NATIONAL BUREAU OF STANDARDS REPORT

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On an Inequality of Hardy, Littlewood and Pólya

by

A. J. Hoffman

U. S. DEPARTMENT OF COMMERCE
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A well-known inequality of Hardy, Littlewood and Pólya (see [3], p. 49) states: Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be points in R^n, and let σ and τ be permutations of the set $1, 2, \dots, n$ such that $\alpha_{\sigma 1} \geqq \alpha_{\sigma 2} \geqq \dots \geqq \alpha_{\sigma n}$ and $\beta_{\tau 1} \geqq \beta_{\tau 2} \geqq \dots > \beta_{\tau n}$. Then in order that

$$\begin{aligned}
&\beta_{\tau 1} \leqq \alpha_{\sigma 1} \\
&\beta_{\tau 1} + \beta_{\tau 2} \leqq \alpha_{\sigma 1} + \alpha_{\sigma 2} \\
&\quad\vdots \\
(1)\qquad &\beta_{\tau 1} + \beta_{\tau 2} + \dots + \beta_{\tau(n-1)} \leqq \alpha_{\sigma 1} + \alpha_{\sigma 2} + \dots + \alpha_{\sigma(n-1)} \\
&\beta_{\tau 1} + \beta_{\tau 2} + \dots + \beta_{\tau n} = \alpha_{\sigma 1} + \alpha_{\sigma 2} + \dots + \alpha_{\sigma n},
\end{aligned}$$

it is necessary and sufficient that $\beta = X\alpha$, where $X = (x_{ij})$ is a matrix satisfying

(2) $\qquad x_{ij} \geqq 0 \qquad$ for all i, j,

(3) $\qquad \sum_j x_{ij} = \sum_i x_{ij} = 1 \qquad$ for all i, j.

Several proofs of this theorem, and of various generalizations, are known (see [1], [2], [4] and [5] for some recent work). The purpose of this note is to present a strengthened

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version of the Hardy, Littlewood and Polya theorem (which we call HLP hereinafter), the proof of which is no more difficult than the proof of HLP.

We shall prove:

Theorem 1. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfy

(4) $$\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n .$$

Then in order that a vector $\beta = (\beta_1, \ldots, \beta_n)$ satisfy

(5) $$\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n$$

and

(6) $$\begin{aligned} \beta_1 &\leqq \alpha_1 \\ \beta_1 + \beta_2 &\leqq \alpha_1 + \alpha_2 \\ &\vdots \\ \beta_1 + \beta_2 + \cdots + \beta_{n-1} &\leqq \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \\ \beta_1 + \beta_2 + \cdots + \beta_n &= \alpha_1 + \alpha_2 + \cdots + \alpha_n , \end{aligned}$$

it is necessary and sufficient that

(7) $$\beta = X\alpha \quad ,$$

where $X = (x_{ij})$ is a matrix satisfying (2), (3)

(8) $x_{i_1 j} \geqq x_{i_2 j}$ for all i_1, i_2, j such that $(j-i_1)(i_1-i_2) \geqq 0$,

and

(9) $$x_{ij} = x_{ji} \quad \text{for all } i,j.$$

Proof of necessity: Consider the set of all vectors β satisfying (5) and (6). They clearly form a bounded, closed, convex set in R^n. Let us designate this set by K. We first show

Lemma. Let $1 \leqq a_1 < a_2 < \dots < a_k = n$. Let $y_1 = y_2 = \dots = y_{a_1}$

$= (\alpha_1 + \alpha_2 + \dots + \alpha_{a_1})/a_1, y_{a_1+1} = y_{a_1+2} = \dots = y_{a_2} =$

$(\alpha_{a_1+1} + \alpha_{a_1+2} + \dots + \alpha_{a_2})/(a_2 - a_1), \dots, y_{a_{k-1}+1} = y_{a_{k-1}+2}$

$= \dots = y_{a_k} = (\alpha_{a_{k-1}+1} + \alpha_{a_{k-1}+2} + \dots + \alpha_{a_k})/(a_k - a_{k-1})$.

Then (i) $y = (y_1, y_2, \dots, y_n)$ is a vertex of K, and (ii) every vertex of K arises in this way.

Proof of lemma: (i) First, we must show that $y \in K$. It is immediate that y satisfies (5). To show that y satisfies (6), we first remark that the last equation of (6) obviously holds, so all that needs to be established are the inequalities. Further, if we prove

(10) $$y_1 + y_2 + \dots + y_s \leqq \alpha_1 + \dots + \alpha_s,$$

for $1 \leqq s \leqq a_1$, it is clear from the definition of y that this argument will establish (10) for any value of s. Now

(10) is equivalent with

$$(11) \qquad s(\alpha_1 + \alpha_2 + \ldots + \alpha_{a_1})/a_1 \leqq \alpha_1 + \ldots + \alpha_s .$$

But by (4), $\alpha_1 + \alpha_2 + \ldots + \alpha_{a_1} \leqq \alpha_1 + \alpha_2 + \ldots + \alpha_s + (a_1 - s)\,\alpha_s$.

Hence,

$$(12) \qquad s(\alpha_1 + \ldots + \alpha_{a_1})/a_1 \leqq \alpha_1 + \ldots + \alpha_s + (a_1 - s)\alpha_s$$
$$- \frac{a_1 - s}{a_1}(\alpha_1 + \ldots + \alpha_s + (a_1 - s)\alpha_s)$$

$$\leqq \alpha_1 + \ldots + \alpha_s + \alpha_s\left[(a_1 - s) - \frac{a_1 - s}{a_1}(s + a_1 - s)\right] \quad \text{(by 4)}$$

$$< \alpha_1 + \ldots + \alpha_s ,$$

which proves (11).

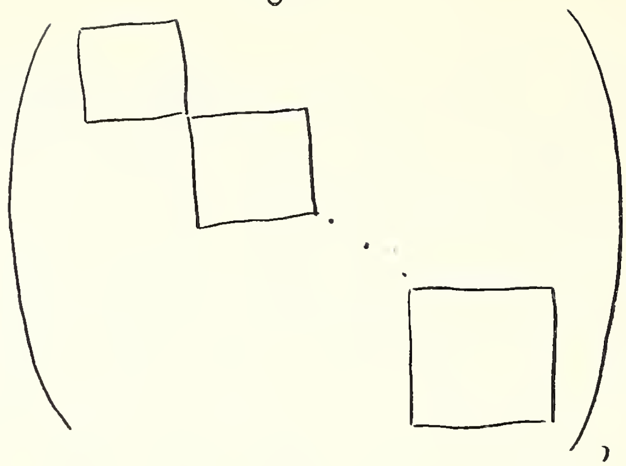
To show that y is a vertex, assume that $y = (y' + y'')/2$, where $y', y'' \in K$. Then if $y'_1 + \ldots + y'_{a_1} > y_1 + \ldots + y_{a_1} = \alpha_1 + \ldots + \alpha_{a_1}$, we would have $y_1'' + \ldots + y_{a_1}'' < \alpha_1 + \ldots + \alpha_{a_1}$, violating (6). Similarly $y_1' + \ldots + y'_{a_j} = y''_1 + \ldots + y''_{a_j} = y_1 + \ldots + y_{a_j}$, $j = 1, \ldots, k$.

Next, suppose for some s, $1 \leqq s < a_1$, we have $y'_s > y_s$. Then by the preceding paragraph, there is an integer t, $1 \leqq t < a_1$ such that $y'_t < y_t$. Note that $y_s = y_t$. Hence $y'_s > y'_t$. If $t < s$, then y' violates (5). If $t > s$, y'' violates (5). Hence

$y'_s = y''_s = y_s$. Clearly this argument extends to any $s = 1,\dots,n$. Thus $y = (y' + y'')/2$ implies $y = y' = y''$. This completes the proof of (i).

To prove (ii), let β satisfy (5) and (6), but not be a vector such as y above. This means that there exists integers $1 \leqq a_1 < a_2 < \dots < a_k = n$ such that $\beta_1 = \beta_2 = \dots = \beta_{a_1} < \beta_{a_1+1} = \dots = \beta_{a_2} < \dots < \beta_{a_{k-1}+1} = \dots = \beta_{a_k}$, but for at least one $j = 1,\dots,k-1$, we have $\beta_1 + {}_2 + \dots + \beta_{a_j} < \alpha_1 + \alpha_2 + \dots + \alpha_{a_j}$. Let ε be any positive number smaller than $\alpha_1 + \dots + \alpha_{a_j} - (\beta_1 + \dots + \beta_{a_j})$, $\beta_{a_{j-1}} - \beta_{a_j}$, $\beta_{a_j} - \beta_{a_{j+1}}$. Define $\beta_1' = \beta_1, \dots, \beta'_{a_{j-1}} = \beta_{a_{j-1}}$, $\beta'_{a_{j-1}+1} = \dots = \beta'_{a_j} = \beta_{a_j} + \varepsilon/(a_j - a_{j-1})$, $\beta'_{a_j+1} = \dots = \beta'_{a_{j+1}} = \beta_{a_{j+1}} - \varepsilon/(a_{j+1} - a_j)$, $\beta'_{a_{j+1}+1} = \beta_{a_{j+1}+1}, \dots, \beta'_n = \beta_n$. Also define $\beta'' = 2\beta - \beta'$. Then it is easy to check that β' and β'' each satisfy (5) and (6), $\beta = (\beta' + \beta'')/2$, $\beta' \neq \beta''$, so β is not a vertex of K. This completes the proof of (ii), thus of the lemma.

We resume our proof of the necessity of theorem 1. Let $\beta \in K$, then β is a convex combination of the vertices of K. By the lemma, each vertex of K is of the form $x_{a_1,\dots,a_k}$, where $X_{a_1,\dots,a_k}$ is the matrix of order n

(12) 

with O's everywhere outside the square boxes along the main diagonal, where the first box is of order a_1, the second box of order $a_2-a_1, \ldots$, the k-th box of order a_k-a_{k-1}; every entry in the first box is $1/a_1$, every entry in the second box is $1/(a_2-a_1), \ldots$, every entry in the k-th box is $1/(a_k-a_{k-1})$. Note that $X_{a_1,\ldots,a_k}$ satisfies (2), (3), (8), (9). Since $\beta = \sum c_i(X^{(i)}\alpha)$, where $c_i \geq 0$, $\sum c_i = 1$, $X^{(i)}$ is a matrix of the type (12), we may write $\beta = (\sum c_i X^{(i)})\alpha$, and observe that any convex combination of matrices satisfying (2), (3), (8), (9) also satisfies these conditions. If we set $X = \sum c_i X^{(i)}$, the proof of the necessity is complete.

<u>Proof of sufficiency</u>: We shall show that if a matrix X satisfies (2), (3), and (8) — (9) is not needed —, and a vector α satisfies (4), then $\beta = X\alpha$ fulfills (5) and (6). We first show

(13) $$\beta_i \geqq \beta_{i+1} \qquad (i = 1,\ldots,n-1)$$

For $\beta_i - \beta_{i+1} = \sum z_j \alpha_j$, where $z_j = x_{ij} - x_{i+1,j}$. By virtue of (8), $z_j \geqq 0$ for $j \leqq i$ and $z_j \leqq 0$ for $j \geqq i + 1$. By (3), $\sum z_j = 0$. It follows from (4) that if we set $u = z_1 + \ldots + z_j$ and $v = z_{j+1} + \ldots + z_n = -u$, $\sum z_j \alpha_j \geqq u\alpha_i + v\alpha_{i+1} = u(\alpha_i - \alpha_{i+1}) \geqq 0$. This proves (5). The proof of (6) - which depends on (2) and (3) - is the same as in HLP.

Bibliography

[1] R. Bellman and A. Hoffman, "On a Theorem of Ostrowski and Taussky," to appear in Archiv der Mathematik.

[2] D. Blackwell, "Equivalent Methods of Comparison," Ann. Math. Stat., June 1953.

[3] G. Hardy, J. Littlewood, and G. Polya, "Inequalities," Cambridge, 1934.

[4] A. Ostrowski, "Sur les Fonctions Convexes et Concaves au sens de Schur", Jour. de Math. Pures et Appl., 31(1952), p. 253.

[5] S. Sherman, "On a Theorem of Hardy, Littlewood, Polya, and Blackwell," Proc. Nat. Acad. Sci., 37(1951), p. 826.

THE NATIONAL BUREAU OF STANDARDS

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The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

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