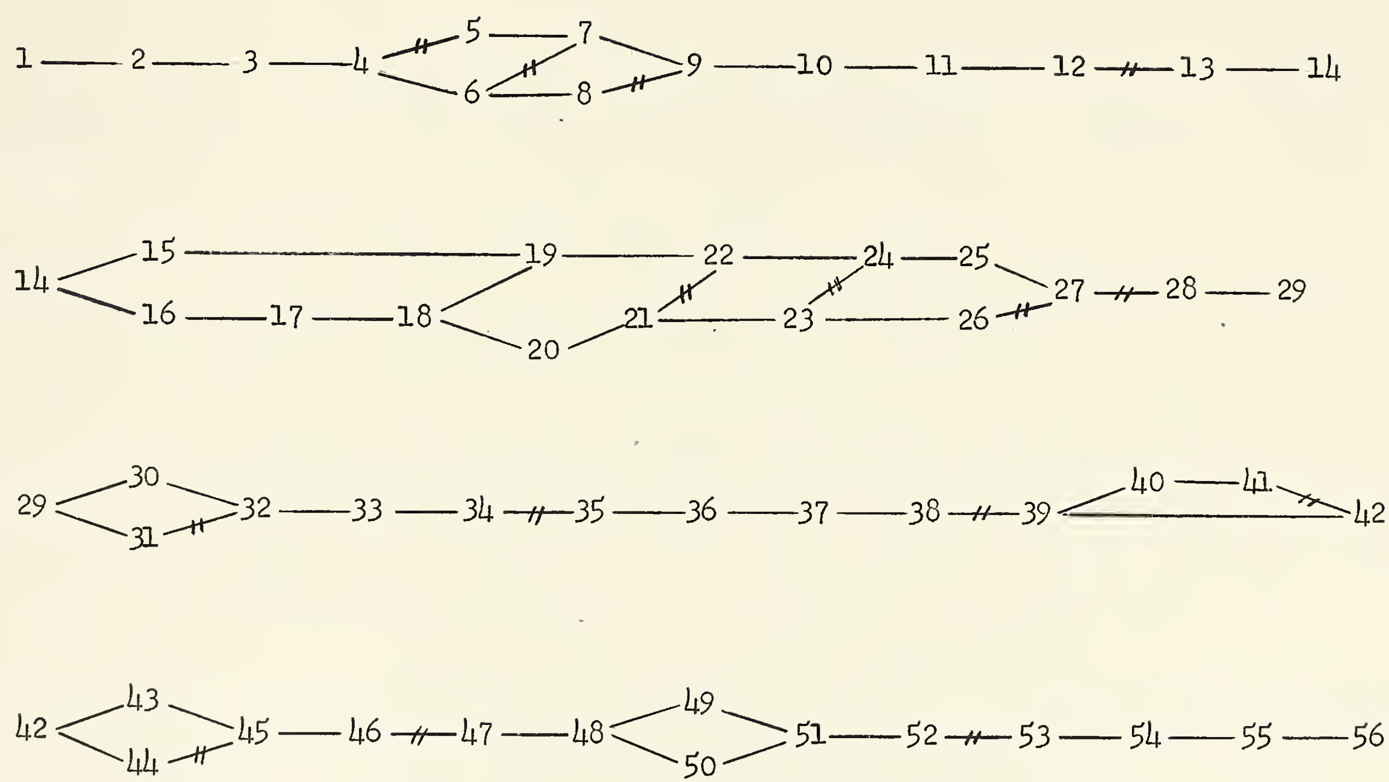


Figure , continued

m = 4   n = 4

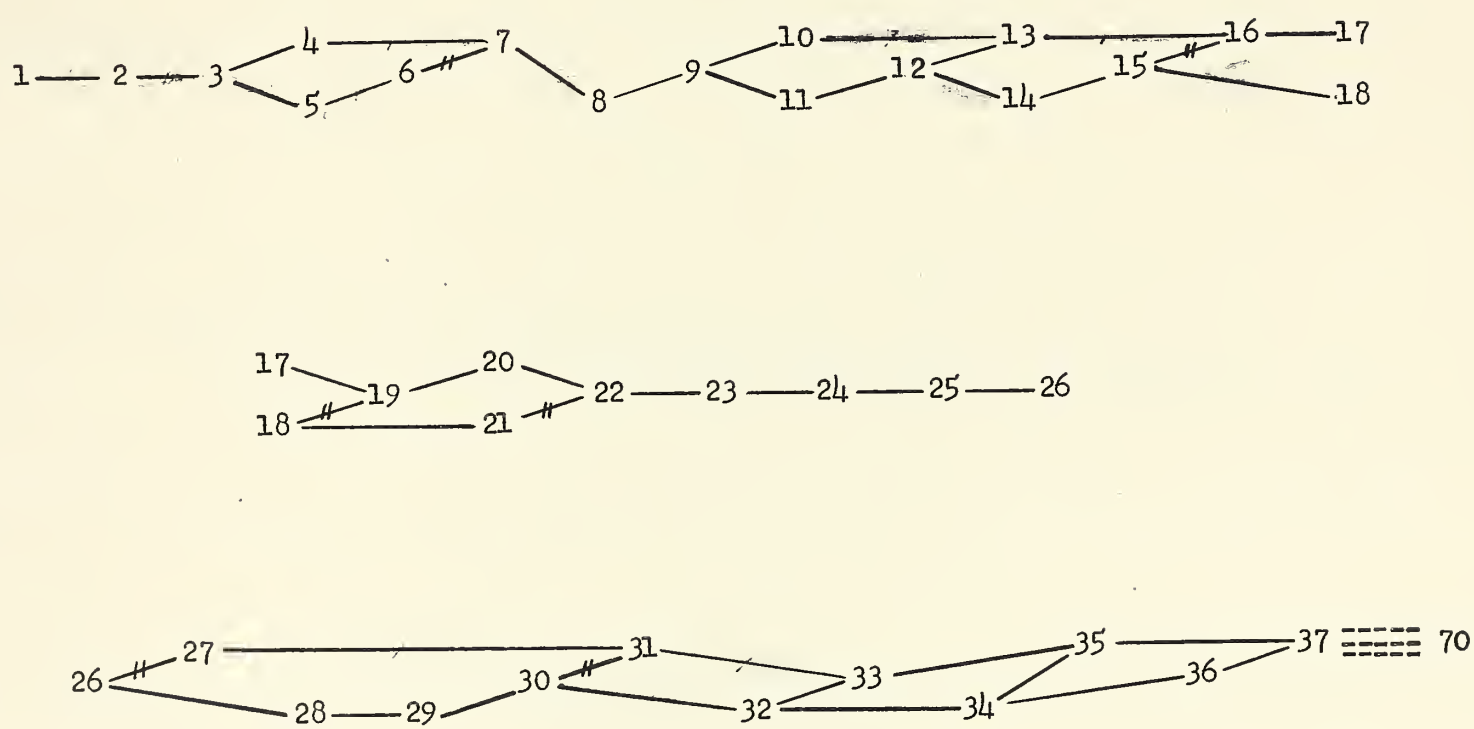


Figure , continued

m = 4   n = 5

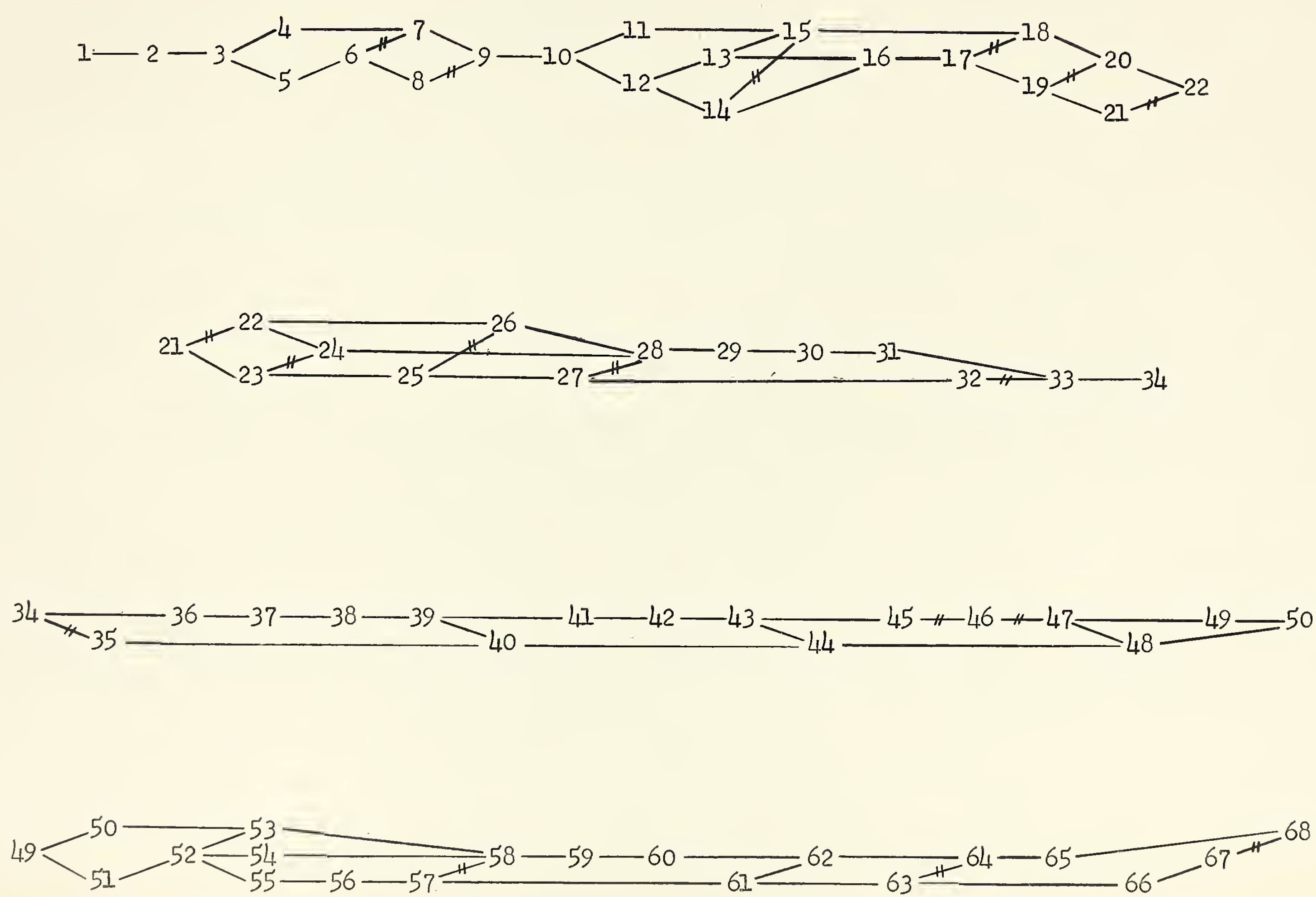


Figure , continued

m = 4   n = 5 (cont'd)

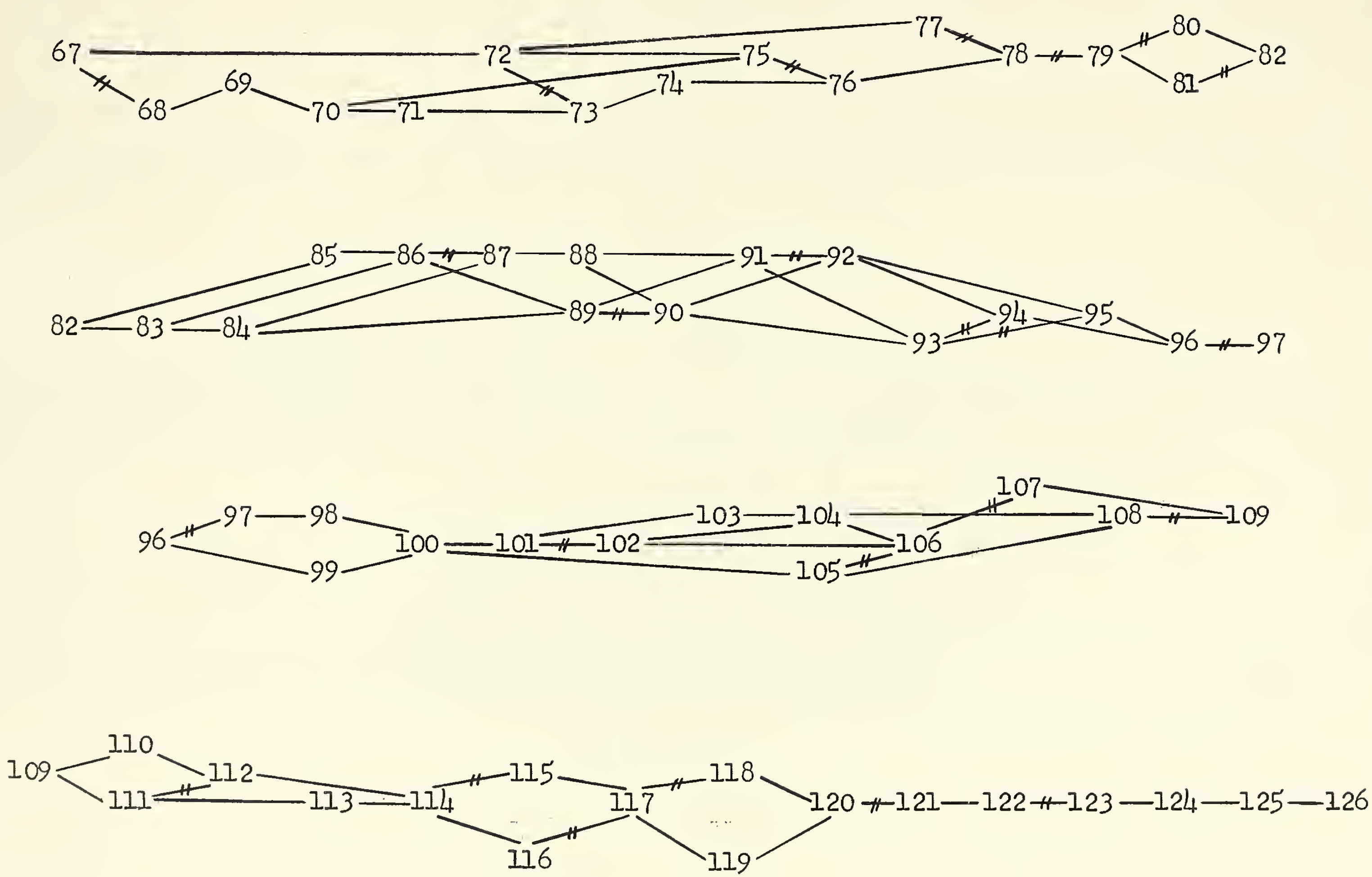
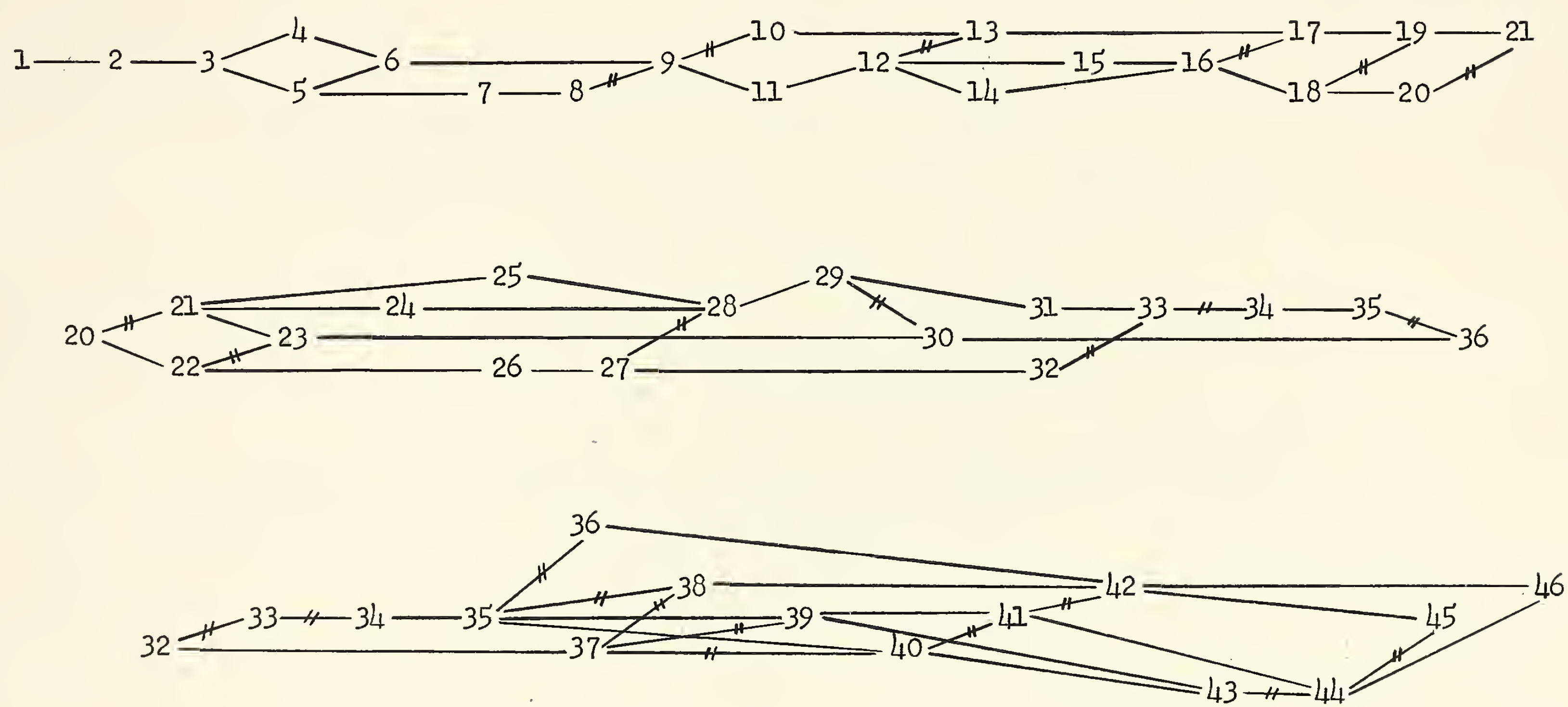


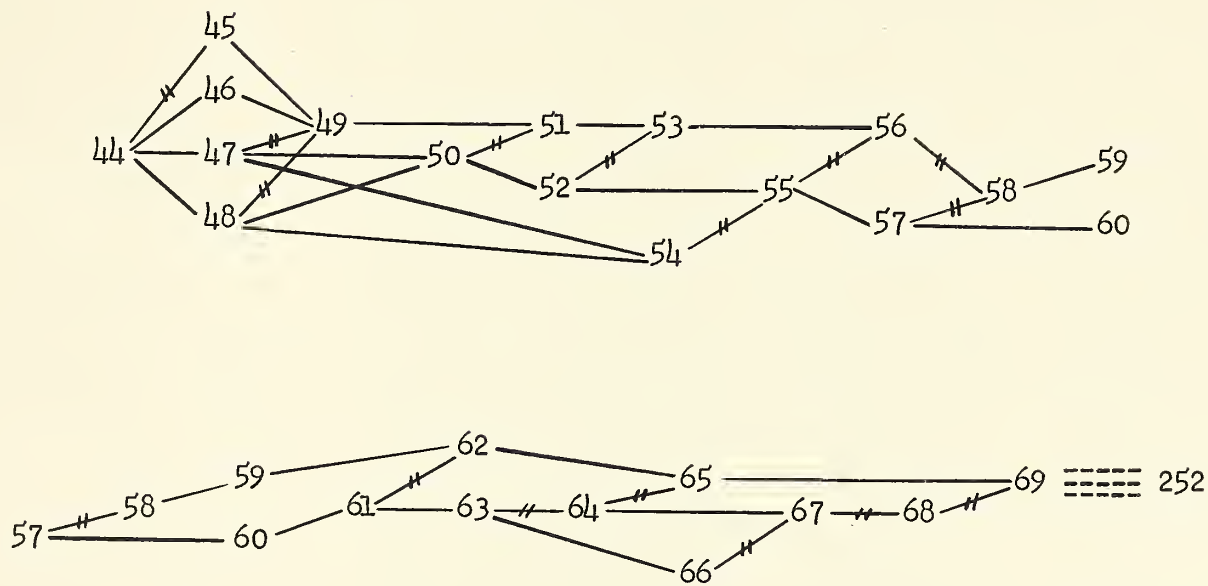
Figure , continued

m = 5   n = 5



Figure, continued

m = 5   n = 5 (cont'd)



The diagram for $m = n = 3$ is the least complicated one where there is not a simple ordering. In this case it would not be possible to construct uniformly most powerful rank order procedures for levels of significance in the intervals (.45, .55) and (.75, .85). Since these are unusual levels there would be no practical difficulty.

The diagram for $m = 3$ and $n = 4$ is like the above in that there does not exist a simple ordering, and for all of the usual levels of significance there exists a uniformly most powerful procedure.

The case of $m = 3$, $n = 5$ illustrates where the lack of simple ordering causes difficulty in finding optimum procedures for a reasonable level of significance, i.e., .10.

Since there are 56 rank orders a randomized test procedure at the .10 level involves the choice of probabilities $a_1, \ldots, a_{56}$ such that their sum is 5.6. Using the results at the end of section 4 we have:

$$\text{(7.c.7)} \qquad \begin{cases} a_1 = \cdots = a_4 = 1 \\ a_7 = a_9 = \cdots = a_{56} = 0 \\ a_5 + a_6 + a_8 = 1.6 \ , \quad 0 \leq a_i \leq 1 \ . \end{cases}$$

Conditions (7.c.7) describe all admissable procedures. We will choose a_5, a_6, and a_8 to obtain the most stringent rank order test. For an admissable test the power may be written as

$$(7.c.8) \qquad B(\delta) = \sum_{i=1}^{4} P(i|\delta) + a_5 P(5|\delta) + a_6 P(6|\delta) + a_8 P(8|\delta) ,$$

where $P(i|\delta)$ is the probability of rank order i for given δ. The effect on the power by the choice of a_5, a_6, and a_8 is given by

$$(7.c.9) \qquad \beta(a_5, a_6, a_8; \delta) = a_5 P(5|\delta) + a_6 P(6|\delta) + a_8 P(8|\delta).$$

The envelope power function then depends on a_5, a_6, and a_8 through

$$(7.c.10) \qquad \beta(\delta) = \max \beta(a_5(\delta), a_6(\delta), a_8(\delta); \delta)$$

(where for each δ, $a_5(\delta)$, $a_6(\delta)$, and $a_8(\delta)$ are subject to (7.c.7) and chosen to give a maximum).

The most stringent procedure is given by the a_5^*, a_6^*, a_8^* which minimizes

$$(7.c.11) \qquad \underset{1 \leq \delta \leq \infty}{\text{maximum}} [\beta(\delta) - \beta(a_5^*, a_6^*, a_8^*; \delta)] .$$

Table 7.c.1 was constructed for finding a_5^*, a_6^*, and a_8^* . We first show that $a_6^*=1$. This will follow if we can show that $a_8^* > 0$, since rank order 6 is always more probable than rank order 8. The maximum of $\beta(\delta)$ - $(\beta(0, 1, .6; \delta)$ occurs at $\delta = 2.2$ and is .0027. The maximum of $\beta(\delta) - \beta(.6, 1, 0; \delta)$ occurs at $\delta = 16$

Table 7.c.1

Table used in constructing the most stringent rank order test when m = 3 and n = 5

δ	$P(5\|\delta)$	$P(6\|\delta)$	$P(8\|\delta)$	$\beta(\delta)$	$\beta(\delta) - \beta(0,1,.6;\delta)$	$\beta(\delta) - \beta(.6,1,0;\delta)$	$\beta(\delta) - \beta(.15,1,.45;\delta)$
1.0	.0179	.0179	.0179	.0286	0	.0	.0
1.9	.0410	.0402	.0371	.0651	.0027	.0004	.0021
2.2	.0452	.0450	.0408	.0722	.0027	0	.0021
4.0	.0487	.0558	.0482	.0850	.0003	0	.0002
4.3	.0476	.0560	.0481	.0848	0	.0002	.0000
7.0	.0350	.0517	.0433	.0777	0	.0050	.0013
14.0	.0158	.0367	.0301	.0547	0	.0085	.0021
16.0	.0131	.0336	.0275	.0501	0	.0086	.0021
18.0	.0110	.0310	.0253	.0462	0	.0086	.0021
20.0	.0094	.0287	.0234	.0427	0	.0084	.0020
30.0	.0048	.0209	.0169	.0310	0	.0072	.0018
40.0	.0029	.0164	.0132	.0243	0	.0062	.0015
50.0	.0020	.0134	.0108	.0199	0	.0053	.0013
60.0	.0014	.0114	.0092	.0169	0	.0047	.0011
70.0	.0011	.0099	.0080	.0147	0	.0041	.0010
80.0	.0008	.0088	.0071	.0131	0	.0038	.0010

and is .0086. The maximum at $\delta = 16$ occurs when $a_5 = .6$, $a_6 = 1$, and $a_8 = 0$ gives the best choice of the a's subject to $a_8 = 0$. Thus it is clear that $a_8^* = 0$ is inadmissable and hence $a_6^* = 1$. We can now find a_5^* and a_8^* by an elementary application of game theory (McKinsey, 1952, Chapter 2). The pure stategies of the statistician are $a_5 = .6$, $a_6 = 1$, $a_8 = 0$ and $a_5 = 0$, $a_6 = 1$, $a_8 = .6$. The pure stategies for nature are the various possible values of δ. The payoff matrix is two functions of δ which are $\beta(\delta) - \beta(.6, 1, 0; \delta)$ and $\beta(\delta) - \beta(0, 1, .6; \delta)$. The game is strictly determined and the optimum strategy for the statistician, i.e., the most stringent procedure is $a_5^* = .15$, $a_6^* = 1$, $a_8^* = .45$, in which case his maximum loss, i.e., the maximum deviation from the envelope power function is .0021 which can occur when $\delta = 2.2$ or 16.

For this particular case there was not much to be gained by the use of the most stringent procedure in contrast to any other admissable procedure. The purpose of the above discussion was to illustrate the possibility of applying the theory of most stringent tests to a nonparametric problem.

When m=n=4 it is possible to construct uniformly most powerful rank order tests with levels of significance in the intervals (0, .043) and (.086, .129). To obtain a test at the

exact .05 level we use the results given at the end of section 4. We then have

$$(7.c.12)\quad \begin{cases} a_1 = a_2 = a_3 = 1 \\ a_6 = \cdots = a_{70} = 0 \\ a_4 + a_5 = .5 \end{cases}$$

To find the most stringent test all that is required is

1. $\underset{1\leq\delta<\infty}{\text{Maximum}} [P(4|\delta) - P(5|\delta)] = .00010$

   which occurs for $\delta = 1.32$

2. $\underset{1\leq\delta<\infty}{\text{Maximum}} [P(5|\delta) - P(4|\delta)] = .03083$

   which occurs for $\delta = 15$.

Then the most stringent procedure is given by $a_4 = .00156$, $a_5 = .49844$, and (7.c.12). The maximum deviation from the envelope power function is .00005 .

When m=4, n=5 it is possible to construct uniformly most powerful rank order tests for levels of significance in the intervals (0, .024) and (.063, .079). If a test at the exact .05 level is desired, we have using the results of section 4:

$$(7.c.13)\quad \begin{cases} a_1 = a_2 = a_3 = a_5 = a_6 = 1 \\ a_9 = \cdots = a_{126} = 0 \\ a_4 + a_7 + a_8 = 1.3 \ , \quad 0 \leq a_i \leq 1 \ . \end{cases}$$

To construct the most stringent rank order test in this case one would proceed as in the two previous examples. This work will not be carried out here for it is apparent that the maximum deviation from the envelope power function resulting from any admissable test will not be appreciably more than that occurring for the most stringent test.

When m=n=5 there exist uniformly most powerful rank order tests with levels of significance in the intervals (0, .012) and (.032, .036). If a test at the .05 level is desired we have from section 4,

$$(7.c.14) \qquad \begin{cases} a_1 = \cdots = a_9 = a_{11} = a_{12} = 1 \\ a_{16} = \cdots = a_{252} = 0 \\ a_{10} + a_{13} + a_{14} + a_{15} = 1.6 \,, \quad 0 \le a_i \le 1 \,. \end{cases}$$

The most stringent test will not be constructed in this case for the same reasons given when m=4, n=5. It is interesting to note that we can obtain a test near the .05 level which would have only half as many points in the set B_2 as a test exactly at the .05 level. Thus, in order to construct a test at the 11/252 = .044 level we have,

$$(7.c.15) \qquad \begin{cases} a_1 = \cdots = a_9 = a_{11} = 1 \,, \\ a_{13} = \cdots = a_{252} = 0 \,, \\ a_{10} + a_{12} = 1 \,, \quad 0 \le a_i \le 1 \,. \end{cases}$$

Clearly for this level of significance the computing would be much easier than for the .05 level.

This then completes the discussion of the construction of exact optimum rank order tests of H_0 against H_L. We have seen that for small sample sizes it is possible to construct the uniformly most powerful rank order tests or the most stringent rank order tests. However, the amount of computing becomes much larger as the sample sizes increase, and these exact methods will not be applicable for most of the situations that arise in practice. The fact that most stringent tests for the cases examined are never much more powerful than any admissable test would lead one to conjecture that it is not necessary to find the best test but some reasonable substitute. In the next subsections we develop the theory of such a test giving evidence that this conjecture is valid.

7.d. Exact distribution of the limiting statistic: The statistic

$$T(z) = \sum_{i=1}^{m+n} v_i/i \qquad (7.d.1)$$

was introduced before theorem 7.c.1. A reinterpretation of that theorem shows that the locally most powerful rank order test of H_0 against H_L is based on small values of $T(z)$. Using this as a motivation, the exact distribution of $T(z)$ under H_0 will be

examined in this subsection. In the next subsection we shall examine its limiting distribution for large samples and consider its optimum properties.

In working with the statistic $T(z)$ it will be convenient to let $m + n = N$ and to use

$$(7.d.2) \qquad D_{Ni} = \sum_{j=i}^{N} j^{-1}$$

Using this notation we obtain theorem 7.d.1 which expresses $T(z)$ in two forms amenable to mathematical treatment.

<u>Theorem</u> 7.d.1. The following expressions for $T(z)$ are equivalent to (7.d.1).

$$(1) \qquad T(z) = \sum_{i=1}^{N} z_i D_{Ni}$$

$$(2) \qquad T(z) = \sum_{i=1}^{n} D_{Ns_i} \quad .$$

<u>Proof</u>. It is clear that (1) and (2) are equivalent so all that needs proof is that (7.d.1) equals (1). Recalling that $v_i = \sum_{j=1}^{i} z_j$ we have

(7.d.3) $$T(z) = \sum_{i=1}^{N} v_i/i$$

$$= \sum_{i=1}^{N} \left(\sum_{j=1}^{i} z_j \right)/i$$

$$= \sum_{j=1}^{N} z_j \left(\sum_{i=j}^{N} i^{-1} \right) ,$$

$$= \sum_{j=1}^{N} z_j D_{Nj} .$$

The numbers D_{Ni} will be used extensively and sums of their powers, i.e.

(7.d.4) $$D_k = \sum_{i=1}^{N} D_{Ni}^k$$

which will be needed later are given by

<u>Lemma</u> 7.d.1. $D_k = k! \, N - \rho_k$

for k = 1,2,3,4 where

$\rho_1 = 0$

$\rho_2 = D_{N1}$

$\rho_3 = \frac{3}{2} D_{N1}^2 + 3D_{N1} + \frac{1}{2} \sum_{j=1}^{N} j^{-2}$

$$\rho_4 = 2D_{N1}^3 + 6D_{N1}^2 + 12D_{N1} + 2 \sum_{j=1}^{N} j^{-2}$$

$$+ \sum_{j=1}^{N} j^{-3} + 2 \sum_{1 \le j < k \le N} \sum (j^2 k)^{-1} .$$

Proof. We need the function

$$\delta(i,j) = \begin{cases} 1 & \text{if } i \leq j \\ 0 & \phantom{\text{if }} i > j \end{cases} . \tag{7.d.5}$$

Then

$$D_1 = \sum_{i=1}^{N} D_{Ni} = \sum_{i=1}^{N} \left(\sum_{j=i}^{N} j^{-1} \right) \tag{7.d.6}$$

$$= \sum_{j=1}^{N} j^{-1} \left[\sum_{i=1}^{N} \delta(i,j) \right] = \sum_{j=1}^{N} 1 = N .$$

$$D_2 = \sum_{i=1}^{N} D_{Ni}^2 = \sum_{i=1}^{N} \left(\sum_{j=i}^{N} j^{-1} \right)^2 \tag{7.d.7}$$

$$= \sum_{i=1}^{N} \left(\sum_{j=i}^{N} j^{-2} + 2 \sum_{i \leq j < k \leq N} \sum j^{-1} k^{-1} \right)$$

$$= \sum_{j=1}^{N} j^{-2} \left[\sum_{i=1}^{N} \delta(i,j) \right] + 2 \sum_{1 \leq j < k \leq N} \sum j^{-1} k^{-1} \left[\sum_{i=1}^{N} \delta(i,j) \right]$$

$$= D_{N1} + 2 \sum_{1 \leq j < k \leq N} \sum k^{-1} = D_{N1} + 2 \sum_{k=1}^{N} (k-1) k^{-1}$$

$$= D_{N1} + 2N - 2D_{N1} = 2N - D_{N1}$$

$$(7.d.8)\quad D_3 = \sum_{i=1}^{N} D_{Ni}^3 = \sum_{i=1}^{N} \left(\sum_{j=i}^{N} j^{-1} \right)^3$$

$$= \sum_{i=1}^{N} \Big[\sum_{j=i}^{N} j^{-3} + 3 \sum_{i \le j < k \le N} \sum j^{-1} k^{-2}$$

$$+ 3 \sum_{i \le j < k \le N} \sum j^{-2} k^{-1} + 6 \sum_{i \le j < k < m \le N} \sum \sum j^{-1} k^{-1} m^{-1} \Big]$$

$$= \sum_{j=1}^{N} j^{-3} \Big[\sum_{i=1}^{N} \delta(i,j) \Big]$$

$$+ 3 \sum_{1 \le j < k \le N} \sum j^{-1} k^{-2} \Big[\sum_{i=1}^{N} \delta(i,j) \Big]$$

$$+ 3 \sum_{1 \le j < k \le N} \sum j^{-2} k^{-1} \Big[\sum_{i=1}^{N} \delta(i,j) \Big]$$

$$+ 6 \sum_{1 \le j < k < m \le N} \sum \sum j^{-1} k^{-1} m^{-1} \Big[\sum_{i=1}^{N} \delta(i,j) \Big]$$

$$= \sum_{j=1}^{N} j^{-2} + 3 \sum_{1 \le j < k \le N} \sum k^{-2}$$

$$+ 3 \sum_{1 \le j < k \le N} \sum j^{-1} k^{-1} + 6 \sum_{1 \le j < k < m \le N} \sum \sum k^{-1} m^{-1} .$$

Now working with the individual sums of the last equation in (7.d.8), we have

$$(7.d.9)\qquad \sum\sum_{1\le j<k\le N} k^{-2} = \sum_{k=1}^{N} (k-1)k^{-2}$$

$$= D_{N1} - \sum_{k=1}^{N} k^{-2} .$$

$$(7.d.10)\qquad \sum\sum_{1\le j<k\le N} j^{-1}k^{-1} = \tfrac{1}{2}[(\sum_{j=1}^{N} j^{-1})^2 - \sum_{j=1}^{N} j^{-2}]$$

$$= \tfrac{1}{2}D_{N1}^2 - \tfrac{1}{2}\sum_{j=1}^{N} j^{-2} .$$

$$(7.d.11)\qquad \sum\sum\sum_{1\le j<k<m\le N} k^{-1}m^{-1} = \sum\sum_{1\le k<m\le N} (k-1)k^{-1}m^{-1}$$

$$= \sum_{m=1}^{N} (m-1)m^{-1} - \sum\sum_{1\le k<m\le N} k^{-1}m^{-1}$$

$$= N - D_{N1} - \tfrac{1}{2}D_{N1}^2 + \tfrac{1}{2}\sum_{m=1}^{N} m^{-2} .$$

By using the results of (7.d.9), (7.d.10), and (7.d.11) in the last equation of (7.d.8), the formula for D_3 is obtained.

(7.d.12) $$D_4 = \sum_{i=1}^{N} D_{Ni}^4 = \sum_{i=1}^{N} \left(\sum_{j=i}^{N} j^{-1} \right)^4$$

$$= \sum_{i=1}^{N} \Big[\sum_{j=i}^{N} j^{-4} + 4 \sum\sum_{i \le j < k \le N} j^{-3}k^{-1}$$

$$+ 4 \sum\sum_{i \le j < k \le N} j^{-1}k^{-3} + 6 \sum\sum_{i \le j < k \le N} j^{-2}k^{-2}$$

$$+ 12 \sum\sum\sum_{i \le j < k < m \le N} j^{-1}k^{-1}m^{-2} + 12 \sum\sum\sum_{i \le j < k < m \le N} j^{-1}k^{-2}m^{-1}$$

$$+ 12 \sum\sum\sum_{i \le j < k < m \le N} j^{-2}k^{-1}m^{-1} + 24 \sum\sum\sum\sum_{i \le j < k < m < n \le N} j^{-1}k^{-1}m^{-1}n^{-1} \Big].$$

We now invert the order of summation in (7.d.12) and perform the summation on i obtaining

(7.d.13) $$D_4 = \sum_{j=1}^{N} j^{-3} + 4 \sum\sum_{1 \le j < k \le N} j^{-2}k^{-1}$$

$$+ 4 \sum\sum_{1 \le j < k \le N} k^{-3} + 6 \sum\sum_{1 \le j < k \le N} j^{-1}k^{-2}$$

$$+ 12 \sum\sum\sum_{1 \le j < k < m \le N} k^{-1}m^{-2} + 12 \sum\sum\sum_{1 \le j < k < m \le N} k^{-2}m^{-1}$$

$$+ 12 \sum\sum\sum_{1 \le j < k < m \le N} j^{-1}k^{-1}m^{-1} + 24 \sum\sum\sum\sum_{1 \le j < k < m < n \le N} k^{-1}m^{-1}n^{-1}.$$

Next we evaluate the individual terms on the right of (7.d.13), obtaining

(7.d.14) $$\sum_{1\le j<k\le N}\sum k^{-3} = \sum_{k=1}^{N}(k-1)k^{-3}$$

$$= \sum_{k=1}^{N} k^{-2} - \sum_{k=1}^{N} k^{-3} .$$

(7.d.15) $$\sum_{1\le j<k<m\le N}\sum\sum k^{-1}m^{-2} = \sum_{1\le k<m\le N}\sum (k-1)k^{-1}m^{-2}$$

$$= \sum_{m=1}^{N}(m-1)m^{-2} - \sum_{1\le k<m\le N}\sum k^{-1}m^{-2}$$

$$= D_{N1} - \sum_{m=1}^{N} m^{-2} - \sum_{1\le k<m\le N}\sum k^{-1}m^{-2} .$$

(7.d.16) $$\sum_{1\le j<k<m\le N}\sum\sum k^{-2}m^{-1} = \sum_{1\le k<m\le N}\sum (k-1)k^{-2}m^{-1}$$

$$= \tfrac{1}{2}D_{N1}^{2} - \tfrac{1}{2}\sum_{k=1}^{N} k^{-2} - \sum_{1\le k<m\le N}\sum k^{-2}m^{-1}$$

(7.d.17) $$\sum_{1\le j<k<m\le N}\sum\sum j^{-1}k^{-1}m^{-1} = \frac{1}{6}\left[D_{N1}^{3} - 3D_{N1}\left(\sum_{j=1}^{N} j^{-2}\right) + 2\sum_{j=1}^{N} j^{-3}\right] .$$

(7.d.18) $$\sum_{1\le j<k<m<n\le N}\sum\sum\sum k^{-1}m^{-1}n^{-1}$$

$$= \sum_{1\le k<m<n\le N}\sum\sum m^{-1}n^{-1} - \sum_{1\le k<m<n\le N}\sum\sum k^{-1}m^{-1}n^{-1}$$

$$= \sum_{1\le m<n\le N}\sum n^{-1} - \sum_{1\le m<n\le N}\sum m^{-1}n^{-1}$$

$$- \sum_{1\le k<m<n\le N}\sum\sum k^{-1}m^{-1}n^{-1}$$

$$= N - D_{N1} - \tfrac{1}{2}D_{N1}^{2} + \tfrac{1}{2}\sum_{j=1}^{N} j^{-2}$$

$$- \frac{1}{6}\left[D_{N1}^{3} - 3D_{N1}\left(\sum_{j=1}^{N} j^{-2}\right) + 2\sum_{j=1}^{N} j^{-3}\right] .$$

We also need the identity

(7.d.19) $$\left(\sum_{i=1}^{N} i^{-1}\right)\left(\sum_{j=1}^{N} j^{-2}\right) = \sum_{i=1}^{N} i^{-3} + \sum_{1\le i<j\le N}\sum i^{-1}j^{-2} + \sum_{1\le i<j\le N}\sum i^{-2}j^{-1}.$$

D_4 is then obtained by substituting the results of (7.d.14) through (7.d.18) in (7.d.13) and simplifying by the use of (7.d.19).

In working with the quantities D_k, the results below are of use in evaluating the remainder terms.

The following formulas are given by Adams (1947, page 27):

(7.d.20) $$\sum_{i=1}^{N} i^{-2} = \frac{\pi^2}{6} - \sum_{j=1}^{\infty} b_j/(N+j)^{[j]} ,$$

where $b_j = (j-1)!/j$,

and

$$A^{[B]} = A(A-1) \cdots (A-B+1), \text{ B an integer.}$$

(7.d.21) $$\sum_{i=1}^{N} i^{-3} = C - \sum_{j=2}^{\infty} c_j/(N+j)^{[j]} ,$$

where $$c_j = \frac{(j-1)!}{j} \left(\sum_{n=1}^{j-1} n^{-1} \right) ,$$

and $$C = 1.202057 \ldots$$

Van der Corput (1952, theorem 14) gives

(7.d.22) $$\sum_{i=1}^{N} i^{-1} = \ln N + \gamma + \frac{1}{2N} + \sum_{j=1}^{J-1} \frac{(-1)^j B_j}{2jN^{2j}} + R_J ,$$

where the remainder R_J has the same sign as and is a fraction of the first neglected term. Here $\gamma = 0.572216\ldots$ is Euler's constant, and B_j is the <u>jth</u> Bernoulli number: $B_1 = 1/6$,

$B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$.

From (7.d.20) and (7.d.22) we obtain

(7.d.23) $$\sum_{1 \leq j < j' \leq N} \sum (j^2 j')^{-1} < \frac{\pi^2}{6} (\ln N + \gamma + \tfrac{1}{2} N^{-1}) .$$

For each rank order z we can form its complement rank order z^c, i.e., if an element of z was 0 (1) the corresponding element of z^c is 1 (0). If z was a rank order for sample sizes m and n then z^c is a rank order for sample sizes n and m. Using $D_1 = N = m+n$ we obtain $T(z) + T(z^c) = N$. Thus it is only necessary to prepare tables of the distribution (under H_0) of T(z) when the first sample size is less than or equal to the second sample size. Also tables for the equal-size-sample case need only be prepared for half of the rank orders. Table IIIa has been prepared for $1 \leq m \leq n \leq 10$. The table was computed from the recursion formulas:

(7.d.24) $$T(z^0) = T(z) + n/(m+n+1)$$

and

(7.d.25) $$T(z^1) = T(z) + (n+1)/(m+n+1) .$$

The rank orders z^0 and z^1 (used at the end of section 7.a) are formed from z by placing an additional element, 0 for z^0 and 1 for z^1, at the extreme right of z. In theorem 7.d.3 it is

shown that the expected value of $T(z)$ is n which was used as a check in the preparation of table IIIa.

When $m = n$ each rank order z can be paired to its complement z^c and we have

$$T(z) - n = n - T(z^c) \ . \tag{7.d.26}$$

Thus the two rank orders in the pair give values of T which have deviations from the mean equal in size and opposite in sign. Under H_0 each rank order has equal probability. Hence under H_0, T is symmetrically distributed.

Theorem 7.d.2 is used in finding the moments of T which are given in theorem 7.d.3. In the proof of theorem 7.d.2 use is made of the fact that z_i^b, where $b > 0$, equals z_i since z_i only takes on the values 0 and 1. Thus whenever a power of z_i different from one occurs in an expression, it can be replaced by z_i. Also we make use of the fact that under H_0

$$E(Z_{i_1} Z_{i_2} \cdots Z_{i_k}) = P(Z_{i_1} = Z_{i_2} = \cdots = Z_{i_k} = 1) = p_k \tag{7.d.27}$$

where $i_1, i_2, \cdots, i_k$ is a subset of the first N integers and

$$p_k = \binom{n}{k} / \binom{N}{k} \ . \tag{7.d.28}$$

<u>Theorem</u>. 7.d.2. Let $c_1, \cdots, c_N$ be arbitrary constants and let

$$C_k = \sum_{i=1}^{N} c_i^k .$$

Under H_0, the moments of

$$X = \sum_{i=1}^{N} c_i Z_i$$

are

I. $m_1 = EX$

$= p_1 C_1$

II. $m_2 = EX^2$

$= p_1 C_2 + p_2 (C_1^2 - C_2)$

III. $m_3 = EX^3$

$= p_1 C_3 + 3 p_2 (C_1 C_2 - C_3)$

$+ p_3 (C_1^3 - 3C_1 C_2 + 2C_3)$

IV. $m_4 = EX^4$

$= p_1 C_4 + 3p_2 (C_2^2 - C_4) + 4p_2 (C_1 C_3 - C_4)$

$+ 6p_3 (C_1^2 C_2 - C_2^2 - 2C_1 C_3 + 2C_4)$

$+ p_4 (C_1^4 - 6C_1^2 C_2 + 3C_2^2 + 8C_1 C_3 - 6C_4)$.

Proof. (Part I)

(7.d.29)
$$m_1 = E(\sum_{i=1}^{N} c_i Z_i)$$

$$= \sum_{i=1}^{N} c_i E Z_i$$

$$= p_1 C_1 \ .$$

(Part II)

(7.d.30)
$$m_2 = E(\sum_{i=1}^{N} c_i Z_i)^2$$

$$= E(\sum_{i=1}^{N} c_i^2 Z_i^2 + 2 \sum_{1 \le i < j \le N} \sum c_i c_j Z_i Z_j)$$

$$= p_1 C_2 + p_2 (C_1^2 - C_2)$$

(Part III)

(7.d.31)
$$m_3 = E(\sum_{i=1}^{N} c_i Z_i)^3$$

$$= E\Big[\sum_{i=1}^{N} c_i^3 Z_i^3 + 3 \sum_{i \neq j} \sum c_i c_j^2 Z_i Z_j^2$$

$$+ 6 \sum_{1 \le i < j < k \le N} \sum \sum c_i c_j c_k Z_i Z_j Z_k\Big]$$

$$= p_1 C_3 + 3p_2 (C_1 C_2 - C_3) + p_3 (C_1^3 - 3C_1 C_2 + 2C_3)$$

(Part IV)

(7.d.32) $$m_4 = E(\sum_{i=1}^{N} c_i Z_i)^4$$

$$= E\Big(\sum_{i=1}^{N} c_i^4 Z_i^4 + 6 \sum_{1\leq i<j\leq N}\sum c_i^2 c_j^2 Z_i^2 Z_j^2$$

$$+ 4\sum_{i\neq j}\sum c_i c_j^3 Z_i Z_j^3 + 6 \sum_{i\neq j\neq k\neq i}\sum\sum c_i c_j c_k^2 Z_i Z_j Z_k^2$$

$$+ 24 \sum_{1\leq i<j<k<m\leq N}\sum\sum\sum c_i c_j c_k c_m Z_i Z_j Z_k Z_m \Big)$$

$$= p_1 C_4 + 3p_2(C_2^2 - C_4) + 4p_2(C_1 C_3 - C_4)$$

$$+ 6p_3(C_1^2 C_2 - C_2^2 - 2C_1 C_3 + 2C_4)$$

$$+ p_4(C_1^4 - 6C_1^2 C_2 + 3C_2^2 + 8C_1 C_3 - 6C_4) .$$

The central moments of X can then be found by the use of

(7.d.33) $$\sigma^2 = E(X-EX)^2$$

$$= EX^2 - (EX)^2$$

(7.d.34) $$\mu_3 = E(X-EX)^3$$

$$= EX^3 - 3(EX)(EX^2) + 2(EX)^3$$

(7.d.35) $$\mu_4 = E(X-EX)^4$$

$$= EX^4 - 4(EX)(EX^3) + 6(EX)^2(EX^2) - 3(EX)^4$$

The moments of T can now be found by using the results of theorem 7.d.2 in the formulas (7.d.33), (7.d.34), and (7.d.35) and the expressions obtained for D_k in lemma 7.d.1. For the third and fourth moments the substitutions and simplifications are extensive. The final results are summarized in

<u>Theorem</u> 7.d.3. Under H_0 the moments of T are

$$ET = n$$

$$\sigma^2 = \frac{mn}{N-1}\left(1 - \frac{D_{N1}}{N}\right)$$

$$\mu_3 = \frac{\lambda(1-\lambda)(1-2\lambda)N^2}{(N-1)(N-2)}(2N - \rho_3 + 3\rho_2)$$

$$\mu_4 = \frac{\lambda(1-\lambda)N}{(N-1)(N-2)(N-3)}\Big[3\lambda(1-\lambda)N^4$$

$$+ 6[9\lambda^2-9\lambda+1-\rho_2\lambda(1-\lambda)]N^3$$

$$+ [12+3\rho_2\lambda(\rho_2+12)(1-\lambda) + (4\rho_3-\rho_4)(6\lambda^2-6\lambda+1)]N^2$$

$$- [\rho_4-4\rho_3+3\rho_2(\rho_2+4)]N + 3\rho_2^2\Big].$$

Where $\lambda = \frac{n}{N} = \frac{n}{m+n}$, and the quantities ρ_i were introduced in lemma 7.d.1. From equations 7.d.22 and 7.d.23 the ρ_i are of the order of magnitude ln N.

Corollary 7.d.3. Under H_0 the variance (σ^2) of T satisfies

$$0 < \frac{mn}{N-1}\left(1 - \frac{\ln N + \gamma + \frac{1}{2}N^{-1}}{N}\right) - \sigma^2 < \frac{1}{12N^2}$$

where N = m + n.

Proof. The corollary follows immediately from theorem 7.d.3 and (7.d.22).

Corollary 7.d.3 gives a method of computing the standard deviation of T which will be correct to three decimal places for N > 20. Table IIIc gives the standard deviation of T for all combinations of sample sizes such that $m + n = N \leq 20$.

The Wilcoxon statistic [Mann and Whitney, 1947] which can be written as

(7.d.36) $$W = \sum_{i=1}^{N} iz_i$$

is used as a test of the hypothesis that two samples come from populations differing only in location. H_{EV}, a special case of H_L, is a hypothesis of this type. Thus T and W will sometimes

be used for the same purpose. Therefore, it is interesting to have some information about their joint distribution.

Theorem 7.d.4. Under H_0 the covariance of T and W is $- mn/4$.

Proof. We have

$$E(TW) = E(\sum_{i=1}^{N} D_{Ni} Z_i)(\sum_{i=1}^{N} iZ_i) \tag{7.d.37}$$

$$= \sum_{i=1}^{N} iD_{Ni} EZ_i + \sum_{i \neq j}\sum jD_{Ni} E(Z_i Z_j)$$

$$= p_1 \sum_{i=1}^{N} iD_{Ni} + p_2 \sum_{i \neq j}\sum jD_{Ni} \quad .$$

Now then,

$$\sum_{i=1}^{N} iD_{Ni} = \sum_{i=1}^{N} i(\sum_{j=i}^{N} j^{-1}) \tag{7.d.38}$$

$$= \sum_{j=1}^{N} j^{-1} [\sum_{i=1}^{N} i\delta(i,j)]$$

$$= \sum_{j=1}^{N} \frac{j+1}{2} = \frac{N(N+1)}{4} + \frac{N}{2}$$

$$= \frac{N}{4} (N+3)$$

and

(7.d.39) $$\sum_{i \neq j}\sum iD_{Nj} = (\sum_{i=1}^{N} i)(\sum_{j=1}^{N} D_{Ni}) - \sum_{i=1}^{N} iD_{Ni}$$

$$= \frac{N^2(N+1)}{2} - \frac{N(N+3)}{4}$$

$$= N(2N+3)(N-1)/4 .$$

Substituting (7.d.38) and (7.d.39) in (7.d.37) and using $E(W) = n(N+1)/2$ and $E(T) = n$; and covariance of $TW = E(TW) - (ET)(EW)$, we obtain the desired result.

Corollary 7.d.4. Under H_0 the correlation between T and W is

$$- \frac{1}{2}\sqrt{\frac{3(N-1)}{(N+1)(1-D_{N1}/N)}}$$

or approximately

$$-\frac{\sqrt{3}}{2} = .8660 \ldots .$$

Proof. The variances of T and W are

(7.d.40) $$\text{Var } T = \frac{mn}{N-1}(1 - D_{N1}/N)$$

and

(7.d.41) $$\text{Var } W = \frac{mn(N+1)}{12}$$

which, combined with theorem 7.d.4, yields the conclusion.

The above work is similar to the study made by Terry [1952, section 9] where he gives the correlation between W and and c_1. In both cases the limiting correlation does not depend on m and n. The limiting correlation in the case considered by Terry is $\sqrt{3/\pi}$ (= .9772) which is somewhat larger (in absolute value) than $-\sqrt{3}/2$ (= $-$ .8660) found in the above case. We will come back to the correlation between T and W in the next subsection.

7.e. <u>Large sample distribution of the limiting statistic</u>. We first show that under H_0 the statistic T has a limiting normal distribution and then indicate that under H_L it also has a normal distribution and is asymptotically most powerful.

We need the result of Epstein and Sobel [1952a, appendix A] that if $X_1, \cdots, X_N$ are independently distributed, and each X has the density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0 \end{cases} \tag{7.e.1}$$

then

$$EX_{Ni} = \sum_{j=i'}^{N} j^{-1} = D_{Ni'}, \quad i' = N - i + 1 \tag{7.e.2}$$

where X_{Ni} is the i-th order statistic in a sample of N and D_{Ni} was introduced in (7.d.2). Also required are the following

results quoted (with minor changes) from Dwass (1953, pp. 303-304):

"Let $(Q_1, \cdots, Q_N)$ be a random vector which takes on each of the N! permutations of the numbers $(1, \cdots, N)$ with equal probability, 1/N! . Sufficient conditions are given for the asymptotic normality of

$$S_N = \sum_{i=1}^{N} a_{Ni} b_{NQ_i} ,$$

where $(a_{N1}, \cdots, a_{NN})$, $(b_{N1}, \cdots, b_{NN})$ are two sets of real numbers given for every N. We will assume that

$$\sum_{i=1}^{N} a_{Ni} = \sum_{i=1}^{N} b_{Ni} = 0 , \quad \sum_{i=1}^{N} a_{Ni}^2 = 1 .$$

Suppose for an integer $k \geq 1$ there is a random variable X satisfying the following conditions:

(a) X has a continuous cdf F(x),

(b) if $X_1, \cdots, X_N$ are independent random variables each with the cdf F(x) and $X_{N1} \leq \cdots \leq X_{NN}$ are the ordered values of $X_1, \cdots, X_N$ then

$$b_{Ni} = EX_{Ni}^k - \sum_{j=1}^{N} EX_{Nj}^k / N$$

for all N and i.

(c) $E|X|^{3k} < \infty$.

(d) Either X_i^k is normal or

(e) $\underset{1 \leq i \leq N}{\text{Maximum}} \, (a_{Ni}) \rightarrow 0$ as $N \rightarrow \infty$.

Then S_N is asymptotically normally distributed."

Theorem 7.e.1. Under H_0 when $N \rightarrow \infty$ in such a way that $n/(m+n)$ tends to a constant λ different from 0 or 1, the random variable

$$t = \frac{T - \lambda N}{\sqrt{\lambda(1-\lambda)N}}$$

has a distribution which approaches a normal distribution with zero mean and unit variance.

Proof. In Dwass' notation let

$$(7.e.3) \quad a_{Ni} = \begin{cases} \sqrt{n/mN} & , \quad i=1, \cdots, m \\ -\sqrt{m/nN} & , \quad i=m+1, \cdots, N \end{cases}$$

then $\sum_{i=1}^{N} a_{Ni} = 0$ and $\sum_{i=1}^{N} a_{Ni}^2 = 1$. Let

(7.e.4) $$b_{Ni} = \sum_{j=i'}^{N} j^{-1} - 1 = D_{Ni'} - 1, \quad i'=N-i+1 \quad .$$

Then by lemma 7.d.1, $\sum_{i=1}^{N} b_{Ni} = 0$. Thus $(a_{N1}, \cdots, a_{NN})$ and $(b_{N1}, \cdots, b_{NN})$ satisfy the preliminary requirements of Dwass. Conditions (a) and (b) are satisfied due to the result of Epstein and Sobel. Since all of the moments exist for the exponential distribution, condition (c) is satisfied. Finally condition (e) is satisfied since

$$\underset{1 \le i \le N}{\text{Maximum}} \left(|a_{Ni}| \right) < \frac{1}{\sqrt{N}} \left(\sqrt{\frac{\lambda}{1-\lambda}} + \sqrt{\frac{1-\lambda}{\lambda}} \right).$$

Thus

(7.e.5) $$S_N = \sum_{i=1}^{N} a_{Ni} b_{NQ_i}$$

$$= \sqrt{\frac{n}{mN}} \sum_{i=1}^{m} b_{nQ_i} - \sqrt{\frac{m}{nN}} \sum_{i=m+1}^{N} b_{NQ_i}$$

$$= \sqrt{\frac{n}{mN}} \left(\sum_{i=1}^{m} [D_{NQ_{N-i+1}} - 1] \right) - \sqrt{\frac{m}{nN}} \left(\sum_{i=m+1}^{N} [D_{NQ_{N-i+1}} - 1] \right)$$

$$= \sqrt{\frac{nN}{m}} - T \left(\sqrt{\frac{n}{mN}} + \sqrt{\frac{m}{nN}} \right).$$

is asymptotically normally distributed. The theorem is proved by noting that S_N as given by the least equality in (7.e.5) is a linear function of t and that t has expected value zero and unit limiting variance.

Tables IIIb and IIIb' compare the exact distribution of X^* with that of the normal approximation for several combinations of sample sizes. The approximation is adequate for finding the critical .10 points for the sample sizes treated. It is necessary to find the exact distribution of T for some sample sizes with $m + n > 10$, if we wish to use the normal approximation in the remaining cases to obtain the .01 and .05 percentage points.

If two test statistics X_1 and X_2 have under H_0 a bivariate normal distribution with means 0, variances 1, and correlation ρ and if H_0 is rejected when $X_1(X_2)$ is greater than λ_α then:
(1) the probability of H_0 being rejected by both X_1 and X_2 is

$$(7.e.6) \qquad P_R = \int_{\lambda_\alpha}^{\infty} \int_{\lambda_\alpha}^{\infty} (2\pi\sqrt{1-\rho^2})^{-1} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right] dx\, dy ,$$

(2) the probability of H_0 being accepted by both X_1 and X_2 is

$$(7.e.7) \qquad P_A = \int_{-\infty}^{\lambda_\alpha} \int_{-\infty}^{\lambda_\alpha} (2\pi\sqrt{1-\rho^2})^{-1} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right] dx\, dy ,$$

---

*

$$X = \frac{T - n}{\sqrt{mn(N-D_{N1})\,/\,N(N-1)}}$$

(3) the probability of H_0 being accepted by one and rejected by the other procedure is

(7.e.8) $$P_D = 1 - P_R - P_A \; .$$

It is reasonable to assume that when T and W and when c_1 and W are properly standardized that the above is applicable for large N. In which case when $\alpha = .05$ and $\lambda_\alpha = 1.645$ we get for T and W where $\rho = \sqrt{3}/2 = .8660$

(7.e.9) $$\begin{cases} P_A = .9288 \\ P_R = .0288 \\ P_D = .0424 \end{cases}$$

and for c_1 and W where $\rho = \sqrt{3/\pi} = .9772$

(7.e.10) $$\begin{cases} P_A = .9405 \\ P_R = .0405 \\ P_D = .0190 \; . \end{cases}$$

(The required values of (7.e.6) and (7.e.7) are given by K. Pearson [1931].)

Consider the ratio of the probability that the two procedures of a pair do not agree to the probability that at least one procedure of the pair rejects H_0. An examination of (7.e.9) and (7.e.10) shows that this ratio is much larger for T and W than it is for c_1 and W.

A rigorous treatment of the limiting distribution of T under H_L would be complicated and will not be given here in view of the fact that we are primarily interested in exact instead of limiting properties. At the present time there is no general theory giving the limiting distribution of such statistics as T under the alternative hypothesis. However, Dwass [1952] treats in detail the limiting distribution of statistics related to T. In particular his example 3, page 64, concerns nonparametric tests of a generalization of H_E. His results are for the alternatives considered in theorem 3.2, i.e.,

$$\delta = \Delta_2/\Delta_1 = 1 + c(m+n)^{-\frac{1}{2}} . \qquad (7.e.11)$$

For these alternatives the power of most procedures is bounded away from 0 and 1, and hence are of statistical interest. Dwass shows that there is a sequence of test statistics which are polynomials in the ranks such that in the limit as the sample sizes tend to infinity in fixed ratio there are elements in the

sequence of test statistics with power arbitrarily close to the best parametric test. His argument also indicates that the limit of the sequence of tests based on polynomials in the ranks is equivalent to the procedure using T. What is not shown is that the procedure T itself has this desirable property. He proved that the elements of the sequence of tests based on polynomials in the ranks have limiting normal distributions, but it was not shown that T has a limiting normal distribution.

The test statistics most closely related to T that have received extensive examinations are those of Terry, [1952] and Wilcoxon (see Mann and Whitney [1947]). Under alternatives like (7.e.11) the Terry statistic was proved asymptotically normally distributed by Hoeffding (1953), and by an entirely different method the Wilcoxon statistic was proved to have a limiting normal distribution under the same alternatives by Lehmann [1951, theorem 3.2]. This evidence also suggests that T has a limiting normal distribution under (7.e.11). Table IV gives the exact power of the test based on T. It is almost as great as the power of the best rank order test, thus supporting the conjecture made at the end of 7.c that there exist procedures which, although not exactly the best possible, will for all practical purposes, give results equivalent to the best rank order test. However, table IV is for small sample sizes only and

does not support the hypothesis that for large samples the test based on T is most powerful. Table IIa gives the exact distribution of T under H_L for alternatives of the form (7.e.11). There is no evidence of the distribution approaching normality but the table was prepared only for small N.

For large samples and alternatives of the form (7.e.11) the following conditions summarize the properties (some proved and others likely conjectures) of the joint distribution of pairs of test statistics such as T and W or c_1 and W.

(1) $EX_1 = EX_2 = 0$, under H_0

(2) $EX_1 = m_1 > 0$, $EX_2 = m_2 \geq 0$ under H_1.

(3) $V(X_1) = V(X_2) = 1$, $\rho(X_1, X_2) = \rho \geq 0$ under H_0 and H_1.

(4) X_1 and X_2 have a bivariate normal distribution under H_0 and H_1.

(5) Of all tests based on large values of a normally distributed test statistic the one using X_1 is most powerful for testing H_0 against H_1.

In the above one can identify X_1 with $(ET-T)/\sqrt{V(T)}$, X_2 with $(W-EW)/\sqrt{V(W)}$, and H_1 with H_L.

<u>Theorem</u> 7.e.2. Under the above conditions

(a) $m_2 = \rho m_1$

(b) If the power of the test at the α level based on X_1 is $1 - \beta$ the power of the test based on X_2 is

$$\int_{\gamma}^{\infty} (2\pi)^{-\frac{1}{2}} \exp[- x^2/2] \, dx$$

where $\gamma = \lambda_\alpha - \rho(\lambda_\alpha - \lambda_{1-\beta})$.

Proof. We will show that if $m_2 \neq \rho m_1$ then there is a test statistic X which is normally distributed with expected value $> m_1$ and variance 1. This would be a contradiction of (5) and hence would imply (a), $m_2 = \rho m_1$. The statistic

(7.e.12) $$X = \frac{X_1 + \lambda X_2}{\sqrt{1+2\rho\lambda+\lambda^2}}$$

is normally distributed with unit variance. All that remains is to show

(7.e.13) $$EX = \frac{m_1 + \lambda m_2}{\sqrt{1+2\rho\lambda+\lambda^2}} \geq m_1$$

for some λ, and the inequality ($\geq$) becomes an equality (=) only when $m_2 = \rho m_1$. It is easily verified that EX takes on its maximum value if

(7.e.14) $$\lambda = \frac{m_2 - \rho m_1}{m_1 - \rho m_2} \quad .$$

Substituting this value of λ into (7.e.13) we obtain the following set of equivalent inequalities.

$$(7.e.15) \qquad \begin{cases} (m_1 + m_2)^2 \geq m_1^2(1+2\rho\lambda+\lambda^2) \\ 2\lambda m_1(m_2-\rho m_1) \geq \lambda^2(m_1^2-m_2^2) \\ 2m_1(m_1-\rho m_2) \geq (m_1-m_2)(m_1+m_2) \quad . \end{cases}$$

From (5) we have $m_1 > m_2$ so that we obtain

$$(7.e.16) \qquad (m_1 - \rho m_2) \geq (m_1 - m_2).$$

The last inequality must hold since $0 \leq \rho \leq 1$. Finally the equality can hold only if $\lambda = 0$ which is equivalent to $m_2 = \rho m_1$.

The second part of the theorem is an immediate consequence of the first result.

Thus in the case that one of the statistics leads to a most powerful procedure there is: (a) a simple relationship between the quantities m_1, m_2, and ρ ; (b) a simple method for finding the power of one of the procedures in terms of the power of the other and their correlation.

ACKNOWLEDGMENTS

For the continuing encouragement and guidance of Professor Howard Levene of Columbia University from the initiation to the completion of this research, I wish to express sincere appreciation.

Staff members of the Statistical Engineering Laboratory have been of great assistance both in providing stimulus and criticism and in the actual preparation of the report. Also, I wish to thank Miss Mary Orr of the Computation Laboratory, National Bureau of Standards, for her diligent care in the computation of the tables.

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INTRODUCTION TO TABLES

Table I. Power functions of parametric tests

Ia. Exponential alternatives. Values of

$$\delta = F^{\alpha}_{2m,2n} \quad F^{\beta}_{2n,2m}$$

are given where $F^{\varepsilon}_{x,y}$ is the upper ε-percentage point of the F-distribution with x and y degrees of freedom. The table is for

$$\alpha = .10; \qquad \beta = .25 \text{ and } .50$$

$$\alpha = .05; \qquad \beta = .05 \text{ and } .25$$

and

$$m = 1(1)5$$

$$n = 1(1)5 .$$

Assume that a first sample of m observations comes from a population with cumulative distribution function F(x) and that a second independent sample of n observations comes from a population with a cumulative distribution function $[F(x)]^{\delta}$, where $\delta \geq 1$. Then when F(x) is known the most powerful test of $\delta = 1$ against $\delta > 1$ at the α level will have power $1 - \beta$ if δ is given by the above formula. Table Ia for $\alpha = .05$ is contained in table 8.3 of Eisenhart, Hastay, and Wallis (1947).

The required percentage points of the F-distribution were obtained from Merrington and Thompson (1943).

Ib. Normal alternatives. Values of

$$\tau = \frac{(\lambda_\alpha - \lambda_{1-\beta})\sqrt{mn}}{\sqrt{m+n}}$$

are given where λ_ε is the upper ε-percentage point of the normal distribution.

The table is for

$$\alpha = .10; \qquad \beta = .25 \text{ and } .50$$
$$\alpha = .05; \qquad \beta = .05 \text{ and } .25$$

and

$$m = 1(1)5$$
$$n = 1(1)5 \quad .$$

Assume that a first sample of m observations comes from a normal distribution with mean θ_1 and variance 1, and that a second independent sample of n observations comes from a normal distribution with mean θ_2 and variance 1, where $\tau = \theta_2 - \theta_1 \geq 0$. Then the best parametric test of $\tau = 0$ against $\tau > 0$ at the α level will have power $1 - \beta$ if τ is given by the above formula. For a slightly different application of this table see theorem 3.2.

Table II. Probabilities of rank orders

IIa. Lehmann alternatives. Values of

$$P(Z=z) = \frac{m!\, n!\, \delta^n}{\prod_{i=1}^{m+n} (u_i+\delta v_i)}$$

are given for

$$m = 1(1)5$$

$$n = 1(1)5$$

and <u>some</u> of the rank orders z. The columns of this table are headed by two symbols, a value of P_β^α and a value of δ. The value of δ is the entry in table Ia corresponding to values of the parameters m, n, α, and β.

The entries in table IIa give the probabilities of rank orders when the first sample of m observations comes from a population with cumulative distribution function F(x), and the second independent sample of n observations comes from a population with cumulative distribution function $[F(x)]^\delta$, where $\delta > 1$. The values of δ used were chosen so that the table would have a simple interpretation in terms of the best parametric test of $\delta = 1$ against $\delta > 1$ at the α level of significance. Thus in the table for $m = n = 3$ where there are 20 rank orders a nonparametric test at the .10 level consists of putting two rank orders into the critical region. So that if we use the

column marked $P_{.50}^{.10}$ we can find the power of the nonparametric tests that result by putting any two rank orders into the critical region by taking their sum. Then we can compare this to the best parametric procedure which has power .50. Such comparisons are made in table IVa. Corollary 7.a.1 and the discussion following it give the source of the above formula and the method for computing this table.

IIa'. Lehmann alternatives. Values are given of

$$P(Z=z) = \frac{m!\, n!\, \delta^n}{\prod_{i=1}^{m+n} (u_i+\delta v_i)} .$$

Table IIa' shows how the probabilities of rank orders for a particular combination of sample sizes (m=n=4) change as the values of δ are increased. In particular this table allows us to compare the power of nonparametric tests of H_L for sample sizes m=n=4 at the .10 level of significance against alternatives which for that level would give power 1 - .25 = .75 when using sample sizes m=n=3 and the most powerful parametric test. Thus parametric and nonparametric tests may be compared in terms of the sample sizes for the two tests which are required to give the same power. This comparison is made in table IVb.

IIb. Lehmann versus normal alternatives for I < II. Values of P(I<II), probability of first sample being less than second sample, are given for the pairs of corresponding alternatives introduced in tables Ia and Ib.

[Note: Upper line corresponds to Lehmann and lower line corresponds to normal alternatives].

The values of P(I<II) for normal alternatives were given by Teichroew (1954).

IIb'. Lehmann versus normal alternatives for n=1. The probabilities of each rank order are given for n=1 and m=1(1)5 for the normal alternative $\theta_2-\theta_1 = 2$ and the Lehmann alternative $\delta = 11.71455$. The choice of parameters is such that the rank order (0,1) is equally-likely for the normal and Lehmann alternatives.

The values of the probabilities of the rank orders for the normal alternative were computed by getting P(I<II) from Teichroew (1954) and then making use of

$$P(S_{m+1}=i) = \frac{1}{m+2-i}\left[(m+1)P(S_m=i) - iP(S_{m+1}=i+1)\right]$$

where S_M is the rank of the observation from the second sample when the first sample contains M observations.

Table III. Distribution of T

IIIa. Values of T for small samples. Values of

$$T = \sum_{i=1}^{m+n} i^{-1} v_i = \sum_{i=1}^{m+n} z_i D_{Ni} = \sum_{i=1}^{n} D_{Ns_i}$$

are given for <u>each</u> rank order when $1 \leq m < n \leq 10$ and for <u>half</u> of the rank orders when m=n=1(1)5. The rank orders are arranged in order of increasing values of T. The table was computed by making use of the recursion formulas

$$T(z^0) = T(z) + n/(m+n+1)$$

$$T(z^1) = T(z) + (n+1)/(m+n+1) .$$

The rank orders z^0 and z^1 (used at the end of section 7.a) are formed from z by placing an additional element, 0 for z^0 and 1 for z^1, at the extreme right of z.

The numbering system of the rank orders used in this table was followed in all of the tables.

IIIb. Normal approximation to the distribution of T. The exact distribution of

$$X = \frac{T - n}{\sqrt{mn(N-D_{N1})/N(N-1)}}$$

is compared with the distribution that X would have if it were

standard normal for m=5, n=4, and m=5, n=5. The column headed P_1 is the exact cumulative probability of X and the column headed P_2 is the cumulative normal probability using the limiting distribution of X.

The justification of this comparison is given by theorem 7.e.1.

IIIb'. Exact significance levels of T using the normal approximation. The exact levels of significance which would be obtained in testing H_0 against H_L if H_0 were rejected when

$$\frac{T - n}{\sqrt{mn(N-D_{N1})/N(N-1)}} > \lambda_\alpha ,$$

where $\lambda_\alpha (\alpha = .05, .10)$ is the upper α-percentage point of the normal distribution, are given for m=4, n=4; m=4, n=5; m=5, n=4; m=4, n=6; m=5, n=5; m=6, n=4.

IIIc. Standard deviation of T. The exact standard deviation of T (see theorem 7.d.4), i.e.

$$\sigma_T = \sqrt{\frac{mn}{m+n-1}\left(1 - \frac{1}{m+n}\sum_{i=1}^{m+n} i^{-1}\right)} ,$$

is given for $1 \leq m \leq n \leq 20$.

For larger values of m+n corollary 7.d.4 approximates σ_T by

$$\sqrt{\frac{mn}{N-1}\left(1 - \frac{\ln N + \gamma + \frac{1}{2}N^{-1}}{N}\right)} \quad ,$$

which is correct to at least 3 decimal places.

Table IV. Power functions of nonparametric tests

IVa. Parametric versus nonparametric tests. For each combination of

$$m = 1(1)5$$
$$n = 1(1)5$$

and

$$\alpha = .10; \quad \beta = .25 \text{ and } .50$$
$$\alpha = .05; \quad \beta = .05 \text{ and } .25 \quad ,$$

the powers of the test based on T, the best nonparametric test, and the c_1 test (Terry, [1952]) corresponding to the Lehmann alternatives with the δ's of the table Ia are given.

IVb. Comparison of nonparametric tests for different alternatives. This table is like IVa except that it is only for m = n = 4 and compares the powers of the nonparametric tests in such a manner as to show how the power increases with the

severity of the alternatives (see the introduction to table IIa').

The alternatives are the same as those used in preparing IIa'.

Table I. POWER FUNCTIONS OF PARAMETRIC TESTS

Table Ia. Exponential alternatives

$\alpha = .10, \beta = .50$

m \ n	1	2	3	4	5
1	9.0000	7.6575	7.2717	7.0891	6.9826
2	5.2202	4.1073	3.7769	3.6173	3.5233
3	4.4413	3.3771	3.0546	2.8968	2.8029
4	4.1133	3.0682	2.7478	2.5893	2.4942
5	3.9335	2.8985	2.5782	2.4188	2.3226

Table Ib. Normal alternatives

$\alpha = .10, \beta = .50$

m \ n	1	2	3	4	5
1	1.813	1.570	1.481	1.433	1.404
2		1.282	1.170	1.110	1.073
3			1.047	.9792	.9362
4				.9065	.8600
5					.8109

Table I. POWER FUNCTIONS OF PARAMETRIC TESTS

Table Ia, continued

$\alpha = .10, \beta = .25$

m \ n	1	2	3	4	5
1	27.0000	18.4858	16.4334	15.5199	15.0031
2	13.9771	8.4783	7.1663	6.5817	6.2518
3	11.4708	6.6052	5.4436	4.9243	4.6300
4	10.4370	5.8387	4.7389	4.2454	3.9646
5	9.8760	5.4242	4.3572	3.8772	3.6030

Table Ib, continued

$\alpha = .10, \beta = .25$

m \ n	1	2	3	4	5
1	2.766	2.397	2.260	2.187	2.142
2		1.956	1.786	1.694	1.637
3			1.597	1.494	1.429
4				1.383	1.312
5					1.237

Table I. POWER FUNCTIONS OF PARAMETRIC TESTS

Table Ia, continued

$\alpha = .05, \beta = .25$

m \ n	1	2	3	4	5
1	57.0000	38.4940	34.0633	32.0958	30.9851
2	22.4440	13.1867	11.0147	10.0534	9.5126
3	17.0351	9.4147	7.6343	6.8455	6.4006
4	14.9492	7.9845	6.3591	5.6371	5.2287
5	13.8552	7.2412	5.6970	5.0099	4.6201

Table Ib, continued

$\alpha = .05, \beta = .25$

m \ n	1	2	3	4	5
1	3.350	2.903	2.737	2.649	2.595
2		2.370	2.163	2.052	1.983
3			1.935	1.810	1.730
4				1.675	1.589
5					1.499

Table I. POWER FUNCTIONS OF PARAMETRIC TESTS

Table Ia, continued

$\alpha = .05, \beta = .05$

m \ n	1	2	3	4	5
1	361.0000	133.6569	99.4200	86.3753	79.5779
2	133.6569	40.8104	27.9416	23.1841	20.7442
3	99.4200	27.9416	18.3518	14.8480	13.0618
4	86.3753	23.1841	14.8480	11.8205	10.2816
5	79.5779	20.7442	13.0618	10.2816	8.8697

Table Ib, continued

$\alpha = .05, \beta = .05$

m \ n	1	2	3	4	5
1	4.793	4.153	3.915	3.790	3.712
2		3.390	3.094	2.936	2.836
3			2.768	2.589	2.476
4				2.397	2.274
5					2.144

Table II. PROBABILITIES OF RANK ORDERS

Table IIa. Lehmann alternatives

N = 2

m = 1, n = 1

i	R.O. \ δ	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	9.0000	27.0000	57.0000	361.0000
1	01	.9000	.9643	.9828	.9972
2	10	.1000	.0357	.0172	.0028

N = 3

m = 1, n = 2

i	R.O. \ δ	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	7.6575	18.4868	38.4940	133.6569
1	011	.8303	.9237	.9622	.9889
2	101	.1084	.0500	.0250	.0074
3	110	.0613	.0264	.0128	.0037

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 3

m = 2, n = 1

i	R.O. \ δ	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
		5.2202	13.9791	22.4440	133.6569
1	001	.7230	.8748	.9182	.9853
2	010	.2325	.1168	.0783	.0146
3	100	.0445	.0084	.0035	.0001

Table IIa, continued

N = 4

m = 1, n = 3

i	R.O. \ δ	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
		7.2717	16.4334	34.0633	99.4200
1	0111	.7865	.8966	.9482	.9818
2	1011	.1082	.0546	.0278	.0101
3	1101	.0615	.0290	.0143	.0051
4	1110	.0438	.0199	.0097	.0034

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 4

m = 2, n = 2

i	R.O. \ δ	$P^{.10}_{.50}$ 4.1073	$P^{.10}_{.25}$ 8.4783	$P^{.05}_{.25}$ 13.1867	$P^{.05}_{.05}$ 40.8104
1	0011	.5408	.7238	.8071	.9305
2	0101	.2118	.1527	.1138	.0445
3	0110	.1404	.0891	.0631	.0231
4	1001	.0516	.0180	.0086	.0011
5	1010	.0342	.0105	.0048	.0006
6	1100	.0213	.0059	.0026	.0003

N = 4

m = 3, n = 1

i	R.O. \ δ	$P^{.10}_{.50}$ 4.4413	$P^{.10}_{.25}$ 11.4708	$P^{.05}_{.25}$ 17.0351	$P^{.05}_{.05}$ 99.4200
1	0001	.5968	.7927	.8503	.9707
2	0010	.2780	.1765	.1340	.0287
3	0100	.1022	.0283	.0149	.0006
4	1000	.0230	.0025	.0009	.0000

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 5

m = 1, n = 4

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	7.0891	15.5199	32.0958	86.3753
1	01111	.7552	.8769	.9378	.9763
2	10111	.1065	.0565	.0292	.0113
3	11011	.0608	.0301	.0151	.0057
4	11101	.0434	.0208	.0104	.0038
5	11110	.0341	.0159	.0079	.0029

N = 5

m = 2, n = 3

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	3.7769	7.1663	11.0147	27.9416
1	00111	.4394	.6277	.7316	.8800
2	01011	.1840	.1537	.1218	.0608
3	01101	.1242	.0919	.0688	.0320
4	01110	.0963	.0667	.0486	.0218
5	10011	.0487	.0214	.0111	.0022
6	10101	.0329	.0128	.0063	.0011
7	10110	.0255	.0093	.0044	.0008
8	11001	.0208	.0073	.0034	.0006
9	11010	.0161	.0054	.0024	.0004
10	11100	.0122	.0038	.0017	.0003

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 5

m = 3, n = 2

i	R.O. \ δ	$p^{.10}_{.50}$	$p^{.10}_{.25}$	$p^{.05}_{.25}$	$p^{.05}_{.05}$
		3.3771	6.6052	9.4147	27.9416
1	00011	.3667	.5604	.6541	.8570
2	00101	.2046	.1954	.1719	.0859
3	00110	.1490	.1234	.1025	.0459
4	01001	.0934	.0514	.0330	.0062
5	01010	.0681	.0324	.0197	.0033
6	01100	.0472	.0197	.0113	.0017
7	10001	.0277	.0078	.0035	.0002
8	10010	.0201	.0049	.0021	.0001
9	10100	.0140	.0030	.0012	.0001
10	11000	.0091	.0017	.0007	.0000

N = 5

m = 4, n = 1

i	R.O. \ δ	$p^{.10}_{.50}$	$p^{.10}_{.25}$	$p^{.05}_{.25}$	$p^{.05}_{.05}$
		4.1133	10.4370	14.9492	86.3753
1	00001	.5070	.7229	.7889	.9557
2	00010	.2851	.2152	.1758	.0428
3	00100	.1399	.0519	.0311	.0015
4	01000	.0547	.0091	.0039	.0000
5	10000	.0133	.0009	.0003	.0000

Table IIa. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 6

m = 1, n = 5

		$p_{.50}^{.10}$	$p_{.25}^{.10}$	$p_{.25}^{.05}$	$p_{.05}^{.05}$
i	R.O. \ δ	6.9826	15.0031	30.9851	79.5779
1	011111	.7312	.8615	.9297	.9718
2	101111	.1047	.0574	.0300	.0122
3	110111	.0598	.0306	.0155	.0062
4	111011	.0428	.0211	.0105	.0041
5	111101	.0336	.0162	.0080	.0031
6	111110	.0278	.0132	.0064	.0025

N = 6

m = 2, n = 4

		$p_{.50}^{.10}$	$p_{.25}^{.10}$	$p_{.25}^{.05}$	$p_{.05}^{.05}$
i	R.O. \ δ	3.6173	6.5817	10.0534	23.1841
1	001111	.3743	.5619	.6777	.8397
2	010111	.1621	.1482	.1226	.0694
3	011011	.1106	.0898	.0700	.0369
4	011101	.0862	.0656	.0497	.0253
5	011110	.0716	.0522	.0388	.0193
6	100111	.0448	.0225	.0122	.0030
7	101011	.0305	.0136	.0070	.0016
8	101101	.0238	.0100	.0050	.0011
9	101110	.0198	.0079	.0039	.0008
10	110011	.0195	.0078	.0038	.0008
11	110101	.0152	.0058	.0027	.0006
12	110110	.0126	.0046	.0021	.0004
13	111001	.0115	.0042	.0019	.0004
14	111010	.0096	.0033	.0015	.0003
15	111100	.0078	.0026	.0011	.0002

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 6

m = 3, n = 3

i	R.O. \ δ	$P^{.10}_{.50}$ 3.0546	$P^{.10}_{.25}$ 5.4436	$P^{.05}_{.25}$ 7.6343	$P^{.05}_{.05}$ 18.3518
1	000111	.2549	.4270	.5305	.7535
2	001011	.1513	.1721	.1652	.1111
3	001101	.1130	.1128	.1017	.0613
4	001110	.0922	.0854	.0746	.0426
5	010011	.0746	.0534	.0383	.0115
6	010101	.0557	.0350	.0236	.0063
7	010110	.0455	.0265	.0173	.0044
8	011001	.0396	.0219	.0140	.0034
9	011010	.0324	.0166	.0102	.0024
10	100011	.0244	.0098	.0050	.0006
11	011100	.0259	.0123	.0074	.0017
12	100101	.0182	.0064	.0031	.0003
13	100110	.0149	.0049	.0023	.0002
14	101001	.0129	.0040	.0018	.0002
15	101010	.0106	.0031	.0014	.0001
16	101100	.0085	.0023	.0010	.0001
17	110001	.0086	.0024	.0010	.0001
18	110010	.0070	.0018	.0007	.0001
19	110100	.0056	.0013	.0005	.0001
20	111000	.0043	.0010	.0004	.0000

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 6

m = 4, n = 2

i	R.O. \ δ	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
		3.0682	5.8387	7.9845	23.1841
1	000011	.2628	.4420	.5328	.7851
2	000101	.1732	.2000	.1940	.1199
3	000110	.1340	.1341	.1226	.0660
4	001001	.1025	.0765	.0583	.0143
5	001010	.0793	.0513	.0368	.0079
6	001100	.0591	.0332	.0225	.0043
7	010001	.0504	.0224	.0130	.0012
8	010010	.0390	.0150	.0082	.0007
9	010100	.0291	.0097	.0050	.0004
10	011000	.0207	.0060	.0029	.0002
11	100001	.0164	.0038	.0016	.0001
12	100010	.0127	.0026	.0010	.0000
13	100100	.0095	.0017	.0006	.0000
14	101000	.0067	.0010	.0004	.0000
15	110000	.0044	.0006	.0002	.0000

N = 6

m = 5, n = 1

i	R.O. \ δ	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
		3.9335	9.8760	13.8552	79.5779
1	000001	.4403	.6639	.7348	.9409
2	000010	.2774	.2392	.2058	.0563
3	000100	.1600	.0743	.0488	.0027
4	001000	.0809	.0188	.0092	.0001
5	010000	.0328	.0035	.0012	.0000
6	100000	.0083	.0003	.0000	.0000

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 7

m = 2, n = 5

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	3.5233	6.2518	9.5126	20.7442
1	0011111	.3286	.5136	.6370	.8076
2	0101111	.1453	.1416	.1212	.0743
3	0110111	.0997	.0865	.0697	.0398
4	0111011	.0780	.0635	.0496	.0274
5	0111101	.0650	.0507	.0388	.0210
6	0111110	.0562	.0424	.0320	.0170
7	1001111	.0412	.0226	.0127	.0036
8	1010111	.0283	.0138	.0073	.0019
9	1011011	.0221	.0102	.0052	.0013
10	1011101	.0184	.0081	.0041	.0010
11	1100111	.0182	.0080	.0040	.0010
12	1011110	.0160	.0068	.0034	.0008
13	1101011	.0142	.0059	.0029	.0007
14	1101101	.0118	.0047	.0023	.0005
15	1110011	.0108	.0042	.0020	.0005
16	1101110	.0103	.0039	.0019	.0004
17	1110101	.0090	.0034	.0016	.0003
18	1110110	.0078	.0028	.0013	.0003
19	1111001	.0074	.0027	.0012	.0002
20	1111010	.0064	.0023	.0010	.0002
21	1111100	.0055	.0019	.0008	.0002

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 7

m = 3, n = 4

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	2.8968	4.9243	6.8455	14.8480
1	0001111	.1911	.3436	.4485	.6739
2	0010111	.1171	.1489	.1521	.1200
3	0011011	.0886	.0996	.0954	.0676
4	0011101	.0729	.0763	.0707	.0475
5	0100111	.0601	.0503	.0388	.0151
6	0011110	.0627	.0626	.0566	.0368
7	0101011	.0455	.0336	.0243	.0085
8	0101101	.0374	.0258	.0180	.0060
9	0110011	.0328	.0214	.0146	.0046
10	0101110	.0322	.0211	.0144	.0046
11	0110101	.0270	.0165	.0108	.0033
12	0110110	.0232	.0135	.0087	.0025
13	1000111	.0207	.0102	.0057	.0010
14	0111001	.0217	.0124	.0079	.0023
15	0111010	.0187	.0102	.0063	.0018
16	1001011	.0157	.0068	.0035	.0006
17	0111100	.0159	.0083	.0050	.0014
18	1001101	.0129	.0052	.0026	.0004
19	1010011	.0113	.0043	.0021	.0003
20	1001110	.0112	.0043	.0021	.0003
21	1010101	.0093	.0033	.0016	.0002
22	1010110	.0080	.0027	.0013	.0002
23	1011001	.0075	.0025	.0012	.0001
24	1100011	.0076	.0026	.0012	.0002
25	1011010	.0065	.0020	.0010	.0001
26	1100101	.0063	.0020	.0009	.0001
27	1011100	.0055	.0016	.0006	.0001
28	1100110	.0054	.0016	.0007	.0001
29	1101001	.0051	.0015	.0007	.0001
30	1101010	.0044	.0012	.0006	.0001

Table II. PROBABILITIES OF RANK ORDERS
Table IIa, continued

N = 7

m = 3, n = 4
(Continued)

i	R.O.	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
31	1110001	.0040	.0011	.0005	.0001
32	1101100	.0037	.0010	.0003	.0001
33	1110010	.0034	.0009	.0004	.0001
34	1110100	.0029	.0007	.0003	.0001
35	1111000	.0024	.0006	.0002	.0000

N = 7

m = 4, n = 3

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	2.7478	4.7389	6.3591	14.8480
1	0000111	.1587	.2976	.3860	.6371
2	0001011	.1103	.1538	.1650	.1428
3	0001101	.0876	.1077	.1087	.0823
4	0010011	.0697	.0685	.0592	.0254
5	0001110	.0740	.0860	.0823	.0583
6	0010101	.0554	.0480	.0390	.0146
7	0010110	.0468	.0383	.0295	.0104
8	0011001	.0425	.0324	.0248	.0082
9	0100011	.0372	.0239	.0161	.0032
10	0011010	.0359	.0254	.0188	.0058
11	0100101	.0296	.0167	.0106	.0018
12	0011100	.0298	.0195	.0140	.0041
13	0100110	.0250	.0133	.0080	.0013
14	0101001	.0227	.0113	.0067	.0010
15	0101010	.0192	.0089	.0051	.0007

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 7

m = 5, n = 2

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	2.8985	5.4242	7.2412	20.7442
1	0000011	.1970	.3562	.4397	.7191
2	0000101	.1428	.1890	.1956	.1453
3	0000110	.1151	.1327	.1295	.0822
4	0001001	.0968	.0897	.0764	.0245
5	0001010	.0781	.0630	.0506	.0138
6	0010001	.0593	.0362	.0248	.0032
7	0001100	.0612	.0429	.0325	.0077
8	0010010	.0478	.0255	.0164	.0018
9	0010100	.0375	.0173	.0106	.0010
10	0100001	.0304	.0113	.0060	.0003
11	0011000	.0284	.0113	.0066	.0005
12	0100010	.0245	.0079	.0040	.0002
13	0100100	.0192	.0054	.0026	.0001
14	0101000	.0146	.0035	.0016	.0000
15	0110000	.0105	.0022	.0010	.0000
16	1000001	.0105	.0021	.0008	.0000
17	1000010	.0085	.0015	.0006	.0000
18	1000100	.0066	.0010	.0004	.0000
19	1001000	.0050	.0006	.0002	.0000
20	1010000	.0036	.0004	.0001	.0000
21	1100000	.0024	.0002	.0001	.0000

Table IIa. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 8

m = 3, n = 5

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	2.8029	4.6300	6.4006	13.0618
1	00011111	.1507	.2872	.3904	.6126
2	00101111	.0941	.1300	.1394	.1220
3	00110111	.0718	.0881	.0885	.0697
4	00111011	.0594	.0680	.0660	.0493
5	01001111	.0495	.0462	.0377	.0174
6	00111101	.0513	.0560	.0531	.0383
7	01010111	.0378	.0314	.0239	.0099
8	00111110	.0455	.0479	.0447	.0314
9	01011011	.0312	.0242	.0178	.0070
10	01100111	.0275	.0203	.0145	.0055
11	01011101	.0270	.0199	.0144	.0054
12	01011110	.0239	.0170	.0121	.0045
13	01101011	.0227	.0156	.0108	.0039
14	01101101	.0196	.0129	.0088	.0030
15	10001111	.0177	.0100	.0059	.0013

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 8

m = 4, n = 4

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	2.5893	4.2454	5.6371	11.8205
1	00001111	.1056	.2156	.2966	.5295
2	00010111	.0756	.1190	.1374	.1429
3	00011011	.0609	.0854	.0928	.0849
4	00100111	.0494	.0572	.0540	.0310
5	00011101	.0519	.0678	.0712	.0610
6	00011110	.0457	.0568	.0583	.0479
7	00101011	.0398	.0410	.0365	.0184
8	00101101	.0339	.0326	.0280	.0132
9	00110011	.0310	.0283	.0237	.0106
10	01000111	.0275	.0218	.0163	.0048
11	00101110	.0299	.0273	.0229	.0104
12	00110101	.0264	.0225	.0182	.0076
13	01001011	.0222	.0156	.0110	.0029
14	00110110	.0233	.0189	.0149	.0060
15	00111001	.0221	.0175	.0137	.0054
16	01001101	.0189	.0124	.0084	.0021
17	01010011	.0173	.0108	.0071	.0017
18	00111010	.0195	.0147	.0112	.0042
19	01001110	.0167	.0104	.0069	.0016
20	01010101	.0147	.0086	.0054	.0012
21	00111100	.0170	.0122	.0091	.0033
22	01010110	.0130	.0072	.0045	.0009
23	01100011	.0128	.0071	.0044	.0010
24	01011001	.0123	.0067	.0041	.0009
25	01100101	.0109	.0057	.0034	.0007
26	01011010	.0109	.0056	.0034	.0007
27	10000111	.0106	.0051	.0029	.0004
28	01100110	.0097	.0047	.0028	.0005
29	01011100	.0095	.0047	.0027	.0005
30	01101001	.0091	.0044	.0026	.0005

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 8

m = 5, n = 3

		$p^{.10}_{.50}$	$p^{.10}_{.25}$	$p^{.05}_{.25}$	$p^{.05}_{.05}$
i	R.O. \ δ	2.5782	4.3572	5.6970	13.0618
1	00000111	.1049	.2140	.2864	.5383
2	00001011	.0897	.1280	.1477	.1578
3	00001101	.0660	.0942	.1026	.0946
4	00010011	.0572	.0696	.0679	.0393
5	00001110	.0571	.0757	.0798	.0682
6	00010101	.0473	.0512	.0472	.0236
7	00010110	.0409	.0608	.0367	.0170
8	00100011	.0375	.0328	.0265	.0078
9	00011001	.0381	.0365	.0318	.0138
10	00100101	.0310	.0242	.0184	.0047
11	00011010	.0330	.0434	.0247	.0100
12	00100110	.0268	.0287	.0143	.0034
13	00011100	.0281	.0343	.0189	.0071
14	00101001	.0250	.0172	.0124	.0027
15	01000011	.0210	.0122	.0079	.0011

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 9

m = 4, n = 5

i	R.O. \ δ	$p^{.10}_{.50}$	$p^{.10}_{.25}$	$p^{.05}_{.25}$	$p^{.05}_{.05}$
		2.4942	3.9646	5.2287	10.2816
1	000011111	.0751	.1645	.2377	.4511
2	000101111	.0547	.0945	.1155	.1359
3	000110111	.0445	.0689	.0792	.0824
4	001001111	.0365	.0475	.0479	.0332
5	000111011	.0382	.0552	.0613	.0598
6	000111101	.0338	.0465	.0505	.0472
7	001010111	.0297	.0347	.0329	.0201
8	000111110	.0305	.0452	.0432	.0391
9	001011011	.0255	.0278	.0254	.0146
10	001100111	.0233	.0243	.0217	.0118
11	010001111	.0209	.0191	.0154	.0059
12	001011101	.0226	.0234	.0210	.0115
13	001101011	.0200	.0195	.0168	.0086
14	001011110	.0204	.0227	.0179	.0096
15	010010111	.0170	.0140	.0106	.0036
16	001101101	.0178	.0164	.0139	.0068
17	001110011	.0168	.0153	.0128	.0062
18	010011011	.0146	.0112	.0082	.0026
19	001101110	.0160	.0159	.0118	.0057
20	010100111	.0133	.0098	.0070	.0021
21	001110101	.0150	.0129	.0106	.0049
22	010011101	.0129	.0094	.0067	.0020
23	001110110	.0135	.0125	.0090	.0041
24	010101011	.0114	.0079	.0054	.0015
25	001111001	.0131	.0108	.0086	.0038

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 9

m = 4, n = 5
(Continued)

i	R.O.	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
26	010011110	.0117	.0091	.0057	.0017
27	001111010	.0118	.0104	.0073	.0032
28	010101101	.0101	.0066	.0044	.0012
29	011000111	.0100	.0065	.0044	.0012
30	010110011	.0096	.0062	.0041	.0011
31	010101110	.0092	.0064	.0038	.0010
32	001111100	.0106	.0090	.0062	.0026

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 9

m = 5, n = 4

i	R.O. \ δ	$p^{.10}_{.50}$ 2.4188	$p^{.10}_{.25}$ 3.8772	$p^{.05}_{.25}$ 5.0099	$p^{.05}_{.05}$ 10.2816
1	000001111	.0626	.1404	.2005	.4152
2	000010111	.0488	.0891	.1113	.1454
3	000011011	.0410	.0673	.0795	.0905
4	000100111	.0360	.0518	.0556	.0438
5	000011101	.0358	.0549	.0627	.0664
6	000101011	.0303	.0391	.0397	.0273
7	000011110	.0321	.0468	.0522	.0527
8	000101101	.0264	.0319	.0313	.0200
9	001000111	.0244	.0264	.0238	.0107
10	000110011	.0248	.0286	.0275	.0165
11	000101110	.0237	.0272	.0261	.0159
12	001001011	.0206	.0200	.0170	.0067
13	000110101	.0216	.0234	.0217	.0121
14	000110110	.0194	.0199	.0181	.0096
15	001001101	.0179	.0163	.0134	.0049

Table II. PROBABILITIES OF RANK ORDERS

Table IIa, continued

N = 10

m = 5, n = 5

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ	2.3226	3.6030	4.6201	8.8697
1	0000011111	.0404	.0982	.1482	.3308
2	0000101111	.0319	.0646	.0860	.1285
3	0000110111	.0270	.0496	.0625	.0820
4	0001001111	.0240	.0391	.0451	.0433
5	0000111011	.0237	.0409	.0498	.0609
6	0001010111	.0203	.0300	.0328	.0276
7	0000111101	.0213	.0351	.0418	.0488
8	0000111110	.0195	.0309	.0362	.0409
9	0001011011	.0179	.0247	.0262	.0205
10	0010001111	.0167	.0209	.0204	.0120
11	0001100111	.0168	.0223	.0231	.0171
12	0001011101	.0161	.0212	.0220	.0164
13	0010010111	.0141	.0160	.0148	.0077
14	0001101011	.0148	.0184	.0185	.0127
15	0001011110	.0147	.0187	.0190	.0138
16	0001101101	.0133	.0158	.0155	.0102
17	0010011011	.0124	.0132	.0119	.0057
18	0001110011	.0128	.0149	.0145	.0093
19	0011000111	.0117	.0119	.0105	.0047
20	0001101110	.0122	.0139	.0134	.0086
21	0010011101	.0112	.0114	.0100	.0045
22	0001110101	.0115	.0128	.0122	.0075
23	0100001111	.0101	.0091	.0073	.0024
24	0010101011	.0103	.0099	.0084	.0035
25	0010011110	.0102	.0100	.0086	.0038
26	0001110110	.0106	.0113	.0105	.0063
27	0001111001	.0103	.0109	.0101	.0060
28	0011001111	.0094	.0085	.0071	.0013
29	0010101101	.0092	.0085	.0070	.0028
30	0100010111	.0085	.0070	.0053	.0016

Table II. PROBABILITIES OF RANK ORDERS
Table IIa, continued

N = 10

m = 5, n = 5
(Continued)

i	R.O.	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
31	0010110011	.0089	.0080	.0060	.0026
32	0001111010	.0095	.0096	.0087	.0050
33	0010101110	.0085	.0074	.0061	.0024
34	0011001011	.0092	.0071	.0057	.0021
35	0010110101	.0080	.0069	.0055	.0021
36	0100011011	.0075	.0057	.0042	.0012
37	0001111100	.0086	.0084	.0075	.0042
38	0100100111	.0070	.0052	.0037	.0010
39	0011001101	.0074	.0061	.0047	.0017
40	0010110110	.0074	.0061	.0046	.0017
41	0011010011	.0071	.0057	.0045	.0016
42	0100011101	.0067	.0050	.0036	.0009
43	0010111001	.0071	.0058	.0046	.0017
44	0011001110	.0068	.0053	.0041	.0014
45	0100101011	.0064	.0043	.0030	.0007
46	0100011110	.0061	.0043	.0031	.0008
47	0011010101	.0064	.0049	.0038	.0013
48	0010111010	.0066	.0051	.0039	.0014
49	0101000111	.0057	.0037	.0025	.0003

Table II. PROBABILITIES OF RANK ORDERS

Table IIa'. Lehmann alternatives

N = 8

m = 4, n = 4

		$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O. \ δ*	2.8968	4.9243	6.8455	14.8480
1	00001111	.1265	.2567	.3567	.5969
2	00010111	.0858	.1296	.1449	.1338
3	00011011	.0673	.0900	.0942	.0771
4	00100111	.0526	.0562	.0491	.0238
5	00011101	.0564	.0701	.0708	.0546
6	00011110	.0491	.0580	.0572	.0425
7	00101011	.0413	.0390	.0319	.0137
8	00101101	.0346	.0304	.0240	.0097
9	00110011	.0312	.0261	.0200	.0077
10	01000111	.0270	.0190	.0125	.0030
11	00101110	.0252	.0196	.0146	.0054
12	00110101	.0262	.0203	.0151	.0055
13	01001011	.0212	.0132	.0081	.0017
14	00110110	.0191	.0131	.0092	.0030
15	00111001	.0216	.0156	.0112	.0039

* The alternatives correspond to m = 3, n = 4.

Table II. PROBABILITIES OF RANK ORDERS

Table IIa', continued

N = 8

m = 4, n = 4

			$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
i	R.O.	δ*	3.0546	5.4436	7.6343	18.3518
1	00001111		.1372	.2861	.3914	.6545
2	00010111		.0906	.1355	.1472	.1226
3	00011011		.0702	.0921	.0937	.0690
4	00100111		.0538	.0546	.0458	.0181
5	00011101		.0583	.0709	.0697	.0484
6	00011110		.0504	.0582	.0559	.0374
7	00101011		.0417	.0371	.0292	.0102
8	00101101		.0347	.0286	.0217	.0072
9	00110011		.0311	.0243	.0180	.0056
10	01000111		.0265	.0169	.0106	.0019
11	00101110		.0249	.0181	.0129	.0039
12	00110101		.0259	.0187	.0134	.0040
13	01001011		.0206	.0115	.0068	.0011
14	00110110		.0186	.0119	.0079	.0022
15	00111001		.0211	.0142	.0098	.0028

* The alternatives correspond to m = n = 3.

Table II. PROBABILITIES OF RANK ORDERS

Table IIb. Lehmann versus normal alternatives for I < II

$\alpha = .10 \quad \beta = .50$

m \ n	1	2	3	4	5
1	.9000 .9001	.8303 .7819	.7865 .6966	.7552 .6325	.7312 .5828
2	.7230 .7819	.5408 .5588	.4394 .4219	.3743 .3336	.3286 .2722
3	.5968 .6966	.3667 .4219	.2549 .2754	.1911 .1918	.1507 .1404
4	.5070 .6235	.2628 .3336	.1587 .1918	.1056 .1193	.0751 .0789
5	.4403 .5828	.1970 .2722	.1049 .1404	.0626 .0789	.0404 .0477

In each case, the first entry corresponds to the Lehmann alternative; the second entry to the normal alternative.

Table II. PROBABILITIES OF RANK ORDERS

Table IIb, continued

$\alpha = .10 \quad \beta = .25$

m \ n	1	2	3	4	5
1	.9643 .9748	.9237 .9204	.8966 .8710	.8769 .8289	.8615 .7933
2	.8748 .9204	.7238 .7637	.6277 .6391	.5619 .5457	.5136 .4754
3	.7927 .8710	.5604 .6391	.4270 .4737	.3436 .3634	.2872 .2875
4	.7229 .8289	.4420 .5458	.2976 .3634	.2156 .2522	.1645 .1821
5	.6639 .7933	.3562 .4754	.2140 .2875	.1404 .1821	.0982 .1211

Table II. PROBABILITY OF RANK ORDERS

Table IIb, continued

$\alpha = .05 \quad \beta = .25$

m \ n	1	2	3	4	5
1	.9828 .9911	.9622 .9633	.9482 .9338	.9378 .9068	.9297 .8826
2	.9182 .9633	.8071 .8561	.7316 .7543	.6777 .6710	.6370 .6040
3	.8503 .9338	.6541 .7543	.5305 .6002	.4485 .4856	.3904 .4006
4	.7889 .9068	.5328 .6710	.3860 .4856	.2966 .3588	.2377 .2726
5	.7348 .8826	.4397 .6040	.2864 .4006	.2005 .2726	.1482 .1921

Table II. PROBABILITIES OF RANK ORDERS

Table IIb, continued

$\alpha = .05 \quad \beta = .05$

m \ n	1	2	3	4	5
1	.9972 .9996	.9889 .9968	.9818 .9922	.9763 .9870	.9718 .9816
2	.9853 .9968	.9305 .9709	.8800 .9316	.8397 .8910	.8076 .8518
3	.9707 .9922	.8570 .9316	.7535 .8458	.6739 .7619	.6126 .6880
4	.9557 .9870	.7851 .8910	.6371 .7619	.5295 .6444	.4511 .5470
5	.9409 .9816	.7191 .8510	.5383 .6880	.4152 .5469	.3308 .4371

Table II. PROBABILITIES OF RANK ORDERS

Table IIb'. Lehmann versus normal alternatives for n = 1.

R.O.	$\tau = \theta_2 - \theta_1 = 2.0000$	$\delta = 11.71455$
	m = 1	
01	.92135	.92135
10	.07865	.07865
	m = 2	
001	.86577	.85417
010	.11116	.13436
100	.02307	.01147
	m = 3	
0001	.82279	.79612
0010	.12894	.17415
0100	.03780	.02739
1000	.01046	.00234
	m = 4	
00001	.78784	.74546
00010	.13980	.20264
00100	.04818	.04434
01000	.01828	.00696
10000	.00587	.00060
	m = 5	
000001	.75845	.70086
000010	.14695	.22300
000100	.05560	.06060
001000	.02470	.01330
010000	.01050	.00205
100000	.00373	.00019

Table III. DISTRIBUTION OF T

Table IIIa. Values of T for small samples

N = 2

m = 1, n = 1

i	R.O.	T
1	01	0.5000

N = 3

m = 1, n = 2

i	R.O.	T
1	011	1.1667
2	101	2.1667
3	110	2.6667

Table III. DISTRIBUTION OF T
Table IIIa, continued

N = 4

m = 1, n = 3

i	R.O.	T
1	0111	1.9167
2	1011	2.9167
3	1101	3.4167
4	1110	3.7500

N = 4

m = 2, n = 2

i	R.O.	T
1	0011	.8333
2	0101	1.3333
3	0110	1.6667

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 5

m = 1, n = 4

i	R.O.	T
1	01111	2.7167
2	10111	3.7167
3	11011	4.2167
4	11101	4.5500
5	11110	4.8000

N = 5

m = 2, n = 3

i	R.O.	T
1	00111	1.4333
2	01011	1.9333
3	01101	2.2667
4	01110	2.5167
5	10011	2.9333
6	10101	3.2667
7	10110	3.5167
8	11001	3.7667
9	11010	4.0167
10	11100	4.3500

Table III. DISTRIBUTION OF T
Table IIIa, continued

N = 6

m = 1, n = 5

i	R. O.	T
1	011111	3.5500
2	101111	4.5500
3	110111	5.0500
4	111011	5.3833
5	111101	5.6333
6	111110	5.8333

N = 6

m = 2, n = 4

i	R.O.	T
1	001111	2.1000
2	010111	2.6000
3	011011	2.9333
4	011101	3.1833
5	011110	3.3833
6	100111	3.6000
7	101011	3.9333
8	101101	4.1833
9	101110	4.3833
10	110011	4.4333
11	110101	4.6833
12	110110	4.8833
13	111001	5.0167
14	111010	5.2167
15	111100	5.4667

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 6

m = 3, n = 3

i	R.O.	T
1	000111	1.1500
2	001011	1.4833
3	001101	1.7333
4	001110	1.9333
5	010011	1.9833
6	010101	2.2333
7	010110	2.4333
8	011001	2.5667
9	011010	2.7667
10	100011	2.9833

N = 7

m = 1, n = 6

i	R.O.	T
1	0111111	4.4071
2	1011111	5.4071
3	1101111	5.9071
4	1110111	6.2404
5	1111011	6.4904
6	1111101	6.6905
7	1111110	6.8571

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 7

m = 2, n = 5

i	R.O.	T
1	0011111	2.8143
2	0101111	3.3143
3	0110111	3.6476
4	0111011	3.8976
5	0111101	4.0976
6	0111110	4.2643
7	1001111	4.3143
8	1010111	4.6476
9	1011011	4.8976
10	1011101	5.0976
11	1100111	5.1476
12	1011110	5.2643
13	1101011	5.3976
14	1101101	5.5976
15	1110011	5.7310
16	1101110	5.7643
17	1110101	5.9310
18	1110110	6.0976
19	1111001	6.1810
20	1111010	6.3476
21	1111100	6.5476

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 7

m = 3, n = 4

i	R.O.	T
1	0001111	1.7214
2	0010111	2.0548
3	0011011	2.3048
4	0011101	2.5048
5	0100111	2.5548
6	0011110	2.6714
7	0101011	2.8048
8	0101101	3.0048
9	0110011	3.1381
10	0101110	3.1714
11	0110101	3.3381
12	0110110	3.5048
13	1000111	3.5548
14	0111001	3.5881
15	0111010	3.7548
16	1001011	3.8048
17	0111100	3.9548
18	1001101	4.0048
19	1010011	4.1381
20	1001110	4.1714
21	1010101	4.3381
22	1010110	4.5048
23	1011001	4.5881
24	1100011	4.6381
25	1011010	4.7548
26	1100101	4.8381
27	1011100	4.9548
28	1100110	5.0048
29	1101001	5.0881
30	1101010	5.2548
31	1110001	5.4214
32	1101100	5.4548
33	1110010	5.5881
34	1110100	5.7881
35	1111000	6.0381

Table III. DISTRIBUTION OF T
Table IIIa, continued

N = 8

m = 1, n = 7

i	R.O.	T
1	01111111	5.2821
2	10111111	6.2821
3	11011111	6.7821
4	11101111	7.1155
5	11110111	7.3654
6	11111011	7.5654
7	11111101	7.7321
8	11111110	7.8750

N = 8

m = 2, n = 6

i	R.O.	T
1	00111111	3.5643
2	01011111	4.0643
3	01101111	4.3976
4	01110111	4.6476
5	01111011	4.8476
6	01111101	5.0143
7	10011111	5.0643
8	01111110	5.1571
9	10101111	5.3976
10	10110111	5.6476
11	10111011	5.8476
12	11001111	5.8976
13	10111101	6.0143
14	11010111	6.1476
15	10111110	6.1571

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 8

m = 2, n = 6

(Continued)

i	R.O.	T
16	11011011	6.3476
17	11100111	6.4810
18	11011101	6.5143
19	11011110	6.6571
20	11101011	6.6810
21	11101101	6.8476
22	11110011	6.9310
23	11101110	6.9905
24	11110101	7.0976
25	11110110	7.2405
26	11111001	7.2976
27	11111010	7.4405
28	11111100	7.6071

N = 8

m = 3, n = 5

i	R.O.	T
1	00011111	2.3464
2	00101111	2.6798
3	00110111	2.9298
4	00111011	3.1298
5	01001111	3.1798
6	00111101	3.2964
7	01010111	3.4298
8	00111110	3.4393
9	01011011	3.6298
10	01100111	3.7631

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 8

m = 3, n = 5

(Continued)

i	R.O.	T.
11	01011101	3.7964
12	01011110	3.9393
13	01101011	3.9631
14	01101101	4.1298
15	10001111	4.1798
16	01110011	4.2131
17	01101110	4.2726
18	01110101	4.3798
19	10010111	4.4298
20	01110110	4.5226
21	01111001	4.5798
22	10011011	4.6298
23	01111010	4.7226
24	10100111	4.7631
25	10011101	4.7964
26	01111100	4.8893
27	10011110	4.9393
28	10101011	4.9630
29	10101101	5.1298
30	10110011	5.2131
31	11000111	5.2631
32	10101110	5.2726
33	10110101	5.3798
34	11001011	5.4631
35	10110110	5.5226
36	10111001	5.5798
37	11001101	5.6298
38	11010011	5.7131
39	10111010	5.7226
40	11001110	5.7726

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 8

m = 3, n = 5
(Continued)

i	R.O.	T
41	11010101	5.8798
42	10111100	5.8893
43	11010110	6.0226
44	11100011	6.0464
45	11011001	6.0798
46	11100101	6.2131
47	11011010	6.2226
48	11100110	6.3560
49	11011100	6.3893
50	11101001	6.4131
51	11101010	6.5560
52	11110001	6.6631
53	11101100	6.7226
54	11110010	6.8060
55	11110100	6.9726
56	11111000	7.1726

N = 8

m = 4, n = 4

i	R.O.	T
1	00001111	1.4619
2	00010111	1.7119
3	00011011	1.9119
4	00100111	2.0452
5	00011101	2.0786
6	00011110	2.2214
7	00101011	2.2452
8	00101101	2.4119
9	00110011	2.4952
10	01000111	2.5452

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 8

m = 4, n = 4
(Continued)

i	R.O.	T
11	00101110	2.5548
12	00110101	2.6619
13	01001011	2.7452
14	00110110	2.8048
15	00111001	2.8619
16	01001101	2.9119
17	01010011	2.9952
18	00111010	3.0048
19	01001110	3.0548
20	01010101	3.1619
21	00111100	3.1714
22	01010110	3.3048
23	01100011	3.3286
24	01011001	3.3619
25	01100101	3.4952
26	01011010	3.5048
27	10000111	3.5452
28	01100110	3.6381
29	01011100	3.6714
30	01101001	3.6952
31	10001011	3.7452
32	01101010	3.8381
33	10001101	3.9119
34	01110001	3.9452
35	10010011	3.9952

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 9

m = 1, n = 8

i	R.O.	T
1	011111111	6.1710
2	101111111	7.1710
3	110111111	7.6710
4	111011111	8.0044
5	111101111	8.2544
6	111110111	8.4544
7	111111011	8.6210
8	111111101	8.7639
9	111111110	8.8889

N = 9

m = 2, n = 7

i	R.O.	T
1	001111111	4.3421
2	010111111	4.8421
3	011011111	5.1754
4	011101111	5.4254
5	011110111	5.6254
6	011111011	5.7921
7	100111111	5.8421
8	011111101	5.9349
9	011111110	6.0599
10	101011111	6.1754
11	101101111	6.4254
12	101110111	6.6254
13	110011111	6.6754
14	101111011	6.7921
15	110101111	6.9254

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 9

m = 2, n = 7
(Continued)

i	R.O.	T
16	101111101	6.9349
17	101111110	7.0599
18	110110111	7.1254
19	111001111	7.2587
20	110111011	7.2921
21	110111101	7.4349
22	111010111	7.4587
23	110111110	7.5599
24	111011011	7.6254
25	111100111	7.7087
26	111011101	7.7683
27	111101011	7.8754
28	111011110	7.8933
29	111101101	8.0183
30	111110011	8.0754
31	111101110	8.1433
32	111110101	8.2183
33	111110110	8.3433
34	111111001	8.3849
35	111111010	8.5099
36	111111100	8.6528

N = 9

m = 3, n = 6

i	R.O.	T
1	000111111	3.0131
2	001011111	3.3464
3	001101111	3.5964
4	001110111	3.7964
5	010011111	3.8464

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 9

m = 3, n = 6
(Continued)

i	R.O.	T
6	001111011	3.9631
7	010101111	4.0964
8	001111101	4.1060
9	001111110	4.2310
10	010110111	4.2964
11	011001111	4.4298
12	010111011	4.4630
13	010111101	4.6060
14	011010111	4.6298
15	010111110	4.7310
16	011011011	4.7964
17	100011111	4.8464
18	011100111	4.8798
19	011011101	4.9393
20	011101011	5.0464
21	011011110	5.0643
22	100101111	5.0964
23	011101101	5.1893
24	011110011	5.2464
25	100110111	5.2964
26	011101110	5.3143
27	011110101	5.3893
28	101001111	5.4298
29	100111011	5.4631
30	011110110	5.5143
31	011111001	5.5560
32	100111101	5.6060
33	101010111	5.6298
34	011111010	5.6810
35	100111110	5.7310
36	101011011	5.7964
37	011111100	5.8238
38	101100111	5.8798
39	110001111	5.9298
40	101011101	5.9393

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 9

m = 3, n = 6
(Continued)

i	R.O.	T
41	101101011	6.0464
42	101011110	6.0643
43	110010111	6.1298
44	101101101	6.1893
45	101110011	6.2464
46	110011011	6.2964
47	101101110	6.3143
48	110100111	6.3798
49	101110101	6.3893
50	110011101	6.4393
51	101110110	6.5143
52	110101011	6.5464
53	101111001	6.5560
54	110011110	6.5643
55	101111010	6.6810
56	110101101	6.6893
57	111000111	6.7131
58	110110011	6.7464
59	110101110	6.8143
60	101111100	6.8238
61	111001011	6.8798
62	110110101	6.8893
63	110110110	7.0143
64	111001101	7.0226
65	110111001	7.0560
66	111010011	7.0798
67	111001110	7.1476
68	110111010	7.1810
69	111010101	7.2226
70	110111100	7.3238
71	111100011	7.3298
72	111010110	7.3476
73	111011001	7.3893
74	111100101	7.4726
75	111011010	7.5143

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 9

m = 3, n = 6
(Continued)

i	R.O.	T
76	111100110	7.5976
77	111101001	7.6393
78	111011100	7.6571
79	111101010	7.7643
80	111110001	7.8393
81	111101100	7.9071
82	111110010	7.9643
83	111110100	8.1071
84	111111000	8.2738

N = 9

m = 4, n = 5

i	R.O.	T
1	000011111	2.0175
2	000101111	2.2675
3	000110111	2.4675
4	001001111	2.6008
5	000111011	2.6341
6	000111101	2.7770
7	001010111	2.8008
8	000111110	2.9020
9	001011011	2.9675
10	001100111	3.0508
11	010001111	3.1008
12	001011101	3.1103
13	001101011	3.2175
14	001011110	3.2353
15	010010111	3.3008

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 9

m = 4, n = 5
(Continued)

i	R.O.	T
16	001101101	3.3603
17	001110011	3.4175
18	010011011	3.4675
19	001101110	3.4853
20	010100111	3.5508
21	001110101	3.5603
22	010011101	3.6103
23	001110110	3.6853
24	010101011	3.7175
25	001111001	3.7270
26	010011110	3.7353
27	001111010	3.8520
28	010101101	3.8603
29	011000111	3.8841
30	010110011	3.9175
31	010101110	3.9853
32	001111100	3.9948
33	011001011	4.0508
34	010110101	4.0603
35	100001111	4.1008
36	010110110	4.1853
37	011001101	4.1937
38	010111001	4.2270
39	011010011	4.2508
40	100010111	4.3008
41	011001110	4.3187
42	010111010	4.3520
43	011010101	4.3937
44	100011011	4.4675
45	010111100	4.4948
46	011100011	4.5008
47	011010110	4.5187
48	100100111	4.5508
49	011011001	4.5603
50	100011101	4.6103

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 9

m = 4, n = 5
(Continued)

i	R.O.	T
51	011100101	4.6437
52	011011010	4.6853
53	100101011	4.7175
54	100011110	4.7353
55	011100110	4.7687
56	011101001	4.8103
57	011011100	4.8282
58	100101101	4.8603
59	101000111	4.8841
60	100110011	4.9175
61	011101010	4.9353
62	100101110	4.9853
63	011110001	5.0103
64	101001011	5.0508
65	100110101	5.0603
66	011101100	5.0782
67	011110010	5.1353
68	100110110	5.1853
69	101001101	5.1937
70	100111001	5.2270
71	101010011	5.2508
72	011110100	5.2781
73	101001110	5.3187
74	100111010	5.3520
75	110000111	5.3841
76	101010101	5.3937
77	011111000	5.4448
78	100111100	5.4948
79	101100011	5.5008
80	101010110	5.5187
81	110001011	5.5508
82	101011001	5.5603
83	101100101	5.6437
84	101011010	5.6853
85	110001101	5.6937

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 9

m = 4, n = 5
(Continued)

i	R.O.	T
86	110010011	5.7508
87	101100110	5.7687
88	101101001	5.8103
89	110001110	5.8187
90	101011100	5.8282
91	110010101	5.8937
92	101101010	5.9353
93	110100011	6.0008
94	101110001	6.0103
95	110010110	6.0187
96	110011001	6.0603
97	101101100	6.0782
98	101110010	6.1353
99	110100101	6.1437
100	110011010	6.1853
101	110100110	6.2686
102	101110100	6.2782
103	110101001	6.3103
104	110011100	6.3282
105	111000011	6.3341
106	110101010	6.4353
107	101111000	6.4448
108	111000101	6.4770
109	110110001	6.5103
110	110101100	6.5792
111	111000110	6.6020
112	110110010	6.6353
113	111001001	6.6437
114	111001010	6.7687
115	110110100	6.7782
116	111010001	6.8437
117	111001100	6.9115
118	110111000	6.9448
119	111010010	6.9687
120	111100001	7.0937

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 9

m = 4, n = 5
(Continued)

i	R.O.	T
121	111010100	7.1115
122	111100010	7.2187
123	111011000	7.2782
124	111100100	7.3615
125	111101000	7.5282
126	111110000	7.7282

N = 10

m = 1, n = 9

i	R.O.	T
1	0111111111	7.0710
2	1011111111	8.0710
3	1101111111	8.5710
4	1110111111	8.9044
5	1111011111	9.1544
6	1111101111	9.3544
7	1111110111	9.5210
8	1111111011	9.6639
9	1111111101	9.7889
10	1111111110	9.9000

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 2, n = 8

i	R.O.	T
1	0011111111	5.1421
2	0101111111	5.6421
3	0110111111	5.9754
4	0111011111	6.2254
5	0111101111	6.4254
6	0111110111	6.5921
7	1001111111	6.6421
8	0111111011	6.7349
9	0111111101	6.8599
10	0111111110	6.9710
11	1010111111	6.9754
12	1011011111	7.2254
13	1011101111	7.4254
14	1100111111	7.4754
15	1011110111	7.5921
16	1101011111	7.7254
17	1011111011	7.7349
18	1011111101	7.8599
19	1101101111	7.9254
20	1011111110	7.9710
21	1110011111	8.0587
22	1101110111	8.0921
23	1101111011	8.2349
24	1110101111	8.2587
25	1101111101	8.3599
26	1110110111	8.4254
27	1101111110	8.4710
28	1111001111	8.5087
29	1110111011	8.5683
30	1111010111	8.6754

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 2, n = 8
(Continued)

i	R.O.	T
31	1110111101	8.6933
32	1110111110	8.8044
33	1111011011	8.8183
34	1111100111	8.8754
35	1111011101	8.9433
36	1111101011	9.0183
37	1111011110	9.0544
38	1111101101	9.1433
39	1111110011	9.1849
40	1111101110	9.2544
41	1111110101	9.3099
42	1111110110	9.4210
43	1111111001	9.4528
44	1111111010	9.5639
45	1111111100	9.6889

N = 10

m = 3, n = 7

i	R.O.	T
1	0001111111	3.7131
2	0010111111	4.0464
3	0011011111	4.2964
4	0011101111	4.4964
5	0100111111	4.5464
6	0011110111	4.6631
7	0101011111	4.7964
8	0011111011	4.8060
9	0011111101	4.9310
10	0101101111	4.9964

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 3, n = 7
(Continued)

i	R.O.	T
11	0011111110	5.0421
12	0110011111	5.1298
13	0101110111	5.1631
14	0101111011	5.3060
15	0110101111	5.3298
16	0101111101	5.4310
17	0110110111	5.4964
18	0101111110	5.5421
19	1000111111	5.5464
20	0111001111	5.5798
21	0110111011	5.6393
22	0111010111	5.7464
23	0110111101	5.7643
24	1001011111	5.7964
25	0110111110	5.8754
26	0111011011	5.8893
27	0111100111	5.9464
28	1001101111	5.9964
29	0111011101	6.0143
30	0111101011	6.0893
31	0111011110	6.1254
32	1010011111	6.1298
33	1001110111	6.1631
34	0111101101	6.2143
35	0111110011	6.2560
36	1001111011	6.3060
37	0111101110	6.3254
38	1010101111	6.3298
39	0111110101	6.3810
40	1001111101	6.4310
41	0111110110	6.4921
42	1010110111	6.4964
43	0111111001	6.5238
44	1001111110	6.5421
45	1011001111	6.5798

Table III. DISTRIBUTION OF T

Table IIIa, continued

m = 3, n = 7
(Continued)

i	R.O.	T
46	1100011111	6.6298
47	0111111010	6.6349
48	1010111011	6.6393
49	1011010111	6.7464
50	0111111100	6.7599
51	1010111101	6.7643
52	1100101111	6.8298
53	1010111110	6.8754
54	1011011011	6.8893
55	1011100111	6.9464
56	1100110111	6.9964
57	1011011101	7.0143
58	1101001111	7.0798
59	1011101011	7.0893
60	1011011110	7.1254
61	1100111011	7.1393
62	1011101101	7.2143
63	1101010111	7.2464
64	1011110011	7.2560
65	1100111101	7.2643
66	1011101110	7.3254
67	1100111110	7.3754
68	1011110101	7.3810
69	1101011011	7.3893
70	1110001111	7.4131
71	1101100111	7.4464
72	1011110110	7.4921
73	1101011101	7.5143
74	1011111001	7.5238
75	1110010111	7.5798
76	1101101011	7.5893
77	1101011110	7.6254
78	1011111010	7.6349
79	1101101101	7.7143
80	1110011011	7.7226

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 3, n = 7
(Continued)

i	R.O.	T
81	1101110011	7.7560
82	1011111100	7.7599
83	1110100111	7.7798
84	1101101110	7.8254
85	1110011101	7.8476
86	1101110101	7.8810
87	1110101011	7.9226
88	1110011110	7.9587
89	1101110110	7.9921
90	1101111001	8.0238
91	1111000111	8.0298
92	1110101101	8.0476
93	1110110011	8.0893
94	1101111010	8.1349
95	1110101110	8.1587
96	1111001011	8.1726
97	1110110101	8.2143
98	1101111100	8.2599
99	1111001101	8.2976
100	1110110110	8.3254
101	1111010011	8.3393
102	1110111001	8.3571
103	1111001110	8.4087
104	1111010101	8.4643
105	1110111010	8.4683
106	1111100011	8.5393
107	1111010110	8.5754
108	1110111100	8.5933
109	1111011001	8.6071
110	1111100101	8.6643
111	1111011010	8.7183
112	1111100110	8.7754
113	1111101001	8.8071
114	1111011100	8.8433
115	1111101000	8.9183

Table III. DISTRIBUTION OF T

Table IIIa, continued)

N = 10

m = 3, n = 7
(Continued)

i	R.O.	T
116	1111110001	8.9738
117	1111101100	9.0433
118	1111110010	9.0849
119	1111110100	9.2099
120	1111111000	9.3528

N = 10

m = 4, n = 6

i	R.O.	T
1	0000111111	2.6175
2	0001011111	2.8675
3	0001101111	3.0675
4	0010011111	3.2008
5	0001110111	3.2341
6	0001111011	3.3770
7	0010101111	3.4008
8	0001111101	3.5020
9	0010110111	3.5675
10	0001111110	3.6131
11	0011001111	3.6508
12	0100011111	3.7008
13	0010111011	3.7103
14	0011010111	3.8175
15	0010111101	3.8353
16	0100101111	3.9008
17	0010111110	3.9464
18	0011011011	3.9603
19	0011100111	4.0175
20	0100110111	4.0675

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 4, n = 6
(Continued)

i	R.O.	T
21	0011011101	4.0853
22	0101001111	4.1508
23	0011101011	4.1603
24	0011011110	4.1964
25	0100111011	4.2103
26	0011101101	4.2853
27	0101010111	4.3175
28	0011110011	4.3270
29	0100111101	4.3353
30	0011101110	4.3964
31	0100111110	4.4464
32	0011110101	4.4520
33	0101011011	4.4603
34	0110001111	4.4841
35	0101100111	4.5175
36	0011110110	4.5631
37	0101011101	4.5853
38	0011111001	4.5948
39	0110010111	4.6508
40	0101101011	4.6603
41	0101011110	4.6964
42	1000011111	4.7008
43	0011111010	4.7060
44	0101101101	4.7853
45	0110011011	4.7937
46	0101110011	4.8270
47	0011111100	4.8310
48	0110100111	4.8508
49	0101101110	4.8964
50	1000101111	4.9008
51	0110011101	4.9187
52	0101110101	4.9520
53	0110101011	4.9937
54	0110011110	5.0298
55	0101110110	5.0631

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 4, n = 6
(Continued)

i	R.O.	T
56	1000110111	5.0675
57	0101111001	5.0948
58	0111000111	5.1008
59	0110101101	5.1187
60	1001001111	5.1508
61	0110110011	5.1603
62	0101111010	5.2060
63	1000111011	5.2103
64	0110101110	5.2298
65	0111001011	5.2437
66	0110110101	5.2853
67	1001010111	5.3175
68	0101111100	5.3310
69	1000111101	5.3353
70	0111001101	5.3687
71	0110110110	5.3964
72	0111010011	5.4103
73	0110111001	5.4282
74	1000111110	5.4464
75	1001011011	5.4603
76	0111001110	5.4798
77	1010001111	5.4841
78	1001100111	5.5175
79	0111010101	5.3434
80	0110111010	5.5393
81	1001011101	5.5853
82	0111100011	5.6103
83	0111010110	5.6464
84	1010010111	5.6508
85	1001101011	5.6603
86	0110111100	5.6643
87	0111011001	5.6782
88	1001011110	5.6964
89	0111100101	5.7353
90	1001101101	5.7853

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 4, n = 6
(Continued)

i	R.O.	T
91	0111011010	5.7893
92	1010011011	5.7937
93	1001110011	5.8270
94	0111100110	5.8464
95	1010100111	5.8508
96	0111101001	5.8782
97	1001101110	5.8964
98	0111011100	5.9143
99	1010011101	5.9187
100	1001110101	5.9520
101	1100001111	5.9841
102	0111101010	5.9893
103	1010101011	5.9937
104	1010011110	6.0298
105	0111110001	6.0448
106	1001110110	6.0631
107	1001111001	6.0948
108	1011000111	6.1008
109	0111101100	6.1143
110	1010101101	6.1187
111	1100010111	6.1508
112	0111110010	6.1560
113	1010110011	6.1603
114	1001111010	6.2060
115	1010101110	6.2298
116	1011001011	6.2437
117	0111110100	6.2810
118	1010110101	6.2853
119	1100011011	6.2937
120	1001111100	6.3310
121	1100100111	6.3508
122	1011001101	6.3687
123	1010110110	6.3964
124	1011010011	6.4103
125	1100011101	6.4187

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 4, n = 6
(Continued)

i	R.O.	T
126	0111111000	6.4238
127	1010111001	6.4282
128	1011001110	6.4798
129	1100101011	6.4937
130	1100011110	6.5298
131	1011010101	6.5353
132	1010111010	6.5393
133	1101000111	6.6008
134	1011100011	6.6103
135	1100101101	6.6187
136	1011010110	6.6464
137	1100110011	6.6603
138	1010111100	6.6643
139	1011011001	6.6782
140	1100101110	6.7298
141	1011100101	6.7353
142	1101001011	6.7437
143	1100110101	6.7853
144	1011011010	6.7893
145	1011100110	6.8464
146	1101001101	6.8687
147	1011101001	6.8782
148	1100110110	6.8964
149	1101010011	6.9103
150	1011011100	6.9143
151	1100111001	6.9282
152	1110000111	6.9341
153	1101001110	6.9798
154	1011101010	6.9893
155	1101010101	7.0353
156	1100111010	7.0393
157	1011110001	7.0448
158	1110001011	7.0770
159	1101100011	7.1103
160	1011101100	7.1143

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 4, n = 6
(Continued)

i	R.O.	T
161	1101010110	7.1464
162	1011110010	7.1560
163	1100111100	7.1643
164	1101011001	7.1782
165	1110001101	7.2020
166	1101100101	7.2353
167	1110010011	7.2437
168	1011110100	7.2810
169	1101011010	7.2893
170	1110001110	7.3131
171	1101100110	7.3464
172	1110010101	7.3687
173	1101101001	7.3782
174	1101011100	7.4143
175	1011111000	7.4238
176	1110100011	7.4437
177	1110010110	7.4798
178	1101101010	7.4893
179	1110011001	7.5115
180	1101110001	7.5448
181	1110100101	7.5687
182	1101101100	7.6143
183	1110011010	7.6226
184	1101110010	7.6560
185	1110100110	7.6798
186	1111000011	7.6937
187	1110101001	7.7115
188	1110011100	7.7476
189	1101110100	7.7810
190	1111000101	7.8187
191	1110101010	7.8226
192	1110110001	7.8782
193	1101111000	7.9238
194	1111000110	7.9298
195	1110101100	7.9476

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 4, n = 6
(Continued)

i	R.O.	T
196	1111001001	7.9615
197	1110110010	7.9893
198	1111001010	8.0726
199	1110110100	8.1143
200	1111010001	8.1282
201	1111001100	8.1976
202	1111010010	8.2393
203	1110111000	8.2571
204	1111100001	8.3282
205	1111010100	8.3643
206	1111100010	8.4393
207	1111011000	8.5071
208	1111100100	8.5643
209	1111101000	8.7071
210	1111110000	8.8738

N = 10

m = 5, n = 5

i	R.O.	T
1	0000011111	1.7718
2	0000101111	1.9718
3	0000110111	2.1385
4	0001001111	2.2218
5	0000111011	2.2813
6	0001010111	2.3885
7	0000111101	2.4063
8	0000111110	2.5175
9	0001011011	2.5313
10	0010001111	2.5552

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 5, n = 5
(Continued)

i	R.O.	T
11	0001100111	2.5885
12	0001011101	2.6563
13	0010010111	2.7218
14	0001101011	2.7313
15	0001011110	2.7675
16	0001101101	2.8563
17	0010011011	2.8647
18	0001110011	2.8980
19	0010100111	2.9218
20	0001101110	2.9675
21	0010011101	2.9897
22	0001110101	3.0230
23	0100001111	3.0552
24	0010101011	3.0647
25	0010011110	3.1008
26	0001110110	3.1341
27	0001111001	3.1659
28	0011000111	3.1718
29	0010101101	3.1897
30	0100010111	3.2218
31	0010110011	3.2313
32	0001111010	3.2770
33	0010101110	3.3008
34	0011001011	3.3147
35	0010110101	3.3563
36	0100011011	3.3647
37	0001111100	3.4020
38	0100100111	3.4218
39	0011001101	3.4397
40	0010110110	3.4675
41	0011010011	3.4813
42	0100011101	3.4897
43	0010111001	3.4992
44	0011001110	3.5508
45	0100101011	3.5647

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 5, n = 5
(Continued)

i	R.O.	T
46	0100011110	3.6008
47	0011010101	3.6063
48	0010111010	3.6103
49	0101000111	3.6718
50	0011100011	3.6813
51	0100101101	3.6897
52	0011010110	3.7175
53	0100110011	3.7313
54	0010111100	3.7353
55	0011011001	3.7492
56	0100101110	3.8008
57	0011100101	3.8063
58	0101001011	3.8147
59	0100110101	3.8563
60	0011011010	3.8603
61	0011100110	3.9175
62	0101001101	3.9397
63	0011101001	3.9492
64	0100110110	3.9675
65	0101010011	3.9813
66	0011011100	3.9853
67	0100111001	3.9992
68	0110000111	4.0052
69	0101001110	4.0508
70	1000001111	4.0552
71	0011101010	4.0603
72	0101010101	4.1063
73	0100111010	4.1103
74	0011110001	4.1159
75	0110001011	4.1480
76	0101100011	4.1813
77	0011101100	4.1853
78	0101010110	4.2175
79	1000010111	4.2218
80	0011110010	4.2270

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 5, n = 5
(Continued)

i	R.O.	T
81	0100111100	4.2353
82	0101011001	4.2492
83	0110001101	4.2730
84	0101100101	4.3063
85	0110010011	4.3147
86	0011110100	4.3520
87	0101011010	4.3603
88	1000011011	4.3647
89	0110001110	4.3841
90	0101100110	4.4175
91	1000100111	4.4218
92	0110010101	4.4397
93	0101101001	4.4492
94	0101011100	4.4853
95	1000011101	4.4897
96	0011111000	4.4948
97	0110100011	4.5147
98	0110010110	4.5508
99	0101101010	4.5603
100	1000101011	4.5647
101	0110011001	4.5825
102	1000011110	4.6008
103	0101110001	4.6159
104	0110100101	4.6397
105	1001000111	4.6718
106	0101101100	4.6853
107	1000101101	4.6897
108	0110011010	4.6937
109	0101110010	4.7270
110	1000110011	4.7313
111	0110100110	4.7508
112	0111000011	4.7647
113	0110101001	4.7825
114	1000101110	4.8008
115	1001001011	4.8147

Table III. DISTRIBUTION OF T

Table IIIa, continued

N = 10

m = 5, n = 5
(Continued)

	R.O.	T
116	0110011100	4.8187
117	0101110100	4.8520
118	1000110101	4.8563
119	0111000101	4.8897
120	0110101010	4.8937
121	1001001101	4.9397
122	0110110001	4.9492
123	1000110110	4.9675
124	1001010011	4.9813
125	0101111000	4.9948
126	1000111001	4.9992

Table III. DISTRIBUTION OF T

Table IIIb. Normal approximation to the distribution of T

m = 5, n = 4

i	X	P_1 *	P_2 **
1	-2.0837	.00794	.01858
2	-1.9310	.01587	.02674
3	-1.8037	.02381	.03562
4	-1.7400	.03175	.04093
5	-1.6946	.03968	.04504
6	-1.6127	.04762	.05337
7	-1.5991	.05556	.05491
8	-1.5036	.06349	.06629
9	-1.4854	.07143	.06877
10	-1.4600	.07937	.07215
11	-1.4082	.08730	.07957
12	-1.3581	.09524	.08723
13	-1.3509	.10317	.08835
14	-1.2555	.11111	.10456
15	-1.2490	.11905	.10583
16	-1.2236	.12699	.11048
17	-1.2054	.13492	.11410
18	-1.1536	.14286	.12425
19	-1.1281	.15079	.12966
20	-1.1036	.15873	.13480
21	-1.0963	.16667	.13654
22	-1.0190	.17460	.15410
23	-1.0144	.18254	.15529
24	-1.0008	.19048	.16446
25	- .9763	.19841	.16627
26	- .9690	.20636	.16627
27	- .9054	.21429	.18263
28	- .8735	.22222	.19130
29	- .8671	.23015	.19294
30	- .8235	.23810	.20511
31	- .8099	.24603	.20900
32	- .7780	.25397	.21828
33	- .7717	.26190	.22015
34	- .7644	.26984	.22231
35	- .7144	.27778	.23749

* Exact cumulative probabilities.

** Normal approximation to the cumulative probabilities.

Table III. DISTRIBUTION OF T

Table IIIb, continued

m = 5, n = 4
(Continued)

i	X	P_1	P_2
36	-.6826	.28571	.24743
37	-.6325	.29365	.26353
38	-.6253	.30159	.26589
39	-.6189	.30952	.26799
40	-.5871	.31746	.27857
41	-.5734	.32540	.28319
42	-.5298	.33333	.29873
43	-.5234	.34127	.30035
44	-.4916	.34921	.31150
45	-.4280	.35714	.33433
46	-.4207	.36508	.33699
47	-.3961	.37302	.34602
48	-.3826	.38095	.35101
49	-.3780	.38889	.35272
50	-.3398	.39683	.36700
51	-.3007	.40476	.38182
52	-.2941	.41270	.38434
53	-.2688	.42063	.39404
54	-.2434	.42857	.40385
55	-.2125	.43651	.41586
56	-.1916	.44444	.42403
57	-.1734	.45238	.43117
58	-.1479	.46032	.44121
59	-.1415	.46825	.44374
60	-.1034	.47619	.45882
61	-.0597	.48413	.47620
62	-.0461	.49206	.48162
63	-.0388	.50000	.48452
64	-.0079	.50794	.49685
65	+.0112	.51587	.50447
66	.0494	.52381	.51970
67	.0630	.53175	.52512
68	.0885	.53968	.53526
69	.1067	.54762	.54249
70	.1312	.55556	.55219

Table III. DISTRIBUTION OF

Table III^{b}, continued

$m = 5, n = 4$
(Continued)

i	X	P_1	P_2
71	.1449	.56349	.55761
72	.1767	.57143	.57013
73	.2022	.57937	.58012
74	.2158	.58730	.58543
75	.2404	.59524	.59500
76	.2722	.60317	.60727
77	.2976	.61111	.61700
78	.3358	.61905	.63149
79	.3431	.62698	.63434
80	.3676	.63592	.64341
81	.3813	.64286	.64851
82	.3858	.65079	.65018
83	.4067	.65873	.65789
84	.4631	.66667	.67835
85	.4949	.67460	.68966
86	.5204	.68254	.69861
87	.5340	.69048	.70333
88	.5722	.69841	.71641
89	.5904	.70635	.72254
90	.6159	.71429	.73102
91	.6222	.72222	.73309
92	.6868	.73016	.75390
93	.7177	.73810	.76353
94	.7250	.74603	.76577
95	.7677	.75397	.77867
96	.7750	.76290	.78083
97	.8268	.76984	.79582
98	.8523	.77778	.80298
99	.8705	.78571	.80799
100	.8768	.79365	.80970
101	.9659	.80159	.83295
102	.9723	.80952	.83455
103	.9796	.81746	.83636
104	1.0041	.82540	.84231
105	1.0614	.83333	.85566

Table III. DISTRIBUTION OF T

Table IIIb, continued

m = 5, n = 4
(Continued)

i	X	P_1	P_2
106	1.0996	.84127	.86433
107	1.1069	.84921	.86585
108	1.1569	.85714	.87636
109	1.1705	.86508	.87900
110	1.2087	.87302	.88667
111	1.2524	.88095	.89472
112	1.2978	.88889	.90286
113	1.3478	.89683	.91117
114	1.3615	.90476	.91340
115	1.4433	.91270	.92549
116	1.4506	.92064	.92661
117	1.4888	.92855	.93176
118	1.5524	.93651	.93967
119	1.6025	.94444	.94542
120	1.6797	.95238	.95352
121	1.6979	.96032	.95525
122	1.8070	.96825	.96462
123	1.8325	.97619	.96652
124	1.9343	.98413	.97344
125	2.0871	.99206	.98156
126	2.2780	1.00000	.98864

Table III. DISTRIBUTION OF T

Table IIIb, continued

m = 5, n = 5

i	X	P_1	P_2
1	-2.3034	.00397	.01064
2	-2.1607	.00794	.01535
3	-2.0418	.01190	.02058
4	-1.9823	.01587	.02373
5	-1.9398	.01984	.02619
6	-1.8634	.02381	.03123
7	-1.8506	.02778	.03208
8	-1.7714	.03175	.03828
9	-1.7614	.03571	.03912
10	-1.7445	.03968	.04058
11	-1.7207	.04365	.05263
12	-1.6723	.04762	.04726
13	-1.6255	.05159	.05197
14	-1.6187	.05556	.05272
15	-1.5930	.05952	.05558
16	-1.5296	.06349	.06301
17	-1.5236	.06746	.06375
18	-1.4998	.07143	.06681
19	-1.4828	.07540	.06904
20	-1.4503	.07937	.07353
21	-1.4344	.08333	.07579
22	-1.4106	.08730	.07912
23	-1.3877	.09127	.08257
24	-1.3809	.09524	.08364
25	-1.3551	.09921	.08771
26	-1.3313	.10317	.09159
27	-1.3087	.10714	.09527
28	-1.3045	.11111	.09612
29	-1.2917	.11508	.09818
30	-1.2688	.11905	.10222
31	-1.2620	.12302	.10347
32	-1.2294	.12698	.10954
33	-1.2124	.13095	.11276
34	-1.2025	.13492	.11468
35	-1.1728	.13889	.12040

Table III. DISTRIBUTION OF T

Table IIIb, continued

m = 5, n = 5
(Continued)

i	X	P_1	P_2
36	-1.1668	.14286	.12161
37	-1.1402	.14683	.12714
38	-1.1261	.15079	.13008
39	-1.1133	.15476	.13285
40	-1.0935	.15873	.13698
41	-1.0836	.16270	.13918
42	-1.0777	.16667	.14052
43	-1.0709	.17063	.14208
44	-1.0340	.17460	.15057
45	-1.0241	.17857	.15292
46	- .9984	.18254	.15904
47	- .9944	.18651	.16001
48	- .9916	.19048	.16070
49	- .9477	.19444	.17164
50	- .9409	.19841	.17338
51	- .9349	.20238	.17492
52	- .9151	.20635	.18007
53	- .9052	.21032	.18268
54	- .9024	.21429	.18342
55	- .8925	.21825	.18606
56	- .8557	.22222	.19608
57	- .8517	.22619	.19719
58	- .8458	.23016	.19883
59	- .8160	.23413	.20725
60	- .8132	.23810	.20805
61	- .7724	.24206	.21994
62	- .7566	.24603	.22461
63	- .7498	.25000	.22669
64	- .7367	.25397	.23065
65	- .7268	.25794	.23367
66	- .7240	.26190	.23453
67	- .7141	.26587	.23758
68	- .7098	.26984	.23891
69	- .6773	.27381	.24911
70	- .6742	.27778	.25009

Table III. DISTRIBUTION OF T

Table IIIb, continued

m = 5, n = 5
(Continued)

i	X	P_1	P_2
71	-.6705	.28175	.25127
72	-.6376	.28571	.26187
73	-.6348	.28968	.26278
74	-.6308	.29365	.26409
75	-.6079	.29762	.27163
76	-.5841	.30159	.27958
77	-.5813	.30556	.28052
78	-.5584	.30952	.28829
79	-.5552	.31349	.28938
80	-.5516	.31746	.29061
81	-.5456	.32143	.29267
82	-.5357	.32540	.29608
83	-.5187	.32937	.30198
84	-.4949	.33333	.31034
85	-.4890	.33730	.31242
86	-.4624	.34127	.32190
87	-.4564	.34524	.32405
88	-.4533	.34921	.32517
89	-.4394	.35317	.33019
90	-.4157	.35714	.33881
91	-.4125	.36111	.33999
92	-.3998	.36508	.34465
93	-.3930	.36905	.34716
94	-.3672	.37302	.35789
95	-.3641	.37698	.35789
96	-.3604	.38095	.35927
97	-.3463	.38492	.36456
98	-.3205	.38889	.37429
99	-.3137	.39286	.37687
100	-.3106	.39683	.37805
101	-.2979	.40079	.38289
102	-.2848	.40476	.38790
103	-.2741	.40873	.39200
104	-.2571	.41270	.39855
105	-.2342	.41667	.40741

Table III. DISTRIBUTION OF T

Table IIIb, continued

m = 5, n = 5
(Continued)

i	X	P_1	P_2
106	-.2245	.42063	.41118
107	-.2214	.42460	.41239
108	-.2186	.42857	.41348
109	-.1948	.43254	.42277
110	-.1917	.43651	.42399
111	-.1778	.44048	.42944
112	-.1679	.44444	.43333
113	-.1552	.44841	.43833
114	-.1421	.45238	.44350
115	-.1322	.45635	.44741
116	-.1294	.46032	.44852
117	-.1056	.46429	.45795
118	-.1026	.46825	.45914
119	-.0787	.47222	.46864
120	-.0759	.47619	.46975
121	-.0430	.48016	.48285
122	-.0362	.48413	.48556
123	-.0232	.48810	.49075
124	-.0133	.49206	.49469
125	-.0037	.49603	.49852
126	-.0006	.50000	.49976

Table III. DISTRIBUTION OF T

Table IIIb'. Exact significance levels of T using the normal approximation

m	n	α	
		.10	.05
4	4	.11429	.04286
4	5	.11111	.04762
5	4	.10317	.03968
4	6	.11905	.06190
5	5	.11508	.04762
6	4	.10001	.02858

Table III. DISTRIBUTION OF T

Table IIIc. Standard deviation of T

m	n	S.D.
N = 1		
0	1	0
1	0	0
N = 2		
0	2	0
1	1	.5000
2	0	0
N = 3		
0	3	0
1	2	.6236
N = 4		
0	4	0
1	3	.6922
2	2	.7993
N = 5		
0	5	0
1	4	.7371
2	3	.9028
N = 6		
0	6	0
1	5	.7692
2	4	.9730
3	3	1.0320
N = 7		
0	7	0
1	6	.7935
2	5	1.0244
3	4	1.1221
N = 8		
0	8	0
1	7	.8126
2	6	1.1484
3	5	1.1895
4	4	1.2285
N = 9		
0	9	0
1	8	.8281
2	7	1.0954
3	6	1.2421
4	5	1.3093

Table III. DISTRIBUTION OF T

Table IIIc, continued

m		n	S.D.
	N = 10		
0		10	0
1		9	.8409
2		8	1.1212
3		7	1.2845
4		6	1.3732
5		5	1.4015
	N = 11		
0		11	0
1		10	.8517
2		9	1.1427
3		8	1.3195
4		7	1.4252
5		6	1.4753
	N = 12		
0		12	0
1		11	.8610
2		10	1.1610
3		9	1.3490
4		8	1.4686
5		7	1.5359
6		6	1.5577
	N = 13		
0		13	0
1		12	.8691
2		11	1.1768
3		10	1.3742
4		9	1.5054
5		8	1.5868
6		7	1.6392
	N = 14		
0		14	0
1		13	.8762
2		12	1.1905
3		11	1.3960
4		10	1.5370
5		9	1.6302
6		8	1.6837
7		7	1.7011
	N = 15		
0		15	0
1		14	.8825
2		13	1.2026
3		12	1.4151

Table III, DISTRIBUTION OF T

Table IIIc, continued

m	n	S.D.
4	11	1.5645
5	10	1.6677
6	9	1.7332
7	8	1.7650
N = 16		
0	16	0
1	15	.8881
2	14	1.1692
3	13	1.4320
4	12	1.5887
5	11	1.7006
6	10	1.7762
7	9	1.8200
8	8	1.8344
N = 17		
0	17	0
1	16	.8931
2	15	1.2230
3	14	1.4470
4	13	1.6101
5	12	1.7295
6	11	1.8139
7	10	1.8681
8	9	1.8946
N = 18		
0	18	0
1	17	.8977
2	16	1.2316
3	15	1.4605
4	14	1.6293
5	13	1.7553
6	12	1.8474
7	11	1.9105
8	10	1.9473
9	9	1.9595
N = 19		
0	19	0
1	18	.8778
2	17	1.2064
3	16	1.4334

Table III. DISTRIBUTION OF T

Table IIIc, continued

m	n	S.D.
4	15	1.6026
5	14	1.7310
6	13	1.8272
7	12	1.8962
8	11	1.9408
9	10	1.9627
N = 20		
0	20	0
1	19	.9056
2	18	1.2466
3	17	1.4837
4	16	1.6621
5	15	1.7992
6	14	1.9041
7	13	1.9819
8	12	2.0356
9	11	2.0672
10	10	2.0776

Table IV. POWER FUNCTIONS OF NONPARAMETRIC TESTS

Table IVa. Parametric versus nonparametric tests

$\alpha = .10 \qquad \beta = .50$

m \ n	1	2	3	4	5
1	.1800 " "	.2771 " "	.3146 " "	.3776 " "	.4387 " "
2	.2624 " "	.3245 " "	.4394 " "	.4553 " "	.4839 " "
3	.2387 " "	.3667 " "	.4062 " "	.4332 " .4268	.4563 .4570 .4482
4	.2535 " "	.3494 " "	.3917 .3938 .3828	.4289 " .4107	.4398 .4402 .4272
5	.2642 " "	.3513 " .3495	.3933 " .3737	.4195 .4199 .3948	.4370 .4375 .4322

In each case, the first entry gives the power of the test based on T; the second entry gives the power of the best rank order test; the third entry gives the power of the test based on c_1.

Table IV. POWER FUNCTIONS OF NONPARAMETRIC TESTS

Table IVa, continued

α = .10 β = .25

m \ n	1	2	3	4	5
1	.1929 " "	.2491 " "	.3586 " "	.4384 " "	.5169 " "
2	.2169 " "	.4343 " "	.6377 " "	.6360 " "	.6638 " "
3	.3171 " "	.5604 " "	.5991 " "	.6302 " .6173	.6531 .6570 .6383
4	.3614 " "	.5420 " "	.5933 .6021 .5738	.6428 " .6078	.6599 .6635 .6259
5	.3983 " "	.5585 " .5349	.6122 .6180 .5693	.6375 .6389 .6264	.6558 .6587 .6179

Table IV. POWER FUNCTIONS OF
NONPARAMETRIC TESTS

Table IVa, continued

α = .05 β = .25

m \ n	1	2	3	4	5
1	.0983 " "	.1443 " "	.1896 " "	.2344 " "	.2789 " "
2	.1377 " "	.2421 " "	.3658 " "	.5083 " "	.6431 " "
3	.1701 " "	.3270 " "	.5305 " "	.5626 " "	.6006 " "
4	.1972 " "	.3996 " "	.5098 " "	.5543 .5582 .5344	.5823 .5871 .5791
5	.2204 " "	.4788 " "	.5162 " .4884	.5650 .5737 .5292	.6030 .6055 .5612

Table IV. POWER FUNCTIONS OF NONPARAMETRIC TESTS

Table IVa, continued

$\alpha = .05 \quad \beta = .05$

m \ n	1	2	3	4	5
1	.0997 " "	.1483 " "	.1964 " "	.2441 " "	.2915 " "
2	.1478 " "	.2792 " "	.4400 " "	.6298 " "	.8114 " "
3	.1941 " "	.4285 " "	.7535 " "	.7639 " "	.7904 " "
4	.2389 " "	.5888 " "	.7442 " "	.7728 .7878 .7458	.8140 .8213 .7843
5	.2823 " "	.7482 " "	.7718 " .7275	.8044 .8252 .7528	.8334 .8382 .7830

Table IV. POWER FUNCTIONS OF NONPARAMETRIC TESTS

Table IVb. Comparison of nonparametric tests for different alternatives

Test procedure	Alternatives	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
T	4, 4	.4289	.6428	.5543	.7728
	3, 4	.4790	.6996	.6204	.8197
	3, 3	.5022	.7345	.6552	.8551
Best R.O.	4, 4	.4289	.6428	.5582	.7878
	3, 4	.4790	.6996	.6312	.8351
	3, 3	.5022	.7345	.6671	.8703
c_1 (Terry)	4, 4	.4107	.6078	.5344	.7458
	3, 4	.4569	.6606	.5978	.7913
	3, 3	.4783	.6932	.6312	.8297

S-E-L File

Addendum of 30 July 1954

to

Contributions to the Theory of Rank Order Statistics
(NBS Report 3262, 3 May 1954)

by

I. Richard Savage

- - - - -

TABLES OF POLYNOMIALS

The table gives the coefficients of the polynomials

$$f_z(\delta) = \prod_{i=1}^{m+n} (u_i + \delta v_i)$$

for all of the rank orders where $1 \leq m < n \leq 5$ and for half of the rank orders where $m = n = 1(1)5$. (The coefficients for $m = n = 5$ have been divided by 5).

On page 34 et seq. of NBS Report 3262, the polynomials $f_z(\delta)$ were introduced and some of their properties were given. In particular equations (7.a.5) give recursion formulas for computing these polynomials.

The diagrams of partial orderings of probabilities of rank orders under Lehmann alternatives (see pages 45-53 and section 7.c in general) were prepared with the aid of these polynomials.

TABLES OF POLYNOMIALS

N = 2

m = 1, n = 1

i	R.O.	0	1
1	01	1	1

N = 3

m = 1, n = 2

i	R.O.	0	1	2	3
1	011	1	3	2	
2	101		1	3	2
3	110			2	4

TABLES OF POLYNOMIALS, Continued

N = 4

m = 1, n = 3

i	R.O.	0	1	2	3	4
1	0111	1	6	11	6	
2	1011		1	6	11	6
3	1101			2	10	12
4	1110				6	18

N = 4

m = 2, n = 2

i	R.O.	0	1	2	3
1	0011	8	12	9	
2	0101	4	10	8	2
3	0110	2	8	10	4

TABLES OF POLYNOMIALS, Continued

N = 5

m = 1, n = 4

i	R.O	0	1	2	3	4	5
1	01111	1	10	35	50	24	
2	10111		1	10	35	50	24
3	11011			2	18	52	48
4	11101				6	42	72
5	11110					24	96

N = 5

m = 2, n = 3

i	R.O.	0	1	2	3	4	5
1	00111	16	48	44	12		
2	01011	8	32	46	28	6	
3	01101	4	22	44	38	12	
4	01110	2	15	40	45	18	
5	10011		8	32	46	28	6
6	10101		4	22	44	38	12
7	10110		2	15	40	45	18
8	11001			8	36	52	24
9	11010			4	26	54	36
10	11100				12	54	54

TABLES OF POLYNOMIALS, Continued

N = 6

m = 1, n = 5

i	R.O.	0	1	2	3	4	5	6
1	011111	1	15	85	225	274	120	
2	101111		1	15	85	225	274	120
3	110111			2	28	142	308	240
4	111011				6	72	282	360
5	111101					24	216	480
6	111110						120	600

N = 6

m = 2, n = 4

i	R.O.	1	2	3	4	5	6	
1	001111	32	160	280	200	48		
2	010111	16	96	220	240	124	24	
3	011011	8	60	176	252	176	48	
4	011101	4	38	140	250	216	72	
5	011110	2	24	110	240	248	96	
6	100111		16	96	220	240	124	24
7	101011		8	60	176	252	176	48
8	101101		4	38	140	250	216	72
9	101110		2	24	110	240	248	96
10	110011			16	104	248	256	96

TABLES OF POLYNOMIALS, Continued

N = 6

m = 2, n = 4
(Continued)

i	R.O.	0	1	2	3	4	5	6
11	110101			8	68	212	288	144
12	110110			4	44	176	304	192
13	111001				24	156	324	216
14	111010				12	108	312	288
15	111100					48	288	384

N = 6

m = 3, n = 3

i	R.O.	0	1	2	3	4	5
1	000111	162	324	198	36		
2	001011	100	270	240	90	12	
3	001101	72	228	264	132	24	
4	001110	48	142	276	168	36	
5	010011	54	189	255	165	51	6
6	010101	36	150	246	198	78	12
7	010110	24	120	204	222	102	18
8	011001	18	102	222	234	120	24
9	011010	12	78	198	246	150	36
10	011100	6	51	165	255	189	54

TABLES OF POLYNOMIALS, Continued

N = 7

m = 2, n = 5

i	R.O.	0	1	2	3	4	5	6	7
1	0011111	64	480	1360	1800	1096	240		
2	0101111	32	272	920	1580	1448	668	120	
3	0110111	16	160	652	1384	1612	976	240	
4	0111011	8	96	470	1200	1682	1224	360	
5	0111101	4	58	340	1030	1696	1432	480	
6	0111110	2	35	245	875	1673	1610	600	
7	1001111		32	272	920	1580	1448	668	120
8	1010111		16	160	652	1389	1612	976	240
9	1011011		8	96	470	1200	1682	1224	360
10	1011101		4	58	340	1030	1696	1432	480
11	1011110		2	35	245	875	1673	1610	600
12	1100111			32	288	1016	1752	1472	480
13	1101011			16	176	764	1636	1728	720
14	1101101			8	108	572	1488	1904	960
15	1101110			4	66	424	1326	2020	1200
16	1110011				48	432	1428	2052	1080
17	1110101				24	276	1164	2136	1440
18	1110110				12	174	924	2130	1800
19	1111001					96	816	2208	1920
20	1111010					48	552	2040	2400
21	1111100						240	1800	3000

TABLES OF POLYNOMIALS, Continued

N = 7

m = 3, n = 4

i	R.O.	0	1	2	3	4	5	6	7
1	0001111	486	1620	1890	900	144			
2	0010111	324	1242	1800	1230	396	48		
3	0011011	216	972	1704	1452	600	96		
4	0011101	144	768	1596	1608	780	144		
5	0100111	162	783	1521	1515	813	222	24	
6	0011110	96	608	1480	1720	944	192		
7	0101011	108	594	1338	1578	1026	348	48	
8	0101101	72	456	1182	1602	1194	462	72	
9	0110011	54	378	1074	1590	1296	552	96	
10	0101110	48	352	1044	1600	1332	568	96	
11	0110101	36	282	906	1530	1434	708	144	
12	0110110	24	212	768	1460	1536	848	192	
13	1000111		162	783	1521	1515	813	222	24
14	0111001	18	177	699	1425	1587	918	216	
15	0111010	12	130	572	1310	1648	1080	288	
16	1001011		108	594	1338	1578	1026	348	48
17	0111100	6	80	426	1160	1704	1280	384	
18	1001101		72	456	1182	1602	1194	462	72
19	1010011		54	378	1074	1590	1296	552	96
20	1001110		48	352	1044	1600	1332	568	96

TABLES OF POLYNOMIALS, Continued

N = 7

m = 3, n = 4
(Continued)

i	R.O.	0	1	2	3	4	5	6	7
21	1010101		36	282	906	1530	1434	708	144
22	1010110		24	212	768	1460	1536	848	192
23	1011001		18	177	699	1425	1587	918	216
24	1100011			108	648	1500	1680	912	192
25	1011010		12	130	572	1310	1648	1080	288
26	1100101			72	492	1320	1740	1128	288
27	1011100		6	80	426	1160	1704	1280	384
28	1100110			48	376	1160	1760	1312	384
29	1101001			36	318	1080	1770	1404	432
30	1101010			24	236	908	1712	1584	576
31	1110001				108	738	1764	1782	648
32	1101100			12	148	704	1616	1792	768
33	1110010				72	564	1596	1944	864
34	1110100				36	372	1368	2112	1152
35	1111000					144	1056	2304	1536

TABLES OF POLYNOMIALS, Continued

N = 8

m = 3, n = 5

i	R.O.	0	1	2	3	4	5	6	7	8
1	00011111	1458	7290	13770	12150	4932	720			
2	00101111	972	5346	11610	12690	7338	2124	240		
3	00110111	648	3996	9972	12876	9060	3288	480		
4	00111011	432	3024	8628	12804	10380	4332	720		
5	01001111	486	3159	8478	12150	10014	4731	1182	120	
6	00111101	288	2304	7480	12560	11432	5296	960		
7	01010111	324	2322	6984	11424	10968	6174	1884	240	
8	00111110	192	1760	6480	12200	12288	6200	1200		
9	01011011	216	1728	5826	10716	11592	7356	2526	360	
10	01100111	162	1404	5112	10140	11838	8136	3048	480	
11	01011101	144	1296	4892	10020	11996	8364	3128	480	
12	01011110	96	976	4120	9340	12244	9244	3700	600	
13	01101011	108	1026	4128	9120	11952	9294	3972	720	
14	01101101	72	756	3364	8220	11908	10224	4816	960	
15	10001111		486	3159	8478	12150	10014	4731	1182	120
16	01110011	54	621	2982	7770	11886	10689	5238	1080	
17	01101110	48	560	2756	7412	11756	10988	5600	1200	
18	01110101	36	450	2366	6790	11494	11480	6264	1440	
19	10010111		324	2322	6984	11424	10968	6174	1884	240
20	01110110	24	328	1890	5950	11046	12082	7200	1800	

TABLES OF POLYNOMIALS, Continued

N = 8

m = 3 n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8
21	01111001	18	270	1678	5610	10912	12360	7552	1920	
22	10011011		216	1728	5826	10716	11592	7356	2526	360
23	01111010	12	194	1310	4790	10238	12776	8600	2400	
24	10100111		162	1404	5112	10140	11838	8136	3048	480
25	10011101		144	1296	4892	10020	11996	8364	3128	480
26	01111100	6	115	910	3850	9394	13195	9850	3000	
27	10011110		96	976	4120	9394	12244	9244	3700	600
28	10101011		108	1026	4128	9120	11952	9294	3972	720
29	10101101		72	756	3364	8220	11908	10224	4816	960
30	10110011		54	621	2982	7770	11886	10689	5238	1080
31	11000111			324	2484	7740	12540	11136	5136	960
32	10101110		48	560	2756	7456	11756	10988	5600	1200
33	10110101		36	450	2366	6790	11494	11480	6264	1440
34	11001011			216	1836	6420	11820	12084	6504	1440
35	10110110		24	328	1890	5950	11046	12082	7200	1800
36	10111001		18	270	1678	5610	10912	12360	7552	1920
37	11001101			144	1368	5360	11080	12736	7712	1920
38	11010011			108	1134	4830	10710	13062	8316	2160
39	10111010		12	194	1310	4790	10238	12776	8600	2400
40	11001110			96	1024	4488	10336	13176	8800	2400

TABLES OF POLYNOMIALS, Continued

N = 8

m = 3, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8
41	11010101			72	828	3904	9676	13312	9648	2880
42	10111100		6	115	910	3850	9394	13195	9850	3000
43	11010110			48	608	3172	8728	13364	10800	3600
44	11100011				324	2754	8982	14166	10854	3240
45	11011001			36	504	2852	8368	13456	11264	3840
46	11100101				216	2052	7608	13812	12312	4320
47	11011010			24	364	2256	7324	13152	12400	4800
48	11100110				144	1536	6444	13296	13500	5400
49	11011100			12	218	1602	6098	12690	13700	6000
50	11101001				108	1296	5964	13176	14016	5760
51	11101010				72	948	4872	12228	15000	7200
52	11110001					432	3888	12192	16128	7680
53	11101100				36	582	3642	11010	16050	9000
54	11110010					288	2928	10704	16800	9600
55	11110100					144	1896	8880	17400	12000
56	11111000						720	6600	18000	15000

TABLES OF POLYNOMIALS, Continued

N = 8

m = 4, n = 4

i	R.O.	0	1	2	3	4	5	6	7
1	00001111	6144	15360	13440	4800	576			
2	00010111	4608	13056	13920	6960	1632	144		
3	00011011	3456	11232	14160	8640	2544	288		
4	00100111	3072	10240	13632	9280	3408	640	48	
5	00011101	2592	9720	14184	10008	3384	432		
6	00011110	1944	8424	14040	11160	4176	576		
7	00101011	2304	8640	13184	10480	4576	1040	96	
8	00101101	1728	7344	12696	11400	5592	1416	144	
9	00110011	1536	6784	12288	11680	6144	1696	192	
10	01000111	1536	6656	11936	11456	6344	2024	344	24
11	00101110	1296	6264	12165	12120	6504	1776	192	
12	00110101	1152	5664	11472	12240	7248	2256	288	
13	01001011	1152	5472	10912	11832	7528	2808	568	48
14	00110110	864	4752	10704	12624	8208	2784	384	
15	00111001	768	4416	10368	12720	8592	3024	432	
16	01001101	864	4536	10020	12048	8496	3504	780	72
17	01010011	768	4160	9536	11984	8912	3920	944	96
18	00111010	576	3648	9456	12816	9552	3696	576	
19	01001110	648	3780	9216	12144	9312	4140	984	96
20	01010101	576	3408	8568	11856	9744	4752	1272	144

TABLES OF POLYNOMIALS, Continued

N = 8

m = 4, n = 4
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8
21	00111100	384	2816	8352	12800	10656	4544	768		
22	01010110	432	2808	7728	11664	10416	5496	1584	192	
23	01100011	384	2656	7600	11728	10576	5584	1600	192	
24	01011001	384	2592	7392	11544	10656	5808	1728	216	
25	01100101	288	2136	6624	11184	11136	6552	2112	288	
26	01011010	288	2112	6552	11136	11184	6624	2136	288	
27	10000111		1536	6656	11936	11456	6344	2024	344	24
28	01100110	216	1728	5808	10656	11544	7392	2592	384	
29	01011100	192	1600	5584	10576	11728	7600	2656	384	
30	01101001	192	1584	5496	10416	11664	7728	2808	432	
31	10001011		1152	5472	10912	11832	7528	2808	568	48
32	01101010	144	1272	4752	9744	11856	8568	5408	576	
33	10001101		864	4536	10020	12048	8496	3504	780	72
34	01110001	96	984	4140	9312	12144	9216	3780	648	
35	10010011		768	4160	9536	11984	8912	3920	944	96

TABLES OF POLYNOMIALS, Continued

N = 9

m = 4, n = 5

i	R.O.	0	1	2	3	4	5	6	7	8
1	000011111	24576	92160	130560	86400	26304	2880			
2	000101111	18432	75264	120960	97440	41328	8736	720		
3	000110111	13824	62208	112800	105360	53376	13872	1440		
4	001001111	12288	56320	105728	105280	60032	19600	3392	240	
5	000111011	10368	51840	105336	110952	63576	18648	2160		
6	000111101	7776	43416	98280	114840	72504	23184	2880		
7	001010111	9216	46080	95936	107840	70704	27040	5584	480	
8	000111110	5832	36450	91530	117450	80478	27540	3600		
9	001011011	6912	38016	87504	109080	79368	33624	7656	720	
10	001100111	6144	34816	83072	108160	82976	37504	9248	960	
11	010001111	6144	34304	81024	105504	82656	39816	11496	1816	120
12	001011101	5184	31536	79992	109320	86616	39624	9648	960	
13	001101011	4608	28416	74208	106320	90192	45264	12432	1440	
14	001011110	3888	26244	73170	108810	92802	45186	11580	1200	
15	010010111	4608	27648	71008	101888	89272	48872	16312	3032	240
16	001101101	3456	23328	66576	104016	95952	52176	15456	1920	
17	001110011	3072	21504	63552	102720	97968	55056	16848	2160	
18	010011011	3456	22464	62760	98292	94224	56496	20640	4188	360
19	001101110	2592	19224	59868	101364	100620	58452	18360	2400	
20	010100111	3072	20480	58944	95616	95668	60240	23376	5104	480

TABLES OF POLYNOMIALS, Continued

N = 9

m = 4, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9
21	001110101	2304	17472	56064	98544	102288	62544	20784	2880		
22	010011101	2592	18360	55764	94656	97968	63120	24636	5304	480	
23	001110110	1728	14256	49632	94356	105540	69228	24540	3600		
24	010101011	2304	16512	51312	90264	98256	67728	28848	6936	720	
25	001111001	1536	13184	47488	92960	106624	71456	25792	3840		
26	010011110	1944	15066	49709	90990	100806	68994	28383	6390	600	
27	001111010	1152	10656	41440	87640	108528	78344	30320	4800		
28	010101101	1728	13392	44952	85296	99984	74064	33816	8688	960	
29	011000111	1536	12544	43680	84912	100944	75216	34320	8768	960	
30	010110011	1536	12288	42528	83136	100344	76512	35952	9504	1080	
31	010101110	1296	10908	39546	80616	100992	79536	38406	10380	1200	
32	001111100	768	8000	34720	81200	110152	86240	35800	6000		
33	011001011	1152	9984	37176	77856	100464	81888	41208	11712	1440	
34	010110101	1152	9888	36768	77304	100416	82416	41664	11832	1440	
35	100001111		6144	34304	81024	105504	82656	39816	11496	1816	120
36	010110110	864	7992	31944	71994	99948	87384	46884	14070	1800	
37	011001101	864	7992	31872	71664	99456	87288	47328	14496	1920	
38	010111001	768	7360	30336	70224	99792	89040	48624	14816	1920	
39	011010011	768	7296	29904	69144	98736	89232	49872	15768	2160	
40	100010111		4608	29648	71008	101888	89272	48872	16312	3032	240
41	011001110	648	6426	27468	66120	98052	91734	52872	17160	2400	
42	010111010	576	5904	26084	64540	98084	93436	54332	17560	2400	
43	011010101	576	5808	25368	62736	96144	93552	56472	19344	2880	
44	100011011		3456	22464	62760	98292	94224	56496	20640	4188	360
45	010111100	384	4384	21360	57960	95676	98196	61020	20900	3000	

TABLES OF POLYNOMIALS, Continued

N = 9

m = 4, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9
46	011100011	384	4416	21480	57948	95136	97584	61200	21492	3240	
47	011010110	432	4644	21642	57120	93408	96936	62358	22750	3600	
48	100100111		3072	20480	58944	95616	95568	60240	23376	5104	480
49	011011001	384	4256	20400	55248	92496	98064	64320	23872	3840	
50	100011101		2592	18360	55764	94656	97968	63120	24636	5304	480
51	011100101	288	3480	17916	51504	90672	100320	68244	26136	4320	
52	011011010	288	3384	17236	49700	88732	100436	70384	27920	4800	
53	100101011		2304	16512	51312	90264	98256	67728	28848	6936	720
54	100011110		1944	15066	49707	90990	100806	68994	28383	6390	600
55	011100110	216	2754	15033	45990	86394	102186	74397	30510	5400	
56	011101001	192	2512	14072	44152	84968	102808	76448	31968	5760	
57	011011100	192	2480	13824	43428	84084	102732	77340	32800	6000	
58	100101101		1728	13392	44952	85296	99984	74064	33816	8688	960
59	101000111		1536	12544	43680	84912	100944	75216	34320	8768	960
60	100110011		1536	12288	43528	83136	100344	76512	35952	9504	1080
61	011101010	144	1980	11714	38990	79926	103390	82456	37080	7200	
62	100101110		1296	10908	39546	80616	100992	79536	38406	10380	1200
63	011110001	96	1496	9816	35496	77544	105024	86304	39424	7680	
64	101001011		1152	9984	37176	77856	100464	81888	41208	11712	1440
65	100110101		1152	9888	36768	77304	100416	82416	41664	11832	1440

TABLES OF POLYNOMIALS, Continued
N = 9

m = 4, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9
66	011101100	96	1432	9200	33250	73934	103558	89210	43200	9000	
67	011110010	72	1170	8062	30830	71698	104000	92008	45440	9600	
68	100110110		864	7992	31944	71994	99948	87384	46884	14670	1800
69	101001101		864	7992	31872	71664	99456	87288	47328	14496	1920
70	100111001		768	7360	30336	70224	99792	89040	48624	14816	1920
71	101010011		768	7296	29904	69144	98736	89232	49872	15768	2160
72	011110100	48	836	6210	25710	64902	102294	98280	52600	12000	
73	101001110		648	6426	27468	66120	98052	91734	52872	17160	2400
74	100111010		576	5904	26048	64540	98084	93436	54332	17560	2400
75	110000111			3072	22016	65344	104480	97408	53024	15616	1920
76	101010101		576	5808	25368	62736	96144	93552	56472	19344	2880
77	011111000	24	490	4215	19950	56826	99750	105375	61250	15000	
78	100111100		384	4384	21360	57960	95676	98196	61020	20900	3000
79	101100011		384	4416	21480	57948	95136	97584	61200	21492	3240
80	101010110		432	4644	21642	57120	93408	96936	62358	22740	3600
81	110001011			2304	17664	56688	99024	101904	61872	20544	2880
82	101011001		384	4256	20400	55248	92496	98064	64320	23872	3840
83	101100101		288	3480	17916	51504	90672	100320	68244	26136	4320
84	101011010		288	3384	17236	49700	88732	100436	70384	27920	4800
85	110001101			1728	14256	49488	93840	105072	69504	25152	3840

TABLES OF POLYNOMIALS, Continued

N = 9

m = 4, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9
86	110010011			1536	13056	46752	91536	105936	72528	27216	4320
87	101100110		216	2754	15033	45990	86394	102186	74397	30510	5400
88	101101001		192	2512	14072	44152	84968	102808	76448	31968	5760
89	110001110			1296	11556	43380	88860	107244	76224	29520	4800
90	101011100		192	2480	13824	43428	84084	102732	77340	32800	6000
91	110010101			1152	10464	40272	85200	107088	80016	32928	5760
92	101101010		144	1980	11714	38990	79926	103390	82456	37080	7200
93	110100011			768	8064	34896	81000	109272	85896	36504	6480
94	101110001		96	1496	9816	35496	77544	105024	86304	39424	7680
95	110010110			864	8424	34860	79380	107436	36436	38280	7200
96	110011001			768	7744	33056	77440	107552	88576	40064	7680
97	101101100		96	1432	9200	33250	73934	103558	89210	43200	9000
98	101110010		72	1170	8062	30830	71698	104000	92008	45440	9600
99	110100101			576	6384	29448	73560	107784	92856	43632	8640
100	110011010			576	6192	28280	71120	106344	94528	46240	9600
101	110100110			432	5076	24990	66990	105798	98574	50220	10800
102	101110100		48	836	6210	25710	64902	102294	98280	52600	12000
103	110101001			384	4640	23504	64800	105136	100480	52416	11520
104	110011100			384	4576	23072	63784	104384	101080	53600	12000
105	111000011				2304	19584	65520	111960	103896	49896	9720

TABLES OF POLYNOMIALS, Continued

N = 9

m = 4, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9
106	110101010			288	3672	19756	58224	101628	105152	59760	14400
107	101111000		24	490	4215	19950	56826	99750	105375	61250	15000
108	111000101				1728	15696	56952	106776	109800	58968	12960
109	110110001			192	2800	16832	54160	100928	109120	63488	15360
110	110101100			192	2672	15728	50772	97096	110020	68400	18000
111	111000110				1296	12636	49698	101574	114246	67230	16200
112	110110010			144	2196	13928	47732	95664	112336	71680	19200
113	111001001				1152	11616	47280	99840	115728	69984	17280
114	111001010				864	9288	40692	93288	118308	78840	21600
115	110110100			96	1576	10844	40576	89228	115360	81200	24000
116	111010001				576	7248	36000	90480	121824	83712	23040
117	111001100				576	6864	33456	85404	120480	89100	27000
118	110111000			48	932	7498	32402	81250	118250	92500	30000
119	111010010				432	5724	30336	82524	121944	93120	28800
120	111100001					2304	22080	77760	128640	101376	30720
121	111010100				288	4152	24228	73272	121140	103800	36000
122	111100010					1728	17712	68208	125472	111360	38400
123	111011000				144	2508	17478	62250	119250	116250	45000
124	111100100					1152	13152	57456	120720	122400	48000
125	111101000					576	8304	45000	114000	135000	60000
126	111110000						2880	30000	105000	150000	75000

TABLES OF POLYNOMIALS, Continued

N = 10

m = 5, n = 5
(coefficients have been divided by 5)

i	R.O.	0	1	2	3	4	5	6	7	8
1	0000011111	75000	225000	255000	135000	32880	2880			
2	0000101111	60000	195000	249000	159000	53304	8880	576		
3	0000110111	48000	170400	243120	178176	70608	14304	1152		
4	0001001111	45000	161250	235500	181500	79728	19986	2652	144	
5	0000111011	38400	149760	236832	193680	85920	19440	1728		
6	0001010111	36000	139800	224940	194412	97500	28380	4440	288	
7	0000111101	30720	132096	230016	206400	99840	24384	2304		
8	0000111110	24576	116736	222720	216960	112704	29184	2880		
9	0001011011	28800	121920	215064	204468	112860	36060	6156	432	
10	0010001111	30000	122500	210750	199500	113652	39900	8430	980	48
11	0001100111	27000	116100	209580	205884	118860	40320	7440	576	
12	0001011101	23040	106752	205536	212304	126480	43248	7824	576	
13	0010010111	24000	105200	196560	204588	129804	51420	12420	1672	96
14	0001101011	21600	100440	197148	211596	133980	49980	10152	864	
15	0001011110	18432	93696	196224	218400	138768	50064	9456	720	
16	0001101101	17280	87264	185712	215568	147120	58896	12768	1152	
17	0010011011	19200	90880	184016	208000	143396	61660	16124	2340	144
18	0001110011	16200	83430	181476	215820	151272	62334	13932	1296	
19	0010100111	18000	86400	178420	207116	147868	66500	18400	2864	192
20	0001101110	13824	76032	175008	218160	158736	67248	15312	1440	

TABLES OF POLYNOMIALS, Continued

N = 10

m = 5, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9
21	0010011101	15360	78848	172608	210048	155088	70992	19632	2992	192	
22	0001110101	12960	71928	168768	216576	163728	76248	12424	1728		
23	0100001111	15000	76250	166625	205125	156576	76776	24165	4705	514	24
24	0010101011	14400	74160	164912	206780	159852	77980	23428	3960	288	
25	0010011110	12288	68608	162048	211008	165312	79632	22992	3632	240	
26	0001110110	10368	62208	157176	216288	174528	82224	20808	2160		
27	0001111001	9720	59616	153792	215856	177480	85104	21888	2304		
28	0011000111	12000	65600	154680	205464	168168	86856	27648	4960	384	
29	0010101101	11520	63936	152896	205616	169936	88304	28144	5024	384	
30	0100010111	12000	64600	150880	200574	167196	90612	31920	7046	884	48
31	0010110011	10800	61020	148794	204372	172788	91980	30066	5508	432	
32	0001111010	7776	51192	141696	213120	187344	95688	26064	2880		
33	0010101110	9216	55296	142016	203776	178544	97744	32624	6064	480	
34	0011001011	9600	55840	140768	200872	177464	99400	34472	6768	576	
35	0010110101	8640	52272	136488	200640	181344	103008	35832	6960	576	
36	0100011011	9600	55040	137448	196008	175698	102528	38892	9232	1242	72
37	0001111100	5832	42282	127980	208980	197928	108018	31140	3600		
38	0100100111	9000	52200	132410	192768	177492	107184	42450	10632	1528	
39	0011001101	7680	47744	128640	196128	184992	110496	40800	8512	768	
40	0010110110	6912	44928	125520	196584	188448	112992	41280	8376	720	
41	0011010011	7200	45480	124716	193872	186816	114240	43284	9288	864	
42	0100011101	7680	47104	125728	191328	182568	113040	45312	11312	1592	96
43	0010111001	6480	42984	122400	195168	190272	115896	42960	8832	768	
44	0011001110	6144	40960	117888	191232	191136	120480	46752	10208	960	
45	0100101011	7200	44280	119536	185846	183316	118916	50704	13694	2124	144

N = 10

m = 5, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9	10
46	0100011110	6144	40448	115328	186528	188160	122472	51312	13312	1936	120	
47	0011010101	5760	38688	112944	187104	192288	125472	50736	11616	1152		
48	0010111010	5184	36720	111528	189312	195936	126240	49272	10608	960		
49	0101000111	6000	38800	110140	180072	186816	127512	57252	16304	2672	192	
50	0011100011	4800	34320	105744	183468	196104	132240	54936	12852	1296		
51	0100101101	5760	37728	108416	179256	187776	129120	58224	16584	2704	192	
52	0011010110	4608	33024	102624	180528	196512	135456	57696	13872	1440		
53	0100110011	5400	35910	104907	176583	188580	132384	61023	17787	2970	216	
54	0010111100	3888	30132	99414	181980	201612	137988	56766	12780	1200		
55	0011011001	4320	31536	99744	178464	197472	138288	59808	14592	1536		
56	0100101110	4608	32256	98656	172896	191160	138144	65184	19344	3472	240	
57	0011100101	3840	28992	94656	174576	198912	143328	63744	15984	1728		
58	0101001011	4800	32720	98304	170820	189168	138432	66936	20620	3672	288	
59	0100110101	4320	30456	94380	168564	190992	142176	69420	21396	3768	288	
60	0011011010	3456	26784	89904	170592	199968	148128	67632	17376	1920		
61	0011100110	3072	24576	85056	166272	200688	153024	71904	19008	2160		
62	0101001101	3840	27712	88192	162384	190560	147744	75648	24656	4640	384	
63	0011101001	2880	23424	82416	163776	200928	155712	74352	19968	2304		
64	0100110110	3456	2590	85224	161052	192516	150720	77136	24828	4548	360	
65	0101010011	3600	26340	85098	159294	190344	150528	78762	26286	5076	432	
66	0011011100	2592	21816	79092	161232	201984	159072	76812	20760	2400		
67	0100111001	3240	24732	82692	158784	192720	153084	79428	25896	4800	384	
68	0110000111	3000	23900	81920	159216	193872	153636	79320	25744	4768	384	
69	0101001110	3072	23552	79424	154560	191184	155808	83616	28480	5584	480	
70	1000001111		15000	76250	166625	205125	156576	76776	24165	4705	514	24

TABLES OF POLYNOMIALS, Continued

N = 10

m = 5, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9	10
71	0011101010	2304	19776	73536	154608	200832	164832	83328	23664	2880		
72	0101010101	2880	22224	75816	150024	189696	158880	88104	31776	6384	576	
73	0100111010	2592	20952	74124	150420	192624	161088	87756	29940	5784	480	
74	0011110001	1920	17536	68640	150432	201888	169824	87360	25088	3072		
75	0110001011	2400	19960	71892	147768	191520	162624	90588	31952	6480	576	
76	0101100011	2400	19560	70032	144606	189786	164172	93588	33894	7074	648	
77	0011101100	1728	15984	63888	143988	199896	174768	93768	28140	3600		
78	0101010110	2304	18816	67824	141576	188520	165984	96576	35784	7656	720	
79	1000010111		12000	64600	150880	200574	167196	90612	31920	7046	884	48
80	0011110010	1536	14720	60672	140448	199584	178080	97248	29632	3840		
81	0100111100	1944	17010	65773	140697	191796	169800	97377	34773	6990	600	
82	0101011001	2160	17928	65640	139104	187968	167880	99048	37200	8064	768	
83	0110001101	1920	16736	63440	137664	188832	170016	100560	37690	8128	768	
84	0101100101	1920	16416	61824	134616	186744	171120	103536	39864	8856	864	
85	0110010011	1800	15870	60954	134268	187332	171942	103938	39936	8856	864	
86	0011110100	1152	11808	52096	129080	196168	186872	108664	35120	4800		
87	0101011010	1728	15120	58344	130248	185280	174048	107880	42504	9648	960	
88	1000011011		9600	55040	137448	196008	175698	102528	38892	9232	1242	72
89	0110001110	1536	14080	56224	128592	185856	176160	109536	43088	9728	960	
90	0101100110	1536	13824	54816	125664	183480	176856	112464	45456	10584	1080	
91	1000100111		9000	52200	132410	192768	177492	107184	42450	10632	1528	96
92	0110010101	1440	13272	53496	124080	182832	177720	114072	46656	11040	1152	
93	0101101001	1440	13152	52920	123096	182352	178320	115032	47160	11136	1152	
94	0101011100	1296	12204	50454	120162	181608	180528	117942	48786	11580	1200	
95	1000011101		7680	47104	124728	191328	182568	113040	45312	11312	1592	96

TABLES OF POLYNOMIALS, Continued

N = 10

m = 5, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9	10
96	0011111000	768	8768	42720	115290	191352	196392	122040	41800	6000		
97	0110100011	1200	11580	48786	117942	180528	181608	120162	50454	12204	1296	
98	0110010110	1152	11136	47160	115032	178320	182352	123096	52920	13152	1440	
99	0101101010	1152	11040	46656	114072	177720	182832	124080	53496	13272	1440	
100	1000101011		7200	44280	119536	185846	183316	118916	50704	13694	2124	144
101	0110011001	1080	10584	45456	112426	176856	183480	125664	54816	13824	1536	
102	1000011110		6144	40448	115328	186528	188160	122472	51312	13312	1936	120
103	0101110001	960	9728	43088	109536	176160	185856	128592	56224	14080	1536	
104	0110100101	960	9648	42504	107880	174048	185280	130248	58344	15120	1728	
105	1001000111		6000	38800	110140	180072	186816	127512	57252	16304	2672	192
106	0101101100	864	8856	39936	103938	171942	187332	134268	60954	15890	1800	
107	1000101101		5760	37728	108416	179256	187776	129120	58224	16584	2704	192
108	0110011010	864	8856	39864	103536	171120	186744	134616	61824	16416	1920	
109	0101110010	768	8128	37696	100560	170016	188832	137664	63440	16736	1920	
110	1000110011		5400	35910	104907	176583	188580	132384	61023	17787	2970	216
111	0110100110	768	8064	37200	99048	167880	187968	139104	65640	17928	2160	
112	0111000011	600	6990	34773	97377	169800	191796	140697	64773	17010	1944	
113	0110101001	720	7656	35784	96576	165984	188520	141576	67824	18816	2304	
114	1000101110		4608	32256	98656	172896	191160	138144	65184	19344	3272	240
115	1001001011		4800	32720	98304	170820	189168	138432	66936	20620	3672	288

TABLES OF POLYNOMIALS, Continued

N = 10

m = 5, n = 5
(Continued)

i	R.O.	0	1	2	3	4	5	6	7	8	9	10
116	0110011100	648	7074	33894	93588	164172	189876	144606	70032	19560	2400	
117	0101110100	576	6480	31952	90588	162624	191520	147768	71892	19960	2400	
118	1000110101		4320	30456	94380	168564	190992	142176	69420	21396	3768	288
119	0111000101	480	5784	29940	87756	161088	192624	150424	74124	20952	2592	
120	0110101010	576	6384	31176	88104	158880	189696	150024	75816	22224	2880	
121	1001001101		3840	27712	88192	162384	190560	147744	75648	24656	4640	384
122	0110110001	480	5584	28480	83616	155808	191184	154560	79424	23552	3072	
123	1000110110		3456	25920	85224	161052	192516	150720	77136	24828	4548	360
124	1001010011		3600	26340	85098	159294	190344	150528	78762	26286	5076	432
125	0101111000	384	4768	25744	79320	153636	193872	159216	81920	23900	3000	
126	1000111001		3240	24732	82692	158784	192720	153084	79428	25896	4800	384

THE NATIONAL BUREAU OF STANDARDS

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