On an Inequality of Hardy, Littlewood and Pólya

by

A. J. Hoffman
THE NATIONAL BUREAU OF STANDARDS

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A well-known inequality of Hardy, Littlewood and Polya (see [3], p. 49) states: Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) be points in \( \mathbb{R}^n \), and let \( \sigma \) and \( \tau \) be permutations of the set \( 1, 2, \ldots, n \) such that \( \alpha_{\sigma_1} \geq \alpha_{\sigma_2} \geq \ldots \geq \alpha_{\sigma_n} \) and \( \beta_{\tau_1} \geq \beta_{\tau_2} \geq \ldots \beta_{\tau_n} \). Then in order that

\[
\beta_{\tau_1} \leq \alpha_{\sigma_1}
\]
\[
\beta_{\tau_1} + \beta_{\tau_2} \leq \alpha_{\sigma_1} + \alpha_{\sigma_2}
\]
\[
\vdots
\]
\[
\beta_{\tau_1} + \beta_{\tau_2} + \ldots + \beta_{\tau_{(n-1)}} \leq \alpha_{\sigma_1} + \alpha_{\sigma_2} + \ldots + \alpha_{\sigma_{(n-1)}}
\]
\[
\beta_{\tau_1} + \beta_{\tau_2} + \ldots + \beta_{\tau_{n}} = \alpha_{\sigma_1} + \alpha_{\sigma_2} + \ldots + \alpha_{\sigma_{n}}
\]

it is necessary and sufficient that \( \beta = X\alpha \), where \( X = (x_{ij}) \) is a matrix satisfying

(2) \( x_{ij} \geq 0 \) for all \( i, j \),

(3) \( \sum_j x_{ij} = \sum_i x_{ij} = 1 \) for all \( i, j \).

Several proofs of this theorem, and of various generalizations, are known (see [1], [2], [4] and [5] for some recent work). The purpose of this note is to present a strengthened

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version of the Hardy, Littlewood and Polya theorem (which we call HLP hereinafter), the proof of which is no more difficult than the proof of HLP.

We shall prove:

Theorem 1. Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) satisfy

\[
\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n. 
\]

Then in order that a vector \( \beta = (\beta_1, \ldots, \beta_n) \) satisfy

\[
\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n 
\]

and

\[
\beta_1 \leq \alpha_1, \\
\beta_1 + \beta_2 \leq \alpha_1 + \alpha_2 \\
\vdots \\
\beta_1 + \beta_2 + \cdots + \beta_{n-1} \leq \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \\
\beta_1 + \beta_2 + \cdots + \beta_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n,
\]

it is necessary and sufficient that

\[
\beta = X\alpha, 
\]

where \( X = (x_{ij}) \) is a matrix satisfying (2), (3)

\[
x_{i_1j} \geq x_{i_2j} \text{ for all } i_1, i_2, j \text{ such that } (j-i_1)(i_1-i_2) \geq 0,
\]

and
(9) \[ x_{ij} = x_{ji} \text{ for all } i, j. \]

**Proof of necessity:** Consider the set of all vectors \( \beta \) satisfying (5) and (6). They clearly form a bounded, closed, convex set in \( R^n \). Let us designate this set by \( K \).

We first show

**Lemma.** Let \( 1 \leq a_1 < a_2 < \ldots < a_k = n \). Let \( y_1 = y_2 = \ldots = y_{a_1} = (\alpha_1 + \alpha_2 + \ldots + \alpha_{a_1})/a_1, y_{a_1+1} = y_{a_1+2} = \ldots = y_{a_2} = \)

\[ (\alpha_{a_1+1} + \alpha_{a_1+2} + \ldots + \alpha_{a_2})/(a_2 - a_1), \ldots, y_{a_{k-1}+1} = y_{a_{k-1}+2} = \]

\[ = \ldots = y_{a_k} = (\alpha_{a_{k-1}+1} + \alpha_{a_{k-1}+2} + \ldots + \alpha_{a_k})/(a_k - a_{k-1}). \]

Then (i) \( y = (y_1, y_2, \ldots, y_n) \) is a vertex of \( K \), and (ii) every vertex of \( K \) arises in this way.

**Proof of lemma:** (i) First, we must show that \( y \in K \).

It is immediate that \( y \) satisfies (5). To show that \( y \) satisfies (6), we first remark that the last equation of (6) obviously holds, so all that needs to be established are the inequalities. Further, if we prove

(10) \[ y_1 + y_2 + \ldots + y_s \leq \alpha_1 + \ldots + \alpha_s, \]

for \( 1 \leq s \leq a_1 \), it is clear from the definition of \( y \) that this argument will establish (10) for any value of \( s \). Now
(10) is equivalent with
\[
(11) \quad s(\alpha_1 + \alpha_2 + \ldots + \alpha_{a_i})/a_1 \leq \alpha_1 + \ldots + \alpha_s.
\]
But by (4), \( \alpha_1 + \alpha_2 + \ldots + \alpha_{a_1} \leq \alpha_1 + \alpha_2 + \ldots + \alpha_s + (a_i - s) \alpha_s \).

Hence,
\[
(12) \quad s(\alpha_1 + \ldots + \alpha_{a_1})/a_1 \leq \alpha_1 + \ldots + \alpha_s + (a_i - s) \alpha_s
\]
\[
- \frac{a_i - s}{a_1} (\alpha_1 + \ldots + \alpha_s + (a_i - s) \alpha_s)
\]
\[
\leq \alpha_1 + \ldots + \alpha_s + \alpha_s \left[(a_i - s) - \frac{a_i - s}{a_1}(s + a_i - s)\right]
\]
\[
< \alpha_1 + \ldots + \alpha_s,
\]
which proves (11).

To show that \( y \) is a vertex, assume that \( y = (y' + y'')/2 \), where \( y', y'' \in K \). Then if \( y_1' \ldots + y_1'a_1 > y_1 + \ldots + y_1'a_1 = \alpha_1 + \ldots + \alpha_{a_1} \), we would have \( y_1'' + \ldots + y_1'a_1 < \alpha_1 + \ldots + \alpha_{a_1} \), violating (6). Similarly \( y_1' + \ldots + y_j'a_j = y_1'' + \ldots + y_j'a_j \), \( j = 1, \ldots, k \).

Next, suppose for some \( s, 1 \leq s < a_i \), we have \( y'_s > y_s \). Then by the preceding paragraph, there is an integer \( t \), \( 1 \leq t < a_i \), such that \( y'_t < y_t \). Note that \( y_s = y_t \). Hence \( y'_s > y'_t \). If \( t < s \), then \( y' \) violates (5). If \( t > s \), \( y'' \) violates (5). Hence
$y'_s = y''_s = y'_s$. Clearly this argument extends to any
$s = 1, ..., n$. Thus $y = (y' + y'')/2$ implies $y = y' = y''$.
This completes the proof of (i).

To prove (ii), let $\beta$ satisfy (5) and (6), but not be a
vector such as $y$ above. This means that there exists in-
tegers $1 \leq a_1 < a_2 < ... < a_k = n$ such that $\beta_1 = \beta_2 = ... = 
\beta_{a_1} < \beta_{a_1+1} = ... = \beta_{a_2} < ... < \beta_{a_{k-1}} = ... = \beta_{a_k}$, but for at
least one $j = 1, ..., k-1$, we have $\beta_{a_j} + 1 < a_{j+1} + 
\beta_{a_j}$, and for at least one $j = 1, ..., k-1$, we have $\beta_{a_j} < a_j + 
\beta_{a_{j+1}}$. Let $\epsilon$ be any positive number smaller than
$\epsilon_1 + ... + \epsilon_{a_j} = (\beta_1 + ... + \beta_{a_j})$, $\beta_{a_j} - \beta_{a_{j-1}}$, $\beta_{a_j} - \beta_{a_{j+1}}$. 
Define $\beta'_1 = \beta_1$, ..., $\beta'_{a_j-1} = \beta_{a_j-1}$, $\beta'_{a_j-1} = ... = \beta'_{a_j}$
$= \beta_{a_j} + \epsilon / (a_j - a_{j-1})$, $\beta'_{a_j+1} = ... = \beta'_{a_{j+1}} = \beta_{a_{j+1}}$
$- \epsilon / (a_{j+1} - a_j)$, $\beta'_{a_{j+1}} = \beta_{a_{j+1}}$, ..., $\beta'_n = \beta_n$. Also define
$\beta'' = 2\beta - \beta'$. Then it is easy to check that $\beta'$ and $\beta''$ each satisfy (5) and (6), $\beta = (\beta' + \beta'')/2$, $\beta' \neq \beta''$, so $\beta$ is not
a vertex of $K$. This completes the proof of (ii), thus of
the lemma.

We resume our proof of the necessity of theorem 1. Let
$\beta \in K$, then $\beta$ is a convex combination of the vertices of
$K$. By the lemma, each vertex of $K$ is of the form $x_{a_1} \cdot \cdot \cdot , 
\cdot \cdot \cdot , x_{a_k}$, where $X_{a_1} \cdot \cdot \cdot , a_k$ is the matrix of order $n$
with 0's everywhere outside the square boxes along the main diagonal, where the first box is of order $a_1$, the second box of order $a_2-a_1$, ..., the $k$-th box of order $a_k-a_{k-1}$; every entry in the first box is $1/a_1$, every entry in the second box is $1/(a_2-a_1)$, ..., every entry in the $k$-th box is $1/(a_k-a_{k-1})$. Note that $X_{a_1,...,a_k}$ satisfies (2), (3), (3), (9). Since $\beta = \sum c_i (X^{(i)} \alpha )$, where $\sum c_i = 1$, $X^{(i)}$ is a matrix of the type (12), we may write $\beta = (\sum c_i X^{(i)}) \alpha$, and observe that any convex combination of matrices satisfying (2), (3), (3), (9) also satisfies these conditions. If we set $X = \sum c_i X^{(i)}$, the proof of the necessity is complete.

Proof of sufficiency: We shall show that if a matrix $X$ satisfies (2), (3), and (3) — (9) is not needed —, and a vector $\alpha$ satisfies (4), then $\beta = X\alpha$ fulfills (5) and (6). We first show

$$(13) \quad \beta_i \geq \beta_{i+1} \quad (i = 1, \ldots, n-1)$$
For $\beta_i - \beta_{i+1} = \sum z_j x_j$, where $z_j = x_{ij} - x_{i+1,j}$. By virtue of (8), $z_j \geq 0$ for $j \leq i$ and $z_j \leq 0$ for $j \geq i + 1$.

By (3), $\sum z_j = 0$. It follows from (4) that if we set $u = z_1 + \ldots + z_j$ and $v = z_{j+1} + \ldots + z_n = -u$, $\sum z_j \alpha_j \geq u \alpha_i + v \alpha_{i+1} = u(\alpha_i - \alpha_{i+1}) \geq 0$. This proves (5). The proof of (6) - which depends on (2) and (3) - is the same as in HLP.

Bibliography


THE NATIONAL BUREAU OF STANDARDS

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