ON THE SOLUTION OF THE CATERER PROBLEM

by

J. W. Gaddum

A. J. Hoffman, D. Sokolowsky
The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.


**Ordnance Development.** These three divisions are engaged in a broad program of research and development in advanced ordnance. Activities include basic and applied research, engineering, pilot production, field testing, and evaluation of a wide variety of ordnance matériel. Special skills and facilities of other NBS divisions also contribute to this program. The activity is sponsored by the Department of Defense.

**Missile Development.** Missile research and development: engineering, dynamics, intelligence, instrumentation, evaluation. Combustion in jet engines. These activities are sponsored by the Department of Defense.

- Office of Basic Instrumentation
- Office of Weights and Measures.
ON THE SOLUTION OF THE CATERER PROBLEM

by

Jerry Gaddum
A. J. Hoffman, D. Sokolowsky

U.S. Air Force
National Bureau of Standards

This work was sponsored (in part) by the Office of Scientific Research, USAF.
ON THE SOLUTION OF THE CATERER PROBLEM

by

J. W. Gaddum*

A. J. Hoffman, D. Sokolowsky**

Introduction

This paper continues the investigation of the Caterer problem formulated and partially solved by Walter Jacobs in [1]. We shall describe and justify an algorithm for solving the problem, indeed for solving a problem that is slightly more general than the original one. A knowledge of [1] is desirable for an interpretation of the abstractly formulated linear program discussed here, but is not necessary for a reading of this paper. We believe, however, that the reader who is interested in the "life cycle" of a linear program - (a) formulation, (b) simplification, (c) development of an algorithm for solution - will learn from the discussion of the caterer problem contained in [1] (which treats (a), (b) and a special case of (c)) and this article an instructive, and in some ways, typical example.

We now introduce the problem to be attacked:

Let $t_1, \ldots, t_m$ be given constants, $r_1, \ldots, r_n$ given non-negative constants, let $X$ be a variable, $z = (z_1, \ldots, z_n)$ be a variable vector lying in the rectangular parallelopiped

---

*U. S. Air Force.

**National Bureau of Standards

This work was sponsored (in part) by the Office of Scientific Research, USAF.
Let \( A_1, \ldots, A_m \) be given vectors in \( n \)-space, the coordinates of each consisting of 1's and 0's such that:

(2) The 1's in each \( A_i \) are consecutive, and

(3) If the coordinates in \( A_i \) which are 1 are \( a, a + 1, \ldots, b \), and the coordinates in \( A_j \) which are 1 are \( c, c+1, \ldots, d \), and if \( a < c \), then \( b \leq d \). We assume tacitly that each \( A_i \) contains at least one 1.

The problem is: for any number \( \lambda \geq 0 \), to find values for \( x \) and \( z \), subject to (1) and

(4) \( (A_i, z) \geq t_i - x \) \( (i = 1, \ldots, m) \)

which minimize the linear form

(5) \( \lambda x + \sum_{i=1}^{n} z_i \)

A comparison of our problem with the transformed caterer problem given in (4.3) of [1] will show that (4.3) is a special case of the above.

2. The solution if \( X \) is specified. Since our object is to study the behavior of \( X \) and \( Z \), it is not unnatural to ask what the solution of our problem would be if \( X \) were specified; i.e., if \( X \) were a given constant. If we set \( u_i = t_i - X \) \( (i = 1, \ldots, m) \), then our problem becomes: for all \( z = (z_1, \ldots, z_n) \) subject to (1) and
Before solving the problem, one should check to see that the inequalities (1) and (4') are consistent. Set \( r = (r_1, \ldots, r_n) \). Then it is clear that the inequalities are consistent if and only if \( (A_i, r) \geq \mu_i \) \( (i = 1, \ldots, m) \), which can be easily ascertained.

Assume now that (1) and (4') are consistent. Let \( S_j \) be the set of all indices \( i \) such that the \( j \)-th coordinate of \( A_i \) is 1. Define \( r^{(0)} = (0, r_2, r_3, \ldots, r_n) \), and

\[
(6) \quad z_1(X) = \max \left\{ 0; \max_{i \in S_1} \left[ \mu_i - (A_i, r^{(0)}) \right] \right\}
\]

If \( z_1(X), \ldots, z_k(X) \) have been defined \( (1 \leq k \leq n) \), let \( r^{(k)} = (z_1(X), \ldots, z_k(X), 0, r_{k+2}, \ldots, r_n) \). Then define

\[
(7) \quad z_{k+1}(X) = \max \left\{ 0; \max_{i \in S_{k+1}} \left[ \mu_i - (A_i, r^{(k)}) \right] \right\}
\]

We contend that the vector \( z(X) = (z_1(X), \ldots, z_n(X)) \) defined inductively above solves the problem of this section.

It is clear that our prescription states that \( z_1(X) \) is the smallest value of \( z \), consistent with (1) and (4'); and that, assuming \( z_1 = z_1(X), \ldots, z_k = z_k(X), z_{k+1}(X) \) is the smallest value of \( z_{k+1} \) consistent with (1) and (4').
We shall prove our contention by showing that

Lemma 1: For every $k$ ($k = 1, \ldots, n$), $z(X)$ minimizes $z_1^+ \ldots + z_k^+$, subject to (1) and $(4')$.

Proof by induction: We have already seen that lemma 1 holds if $k = 1$. Assume it holds for $k - 1$, we wish to show it holds for $k$. If $z_k(X) = 0$, this is obvious from the induction hypothesis. If $z_k(X) \neq 0$, then by (7), there exists a vector $A_i$ such that

(8) \[ z_k(X) + (A_i, r^{(k-1)}) = u_i, \quad i \in S_k \]

Let us write out the inequality

(9) \[ (A_i, z) \geq u_i \]

as

(9') \[ z_{k-\alpha}^+ + \ldots + z_{k-1}^+ + z_k^+ + z_{k+1}^+ \ldots + z_{k+\beta}^+ \geq \mu_i \]

(Recall (2)). Of course $\alpha$ or $\beta$ or both may be 0. Then (8) says that

(10) \[ z_{k-\alpha}(X) + \ldots + z_k(X) + r_{k+1} + \ldots + r_{k+\beta} = \mu_i \]

Now (9') is one of the inequalities $(4')$. Hence, by (10) and (1), $z_{k-\alpha}(X) + \ldots + z_k(X) = \min z_{k-\alpha}^+ \ldots + z_k^+$, for all $z$ subject to (1) and $(4')$. But by the induction hypothesis, $z_1(X) + \ldots + z_{k-\alpha-1}(X) = \min z_1^+ \ldots + z_{k-\alpha-1}^+$. Combining these last two statements proves the induction step.
Note that our proof uses (2), but not (3). The theorem is false if (2) does not hold.

Now, it is clear from (4) that the set of X for which the inequalities (1) and (4) of our main problem are consistent form a closed half-line containing arbitrarily large positive numbers. Let e be the end point of . For each X ≥ e, let S(X) = min (z₁ +...+ zₙ), subject to (1) and (4). We have just seen how to compute S(X) and we now prove

3. For X ≥ e, S(X) is a convex non-increasing function of X.

Proof: Let X₁ > X₂; then the convex region which is the intersection of the half-spaces corresponding to X=X₁ in (4) contains the convex region corresponding to X=X₂; hence S(X₁) ≤ S(X₂). To prove S(X) is convex, we first prove a more general lemma.

Let A and B be two real linear spaces, and let K be a convex set in A ⊗ B. Let V be the projection of K on A; i.e.,

V = \{v | v ∈ A, ∃ w ∈ B such that v ⊗ w ∈ K\}; let W be the projection of K on B. It is easy to see that V and W are convex sets. Let f be a convex function bounded from below defined on V.

Lemma 3. The function

(11)  g(w) = \inf_{v \otimes w \in K} f(v)

is a convex function on W.

Proof: We must show that w₁ ∈ W, w₂ ∈ W, α ≥ 0, β ≥ 0, α+β = 1, imply
Let $E$ be any positive number. Then by (11), there exist $v_1, v_2$ such that

$$v_1 \odot w_1 \in K, v_2 \odot w_2 \in K, \text{ and}$$

$$f(v_1) \leq g(w_1) + E, \quad f(v_2) \leq g(w_2) + E.$$ 

Since $K$ is convex, it follows from (13) that

$$\alpha (v_1 \odot w_1) + \beta (v_2 \odot w_2) = (\alpha v_1 + \beta v_2) \odot (\alpha w_1 + \beta w_2) \in K.$$ 

Therefore, since $f$ is convex, $$g(\alpha w_1 + \beta w_2) \leq f(\alpha v_1 + \beta v_2) \leq \alpha f(v_1) + \beta f(v_2) \leq \alpha g(w_1) + \beta g(w_2) + E,$$ by (14).

Since this inequality holds for every $E > 0$, we have (12).

Apply lemma 3 to the present situation as follows: Let $z = v, z_1 + ... + z_n = f(v), X = w, S(X) = g(w).$ This completes the proof of 3.

4. **Summary of the situation to date.**

Let us assume, temporarily that $S(X)$ is explicitly known. Now it is manifest from (6) and (7) that $z_j(X)$ is a polygonal curve. Hence $S(X) = \sum_{j=1}^{n} z_j(X)$ is a polygonal curve. Since $S(X)$ is convex, the half-line $L$ will break up into intervals $(e = e_0, e_1), (e_1, e_2), ... , (e_d, \infty),$ such that the slope of $S(X)$ is $-m_i$ in $(e_0, e_1), -m_2$ in $(e_1, e_2), ... , -m_d$ in $(e_{d-1}, e_d), 0$ in $(e_d, \infty), m_1 > m_2 > ... > m_d > 0$. It is obvious
that \( e_d = \max_i t_i \) and that in \((e_d, \infty)\), \(S(X) = 0\). It is also clear that \( e_o = \max_i \left\{ t_i - (A_i, r) \right\} \). Assuming that all \( e_o \) and \( m_o \) were known, it is obvious that the minimum of (5) (which, as the sum of two convex polygonal functions of \( X \), is also a convex polygonal function of \( X \)) could be obtained as follows:

if \( \lambda \geq m_1 \), the minimum of (5) is attained at \( X = e_o \),

if \( m_1 \geq \lambda \geq m_2 \), the minimum of (5) is attained at \( X = e_1 \),

(15) if \( m_{d-1} \geq \lambda \geq m_d \), the minimum of (5) is attained at \( X = e_{d-1} \),

if \( m_d \geq \lambda \geq 0 \), the minimum of (5) is attained at \( X = e_d \),

if \( 0 > \lambda \), (5) has no minimum.

Since we have already seen in paragraph 2 how to obtain the \( z_j \) which minimize (5) once the correct \( X \) is known, formulas (15) solve our problem once the numbers \( e_o, e_1, \ldots, e_d \) and \( m_1, \ldots, m_d \) are determined.

We already know \( e_o \). We shall show below how, given \( e_{o-1} \) one may obtain \( e_o \) and \( m_o \) by a reasonably efficient procedure. Since \( e_d \) is also known, this procedure, in view of (15), will solve our problem.

5. The algorithm.

Recall now the procedure described in paragraph 2, particularly formulas (6) and (7).
An inequality \((A_i, z) \geq t_i - X\) is said to be bounding for \(z_j\) at \(X\) if \(i \in S_j\) and \(z_j(X) = \lambda_i - (A_i, r(j-1))\). Let \(B_j\) be the set of all indices \(i\) such that \((A_i, z) \geq t_i - X\) is bounding for \(z_k\) at \(X = e_{o-1}\). Note that

\[(16) \quad z_j(e_{o-1}) > 0\]

implies that \(B_j\) is not empty. Let \(T_i\) be the set of all indices \(j\) such that the \(j\)-th coordinate of \(A_i\) is 1. Then

\[(17) \quad i \in B_j, k > j, k \in T_i \text{ imply } i \in B_k, z_k(e_{o-1}) = r_k\]

We now divide all indices \(i = 1, \ldots, n\) into two classes \(M\) (moving) and \(S\) (stationary) as follows: if \(z_1(e_{o-1}) = 0\), assign 1 to \(S\); if \(z_1(e_{o-1}) > 0\), assign 1 to \(M\); if 1, 2, \ldots, \(k\) have been assigned, and \(z_{k+1}(e_{o-1}) = 0\), assign \(k+1\) to \(S\); if \(z_{k+1}(e_{o-1}) > 0\), and no index in \(\bigcup_{i \in B_{k+1}} T_i\) (recall (16)) has been assigned to \(M\), we assign \(k+1\) to \(M\), but if an index in \(\bigcup_{i \in B_{k+1}} T_i\) has been assigned to \(M\), we put \(k+1\) in \(S\).

It will turn out that \(m_i\) is the number of indices assigned to \(M\), but before we show this, we first prove

**Lemma 4.** (a) If \(i \in \bigcup_j B_j\), then there is at most one index in the intersection of \(T_i\) and \(M\); (b) for any \(i\), there are at most two indices in the intersection of \(T_i\) and \(M\).

**Proof of (a):** Assume \(i \in B_j\); \(k, l \in T_i\), \(k, l \in M\), \(k < l\). If \(l \geq j\), then (17) and the definition of \(M\) contradict \(l \in M\).
Therefore, we may assume \( k < l < j \).

Let \( i_0 \in B_l \) (we know from (16) that \( l \in M \) implies \( B_l \) is not empty). Then since \( k \in M, k < l \), it follows that \( k \notin T_{i_0} \). But \( k, l, j \in T_i \); hence by (2) and (3), \( p > l \), \( p \in T_i \) imply \( p \in T_{i_0} \). But by (17), \( p \in T_{i_0} \), \( p > l \) imply \( z_p(e_{o-1}) = r_p \). Together with \( i \in B_j \), these facts imply \( i \in B_{i_0} \). We know, however, that \( k \in M, k \in T_i \). Combined with \( l \in T_i \), we have a contradiction of \( l \in M \).

**Proof of (b):** Assume \( j < k < l \); \( j, k, l \in T_i \), \( j, k, l \in M \).

Let \( p \in B_k \). Then \( j \notin T_p \), and also, by (17), \( l \notin T_p \). A comparison of \( T_i \) and \( T_p \) exhibits a violation of (2) and (3).

6. **Going from** \( e_{o-1} \) **to** \( e_o \): Resuming our main discussion, let \( K \) be the set of all \( i \) such that the intersection of \( T_i \) and \( M \) consists of two indices (\( K \) may be empty). Let \( \beta = \min_{i \in K} [(A_{i_o}, z(e_{o-1})) - t_i + e_{o-1}] \) if \( K \) is not empty, \( \beta = +\infty \) otherwise. Then \( \beta > 0 \). For if \( \beta = 0 \), i.e., there is an \( i_o \in K \) such that \( (A_{i_o}, z(e_{o-1})) = t_i - e_{o-1} \), then if \( k \) is the largest index in \( T_{i_o} \), clearly \( i_o \in B_k \), and we would have a violation of lemma 4(a).

Let \( \delta = \min_{i \in M} z_i(e_{o-1}) \). By the definition of \( M \), \( \delta > 0 \).

Finally, let \( \alpha = \min (\beta, \delta) > 0 \). We contend

**Lemma 5.** For all \( X \), \( e_{o-1} \leq X \leq \alpha + e_{o-1} \) and each \( j = 1, \ldots, n \), we have
Before proving the lemma, let us informally state its content. If \( X \) lies in the prescribed range, then: if \( j \in S \), \( z_j(X) = z_j(e_0 - 1) \); i.e., \( z_j(X) \) is stationary. If \( j \in M \), \( z_j(X) \) moves by precisely the amount by which \( X \) differs from \( e_0 - 1 \).

Proof: Obviously the lemma holds for \( z_1(X) \). Assuming it holds for \( z_1, \ldots, z_{k-1}(X) \), we shall show it holds for \( z_k(X) \). By virtue of lemma 1, we must show that if \( z_1 = z_1(X), \ldots, z_{k-1} = z_{k-1}(X) \) satisfy (13), \( z_{k+1} = r_{k+1}, \ldots, z_n = r_n \), then (i) if we set \( z_k = z_k(X) \) according to (13), the vector \( z \) satisfies (1) and (4); but (ii) if we set \( z_k < z_k(X) \) the vector \( z \) will not satisfy (1) and (4).

Suppose \( k \in M \). We first prove (i). For each \( i \in S_k, T_i \) may contain no indices in the range \( 1, \ldots, k-1 \) in \( M \) or 1 index in the range \( 1, \ldots, k-1 \) in \( M \) (lemma 4(b)). It is clear from the induction hypothesis that, in the former case, \( (A_i, z) \geq t_i - X \). In the latter case, again from the induction hypothesis, we have

\[
\text{19) } (A_i, z) \geq (A_i, z(e_0 - 1)) - 2(x - e_0 - 1))
\]

But, by the definition of \( \alpha \) and \( \beta \), for this range of \( X \),

\[
\text{20) } X-e_0-1 \leq (A_i, z(e_0 - 1)) - t_i + e_0 - 1.
\]
Combining (19) and (20), we obtain again \((A_1, z) \geq t_i - X\). 

Next, let \(i \in B_k\). Then clearly \((A_i, z) = t_i - X\), which proves (ii).

Suppose \(k \in S\). The proof of (i) involves no new ideas, and is omitted. To prove (ii), if \(z_k(e_{i-1}) = 0\), (ii) is immediate; hence, assume \(z_k(e_{i-1}) > 0\), which, by (16), (17), and the construction of \(S\), implies that there exists an \(i \in B_k\), \(j \in S\), \(j < k\) such that \(j \in M\). It follows from (17) and (7) that \((A_i, z) = t_i - X\), which proves (ii).

We observe the following immediate consequences of lemma 5.

**Corollary 1.** The function \(z_j(X)\) is a monotonic, non-increasing function of \(X\).

This implies that in going from \(e_{i-1}\) to \(e_i\), we may substitute for (i) the inequalities

\[
0 \leq z_j(X) \leq z_j(e_{i-1})
\]

**Corollary 2.** The slope \(m_i\) is equal to the number of indices in \(M\).

To complete our algorithm, when we arrive at \(X = \alpha + e_{i-1}\) we replace \(e_{i-1}\) by \(e_{i-1} + \alpha\) and repeat our construction of the new \(M\) and \(S\), say \(M'\) and \(S'\). By virtue of the convexity of \(S(X)\), the number of indices in \(M'\) is less than or equal to the number of indices in \(M\). If it is less than, then \(\alpha + e_{i-1} = e_i\). If it is equal, we proceed in the same manner.
It is clear, from the fact that each $z_j(x)$ is a polygonal function of $X$ with a finite number of vertices, and (from the definition of $\alpha$) that each step takes us from an $X$ corresponding to a vertex of some $z_j(X)$ to the next largest $X$ corresponding to a vertex of (possibly some other) $z_j(X)$, that we arrive at $e_\alpha$ in a finite number of steps.

BIBLIOGRAPHY

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.