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ON THE SOLUTION OF THE CATERER PROBLEM

by

J. W. Gaddum

A. J. Hoffman, D. Sokolowsky

U. S. DEPARTMENT OF COMMERCE
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Introduction

This paper continues the investigation of the Caterer problem formulated and partially solved by Walter Jacobs in [1]. We shall describe and justify an algorithm for solving the problem, indeed for solving a problem that is slightly more general than the original one. A knowledge of [1] is desirable for an interpretation of the abstractly formulated linear program discussed here, but is not necessary for a reading of this paper. We believe, however, that the reader who is interested in the "life cycle" of a linear program - (a) formulation, (b) simplification, (c) development of an algorithm for solution - will learn from the discussion of the caterer problem contained in [1] (which treats (a), (b) and a special case of (c)) and this article an instructive, and in some ways, typical example.

We now introduce the problem to be attacked:

Let $t_1, \ldots, t_m$ be given constants, $r_1, \ldots, r_n$ given non-negative constants, let X be a variable, $z = (z_1, \ldots, z_n)$ be a variable vector lying in the rectangular parallelopiped

*U. S. Air Force.
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(1) $$0 \leqq z_i \leqq r_i \qquad (i = 1,\ldots,n)$$

Let $A_1,\ldots,A_m$ be given vectors in n-space, the coordinates of each consisting of 1's and 0's such that:

(2) The 1's in each A_i are consecutive, and

(3) If the coordinates in A_i which are 1 are $a, a+1,\ldots,b$, and the coordinates in A_j which are 1 are $c, c+1,\ldots,d$, and if $a < c$, then $b \leqq d$. We assume tacitly that each A_i contains at least one 1.

The problem is: for any number $\lambda \geqq 0$, to find values for X and z, subject to (1) and

(4) $$(A_i, z) \geqq t_i - X \qquad (i = 1,\ldots,m)$$

which minimize the linear form

(5) $$\lambda X + \sum_{i=1}^{n} z_i$$

A comparison of our problem with the transformed caterer problem given in (4.3) of [1] will show that (4.3) is a special case of the above.

2. <u>The solution if</u> X <u>is specified</u>. Since our object is to study the behavior of X and Z, it is not unnatural to ask what the solution of our problem would be if X were specified; i.e., if X were a given constant. If we set $u_i = t_i - X$ $(i = 1,\ldots,m)$, then our problem becomes: for all $z = (z_1,\ldots,z_n)$ subject to (1) and

(4') $(A_i, z) \geqq \mu_i \qquad (i = 1,\ldots,m),$

minimize

(5') $z_1 + \ldots + z_n.$

Before solving the problem, one should check to see that the inequalities (1) and (4') are consistent. Set $r = (r_1,\ldots,r_n)$. Then it is clear that the inequalities are consistent if and only if $(A_i, r) \geqq \mu_i$ $(i = 1,\ldots,m)$, which can be easily ascertained.

Assume now that (1) and (4') are consistent. Let S_j be the set of all indices i such that the j-th coordinate of A_i is 1. Define $r^{(0)} = (0, r_2, r_3,\ldots,r_n)$, and

(6) $$z_1(X) = \max \left\{0; \max_{i \in S_1} [\mu_i - (A_i, r^{(0)})]\right\}$$

If $z_1(X),\ldots,z_k(X)$ have been defined $(1 \leqq k \leqq n)$, let $r^{(k)} = (z_1(X),\ldots,z_k(X), 0, r_{k+2},\ldots,r_n)$. Then define

(7) $$z_{k+1}(X) = \max \left\{0; \max_{i \in S_{k+1}} [\mu_i - (A_i, r^{(k)})]\right\}$$

We contend that the vector $z(X) = (z_1(X),\ldots,z_n(X))$ defined inductively above solves the problem of this section.

It is clear that our prescription states that $z_1(X)$ is the smallest value of z, consistent with (1) and (4'); and that, assuming $z_1 = z_1(X),\ldots,z_k = z_k(X)$, $z_{k+1}(X)$ is the smallest value of z_{k+1} consistent with (1) and (4').

We shall prove our contention by showing that

<u>Lemma</u> 1: For every k ($k = 1,\ldots,n$), $z(X)$ minimizes $z_1+\ldots+z_k$, subject to (1) and (4').

Proof by induction: We have already seen that lemma 1 holds if $k = 1$. Assume it holds for $k - 1$, we wish to show it holds for k. If $z_k(X) = 0$, this is obvious from the induction hypothesis. If $z_k(X) \neq 0$, then by (7), there exists a vector A_i such that

(8) $$z_k(X) + (A_i, r^{(k-1)}) = u_i, \; i \in S_k$$

Let us write out the inequality

(9) $$(A_i, z) \geqq \mu_i$$

as

(9') $$z_{k-\alpha}+\ldots+z_{k-1}+z_k+z_{k+1}+\ldots+z_{k+\beta} \geqq \mu_i$$

(Recall (2)). Of course α or β or both may be 0). Then (8) says that

(10) $$z_{k-\alpha}(X) +\ldots+ z_k(X) + r_{k+1} +\ldots+ r_{k+\beta} = \mu_i$$

Now (9') is one of the inequalities (4'). Hence, by (10) and (1), $z_{k-\alpha}(X) +\ldots+ z_k(X) = \min z_{k-\alpha}+\ldots+ z_k$, for all z subject to (1) and (4'). But by the induction hypothesis, $z_1(X) +\ldots+ z_{k-\alpha-1}(X) = \min z_1 +\ldots+ z_{k-\alpha-1}$. Combining these last two statements proves the inductio. step.

Note that our proof uses (2), but not (3). The theorem is false if (2) does not hold.

Now, it is clear from (4) that the set of X for which the inequalities (1) and (4) of our main problem are consistent form a closed half-line $\mathcal{L}$ containing arbitrarily large positive numbers. Let e be the end point of $\mathcal{L}$. For each $X \geqq e$, let $S(X) = \min (z_1 + \ldots + z_n)$, subject to (1) and (4). We have just seen how to compute S(X) and we now prove

3. <u>For</u> $X \geqq e$, S(X) <u>is a convex non-increasing function of</u> X.

Proof: Let $X_1 > X_2$; then the convex region which is the intersection of the half-spaces corresponding to $X=X_1$ in (4) contains the convex region corresponding to $X=X_2$; hence $S(X_1) \leqq S(X_2)$. To prove S(X) is convex, we first prove a more general lemma.

Let A and B be two real linear spaces, and let K be a convex set in $A \otimes B$. Let V be the projection of K on A; i.e., $V = \{v \mid v \in A, \exists\, w \in B$ such that $v \otimes w \in K$; let W be the projection of K on B. It is easy to see that V and W are convex sets. Let f be a convex function bounded from below defined on V.

<u>Lemma</u> 3. The function

$$g(w) = \inf_{v \otimes w \in K} f(v) \tag{11}$$

is a convex function on W.

<u>Proof</u>: We must show that $w_1 \in W$, $w_2 \in W$, $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta = 1$, imply

(12) $$g(\alpha w_1 + \beta w_2) \leqq \alpha g(w_1) + \beta g(w_2).$$

Let ε be any positive number. Then by (11), there exist v_1, v_2 such that

(13) $$v_1 \otimes w_1 \in K, \; v_2 \otimes w_2 \in K, \text{ and}$$

(14) $$f(v_1) \leqq g(w_1) + \varepsilon \;, \; f(v_2) \leqq g(w_2) + \varepsilon \,.$$

Since K is convex, it follows from (13) that $\alpha(v_1 \otimes w_1) + \beta(v_2 \otimes w_2) = (\alpha v_1 + \beta v_2) \otimes (\alpha w_1 + \beta w_2) \in K$. Therefore, since f is convex, $g(\alpha w_1 + \beta w_2) \leqq f(\alpha v_1 + \beta v_2) \leqq \alpha f(v_1) + \beta f(v_2) \leqq \alpha g(w_1) + \beta g(w_2) + \varepsilon$, by (14). Since this inequality holds for every $\varepsilon > 0$, we have (12).

Apply lemma 3 to the present situation as follows: Let $z = v$, $z_1 + \ldots + z_n = f(v)$, $X = w$, $S(X) = g(w)$. This completes the proof of 3.

4. <u>Summary of the situation to date</u>.

Let us assume, temporarily that S(X) is explicitly known. Now it is manifest from (6) and (7) that $z_j(X)$ is a polygonal curve. Hence $S(X) = \sum_{j=1}^{n} z_j(X)$ is a polygonal curve. Since S(X) is convex, the half-line $\mathcal{L}$ will break up into intervals $(e = e_0, e_1)$, $(e_1, e_2), \ldots, (e_d, \infty)$, such that the slope of S(X) is $-m_1$ in (e_0, e_1), $-m_2$ in (e_1, e_2), ..., $-m_d$ in (e_{d-1}, e_d), 0 in (e_d, ∞), $m_1 > m_2 > \ldots > m_d > 0$. It is obvious that $e_d = \max_i t_i$ and that in (e_d, ∞), $S(X) = 0$. It is also clear that $e_o = \max_i \{t_i - (A_i, r)\}$. Assuming that all e_δ and m_δ were known, it is obvious that the minimum of (5) (which, as the sum of two convex polygonal functions of X, is also a convex polygonal function of X) could be obtained as follows:

if $\lambda \geqq m_1$, the minimum of (5) is attained at $X = e_o$,

if $m_1 \geqq \lambda \geqq m_2$, the minimum of (5) is attained at $X = e_1$,

. . .

(15) if $m_{d-1} \geqq \lambda \geqq m_d$, the minimum of (5) is attained at $X = e_{d-1}$,

if $m_d \geqq \lambda \geqq 0$, the minimum of (5) is attained at $X = e_d$,

if $0 > \lambda$, (5) has no minimum.

Since we have already seen in paragraph 2 how to obtain the z_j which minimize (5) once the correct X is known, formulas (15) solve our problem once the numbers $e_o, e_1, \ldots, e_d$ and $m_1, \ldots, m_d$ are determined.

We already know e_o. We shall show below how, given $e_{\delta-1}$ one may obtain e_δ and m_δ by a reasonably efficient procedure. Since e_d is also known, this procedure, in view of (15), will solve our problem.

5. <u>The algorithm.</u>

Recall now the procedure described in paragraph 2, particularly formulas (6) and (7).

An inequality $(A_i,z) \geqq t_i - X$ is said to be <u>bounding</u> for z_j at X if $i \in S_j$ and $z_j(X) = \mu_i - (A_i, r^{(j-1)})$. Let B_j be the set of all indices i such that $(A_i,z) \geqq t_i - X$ is bounding for z_k at $X = e_{\delta-1}$. Note that

$$(16) \qquad z_j(e_{\delta-1}) > 0$$

implies that B_j is not empty. Let T_i be the set of all indices j such that the j-th coordinate of A_i is 1. Then

$$(17) \qquad i \in B_j,\ k > j,\ k \in T_i \text{ imply } i \in B_k,\ z_k(e_{\delta-1}) = r_k$$

We now divide all indices i = 1,...,n into two classes M (moving) and S (stationary) as follows: if $z_1(e_{\delta-1}) = 0$, assign 1 to S; if $z_1(e_{\delta-1}) > 0$, assign 1 to M; if 1,2,...,k have been assigned, and $z_{k+1}(e_{\delta-1}) = 0$, assign k+1 to S; if $z_{k+1}(e_{\delta-1}) > 0$, and no index in $\bigcup_{i \in B_{k+1}} T_i$ (recall (16)) has been assigned to M, we assign k+1 to M, but if an index in $\bigcup_{i \in B_{k+1}} T_i$ has been assigned to M, we put k+1 in S.

It will turn out that m_i is the number of indices assigned to M, but before we show this, we first prove

<u>Lemma</u> 4. (a) If $i \in \bigcup_j B_j$, then there is at most one index in the intersection of T_i and M; (b) for any i, there are at most two indices in the intersection of T_i and M.

<u>Proof of</u> (a): Assume $i \in B_j$; $k,\ell \in T_i$, $k,\ell \in M$, $k < \ell$. If $\ell \geq j$, then (17) and the definition of M contradict $\ell \in M$.

Therefore, we may assume $k < \ell < j$.

Let $i_o \in B_\ell$ (we know from (16) that $\ell \in M$ implies B_ℓ is not empty). Then since $k \in M$, $k < \ell$, it follows that $k \notin T_{i_o}$. But $k, \ell, j \in T_i$; hence by (2) and (3), $p > \ell$, $p \in T_i$ imply $p \in T_{i_o}$. But by (17), $p \in T_{i_o}$, $p > \ell$ imply $z_p(e_{\delta-1}) = r_p$. Together with $i \in B_j$, these facts imply $i \in B_\ell$. We know, however, that $k \in M$, $k \in T_i$. Combined with $\ell \in T_i$, we have a contradiction of $\ell \in M$.

<u>Proof of</u> (b): Assume $j < k < \ell$; $j, k, \ell \in T_i$, $j, k, \ell \in M$. Let $p \in B_k$. Then $j \notin T_p$, and also, by (17), $\ell \notin T_p$. A comparison of T_i and T_p exhibits a violation of (2) and (3).

6. <u>Going from</u> $e_{\delta-1}$ <u>to</u> e_δ: Resuming our main discussion, let K be the set of all i such that the intersection of T_i and M consists of two indices (K may be empty). Let $\beta = \min_{i \in K} \{(A_i, z(e_{\delta-1})) - t_i + e_{\delta-1}\}$ if K is not empty, $\beta = +\infty$ otherwise. Then $\beta > 0$. For if $\beta = 0$, i.e., there is an $i_o \in K$ such that $(A_{i_o}, z(e_{\delta-1})) = t_i - e_{\delta-1}$, then if k is the largest index in T_{i_o}, clearly $i_o \in B_k$, and we would have a violation of lemma 4(a).

Let $\gamma = \min_{i \in M} z_i(e_{\delta-1})$. By the definition of M, $\gamma > 0$. Finally, let $\alpha = \min(\beta, \gamma) > 0$. We contend

<u>Lemma</u> 5. For all X, $e_{\delta-1} \leqq X \leqq \alpha + e_{\delta-1}$ and each $j = 1, \ldots, n$, we have

$$(18)\qquad z_j(X) = \overline{z_j(X)} \equiv \begin{cases} z_j(e_{\delta-1}) & \text{if } j \in S \\ z_j(e_{\delta-1}) - (X-e_{d-1}) & \text{if } j \in M \end{cases}$$

Before proving the lemma, let us informally state its content. If X lies in the prescribed range, then: if $j \in S$, $z_j(X) = z_j(e_{\delta-1})$; i.e., $z_j(X)$ is <u>stationary</u>. If $j \in M$, $z_j(X)$ <u>moves</u> by precisely the amount by which X differs from $e_{\delta-1}$.

<u>Proof</u>: Obviously the lemma holds for $z_1(X)$. Assuming it holds for $z_1,\ldots,z_{k-1}(X)$, we shall show it holds for $z_k(X)$. By virtue of lemma 1, we must show that if $z_1 = z_1(X),\ldots,z_{k-1} = z_{k-1}(X)$ satisfy (18), $z_{k+1} = r_{k+1},\ldots,z_n = r_n$, then (i) if we set $z_k = \overline{z_k(X)}$ according to (18), the vector z satisfies (1) and (4); but (ii) if we set $z_k < \overline{z_k(X)}$ the vector z will not satisfy (1) and (4).

Suppose $k \in M$. We first prove (i). For each $i \in S_k$, T_i may contain no indices in the range $1,\ldots,k-1$ in M or 1 index in the range $1,\ldots,k-1$ in M (lemma 4(b)). It is clear from the induction hypothesis that, in the former case, $(A_i,z) \geqq t_i - X$. In the latter case, again from the induction hypothesis, we have

$$(19)\qquad (A_i,z) \geqq (A_i,z(e_{\delta-1})) - 2(x - e_{\delta-1}))$$

But, by the definition of α and β, for this range of X,

$$(20)\qquad X-e_{\delta-1} \leqq (A_i,z(e_{\delta-1})) - t_i + e_{\delta-1}.$$

Combining (19) and (20), we obtain again $(A_i,z) \geqq t_i - X$. Next, let $i \in B_k$. Then clearly $(A_i,z) = t_i - X$, which proves (ii).

Suppose $k \in S$. The proof of (i) involves no new ideas, and is omitted. To prove (ii), if $z_k(e_{\delta-1}) = 0$, (ii) is immediate; hence, assume $z_k(e_{\delta-1}) > 0$, which, by (16), (17), and the construction of S, implies that there exists an $i \in B_k$, $j \in S_i$, $j < k$ such that $j \in M$. It follows from (17) and (7) that $(A_i,z) = t_i - X$, which proves (ii).

We observe the following immediate consequences of lemma 5.

Corollary 1. The function $z_j(X)$ is a monotonic, non-increasing function of X.

This implies that in going from $e_{\delta-1}$ to e_δ, we may substitute for (1) the inequalities

(1')
$$0 \leqq z_j(X) \leqq z_j(e_{\delta-1})$$

Corollary 2. The slope m_1 is equal to the number of indices in M.

To complete our algorithm, when we arrive at $X = \alpha + e_{\delta-1}$ we replace $e_{\delta-1}$ by $e_{\delta-1} + \alpha$ and repeat our construction of the new M and S, say M' and S'. By virtue of the convexity of S(X), the number of indices in M' is less than or equal to the number of indices in M. If it is less than, then $\alpha + e_{\delta-1} = e_\delta$. If it is equal, we proceed in the same manner.

It is clear, from the fact that each $z_j(x)$ is a polygonal function of X with a finite number of vertices, and (from the definition of α) that each step takes us from an X corresponding to a vertex of some $z_j(X)$ to the next largest X corresponding to a vertex of (possibly some other) $z_j(X)$, that we arrive at e_δ in a finite number of steps.

BIBLIOGRAPHY

[1] The Caterer Problem, by W. Jacobs, to appear.

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