NATIONAL BUREAU OF STANDARDS REPORT

2942

Progress Report
on
Air Conditioning in Underground Structures

to
Office of the Chief of Engineers
Department of the Army

NBS

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

U. S. DEPARTMENT OF COMMERCE

Sinclair Weeks, *Secretary*

NATIONAL BUREAU OF STANDARDS

A. V. Astin, *Director*

THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.

Electricity. Resistance Measurements. Inductance and Capacitance. Electrical Instruments. Magnetic Measurements. Applied Electricity. Electrochemistry.

Optics and Metrology. Photometry and Colorimetry. Optical Instruments. Photographic Technology. Length. Gage.

Heat and Power. Temperature Measurements. Thermodynamics. Cryogenics. Engines and Lubrication. Engine Fuels. Cryogenic Engineering.

Atomic and Radiation Physics. Spectroscopy. Radiometry. Mass Spectrometry. Solid State Physics. Electron Physics. Atomic Physics. Neutron Measurements. Infrared Spectroscopy. Nuclear Physics. Radioactivity. X-Rays. Betatron. Nucleonic Instrumentation. Radiological Equipment. Atomic Energy Commission Instruments Branch.

Chemistry. Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Gas Chemistry. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

Mechanics. Sound. Mechanical Instruments. Aerodynamics. Engineering Mechanics. Hydraulics. Mass. Capacity, Density, and Fluid Meters.

Organic and Fibrous Materials. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Organic Plastics. Dental Research.

Metallurgy. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion.

Mineral Products. Porcelain and Pottery. Glass. Refractories. Enameled Metals. Concreting Materials. Constitution and Microstructure. Chemistry of Mineral Products.

Building Technology. Structural Engineering. Fire Protection. Heating and Air Conditioning. Floor, Roof, and Wall Coverings. Codes and Specifications.

Applied Mathematics. Numerical Analysis. Computation. Statistical Engineering. Machine Development.

Electronics. Engineering Electronics. Electron Tubes. Electronic Computers. Electronic Instrumentation.

Radio Propagation. Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Frequency Utilization Research. Tropospheric Propagation Research. High Frequency Standards. Microwave Standards.

Ordnance Development.
Electromechanical Ordnance.
Ordnance Electronics.
These three divisions are engaged in a broad program of research and development in advanced ordnance. Activities include basic and applied research, engineering, pilot production, field testing, and evaluation of a wide variety of ordnance matériel. Special skills and facilities of other NBS divisions also contribute to this program. The activity is sponsored by the Department of Defense.

Missile Development. Missile research and development: engineering, dynamics, intelligence, instrumentation, evaluation. Combustion in jet engines. These activities are sponsored by the Department of Defense.

- Office of Basic Instrumentation
- Office of Weights and Measures.

NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT 1003-10-4831

November 27, 1953

NBS REPORT 2942

Progress Report

on

Air Conditioning in Underground Structures

by

B. A. Peavy
H. E. Robinson
R. S. Dill

to

Office of the Chief of Engineers
Department of the Army

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

PROGRESS REPORT

AIR CONDITIONING IN UNDERGROUND STRUCTURES

I. INTRODUCTION

During the period from May 1 to November 15, 1953, tests have been continued in the underground test chamber at Mount Weather, testing at the Fort Ritchie site was initiated, and tests were made on the tunnel ventilation and underground spray pond projects at Mount Weather, Va. Data from previous tests were analyzed and two reports are included, one on an initial warm-up period and the other on tunnel ventilation. Also included are the method for determining thermal conductivity of rock core samples, a theoretical heat transfer study of the vehicular tunnel and underground reservoir at the Fort Ritchie site, a trip to other underground sites, and a mathematical treatise on the region bounded internally by a circular cylinder.

II. TESTS PERFORMED AT MT. WEATHER, VA.

1. Underground Chamber

Test condition 12 - Steady state heating at 75° F air temperature from February 11 to April 13, 1953.
No dehumidification or ventilation air.

Test condition 13 - Steady state heating at 75° F air temperature with ventilation air from April 14 to May 7, 1953.

Test condition 14 - Cooling of chamber with negligible heat added and no ventilation air or dehumidification from May 8 to May 27, 1953.

The remainder of the report period was spent with intermittent refrigeration of the chamber air in order to reduce the temperature in the rock to approximately the original temperature of 53° F.

2. Underground Spray Pond Heat Exchanger

Test condition 1 - A constant heat input rate of approximately 55,000 BTU/hr was added to the spray water until the temperature of the pond water reached 100° F - May 18 to June 7, 1953.

Test condition 2 - The temperature of the spray water was maintained at approximately 100° F until there was perceptible temperature rise at the 20-foot rock depth.

The remainder of the report period was spent with refrigeration of the pond water in order to reduce the temperature in the rock approximately to the original temperature of 53° F. During the next reporting period, test conditions 3 and 4 will be run which are a repeat of conditions 1 and 2 except recirculation will be made to a stagnant pond instead of using sprays.

3. Tunnel Ventilation

Test condition 1 - Summer ventilation air was induced through the 1400-foot tunnel at an average air-flow rate of 24,700 cfm for the period July 8 to July 20, 1953. Temperatures and relative humidity were recorded for the air entering and leaving the tunnel.

No more tests were made during this reporting period because of Bureau of Mines activity in the main tunnel. During the next reporting period, tests will be made using winter ventilation air.

III. TEMPERATURE STUDIES AT SITE NEAR FT. RITCHIE, MD.

1. Ventilation shaft

Thermocouples were placed at selected positions in the east ventilation shaft and temperatures are measured by them in conjunction with a recording potentiometer located at the bottom of the shaft. The intended operation is that the east ventilation shaft will be used every other week. Recordings will be made of the air and rock temperatures currently during use of the east ventilation shaft.

2. Structure

Thermocouples were placed at selected positions in the rock, rock surface, ventilation air stream to structure and exhaust air stream from the structure. Temperatures are measured by the thermocouples in conjunction with a portable precision potentiometer. Readings of these temperatures are being made on a bi-weekly basis due to the very limited occupancy of the structure.

3. Cooling Water Heat Exchange to Earth

Temperature sensitive instruments have been ordered for installation in the 8-inch cooling water pipe. Temperatures will be measured at the cooling tower bypass and on the feed and return lines at the portal, blast gate, and power room.

IV. INITIAL WARM-UP PERIOD OF UNDERGROUND CHAMBER MT. WEATHER, VIRGINIA - April 23 to May 15, 1953

Object

The object of this test was to determine the time needed to bring the temperature of an underground chamber up to a temperature comfortable for human occupancy by means of a constant heat input rate, and also to determine the applicability of the theoretical equations of heat transfer to the experimental data gathered.

Description of Underground Chamber and Equipment

The underground chamber, the dimensions of which are 100' x 35' x 10' high, is adjacent to and connected with an experimental mine operated by the Bureau of Mines at Mount Weather, Virginia. The chamber is approximately 215 feet below the surface of the ground and 1,200 feet from the surface in a horizontal direction. The rock mass bounding the chamber consists mainly of greenstone with a scattering of epidote and quartz, and traces of various other minerals. Petrographically, the greenstone is a metamorphic basalt partially colored green by the presence of chlorite.

Measurements of the surface area of the walls, floor, and ceiling were made and showed that the projected surface area was approximately 10,000 square feet. Physical determinations of greenstone rock from the excavation were made at this Bureau and the results were:

Density, p	= 186 lb/ft^3
Specific heat, c	= 0.2 BTU/lb °F
Thermal conductivity (cores) k	= 1.45 BTU/hr ft °F
Thermal diffusivity,	= 0.039 ft^2/hr

The apparent porosity of greenstone samples tested by the Bureau of Mines was 0.50 percent.

The walls and ceiling are painted white and the floor paved with concrete. Figure 1 shows plan and elevation views of the chamber and arrangement of mechanical equipment and air distribution ductwork. Air was forced by the circulating fan (1) into the ductwork past electric strip heaters (5) to the diffusers (6) and the air from the chamber was returned to the circulating fan through the air return filters (7) and the plenum chamber.

Fifteen 12-foot long thermocouple poles were placed at selected positions (figure 2) in the rock. Thermocouples had been previously attached to these poles at intervals--one-half-foot intervals up to 6 feet and one-foot intervals from 6 to 12 feet. Room air temperatures were measured at heights of 2, 30, 60, and 90 inches above the floor at twenty stations in the chamber. The thermocouples were copper-constantan and temperatures were measured by them in conjunction with an indicating potentiometer located in an instrument room built within the chamber.

Theory

For analytical purposes, the heat transfer in the test chamber was compared with that occurring in three elementary geometric shapes; namely, a cylinder, a sphere, and a group of plane surfaces. The boundary conditions were that heat was transferred to a solid bounding an elementary shape with a constant heat flux at the surface of the shape and the initial temperature of the solid was assumed equal to zero as an arbitrary datum plane. For the three cases under consideration, the temperature at a distance from the surface is:

plane
$$v = \frac{Q}{K}\sqrt{\alpha t}\left(2 \text{ ierfc } \frac{x}{2\sqrt{\alpha t}}\right) \tag{1}$$

cylinder
$$v = \frac{Q}{K}\left(\frac{a}{r}\right)^{1/2}\sqrt{\alpha t}\left[2 \text{ ierfc } \frac{r-a}{2\sqrt{\alpha t}} - \sqrt{\alpha t}\,\frac{(a+3r)}{8ar}\,4\, i^2\text{erfc } \frac{r-a}{2\sqrt{\alpha t}} + \frac{\alpha t(33r^2+6ar+9a^2)}{16a^2r^2}\, i^3\text{erfc } \frac{r-a}{2\sqrt{\alpha t}}\right] \tag{2}$$

sphere
$$v = \frac{Q\,a^2}{Kr}\left[\text{erfc } \frac{r-a}{2\sqrt{\alpha t}} - e^{\frac{r-a}{a}+\frac{\alpha t}{a^2}}\,\text{erfc}\left(\frac{r-a}{2\sqrt{\alpha t}} + \sqrt{\frac{\alpha t}{a^2}}\right)\right] \tag{3}$$

and the temperature at the surface of the solid is:

plane
$$v = 1.1284\,\frac{Q}{K}\sqrt{\alpha t} \tag{4}$$

cylinder
$$v = \frac{Qa}{K}\left[\frac{2}{\sqrt{\pi}}\left(\frac{\alpha t}{a^2}\right)^{1/2} - \frac{1}{2}\,\frac{\alpha t}{a^2} + \frac{1}{2\sqrt{\pi}}\left(\frac{\alpha t}{a^2}\right)^{3/2} - \frac{3}{16}\left(\frac{\alpha t}{a^2}\right)^{2}\right] \tag{5}$$

sphere
$$v = \frac{Qa}{K}\left[1 - e^{\alpha t/a^2}\,\text{erfc}\sqrt{\frac{\alpha t}{a^2}}\right] \tag{6}$$

where the equations for the cylinder are applicable for small values of time (satisfactory for the duration of the test in question). Development of the above equation can be found in "Conduction of Heat in Solids", by Carslaw and Jaeger, Oxford Press, 1948.

Nomenclature and Units

- v = temperature above arbitrary datum plane, °F.
- Q = constant heat flux at surface of solid, BTU/hr ft^2.
- t = time, hours.
- α = thermal diffusivity of solid, ft^2/hr.
- x = distance into solid from plane surface, ft.
- a = radius of cylinder on sphere, ft.
- r = radius of concentric cylinder or sphere composed of solid for r greater than a, ft.
- K = thermal conductivity of solid, BTU/hr ft deg F.
- erfc y is the complement of the error integral and ierfc y, i^2erfc y and i^3erfc are successive integrations of erfc y.

Test Procedure

With the initial temperature in the rock to 12 feet in depth practically uniform at 53.0 °F, a constant heat input of 17.8 kilowatt or 60,800 BTU/hr was supplied to the test chamber. At regular intervals during the test, temperatures of the rock surface, the rock at selected depths, room air, wet and dry bulb of the room were recorded as well as the electric energy input as measured by kilowatt-hour meters.

The test was arbitrarily terminated at the time when the average of the plane rock surface temperatures at poles 1, 2, 3, 4, 5, 9, 11, and 15 on figure 2 reached 70° F. This time was 522 hours or 21.75 days.

Results

Figure 3 is a plot of the average rock temperature (computed from the average of the temperatures on the fifteen poles) against depth in the rock, with time as a parameter. Figures 4 to 8 show temperature distributions at various cross sections in the rock after 521 hours of test.

Following is a table showing the heat input to the chamber from readings of watt hour meters compared with the heat stored in the rock computed from the mean temperature rise at various times from the start of the test. Also the depth of perceptible heat penetration is noted.

Time	Electric heat input	Heat in rock*	Heat penetration
hr	BTU	BTU	ft
49	3,010,300	2,979,300	5.0
100	5,960,000	6,299,000	7.8
170	10,256,000	11,243,000	8.8
290	17,928,000	18,581,000	11.0

*No attempt was made to account for irregularity in heat flow at compressor room and chamber entrance in computing heat absorbed by rock. The mean temperature was determined by finding area under curves of figure 3, and dividing the area by the depth of perceptible heat penetration. The specific heat of the rock was 37.2 BTU/ft^3 °F.

The average room air temperature was approximately 6° F above that of the rock surface temperature throughout the test and the heat flux calculated from the kilowatt-hour meter readings averaged 6.08 BTU/hr ft^2. From these data, the heat transfer from the room air to the rock surface appeared to obey Newton's law of cooling:

$$Q = h \Delta T \qquad (7)$$

where the surface conductance coefficient h was approximately 1.0 BTU/hr ft^2 °F.

Referring to equations (4), (5), and (6), the heat flux, Q, was computed from the experimental surface temperature data corresponding to various times after the start of the test and plotted on figure 9. The equivalent radii for the cylinder and sphere were computed on the basis of surface areas equal to that of the space, yielding radii of 15.9 ft for the cylinder and 28.2 ft for the sphere. The values of heat flux computed from equations (4), (5), and (6) were used in computing the temperatures at 0.5-, 1.0-, 2.0-, 3.0-, and 4.0-foot depths in the rock by equations (1), (2), and (3). These temperatures are plotted on figure 10 along with the experimental values for comparison purposes.

Figures 11 and 12 show views of the underground chamber.

Discussion and Conclusions

1. The heat balance showed that the heat stored in the rock as computed from the observed temperatures of the rock agreed with the measured electrical heat input to within 10 percent. This comparison was made for the first 290 hours of the test because the heat penetration exceeded 12 feet soon after that time.

   The computed heat input was greater than the measured heat input, which seems to show that the average temperatures computed from temperatures in the rock from the fifteen poles were higher than the actual average temperatures in the rock.

2. Figures 4 through 8 show isotherms in the rock at various cross sections. The isotherms tend to elliptical shape, especially in smaller cross sections and at greater depths.

3. The coefficient of heat transfer between the air and rock surface was approximately 1.0 BTU/hr ft^2 deg F. This value is approximately what would be expected for this case wherein heat transfer was by natural convection from nearly still air, with negligible radiation because all surfaces were approximately the same temperature.

4. The constant rate of energy input to the chamber, as measured electrically averaged 60,800 BTU/hr, which gives an average heat flux of 6.08 BTU/hr ft^2 for the measured 10,000 square feet of projected surface area. The discrepancy between the measured and calculated heat flux (figure 9) is believed to be due primarily to the difference between the actual shape of the chamber and the shapes assumed for mathematical analysis; namely, that heat flow from the chamber surfaces took place in three dimensions (diverging similar to radii of a sphere) whereas the equations apply strictly to one-dimensional flow for the plane surface, two-dimensional flow for the cylinder equation, or three-dimensional flow for the sphere equation. The difference between the measured input flux and that calculated from the plane and cylinder equations is considered due to the extra bulk of rock beyond corners and edges and that from the sphere equation is considered due mainly to the difference between the configuration of the chamber and a sphere. Other sources of error that may help to explain the discrepancy of the measured and computed values are air leaks, heated water migration from drips into the chamber via the aggregate and concrete floor to the tunnel outside, water vapor migration to the tunnel outside and heat transfer through walls at chamber entrance and compressor room.

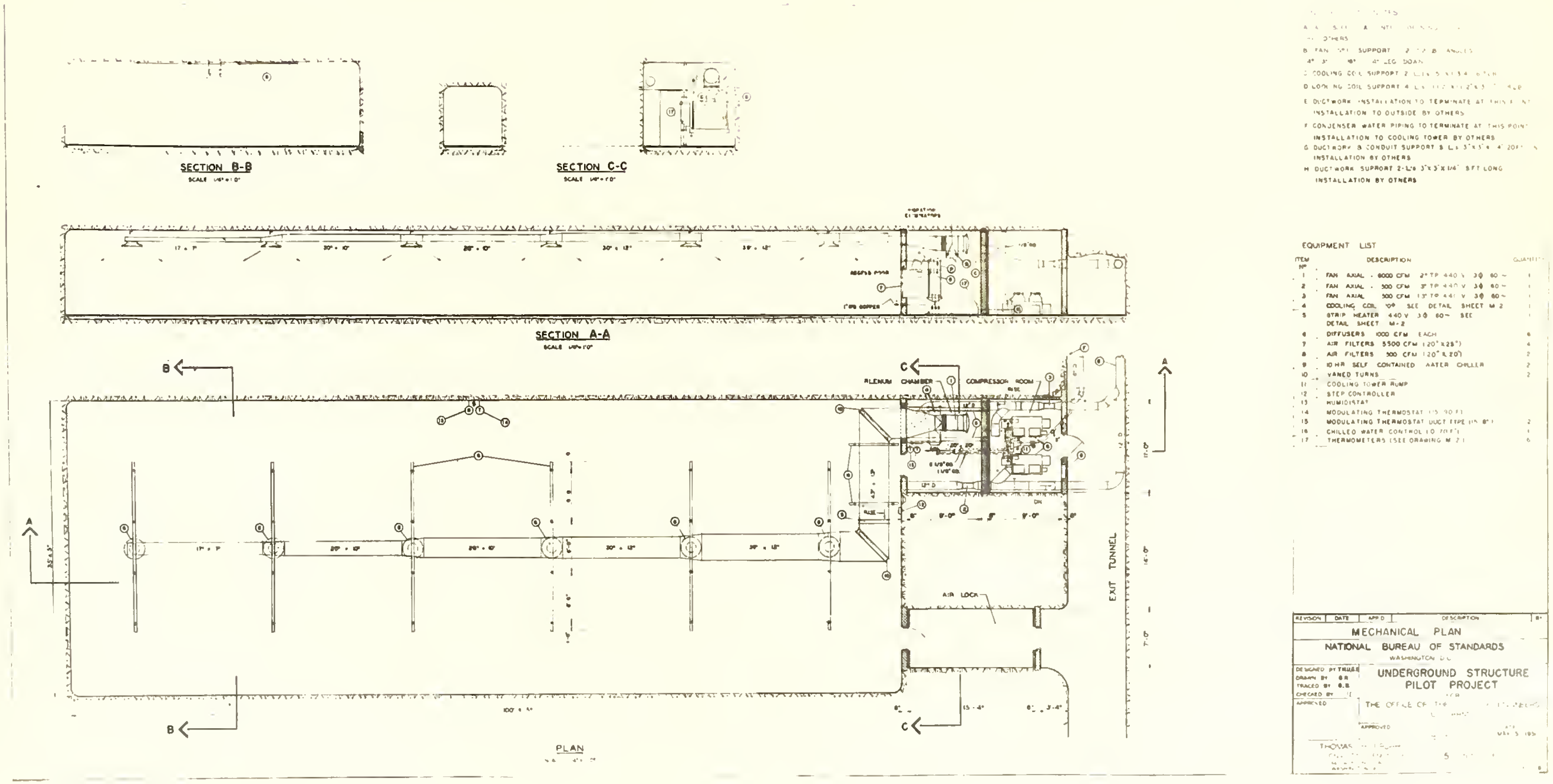
5. In order to determine the applicability of the theoretical equations of heat transfer to the experimental data, the measured rock surface temperatures were used in equations (4), (5), and (6), and resultant values of heat flux were plotted on figure 9. For the 522 hours of testing, the computed values decreased almost linearly with time. Percentage decreases in 522 hours are: plane, 14 percent; cylinder, 9.1 percent, and sphere, 7.4 percent. Since the actual heat flux was constant within ±2 percent, the change in heat flux indicated by equations (4), (5), and (6) for planes, cylinders, and spheres of equal surface area indicate the degree to which these simple geometric shapes cannot be used to accurately predict heat flow in a chamber of another shape. Equations (4), (5), and (6) are theoretically based on constant heat flux to the surface of the solid from which heat flows into the solid whose volume at a given distance from the surface is known. A comparison is given in the following table showing the volume of solid surrounding the actual chamber and that of the three geometric shapes at various depths.

Volume of rock between the surface and various depths, cu ft

Chamber shape	Depth below surface, ft					
	1	2	3	4	6	10
Actual chamber	10,465	21,876	34,258	47,637	77,485	150,240
Sphere: Radius = 28.2'	10,351	21,438	33,283	45,912	73,622	139,570
Cylinder: Radius = 15.9'	10,304	21,237	32,798	44,990	71,251	131,320
Plane	10,000	20,000	30,000	40,000	60,000	100,000

The table shows that the volumes of rock in the sphere, cylinder, and plane diverge from the actual volume with increase in depth. The sphere is the closest approximation to the actual volume. The divergence between the amount of rock being heated in the actual chamber and that which would be heated for the plane surface, cylindrical, and spherical cases for measuring depths probably explains why the computed heat flux decreased with time as shown in figure 9.

6. Figure 10 shows the temperatures in the rock at 0.5-, 1-, 2-, 3-, and 4-foot depths and the temperatures computed from equations (1), (2), and (3). As was noted in the results, values of the heat flux were picked from figure 9 in order to determine the temperature in the rock at any given time. This seems to adjust the temperatures in the rock to allow for the increase in volume of the actual over the volumes represented by the plane, cylinder, and sphere. The computed values give fair agreement to the experimental values, especially at the 2-, 3-, and 4-foot depths.



SECTION B-B
SECTION C-C
SECTION A-A
PLAN
PLENUM CHAMBER
COMPRESSOR ROOM
AIR LOCK
EXIT TUNNEL
B ANGLES
B FAN SUPPORT
C COOLING COIL SUPPORT
D COOLING COIL SUPPORT
E DUCTWORK INSTALLATION TO TERMINATE AT THIS POINT INSTALLATION TO OUTSIDE BY OTHERS
F CONDENSER WATER PIPING TO TERMINATE AT THIS POINT INSTALLATION TO COOLING TOWER BY OTHERS
G DUCTWORK & CONDUIT SUPPORT INSTALLATION BY OTHERS
H DUCTWORK SUPPORT 2-L's 3"x3"x1/4" 5FT LONG INSTALLATION BY OTHERS
EQUIPMENT LIST
ITEM Nº
DESCRIPTION
1 FAN AXIAL - 6000 CFM
2 FAN AXIAL - 500 CFM
3 FAN AXIAL 500 CFM
4 COOLING COIL SEE DETAIL SHEET M 2
5 STRIP HEATER 440 V 3Ø 60~ SEE DETAIL SHEET M-2
6 DIFFUSERS 1000 CFM EACH
7 AIR FILTERS 5500 CFM (20"x25")
8 AIR FILTERS 500 CFM (20"x20")
9 10 HP SELF CONTAINED WATER CHILLER
10 VANED TURNS
11 COOLING TOWER PUMP
12 STEP CONTROLLER
13 HUMIDISTAT
14 MODULATING THERMOSTAT
15 MODULATING THERMOSTAT DUCT TYPE
16 CHILLED WATER CONTROL
17 THERMOMETERS (SEE DRAWING M 2)
REVISION
DATE
DESCRIPTION
MECHANICAL PLAN
NATIONAL BUREAU OF STANDARDS
WASHINGTON D.C.
UNDERGROUND STRUCTURE
PILOT PROJECT
DESIGNED BY
DRAWN BY
TRACED BY
CHECKED BY
APPROVED

Fig. 1

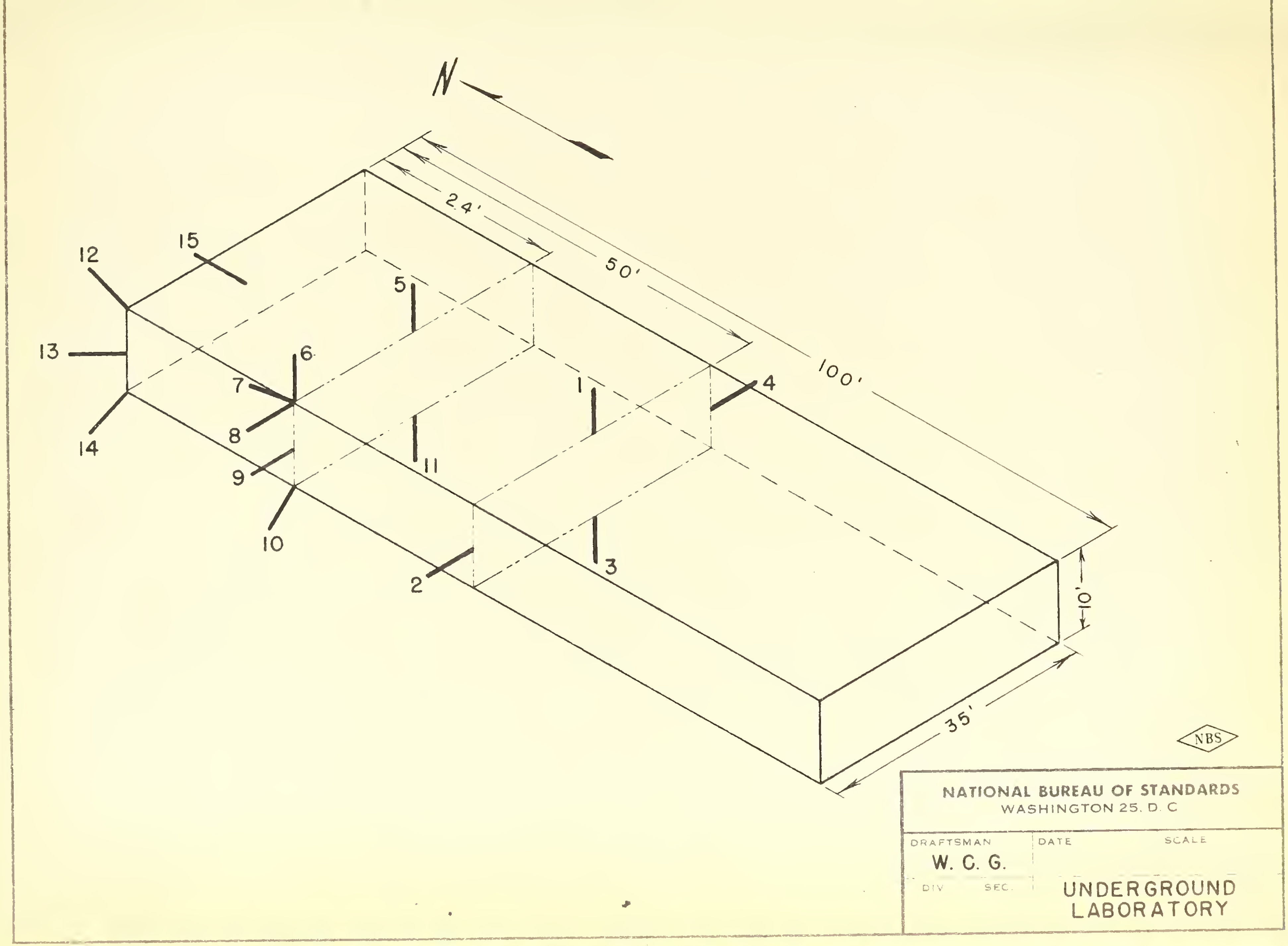

N
24'
50'
100'
10'
35'
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
NBS
NATIONAL BUREAU OF STANDARDS
WASHINGTON 25, D. C.
DRAFTSMAN
W. C. G.
DATE
SCALE
DIV
SEC.
UNDERGROUND LABORATORY

FIG. 2

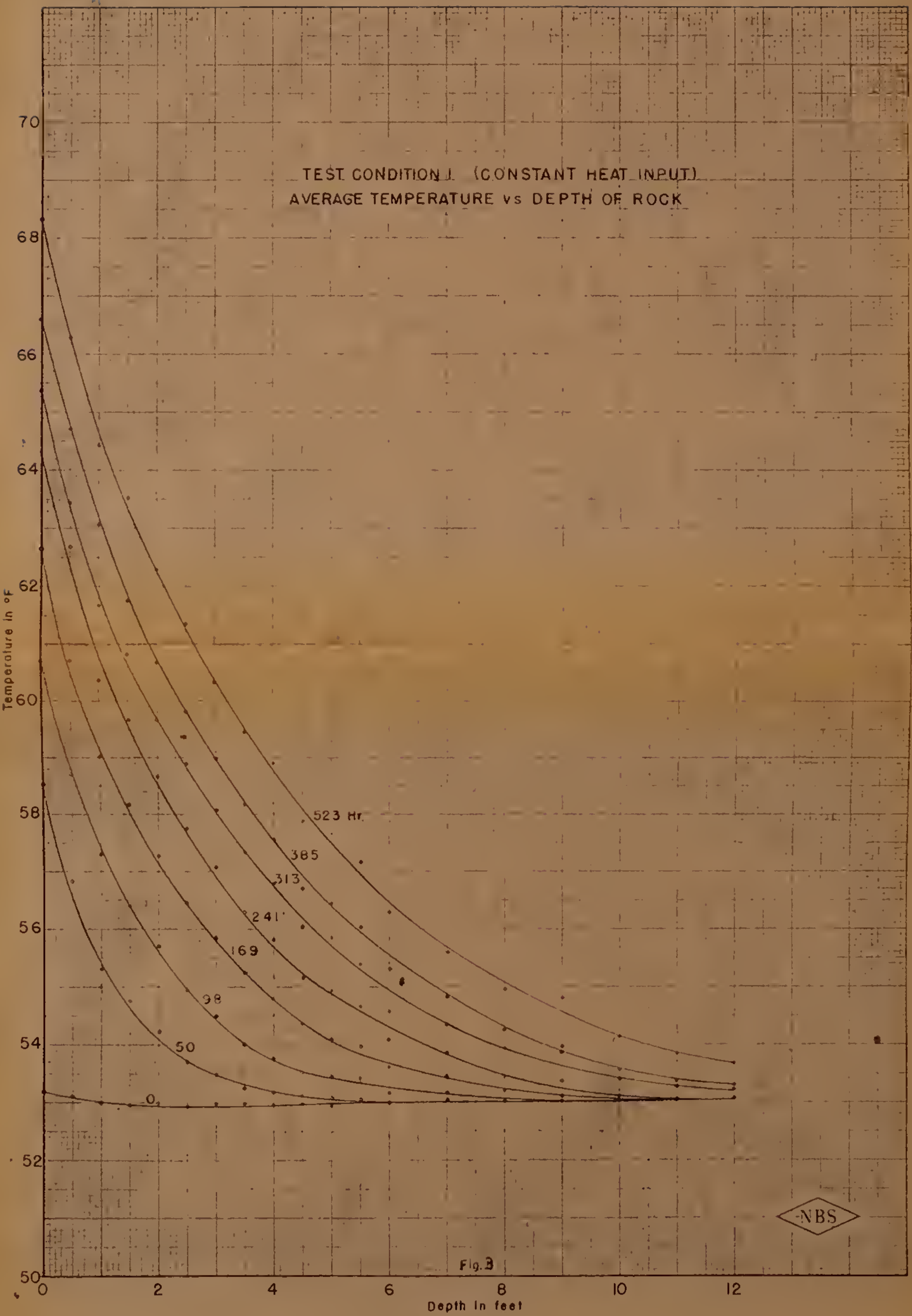

TEST CONDITION J (CONSTANT HEAT INPUT)
AVERAGE TEMPERATURE vs DEPTH OF ROCK
Temperature in °F
70
68
66
64
62
60
58
56
54
52
50
523 Hr.
385
313
241
169
98
50
0
0
2
4
6
8
10
12
Depth in feet
NBS

Fig. 3

NBS

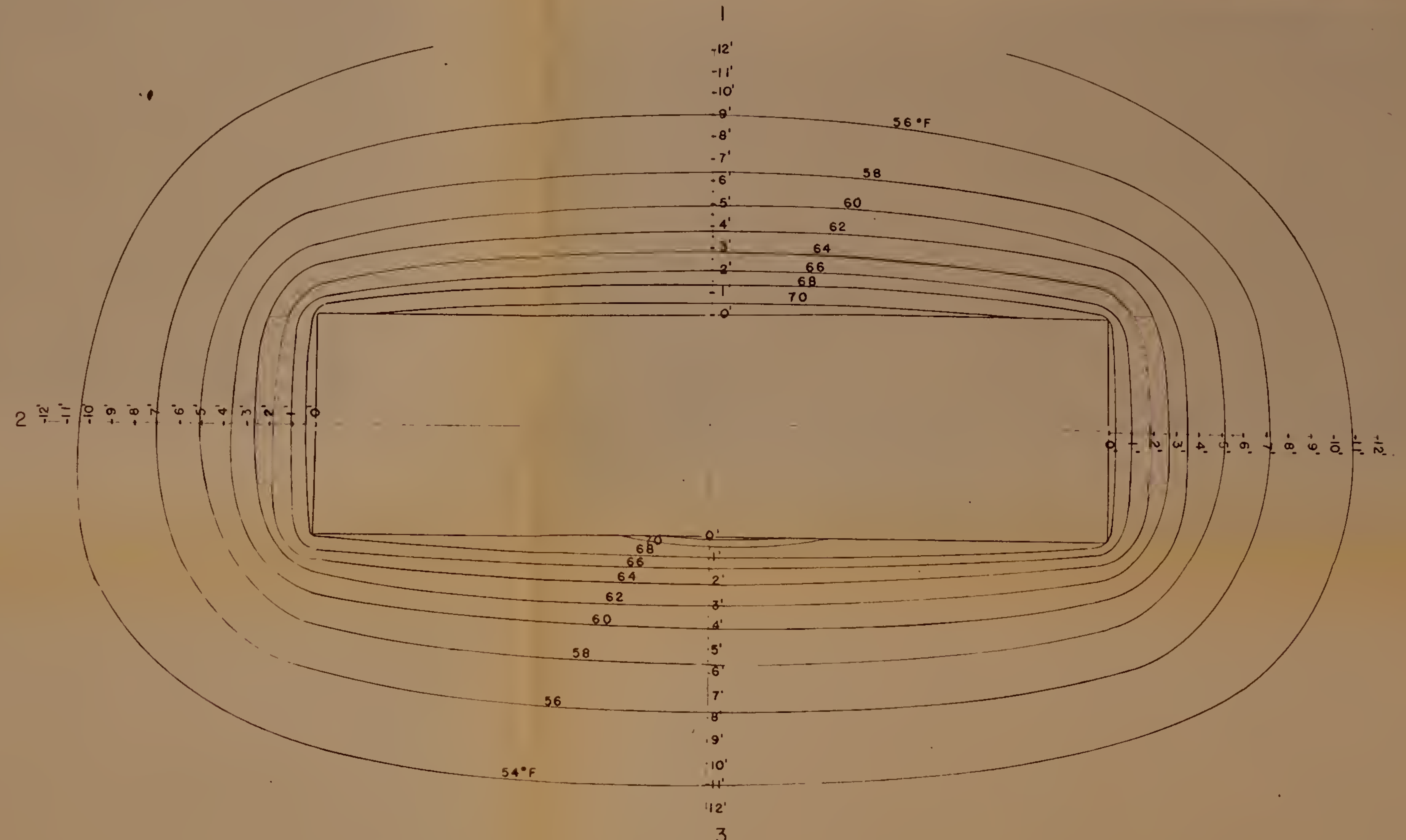


FIG. 4

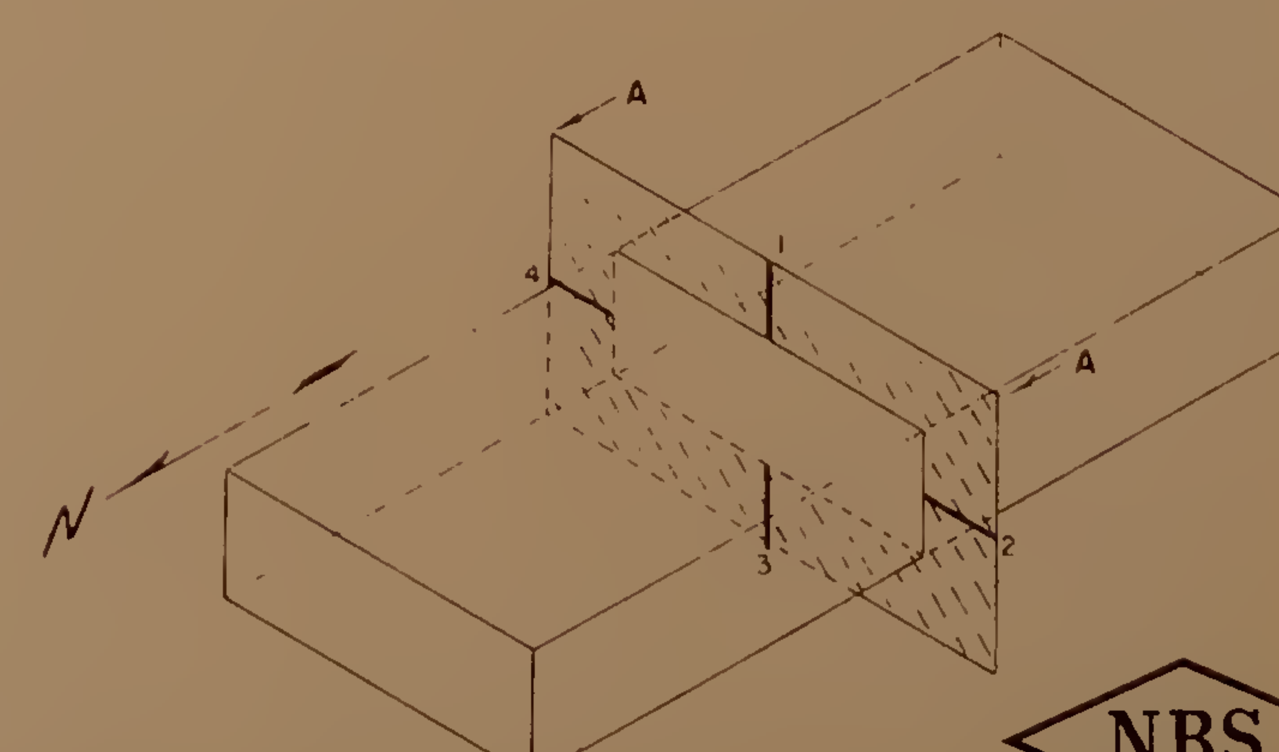
SECTION A-A

SCALE: 1/4" = 1'-0"

TEST CONDITION No. I - HEATING UP PHASE

TEMPERATURE DISTRIBUTION IN ROCK

AFTER 521 HOURS



NBS

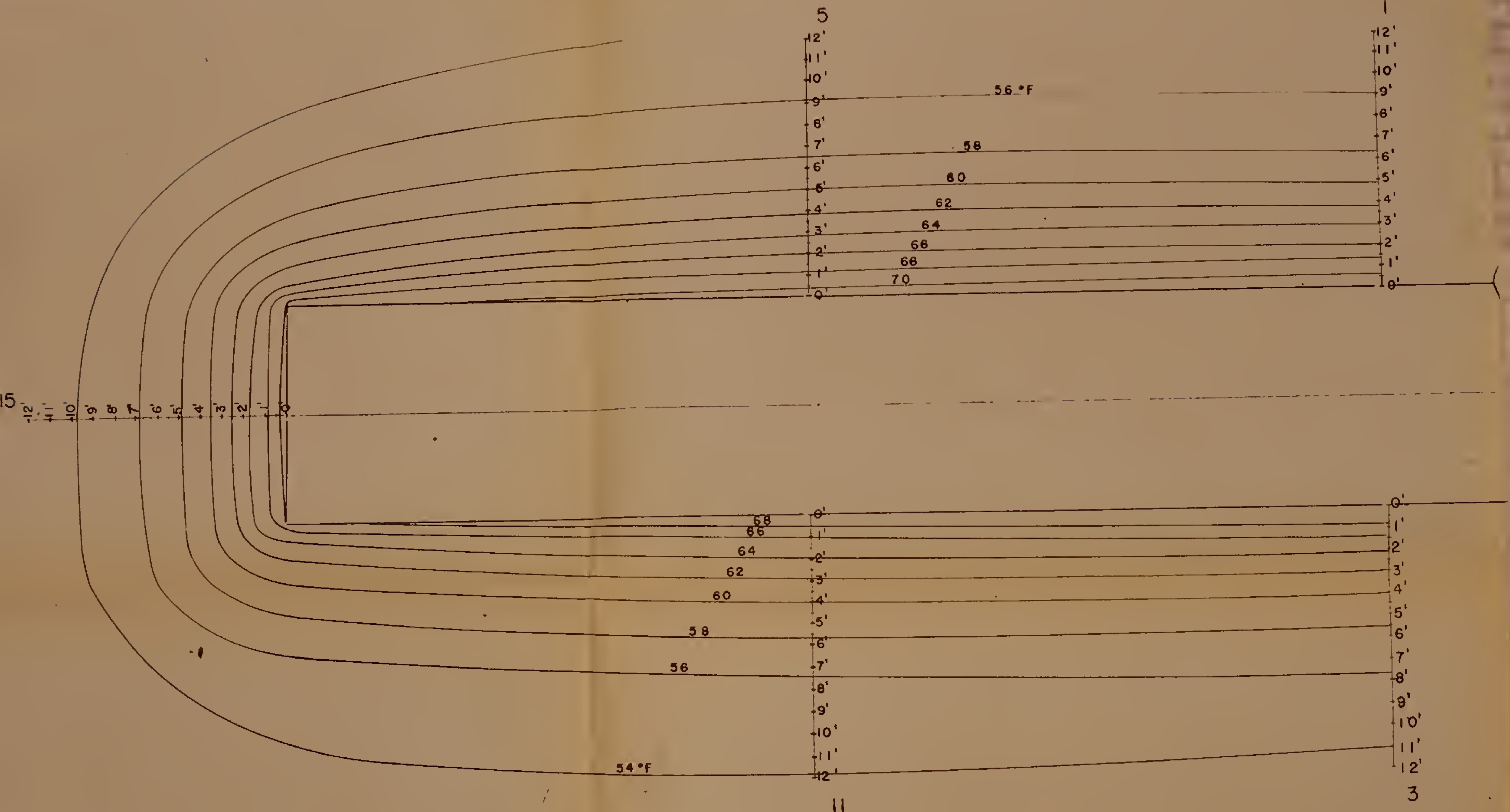


FIG. 5

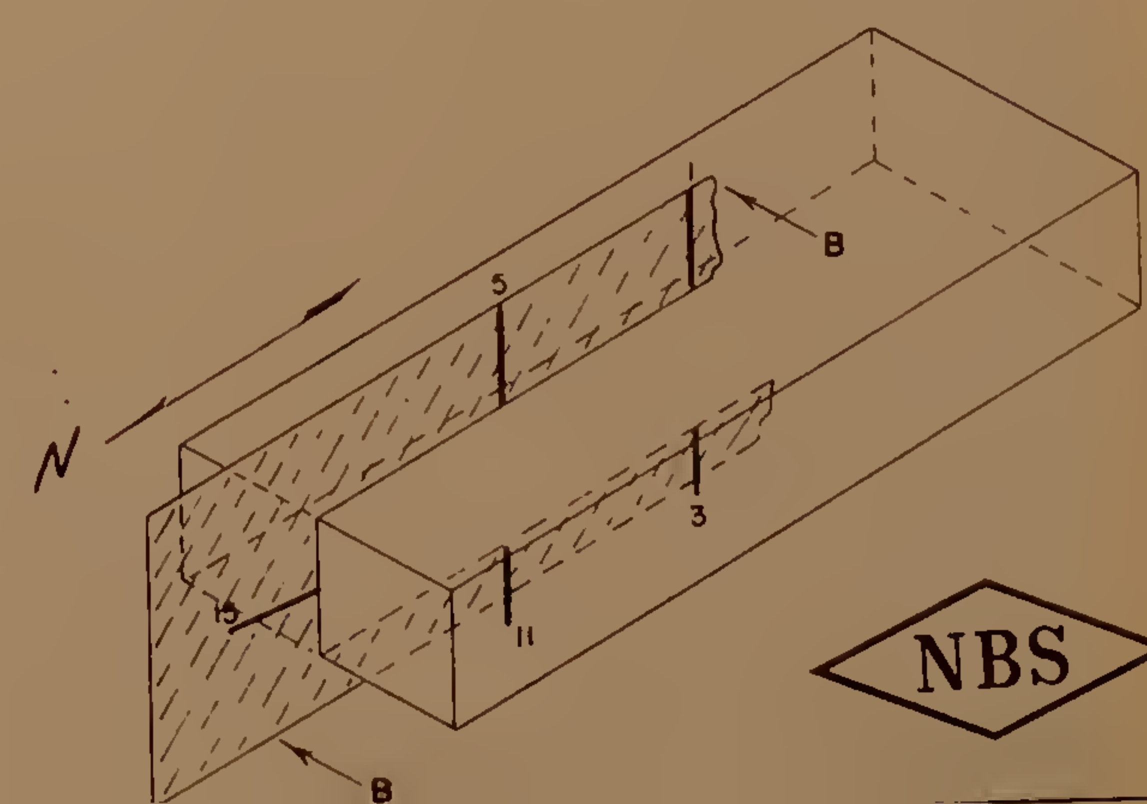
SECTION B-B

SCALE: 1/4" = 1'-0"

TEST CONDITION No. 1 – HEATING UP PHASE

TEMPERATURE DISTRIBUTION IN ROCK

AFTER 521 HOURS



NBS

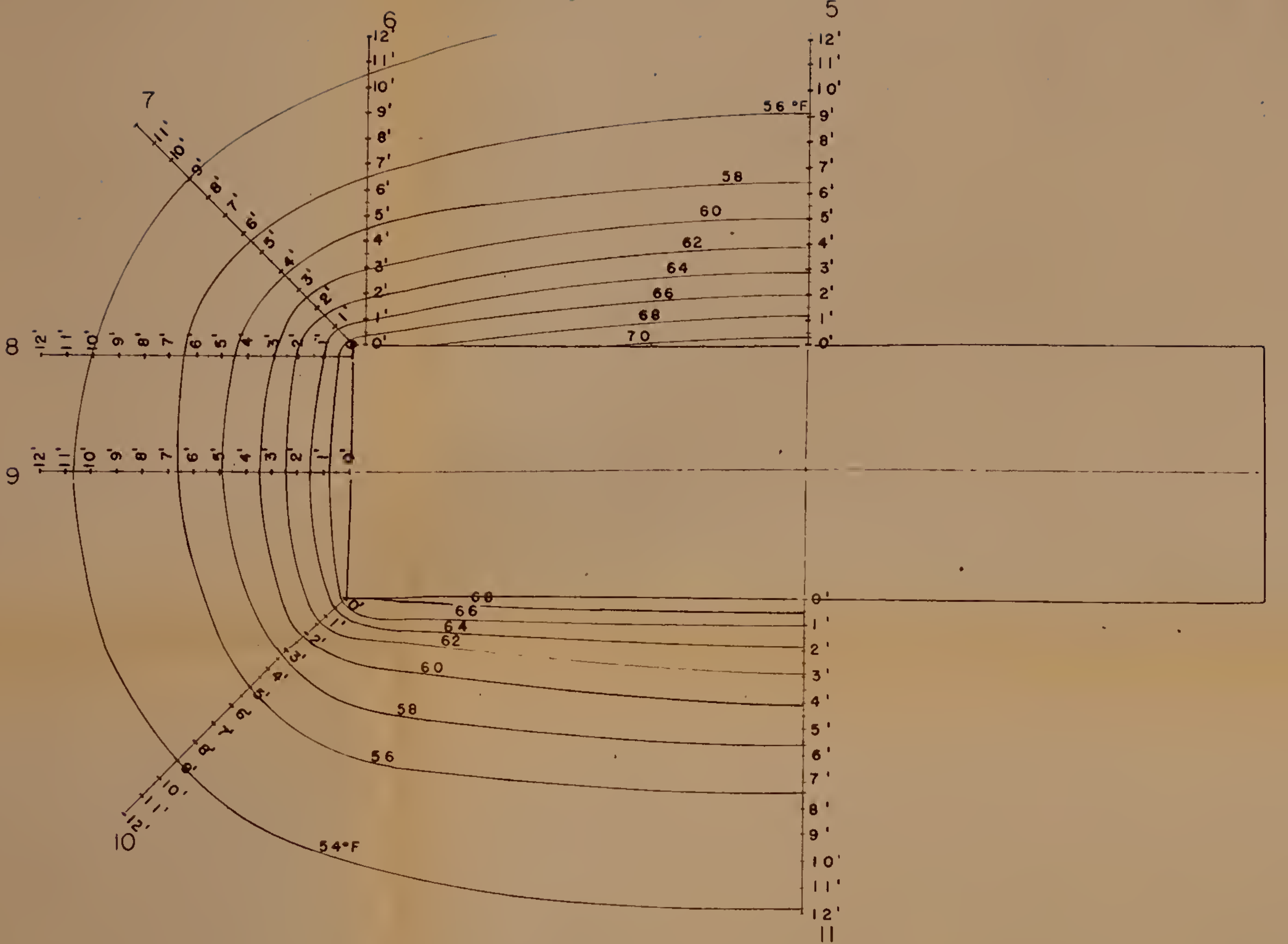


FIG. 6

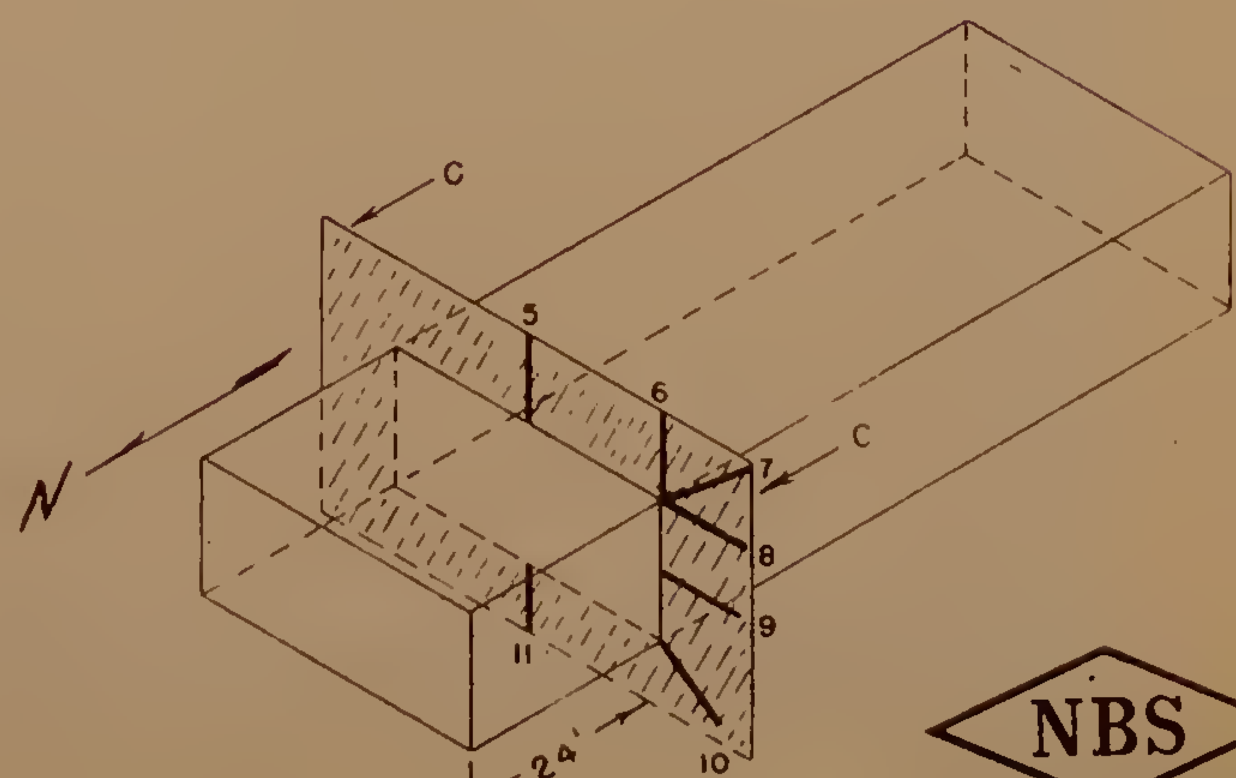
SECTION C-C

SCALE: 1/4" = 1'-0"

TEST CONDITION No. 1 – HEATING UP PHASE

TEMPERATURE DISTRIBUTION IN ROCK

AFTER 521 HOURS



NBS

NBS

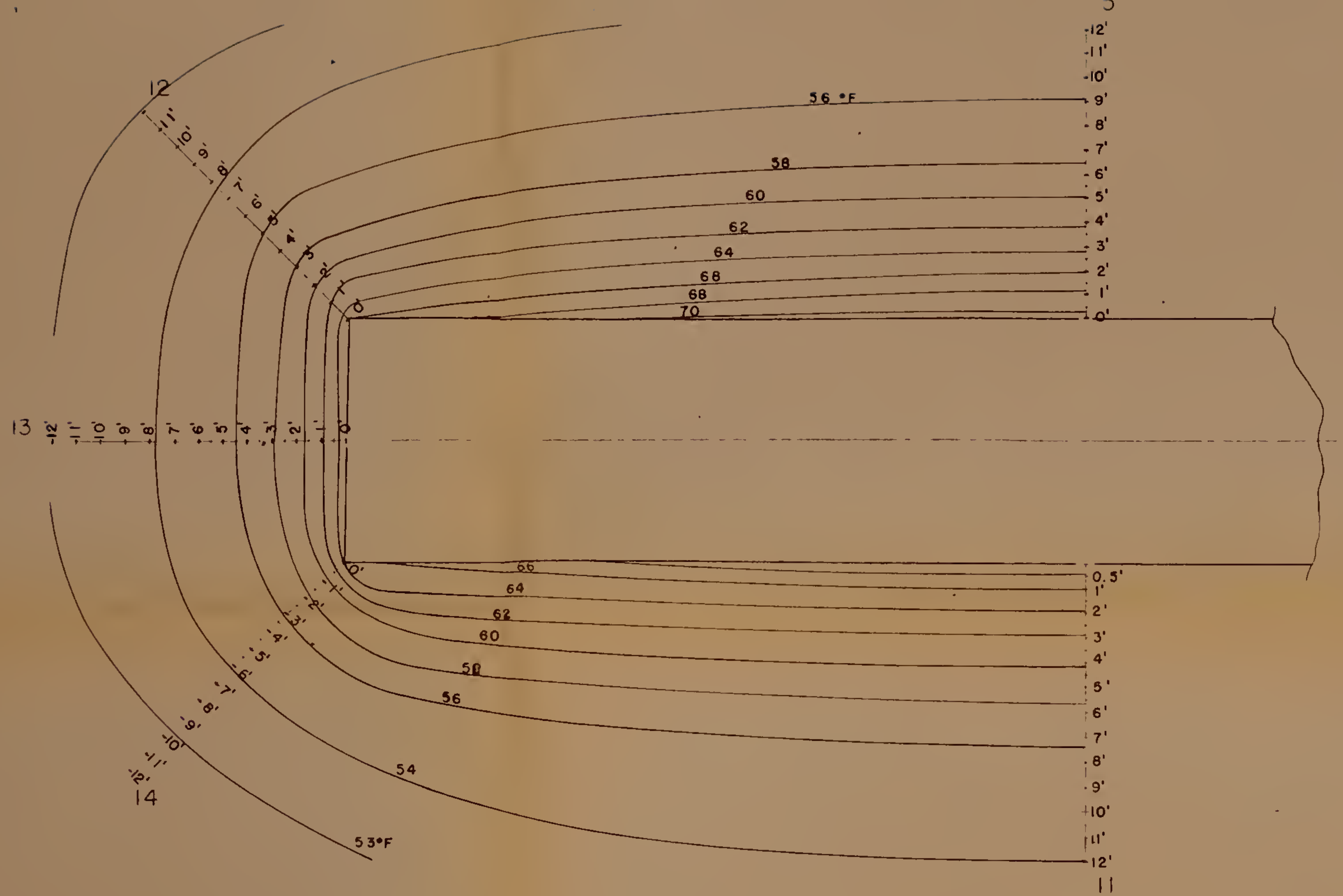
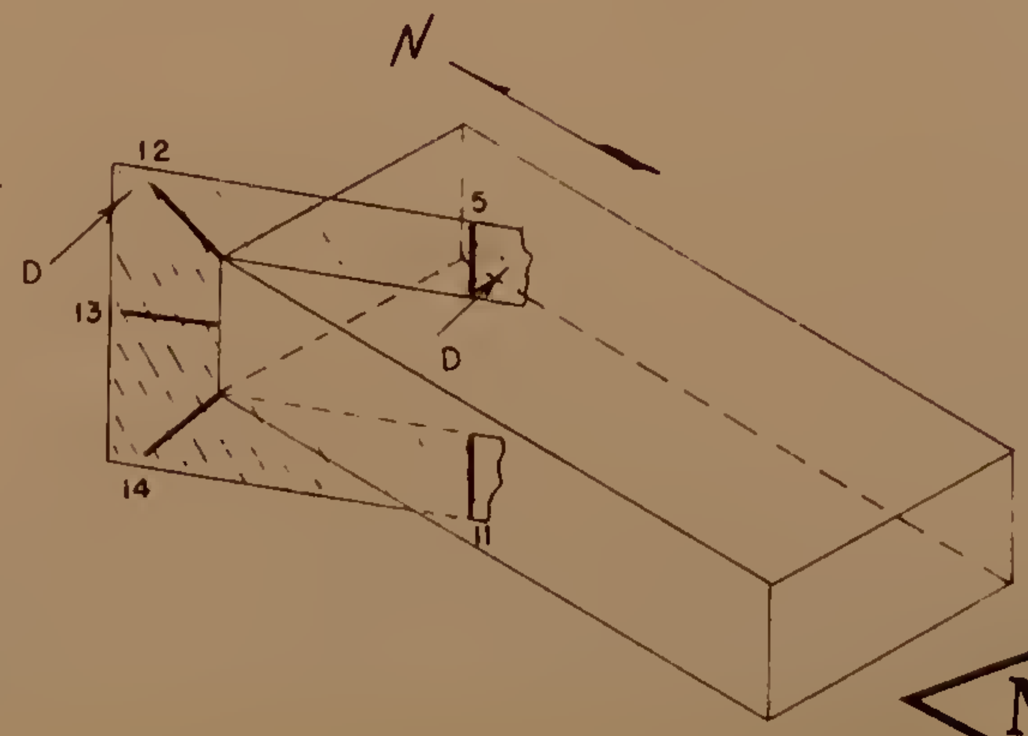


FIG. 7

SECTION D-D

SCALE: 1/4" = 1'-0"

TEST CONDITION No. 1 – HEATING UP PHASE
TEMPERATURE DISTRIBUTION IN ROCK
AFTER 521 HOURS



NBS

11
12
9 NBS 2

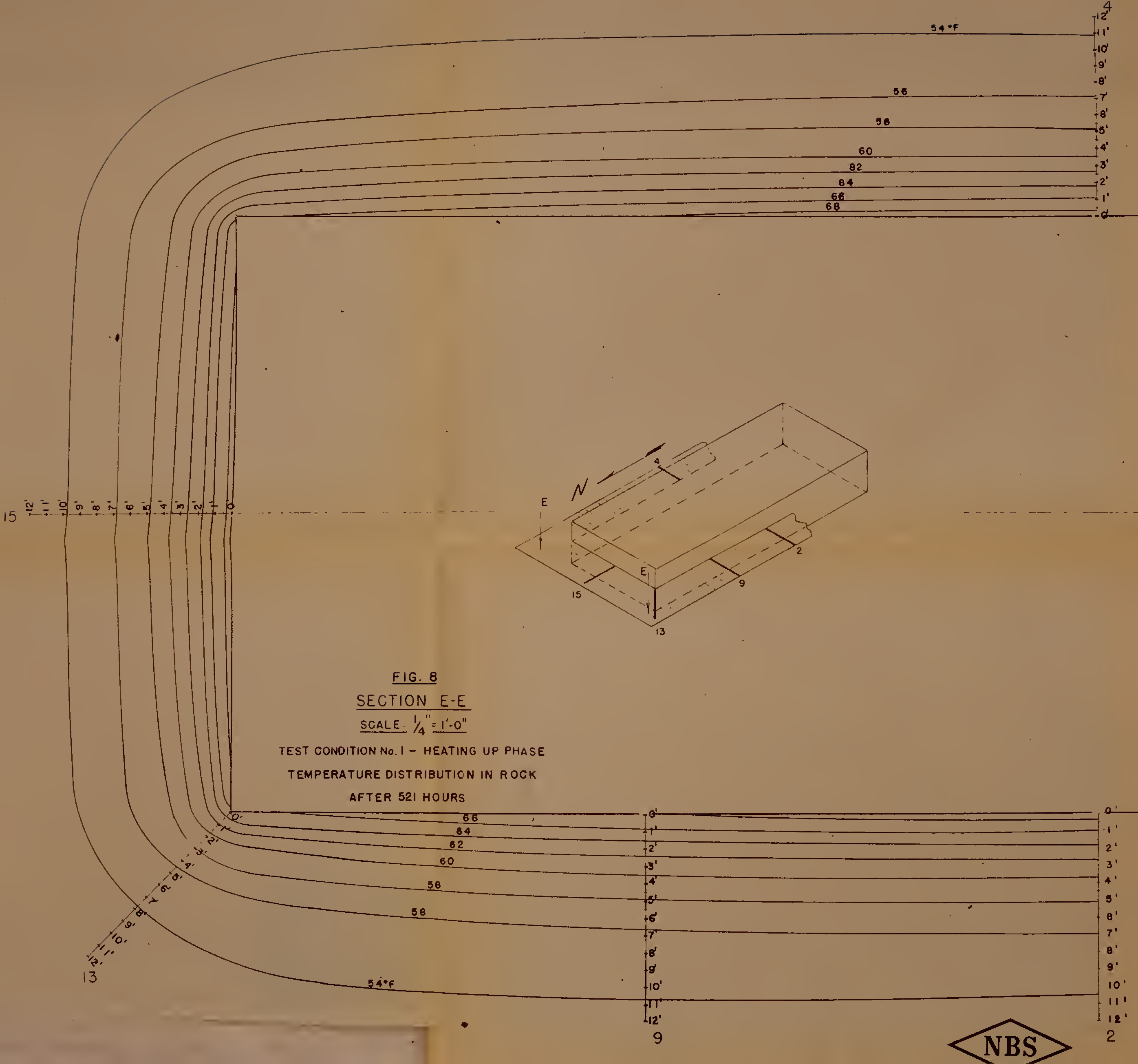


FIG. 8

SECTION E-E

SCALE 1/4" = 1'-0"

TEST CONDITION No. 1 – HEATING UP PHASE

TEMPERATURE DISTRIBUTION IN ROCK

AFTER 521 HOURS

ELECTRICAL AND COMPUTED HEAT FLUX, TEST CONDITION # 1

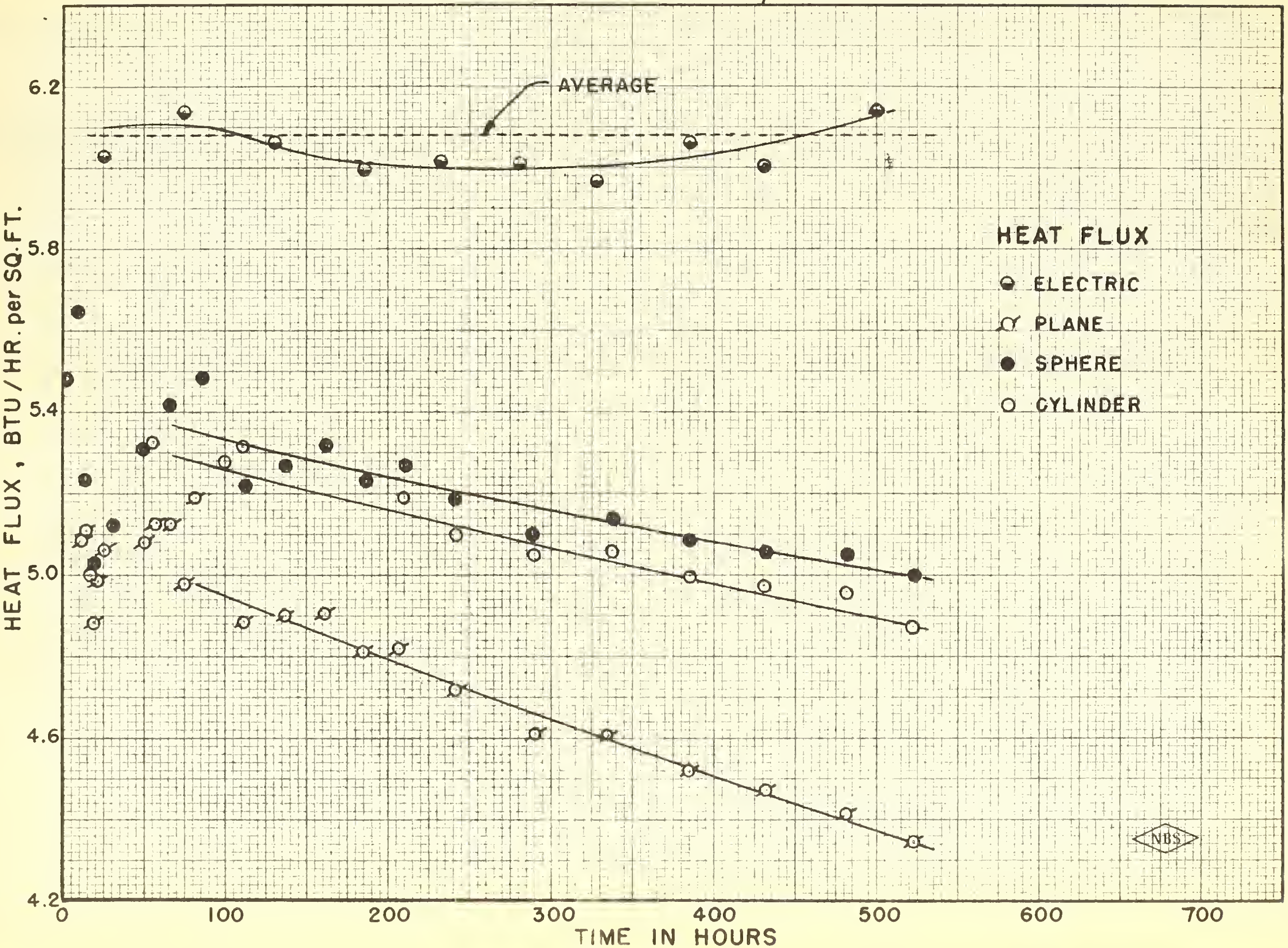
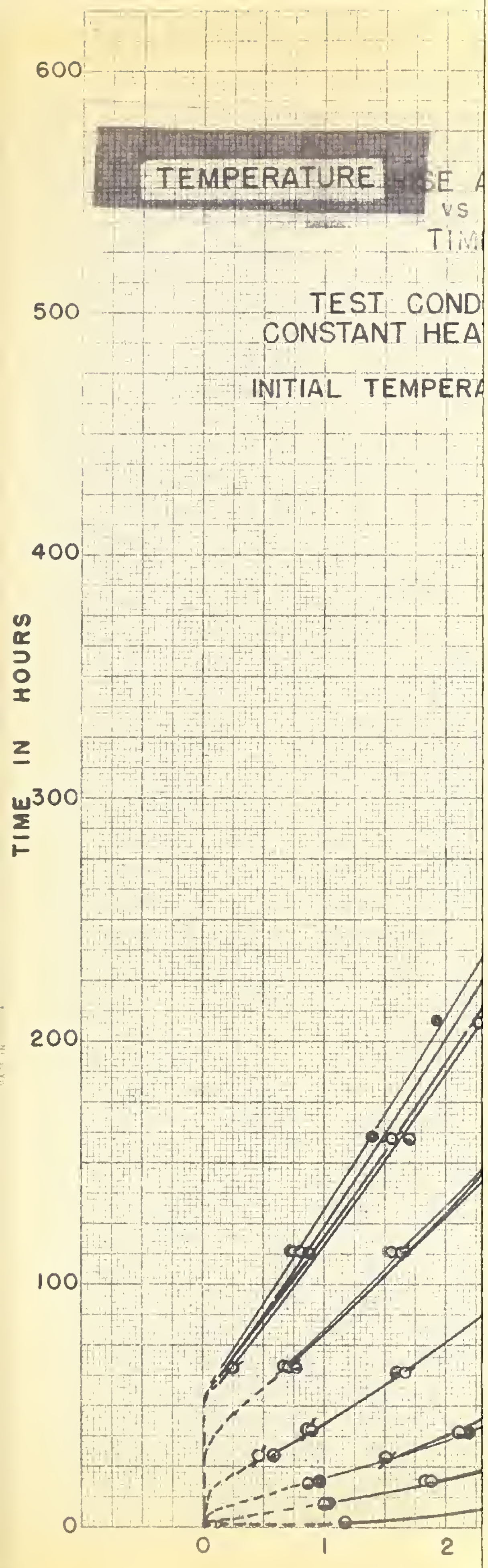


FIG. 9



TEMPERATURE
SE A
VS
TIM
TEST COND
CONSTANT HEA
INITIAL TEMPERA
TIME IN HOURS
600
500
400
300
200
100
0
0
1
2

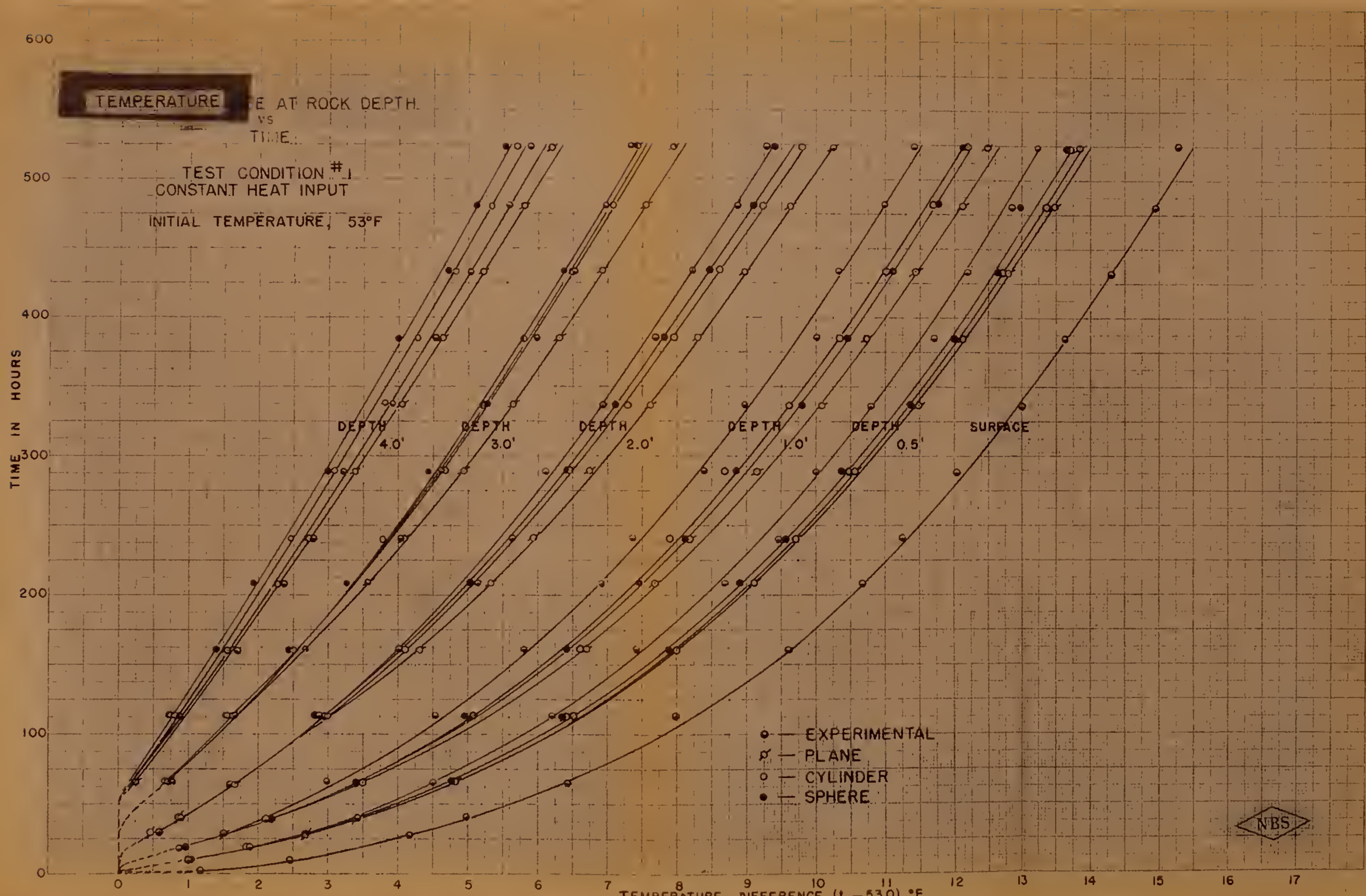

TEMPERATURE
E AT ROCK DEPTH
VS
TIME
TEST CONDITION #1
CONSTANT HEAT INPUT
INITIAL TEMPERATURE, 53°F
DEPTH 4.0'
DEPTH 3.0'
DEPTH 2.0'
DEPTH 1.0'
DEPTH 0.5'
SURFACE
EXPERIMENTAL
PLANE
CYLINDER
SPHERE
TIME IN HOURS
TEMPERATURE DIFFERENCE (t_x − 53.0), °F
NBS

FIG. 10

Figure 11. Underground Chamber - Entrance door on right, door to plenum chamber on left.

Figure 12 - Underground Chamber - Rear

V. TUNNEL COOLING OF VENTILATION AIR

Object

The object of this test was to determine the temperature drop and heat loss of ventilation air in an underground tunnel during summer conditions at Mount Weather, Virginia, and to obtain data that would help to predict what might be expected in other tunnels used for ventilation of underground spaces.

Description of Underground Tunnel and Equipment

The underground tunnel (figure 1) was approximately 1,400 feet long with nominal dimensions of seven by seven feet. At the end of the tunnel was a 230-foot shaft (at L5 on figure 1) upward to the surface. Air was drawn through the tunnel from its entrance at the surface and up the shaft by a centrifugal blower mounted over the shaft. Canvas barriers were placed across the tunnels R1 and R2-1 for the test as shown in figure 1 to prevent appreciable by-passing of the main tunnel.

Hygrothermographs were placed at the tunnel entrance and at the end of the tunnel at L5 to measure dry-bulb temperature and relative humidity. Three times daily, temperature readings of the air, rock, and flowing water were taken at various cross sections (1-13 on figure 1) in the tunnel. Other readings were air-flow rate, static pressure drop across the blower, and pressure drop in a selected 100-foot section of tunnel.

Results

An average air-flow rate of 24,700 cubic feet per minute was maintained in the tunnel for a period of ten days for which the last five days are analyzed in this report.

Figure 2 shows the outside air conditions for the period July 14-18 and the average daily dry-bulb temperature and humidity computed from the areas under the curves. For this period, the temperature at the end of the tunnel at L5, was nearly constant at 59° F and 100 percent relative humidity. Table 1 shows the average daily heat and moisture release of the ventilation air, as well as the heat loss rate per foot of tunnel and temperature difference between the entering air and leaving air.

Figure 3 shows the air temperatures at the numerically designated cross sections of the tunnel for times 0630, 1230, and 1700 of July 15. Tables 2 and 3 show the air, rock surface, and water temperatures at the thirteen stations and the rate of heat loss of the air per foot of tunnel between the stations for the times 1700 and 0630 of July 15, respectively.

The coefficient of heat transfer between the air and the rock surface was computed for the time 1700 July 15 and for the distances between the entrance and the various stations up to 1,200 feet from the entrance by the relationship

$$h = \frac{Q}{A \, \Delta t_m}$$

where h = coefficient of heat transfer between the air and rock surface, BTU/hr ft^2 deg F

A = projected surface area of tunnel, ft^2

Δt_m = logarithmic mean temperature difference of air and rock surface, deg F

Q = heat transfer rate to surface A, BTU/hr

The average value for these computations was 4.73 BTU/hr ft^2 deg F.

The static pressure drop across the blower was 3.5 inches of water and the pressure drop in the selected 100-foot section of tunnel (between stations 11 and 12) was 0.04 inches of water.

Discussion and Conclusions

1. Table 1 shows that average heat loss rates up to 932,000 BTU/hr can be accomplished by this tunnel. No conclusions can be made at this time as to whether this rate could be sustained over a long duration of average outside air conditions of 82 deg F and 56 percent relative humidity. It is hoped that a test can be made during a greater portion of the 1954 summer season to determine more conclusively the effectiveness of the rock surrounding the tunnel as a heat sink for an extended period of time. Also, it is suspected that additional benefit will accrue from the seasonal changes, e.g., the rock will act as a heat sink in the winter and a heat source in the summer.

2. Table 2 shows that more than half of the heat lost from the air was lost in the first 200 feet of tunnel. This was a case of sensible cooling of the air with possible evaporation of moisture from the tunnel walls. Drying of the tunnel walls up to R1 was observed. A case of the combination of both sensible and latent cooling is shown in table 3 where less than 20 percent of the heat was taken from the air in the first 200 feet. This condition will not always be true, but serves to show that where air temperatures in the tunnel go below the entering air dew-point temperature, more of the tunnel area can be utilized as a heat sink by dissipating the latent heat of condensation.

3. Air flowing by diffusion or force through side passages added to the effective area for heat transfer. This was evident by observation of air flow around the edges of the canvas barriers placed in R1

and R2-1 and by table 2, where the heat loss between stations 6 and 7 increased over the two preceding stations (the temperature drop was greater but actually the air-flow rate was smaller). Water flow from the side passage R1 was more than that in the main tunnel above R1 and was also colder; as seen on tables 2 and 3, comparing water temperatures at stations 6 and 7.

4. The average coefficient of heat transfer between the air stream and the rock surface, when the heat transfer was for all practical purposes sensible cooling, was computed to be 4.73 BTU/hr ft^2 deg F. This value is in fair agreement with those given in the "Guide", handbook of the American Society of Heating and Ventilating Engineers, 1953, figure 4, page 179, for the rougher surfaces with an air velocity of 5.7 miles per hour, equal to that in the tunnel during the test. The average coefficient of heat transfer when both sensible and latent cooling has occurred was computed from temperatures in table 3 to be 5.57 BTU/hr ft^2 deg F.

5. When only sensible cooling of the air occurs in traversing the tunnel, the unit heat loss in BTU/hr (ft of tunnel)(°F temp change in the 1,400 feet of tunnel) would have a value of 19 for an air ventilation rate of 24,700 c.f.m. Higher values would indicate removal of moisture from the air and lower values would indicate an increase in moisture content of the air during its travel through the tunnel. Thus, in table 1, the values of this unit heat loss for July 15, 16, 17, and 18 indicate drying of the air stream and that for July 14 indicates addition of moisture to the air stream. These changes in moisture content are directly related to positive and negative values of moisture loss shown in column 4 of table 1.

6. Figure 4 is a plot on semi-logarithmic graph paper of data from tables 2 and 3 for the temperature difference between the temperatures at the various stations and the temperature at the end of the tunnel against the first 1,000 feet of tunnel. This plot seems to show that within a restricted length of tunnel, the empirical relationship of the temperature difference to distance in the tunnel may be represented by

$$\Delta T = e^{a + b^x}$$

where ΔT = temperature difference
x = distance in tunnel from entrance
e = base for natural logarithms
a, b = constants

Determinations of the constants a and b were made, but will be held in abeyance until further substantiating data are gathered.

Table 1. Average daily heat and moisture loss of tunnel ventilation air

Average air flow 24,700 cfm based on 59° F and 100 percent relative humidity at end of tunnel

Date	Outside air conditions			Moisture loss	Heat loss		
	Dry-bulb temperature	Relative humidity	Dew point		BTU/hr	Tons refrigeration	$\frac{BTU}{hr\text{-}ft\text{-}\Delta T^*}$
	°F	Percent	°F	lb/hr			
7/14/53	72.9	58.7	57.5	-75	320,000	26.6	16.4
7/15/53	76.0	61.0	61.5	+107	640,000	53.2	26.9
7/16/53	79.0	54.6	61.2	84	696,000	58.0	24.8
7/17/53	80.9	52.4	61.8	118	786,000	65.4	25.6
7/18/53	81.9	55.7	64.5	252	932,000	77.7	29.1

* ΔT = temperature difference between temperature of entering air and air at L5.

Table 2. Temperatures at stations and lineal heat loss between stations - 1700 July 15, 1953

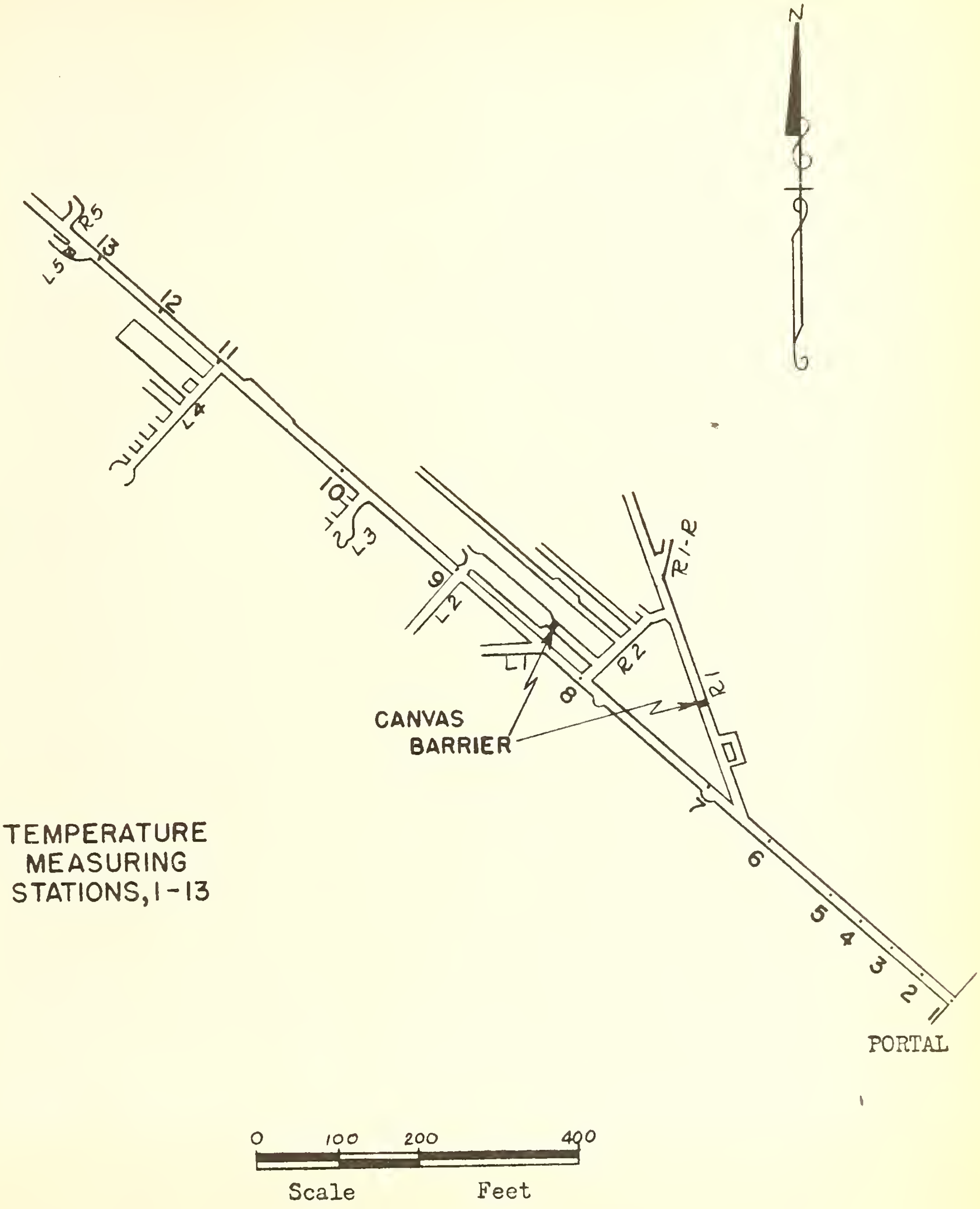
Station	Distance in tunnel	Air temperature	Heat loss (sensible) BTU/hr ft tunnel	Average surface temperature	Water temperature
No.	ft	°F		°F	°F
1	0	85.0		---	58.3
			1,752		
2	50	82.2		66.2	57.6
			2,960		
3	100	76.8		67.5	57.2
			2,020		
4	150	73.0		62.9	56.7
			674		
5	200	71.8		61.9	56.2
			505		
6	300	70.1		60.0	55.4
			898		
7	400	66.8		60.7	58.4
			618		
8	600	62.3		60.0	58.3
			96		
9	800	61.6		59.4	58.3
			204		
10	1,000	60.1		59.2	57.3
			123		
11	1,200	59.2		57.9	56.9

12	1,300	59.3		57.6	56.6

13	1,400	59.1		57.1	56.7

Table 3. Temperature at stations and lineal heat loss between stations - 0630 July 15, 1953

Station	Distance in tunnel	Air temperature	Heat loss BTU/hr ft of tunnel: Sensible	Heat loss BTU/hr ft of tunnel: Latent	Average surface temperature	Water temperature
No.	ft	°F			°F	°F
1	0	65.2			---	58.0
			449	---		
2	50	64.5			60.7	57.9
			180	---		
3	100	64.1			60.3	57.8
			494	---		
4	150	63.2			59.7	56.5
			270	---		
5	200	62.7			59.8	56.2
			146	67		
6	300	62.1			59.2	55.5
			281	562		
7	400	61.0			59.2	58.2
			124	258		
8	600	60.0			58.6	58.1
			84	141		
9	800	59.4			58.5	57.4
			39	79		
10	1,000	59.1			57.9	57.7
			67	123		
11	1,200	58.6			57.9	57.5
			67	90		
12	1,300	58.4			57.6	56.9
			---	---		
13	1,400	58.7			57.7	57.4



MOUNT WEATHER TESTING ADIT

FIGURE 1

NBS

7-11-52

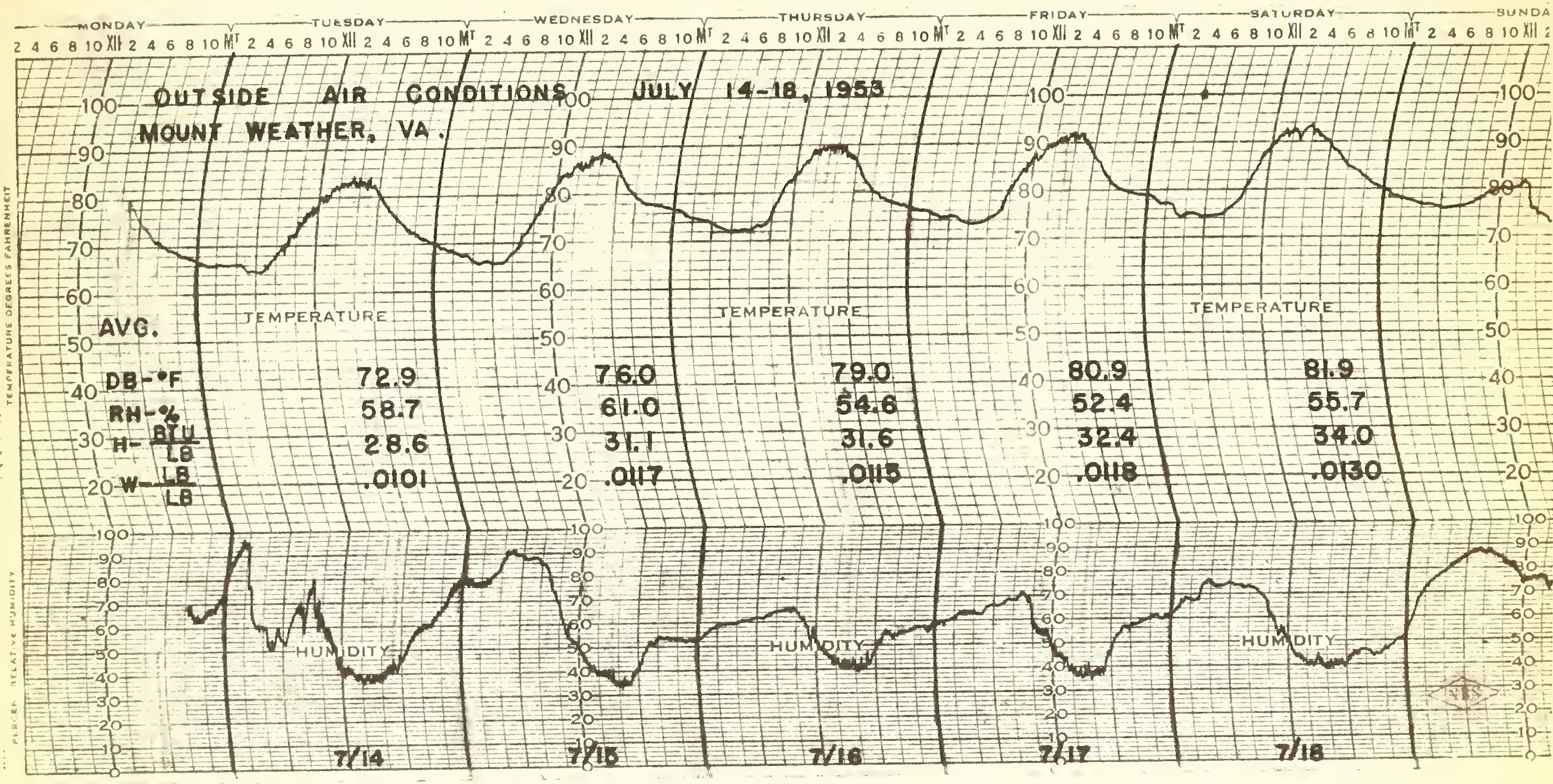

OUTSIDE AIR CONDITIONS JULY 14-18, 1953
MOUNT WEATHER, VA.
MONDAY
TUESDAY
WEDNESDAY
THURSDAY
FRIDAY
SATURDAY
AVG.
DB-°F
RH-%
H-BTU/LB
W-LB/LB
TEMPERATURE
HUMIDITY
72.9
58.7
28.6
.0101
76.0
61.0
31.1
.0117
79.0
54.6
31.6
.0115
80.9
52.4
32.4
.0118
81.9
55.7
34.0
.0130
7/14
7/15
7/16
7/17
7/18
TEMPERATURE DEGREES FAHRENHEIT
PERCENT RELATIVE HUMIDITY
NBS

FIG. 2

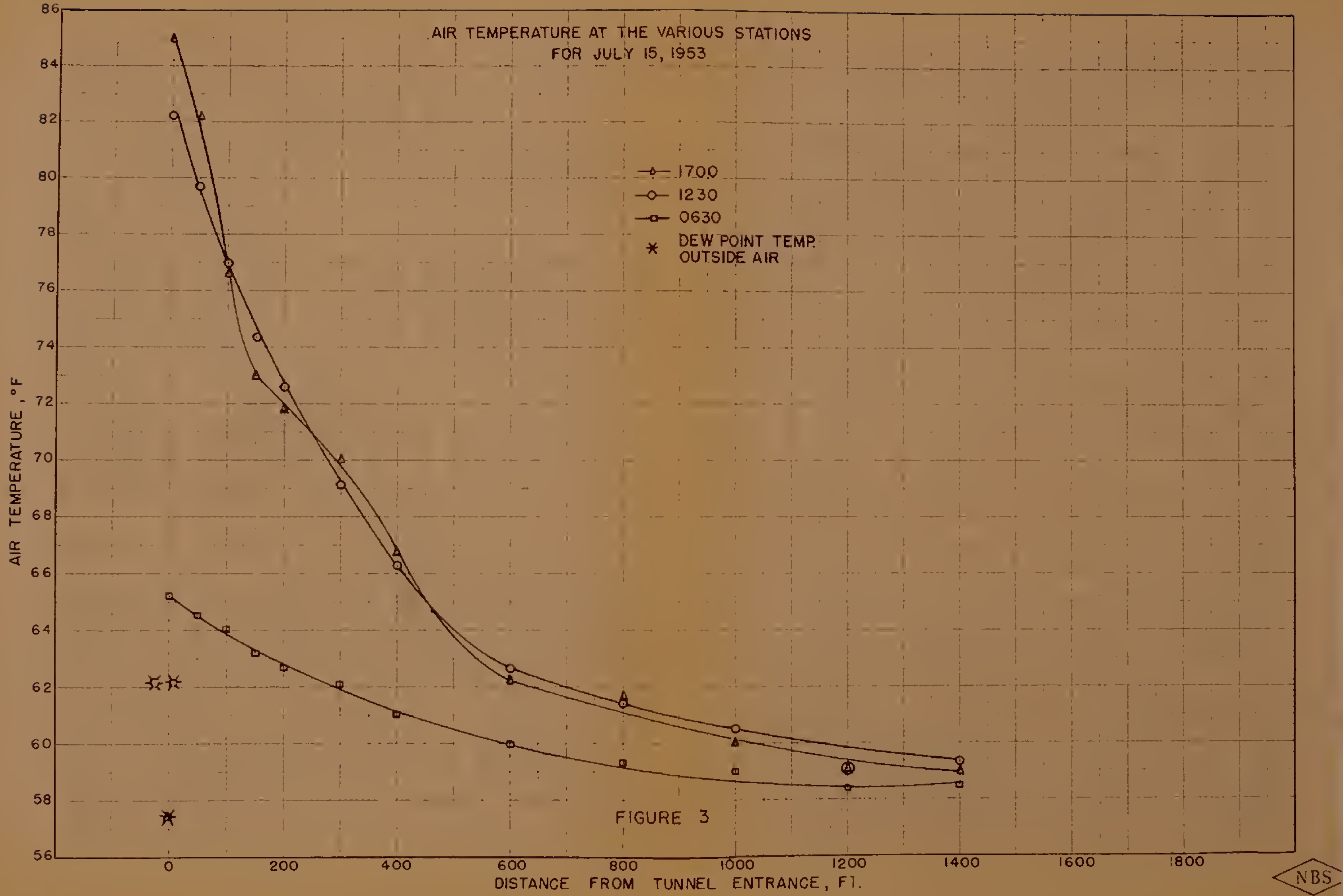

AIR TEMPERATURE AT THE VARIOUS STATIONS
FOR JULY 15, 1953
1700
1230
0630
DEW POINT TEMP.
OUTSIDE AIR
AIR TEMPERATURE, °F
86
84
82
80
78
76
74
72
70
68
66
64
62
60
58
56
0
200
400
600
800
1000
1200
1400
1600
1800
DISTANCE FROM TUNNEL ENTRANCE, FT.
NBS

FIGURE 3

VI. DETERMINATION OF THE THERMAL CONDUCTIVITY OF ROCK CORE SAMPLES

H. E. Robinson

Various physical properties of rock of interest for engineering purposes can be determined by tests made on rock core samples obtained in exploratory test drilling of proposed sites, provided that test methods are available for application to cylindrical specimens. To take advantage of this possibility, an apparatus was assembled for determining the thermal conductivity of rock in the form of cylindrical core samples from 3/4 to 1 1/2 inches in diameter and about 18 inches long.

The thermal conductivities of a few cylindrical samples were determined with this apparatus, by comparative method using a rod of stainless steel of measured conductivity as a reference.

Theory

If a long uniform rod is placed in an ambient of uniform temperature, and one end of the rod is raised in temperature, the termperatue distribution along the rod after a steady state of temperature is attained is given by the general equation:

$$T = Ae^{-mx} + Be^{mx} \qquad (1)$$

where

- T is the temperature rise above the ambient temperature, at a point x
- x is the distance along the rod from a point taken as $x = o$
- e is the base of the system of natural logarithms (2.7183 ...)
- $m = \left(\frac{hP}{kA}\right)^{1/2}$, where h is the coefficient of heat transfer between the convex surface of the rod and the ambient, P and A are the perimeter and area of a cross-section of the rod, respectively, and k is its thermal conductivity. These quantities should be expressed in consistent units.
- A and B are constants determined by the imposed conditions.

If the rod were infinitely long, B would be equal to zero and equation (1) would become

$$T = T_o e^{-mx} \qquad (2)$$

if $T = T_o$ at $x = o$.

Taking natural logarithms on both sides of equation (2)

$$\ln T = \ln T_0 - mx \tag{3}$$

If values of the logarithm of observed values of T at different positions x are plotted against x, a straight line should be obtained, the slope of which would give the value of -m. If data were obtained for two different rods under conditions where h is the same, the rods having the respective properties P_1, A_1, k_1, and P_2, A_2, k_2, and m_1 and m_2 were determined from the slopes of the respective lines, then

$$\frac{m_1}{m_2} = \sqrt{\frac{h\ P_1}{k_1 A_1} \cdot \frac{k_2\ A_2}{h\ \ P_2}} = \sqrt{\frac{k_2\ D_2}{k_1\ D_1}} \tag{4}$$

where D_1 and D_2 are the diameters of the rods, respectively.

Hence, if k_1 were known,

$$k_2 = \left(\frac{m_1}{m_2}\right)^2 \frac{D_1}{D_2} k_1 \tag{5}$$

In an actual case, where the rod is not infinitely long, but where the part of the rod used for the measurements of T and x ($o < x < x_m$) is somewhat less than the length of the rod, L, (i.e., $x_m < L$), equation (1) can be put into the form

$$T = T_0 e^{-mx} \left(\frac{1 + e^{-2mL(1-x/L)}}{1 + e^{-2mL}}\right) \tag{6}$$

It will be observed that equation (6) is identical with equation (2) except for the coefficient of $T_0 e^{-mx}$ in equation (6). The value of this coefficient is nearly unity for moderate and large values of mL, provided the value of x/L (which has a maximum value of x_m/L) is not too large. It can be shown that the coefficient in equation (6) will not differ from unity by more than one percent if

$$x_m/L \leq 1 - \frac{4.6}{2mL} \tag{7}$$

Consequently, the relation given in equation (5) can be used for a rod of practical length (L) with only small error, provided that the measurements of T are made at values of x less than x_m, where x_m is limited in accordance with equation (7). The value of x_m can usually be estimated in advance of a test to keep the values of x used in the test within safe limits.

Description of the Apparatus

The thermal conductivity of the rock core sample was determined by comparison with that of a reference rod of Type 303 stainless steel, 0.85 inch in diameter and 17 inches long, the conductivity of which had previously been determined, by another method, to be 8.91 Btu/hr ft (deg F) at approximately the mean temperature at which the rod was used.

The test sample and reference rod were painted with the same black paint to assure that their surfaces would have the same emissivity and that the corresponding values of h would be equal. They were hung horizontally in line axially in an enclosed chamber 3 x 1 1/2 x 1 1/2 ft in size. A few turns of resistance wire were wrapped around the adjoining end of each rod, so that these ends could be heated to desired temperatures by passing an electrical current through the windings. Butt-welded thermocouples were placed on each rod in close contact with the rod surface at four accurately measured positions spaced from 3 to 4 cms apart. These positions were all well within the limits established by equation (7).

The ambient temperature was measured by two fine wire thermocouples placed in corresponding positions two inches away from each rod. All thermocouples were read with their cold junctions at 32° F, using a precision potentiometer.

Procedure and Results

The adjoining ends of the rods were heated electrically with constant voltages impressed, until all temperature measurements became steady, at which time observations were recorded. The heater voltages were selected to make the steady temperature of each rod at the point $x = o$ from 20 to 40 degrees F above the ambient air temperature.

For the temperatures used, the difference of the emfs of the rod and air thermocouples could be taken as substantially proportional to the temperature difference between rod and air. The emf differences were plotted as ordinates on semi-logarithm paper against the corresponding values of x on the linear abscissa scale, for each rod. It was found that the points for each rod conformed closely to straight lines (see Figure 1), the slopes of which were determined to yield the values of "m" used in equation (5) to calculate the thermal conductivity of the test sample (see attached sample calculation). Table 1 shows the thermal conductivities of several samples determined in this manner. For comparison, the last column of the table gives values for similar materials from a compilation by Francis Birch.

Table 1

Material	Avg. specimen temperature °F	Thermal conductivity, Btu/hr ft (deg F) Observed	Published data[1]
"Greenstone"[2]	82	1.44	1.33 (Md. diabase)
Granite[3]	88	1.67	1.22-1.74
Gneiss	85	1.48	1.04-1.62
Pyrex glass	79	0.58	0.59

[1] From Handbook of Physical Constants, Special Paper No. 36, Geological Society of America, pp. 251-260.

[2] From Mt. Weather, Va.

[3] From Woodstock, Md.

Discussion

The observed thermal conductivity values are near, or within the range of, values published for similar materials. Because of the incompleteness as regards petrographic identity of the materials involved in the comparisons, a sharper comparison is not feasible.

The test method and apparatus are relatively simple to set up and operate, and with good laboratory techniques it seems possible to obtain results in successive tests on a specimen agreeing within about ± 5 percent. This includes the element of judgment involved in drawing the lines illustrated in Figure 1.

Sample Calculation for Gneiss (Figure 1)

From Equation 5

$$k_2 = \left(\frac{m_1}{m_2}\right)^2 \frac{D_1}{D_2} k_1$$

$k_1 = 8.91 \frac{\text{Btu}}{(\text{hr ft (def F)}}$ = thermal conductivity of stainless steel reference

$\frac{D_1}{D_2} = \frac{0.85}{0.86} = 0.99$ ratio of diameters

$m_1 = \frac{8.15 \text{ cm}}{12.75 \text{ cm}} = 0.64$ proportional slope for stainless steel as measured from line in Figure 1

$m_2 = \frac{15.58 \text{ cm}}{10.0 \text{ cm}} = 1.56$ proportional slope for gneiss as measured from line in Figure 1

$$k_2 = \left(\frac{0.64}{1.56}\right)^2 (0.99)(8.91) = 1.48 \frac{\text{Btu}}{\text{hr ft °F}}$$

VII. HEAT TRANSFER PROBLEMS AT THE FORT RITCHIE UNDERGROUND SITE

B. A. Peavy

Acting under a verbal request from Messrs. Kirkpatrick and Letts of the Office of the Chief of Engineers, a study has been made of two heat transfer problems at an underground site near fort Ritchie. In general, these problems are:

1. Determine the effectiveness of the rock bounding a vehicular tunnel as a heat sink.

2. Determine the effectiveness of the rock bounding a water reservoir as a heat sink.

Heat Transfer to Rock Surrounding a Vehicular Tunnel

In accordance with present plans, during an emergency and with cooling-tower services impaired, air from the structure will be drawn along the vehicular tunnel by a 30,000 cfm alert fan located in the fan room 1,000 feet from the structure to which it is returned through a duct. The purpose of this is to dissipate heat from the warm air to the rock of the tunnel in order to reduce the load on the air cooling equipment. No definite schedule of operation has as yet been established for the above equipment during emergency periods, consequently, in what follows, necessary assumptions have been made.

It is assumed that during the time that the vehicular tunnel is to be used as a heat sink, the heat input to the structure will be constant and that the cooling load will be accomplished by the combined effects of the air cooling equipment and the loss of heat to the surface of the vehicular tunnel. It is further assumed that the mean temperature of the air in the tunnel will be 75° F and the initial temperature of the rock bounding the vehicular tunnel 58° F. This assumption implies that the heat flux at the rock surface will decrease with time and the change in rate of heat absorption by the tunnel rock will be made good by the air cooling equipment.

The heat balance for heat transfer from the air to the rock surface of the tunnel is:

$$Q = H(u - v) = -K\left(\frac{dv}{ar}\right) \qquad r = a$$

where
Q = heat flux, BTU/hr ft^2
H = coefficient of heat transfer from air to the rock surface BTU/hr ft^2 deg F
u = constant mean air temperature in tunnel, deg F
v = mean rock surface temperature, deg F
K = thermal conductivity of rock, BTU/hr ft deg F
r = position in the rock from the axis of the tunnel, ft
a = equivalent cylindrical radius of tunnel, ft.

The rate of heat flow into the rock bounding a tunnel of radius a is given for the conditions assumed above, and for moderate values of time, by the following equation:

$$Q = \frac{K(u - 58)}{2a}\left[1 - h\left(\frac{\alpha t}{\pi}\right)^{1/2} - (1 - 2ah + n)\, e^{n}\operatorname{erfc}\sqrt{n}\right]$$

where α = thermal diffusivity, ft^2/hr
$h = H/K$, ft^{-1}
t = time, hrs
$n = \alpha t h^2$, dimensionless
$e^{n}\operatorname{erfc}\sqrt{n}$ is plotted on figure 4

Substituting values K = 1.45 BTU/hr ft deg F, H = 1.0 BTU/hr ft^2 deg F, α = 0.039 ft^2/hr, a = 15.9 ft, and u = 75 deg F, numerical results obtained from the above equation for various values of time are shown in Figure 1 for which the surface area of the 1,000-ft tunnel is taken to be 100,000 ft^2.

It should be noted carefully, however, that although the rock tunnel is capable, under the assumed temperature conditions, of absorbing heat initially at the rate of about 1.7 x 10^6 BTU/hr, the maximum heat delivery to it by 30,000 cfm of air changing temperature from 75 deg to 58 deg F is only about 0.55 x 10^6 BTU/hr. If the fan delivery were increased three times, maintaining the assumed temperature conditions, heat delivery to the rock might approach that shown in figure 1.

The heat absorbing capacity of the tunnel is, therefore, less governed by the rock than by the volume and average temperature of the air in excess of rock temperature. The total heat absorbing capacity of the rock under the assumed conditions, over the initial period of seven days, as taken from figure 1, amounts to 102 x 10^6 BTU, corresponding to a level average rate for the period of 0.61 x 10^6 BTU/hr. Since the heat-absorbing capacity of the rock of the tunnel increases as the rate of heat input is reduced, the rock should be capable of absorbing heat at the rate of 0.6 x 10^6 BTU/hr for a period in excess of seven days, or of absorbing substantially more for periods less than seven days.

Heat Transfer to Rock Surrounding Underground Reservoir

The heat transfer to the rock surrounding an underground reservoir was computed by the architect engineer under W. D. Contract No. 57 (communication of H. A. Foster dated March 3, 1953). Another approach to this problem with the present assumptions is that there is a constant volume of water in the reservoir and the entire surface area (excluding the dam face) of the reservoir is in contact with the water.

The physical problem is to determine the heat transferred to the rock surface of a 230 by 34 by 29 ft high water reservoir holding approximately 1,525,000 gallons. Heat is transferred to water flowing through the heat exchangers of refrigeration condensers and Diesel lubricating oil and jacket coolers at assumed rates of 5, 6, and 7 million BTU/hr. The water is returned to the reservoir, heating its contents. At the same time, some of the heat is transferred to the rock surface which has an initial temperature of 52° F. The water is constantly recirculated in this system. After a period of time, the water will not be useful as a heat exchange medium due to an excessive rise in temperature and it is this time which is to be determined. The heat balance for heat transfer to the rock surface of the reservoir is:

$$Mc'\frac{dv}{dt} - 2\pi aK\,\frac{dv}{dr} = Q, \quad \text{at } r = a$$

which assumes that at any time the mean temperature of the water and rock surface are equal and where:

M = weight of water per unit length of cylinder, lbs/ft.
c' = specific heat of water, BTU/lb deg F.
K = thermal conductivity of rock, BTU/hr ft deg F.
Q = heat transfer rate to water per unit length of cylinder (constant) BTU/hr ft.
Q' = heat transfer rate to rock surface per unit length of cylinder, BTU/hr ft.
v = mean temperature of water and rock surface above the initial temperature of water and rock at $t = o$, deg F.
t = time, hrs.
a = equivalent cylindrical radius of reservoir, ft.
α = thermal diffusivity of rock, ft^2/hr.
$h' = \frac{2\pi aK}{Mc'\alpha}$, ft^{-1}
$n = \alpha t h'^2$

$e^n \operatorname{erfc}\sqrt{n}$ is plotted on figure 4.

The relationship derived from assuming the rock is internally bounded by a cylinder of radius a and for moderately short values of time is:

$$v = \frac{Q}{2\pi aK}\left[\left(\frac{2ah' + 3}{h'}\right)\left(\frac{\alpha t}{\pi}\right)^{1/2} - \left(\frac{2ah' + 3}{2h'^2}\right)(1 - e^n \operatorname{erfc}\sqrt{n}) - \frac{\alpha t}{2}(1 + 2\, e^n \operatorname{erfc}\sqrt{n})\right]$$

$$Q' = Q\left[\left(1 + \frac{1}{2ah'}\right)\left(1 - e^n \operatorname{erfc}\sqrt{n}\right) - \frac{1}{a}\left(\frac{\alpha t}{\pi}\right)^{1/2} + \frac{\alpha t h'}{a}\, e^n \operatorname{erfc}\sqrt{n}\right]$$

Substituting the following values for the constants:

$K = 1.45$ BTU/hr ft deg F

$\alpha = 0.039\ ft^2/hr$

$a = \frac{230(68 + 58) + 34 \times 29}{2\pi 230} = 20.7$ ft

$M = \frac{1.525 \times 10^6 \times 8.34}{230} = 55,300$ lb/ft of reservoir

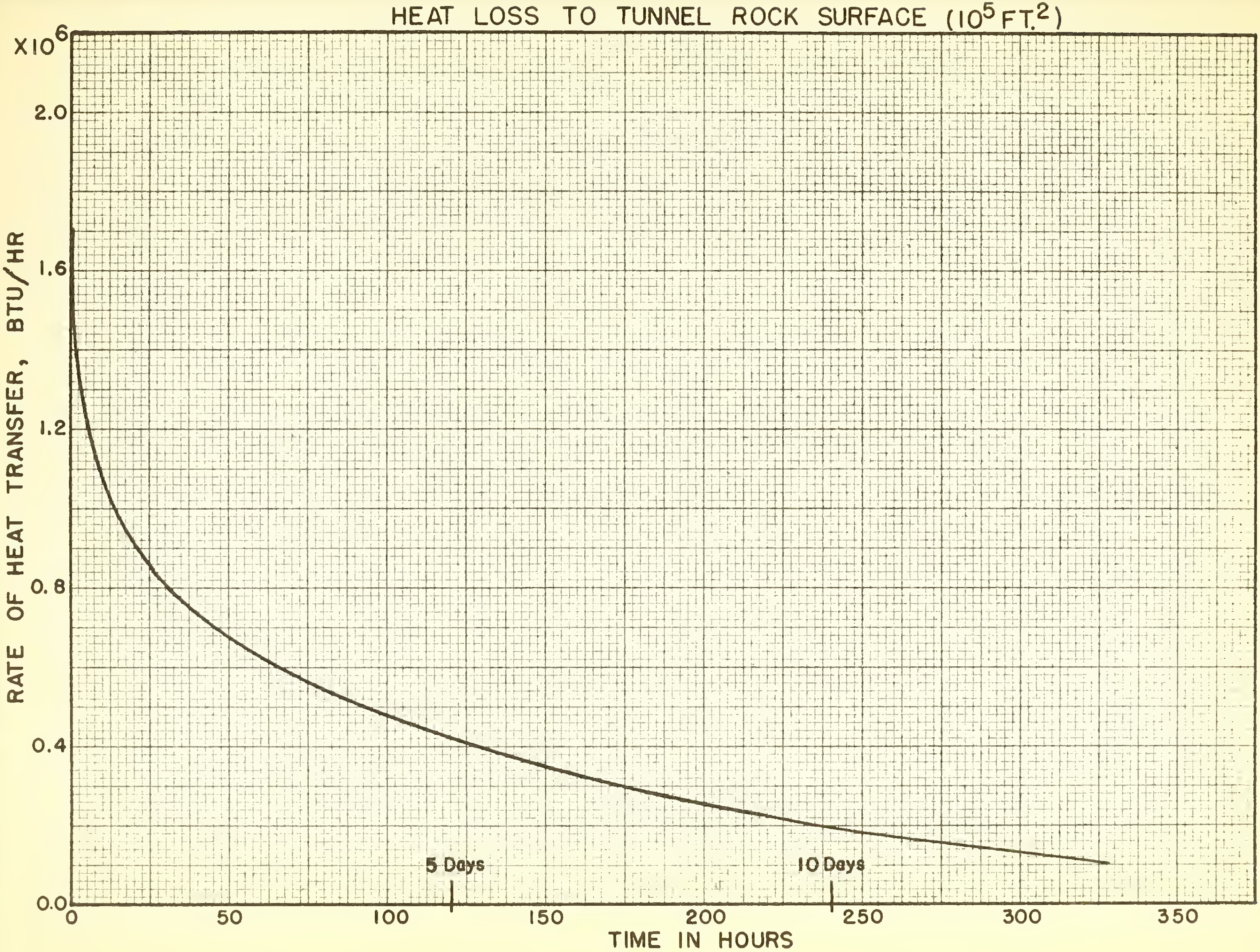
$Q = 21,720;\ 26,100;\ 30,420$ BTU/hr ft of reservoir

$h = \frac{2\pi \times 20.7 \times 1.45}{55,300 \times 1.0 \times 0.039} = 0.0876/ft$

In the above equations, values at various times of rock surface temperature and the rate of heat transfer to the rock surface of the reservoir are shown in figures 2 and 3 for heat inputs of 5, 6, and 7 million BTU/hr. Also plotted on figure 2 is the temperature rise of the water with time, assuming that no heat is transferred to the rock.

With a heat input of 6×10^6 BTU/hr to water, assuming that the maximum allowable temperature entering the refrigeration condenser will be 90° F, the mean temperature of the water in the well-stirred reservoir should be 90° F or, referring to figure 2, a time of 88 hours or 3.7 days will elapse before the reservoir will no longer be useful as a heat exchanger. The gain in time over what would be expected if heat is transferred to the water only will be 8 hours. If higher water temperatures are allowable, the gain in time would be increased.

With a heat input of 5×10^6 BTU/hr and the above assumptions, the elapsed time will be 106 hours or 4.4 days before the reservoir will no longer be useful as a heat exchanger and the gain in time over what would be expected if heat is transferred to the water only is 9 hours.



HEAT LOSS TO TUNNEL ROCK SURFACE (10^5 FT.^{2})
X10^6
2.0
1.6
1.2
0.8
0.4
0.0
RATE OF HEAT TRANSFER, BTU/HR
0
50
100
150
200
250
300
350
TIME IN HOURS
5 Days
10 Days

FIG. 1

TEMPERATURE OF RESERVOIR ROCK SURFACE

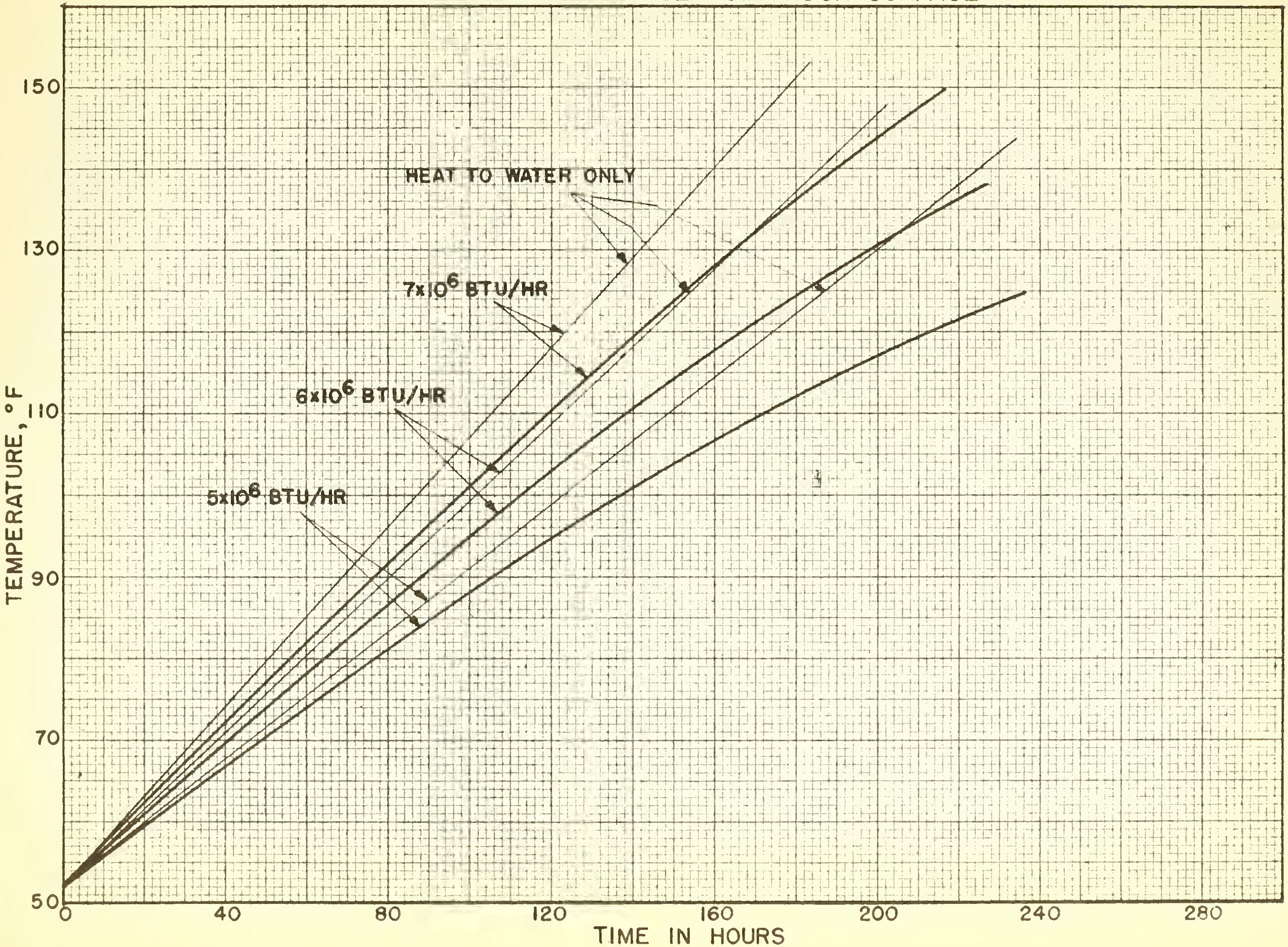
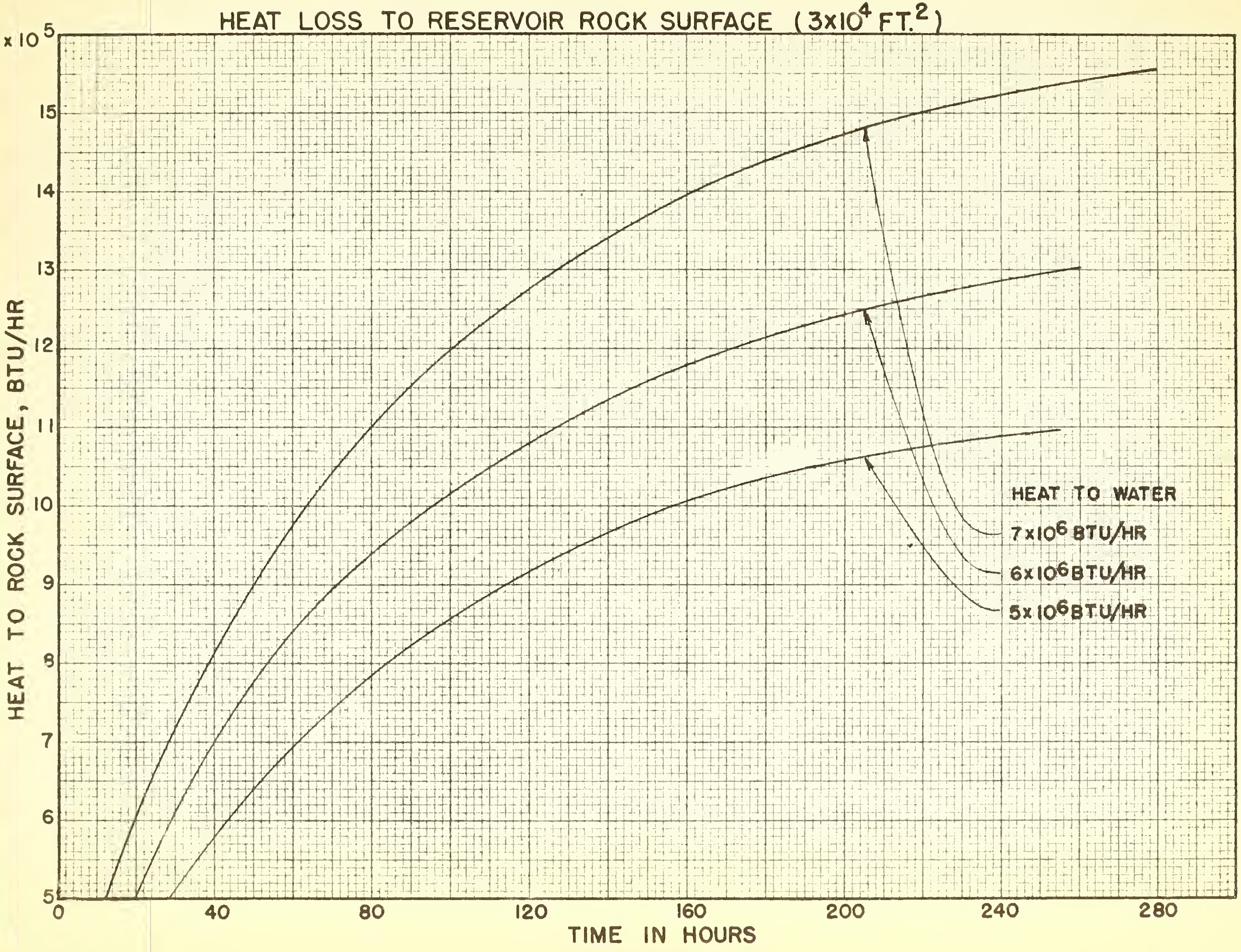


FIG. 2



HEAT LOSS TO RESERVOIR ROCK SURFACE (3x10⁴ FT.²)
x 10⁵
HEAT TO ROCK SURFACE, BTU/HR
15
14
13
12
11
10
9
8
7
6
5
0
40
80
120
160
200
240
280
TIME IN HOURS
HEAT TO WATER
7x10⁶ BTU/HR
6x10⁶ BTU/HR
5x10⁶ BTU/HR

FIG. 3

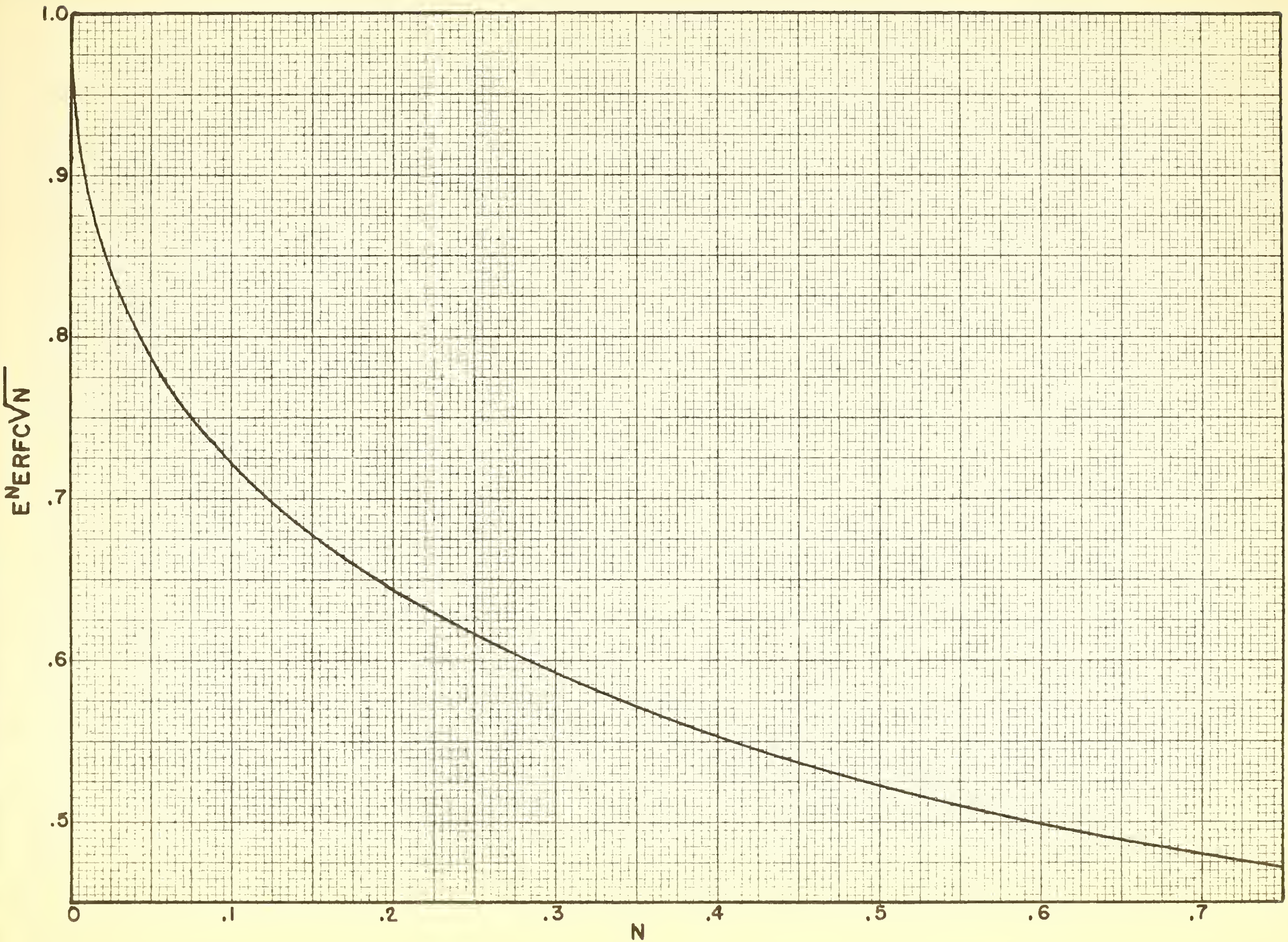


FIG. 4

VIII. REPORT OF OBSERVATIONS ON AIR CONDITIONING EQUIPMENT IN UNDERGROUND INSTALLATIONS NEAR ATCHISON, KANSAS AND ALBUQUERQUE, NEW MEXICO

J. C. LETTS (OCE) AND R. S. DILL (NBS)

1. ITINERARY

On 30 August, J. C. Letts and R. S. Dill left Washington by B&O RR for the purpose of gathering data and information on the performance of air conditioning equipment now in use in underground protective structures near Atchison, Kansas, Albuquerque, New Mexico, and Temple, Texas. The first stop was at Kansas City, Missouri, for consultation with representatives of Black and Veatch Company, Architect Engineers for the sites near Albuquerque and Temple. Due to cancellation of the airline flight from Albuquerque, the visit to Temple was omitted and data gathered on that installation are those obtained from the Architect-Engineers. Letts and Dill returned to Washington 11 September.

2. Persons Interviewed

The following persons were consulted concerning the design and operation of the air conditioning systems of interest:

At Kansas City

Mr. B. F. Steves, Engineer, Black & Veatch
Mr. J. C. Robbins, " "
Mr. D. P. Groshens, " Tulsa District, Engineer Corps

At Atchison

Mr. Van Marter, Manager, Page Airways, Inc.
Mr. D. V. Case, Engineer, USDA, Retired

At Albuquerque

Lt. Col. J. C. Hogle. Engineer, Construction Branch, Headquarters, Field Command
Capt. John C. Ward
Mr. J. T. Fallon
Mr. Clives, Plant Operator
Mr. H. W. Fenn, Plant Operator

3. Observations at Atchison

The installation near Atchison is under Contract Operation by Page Airways, Incorporated of Rochester, N.Y. It is described in a report, undated, signed by Lt. Col. E. M. Serrem, Ordnance Corps, which forms part of a brochure received from Mr. Van Marter, manager for Page Airways, Inc.

Limestone was mined from this site, beginning about 1886, resulting in a cavity with a fairly level floor, about 15 acres in extent, with a ceiling height of about 12 feet and a rock and earth over-lay of from 40 to 100 feet. During a food storage shortage incident to World War II, the site was taken over by the Department of Agriculture, a large brine refrigerating system was installed, and the space was maintained at 32° F for about seven years beginning in 1945. Installation and operation of the refrigerating equipment was under the supervision of Mr. D. V. Case.

The original refrigerating equipment consisted of three 400-ton ammonia compressors, each with a 600-HP motor, a cooling tower for condenser water and heat exchangers for chilling the brine. The space was cooled by means of 48 unit coolers to which the brine was pumped from the heat exchangers through a circuit of 8-inch pipe. Mr. Case reported that 26 days were required to cool the space to 32° F from the initial temperature of 57° F. Thereafter, the cooling equipment was found to be considerably excessive in capacity, so much so that the three original compressors and motors were exchanged for two smaller units each having a 320-HP motor.

After seven years' operation as a cold storage chamber, the plant was shut down and lay idle until about the fall of 1952 when operations commenced to warm it for use as a machinery storage space. Heat was supplied by a boiler, presumably through a heat exchanger to the brine circulated through some of the unit coolers. Much water resulted from the melting of the ice on the brine lines and, apparently, on and in the rock and this water was removed by means of sump pumps. Later, the refrigerating machines were started and dehumidification was accomplished by means of part of the unit coolers. This operation proved expensive so it was changed such that now part of the unit coolers are used for dehumidification while another part are used to cool the condenser water for the refrigerating machine. The cooling tower was valved out of the circuit. The system was operating in this manner on the occasion of this visit.

Since the underground site served as both heat source and sink, the refrigerating machine would be expected to deliver more heat to the site than it removed from it in the manner of a heat pump. The temperature in the site should therefore rise except for the heat absorbed by the surrounding rock and that removed by ventilating air. Engineering data on the installation under this regime of operation were gathered as follows:

Table 1. OPERATING DATA, TYPICAL DAYS - ATCHISON

Operating data	Date, 1953		
	March 19	June 18	August 28
Outside temperature, approx., avg. °F--	45	78	80
Inside temperature, Chamber "A" °F--	66	73	78
" " " "B" °F--	72	74	74
" " " "C" °F--	57	55	74
" wet bulb temp., Chamber "A" °F--	53	60	60
" " " " " "B" °F--	56	60	58
" " " " " "C" °F--	51	52	55
Units used for cooling No.--	12	11	9
" " " heating No.--	32	32	29
Brine circulating rate, each unit GPM--	50	50	50
Refrigerating effect Tons--	203	184	135
Heating effect, refrig. machines Tons--	240	240	220
Net " " " " Tons--	37	56	85
" " " " " Btu hr^{-1}--	440,000	672,000	1,020,000
Heat input, electric motors, etc. " --	1,065,000	939,000	895,000
Total heat input " --	1,509,000	1,611,000	1,915,000
" " " , average " --		1,678,000	
Heat absorbed by structure, Btu hr^{-1} ft^{-2}--		1.7	

The data in Table 1 are considered indicative although high accuracy cannot be claimed for them because the observations on which they are based were made by plant personnel with uncalibrated instruments, temperatures were instantaneous or "spot" readings, etc. The refrigerating effect was estimated on the nameplate brine circulating rate for the unit coolers in use as dehumidifiers and the heating effect of the refrigerating machines within the underground site was estimated in the same way. The heat input due to electric motors was estimated from the readings of a watthour meter which measured the power received by them.

The water condensed from the air in the dehumidification process is gathered in a measuring tank on a truck and the plant log sheets showed consistently that the rate is slightly in excess of 1,000 gallons per day. This accounts for about 30 tons of the refrigerating capacity now being utilized.

The ventilation of the site is held at a minimum. Fresh air is admitted for short periods and only when there is reason to suppose the air in the structure is becoming vitiated. For this reason, ventilation is ignored in the above heat transfer estimate.

On the occasion of this visit, operating conditions were observed as shown in Table 2.

Table 2

Head pressure		180 psig
Suction pressure		25 "
Brine temperature,	supply	31° F
" " ,	return	39° F
Condenser water,	supply	87° F
" " ,	return	92° F

The data in Table 2 indicate that the refrigerating machines are operating through a pressure range almost as great as if they were making ice. There is reason to believe that the power consumption could be considerably reduced if the cold air from the unit coolers could be used to extract heat from the units now being used for cooling the condenser water. It might be possible to accomplish this by setting units being used for dehumidification next to units being used to cool condenser water and connecting them by means of a duct so that the cooled air from the one would cool the condenser water flowing through the other. This arrangement would permit lowering the head pressure against which the refrigerating machines must work and would, therefore, be expected to save power. The arrangement would not be as effective as dehumidification units containing both a dehumidification coil and a condenser cooling water coil in the same cabinet and with a single motor-driven fan for both. A choice between these procedures must be based upon a detailed analysis of the factors involved.

With either of these arrangements, it is unlikely that the heat due to the dehumidification process will exactly equal the heat requirements of the space. In fact, gradual temperature rise can be expected in this installation and, for this reason, dismantling of the present refrigerating equipment is not recommended. This equipment, including the cooling tower, may be essential for controlling the temperature within the site if it is in use for a few years.

4. Observations at Monsanto Base

The installation examined at Monsanto Base occupies a series of interconnected tunnels in hard rock with a number of enlargements which serve as work rooms. The rock cover appears to vary in depth from about 100 to 200 feet or more. The engine room inspected, which is also underground, contained a Diesel engine-generator set of 400 KW capacity and air-conditioning equipment for the installation including a bank of four absorption-type (Bryant "29R") dehumidifiers, a 30-ton mechanical air-conditioning machine, an oil-burning steam boiler, an evaporation cooler for the engine jacket water, and necessary fans or blowers and auxiliary equipment. Air is brought down to the engine room from the surface of the mountain through a vertical shaft 36 inches in diameter and 112 feet high. This air is used for combustion in the engine and boiler when in operation, for cooling the condenser of the air conditioner and the engine jacket water, and for fresh air supply to the installation.

The Diesel engine is now operated only about 8 hours per week for maintenance purposes and the bank of dehumidifiers is operated only when the relative humidity in the installation approaches the limit, 50 percent, set by the operating instructions. Table 4 "Operating Conditions" shows that it was necessary to operate the dehumidifiers only 458 hours during the calendar year 1953. The humidity in the installation appears to depend more than the temperature upon external conditions. The air conditioner is operated only when either the Diesel engine or the dehumidifier-boiler combination is operated and then for the purpose of keeping the engine room tolerably cool, not for cooling the working portion of the installation. The boiler supplies heat for reactivating

the dehumidifiers so that operation of the boiler is necessary when the dehumidifiers operate.

The data gathered at the Monsanto Base, and shown in Tables 3 and 4, indicate that the surrounding rock is absorbing all the heat generated in the installation, except that removed by the ventilating air, without material assistance from the air-conditioning machine. Absorption of this heat is gradually raising the temperature in the structure so that removal of some heat by mechanical means can be expected to become essential in the future. The net heat generated in the installation, that is, the difference between the total power consumed and the heat removed by the ventilating air, is less than the capacity of the air-conditioning machine, on a yearly average basis. Therefore, the installation probably can be kept from becoming intolerably warm by means of the present air conditioner and the ventilating air. This would not be so if the Diesel engine were operated for a long period. In that case, the present air conditioner would have to be used to cool the engine room and other means would be necessary to cool the working space of the installation. Also, it would not be true if it were necessary to operate the dehumidifiers for a long period. However, the climate at this site, with the usual low humidity and dry earth, is so favorable that continuous dehumidification for long periods is not likely to be required.

<u>Itinerary</u>

Arrived at base	September 3
Visited plant	" 4 and 5
Departed from base	" 6

Table 3. ENGINEERING DATA - PLANT NO. 1

Data	Source
Floor area: 65,912 ft^2	Ward
Wall and ceiling area: 277,000 ft^2	Comp. No. 1
Total internal area: 343,000 ft^2	
Air flow: Supply 19,475 CFM	
Fresh 5,000 CFM	
Return 14,475 CFM	
Air filters (total air) 18 - Owens Corning 20 x 20 in.	Fenn
Dehumidifiers - 4 Bryant "29R" - 2900 CFM	Nameplate
Plant was put in operation, 1948	Ward
Max. electric load in plant (sum of capacities) 280 KW	Ward
Power input - typical month - 130 KW, average	Ward
Boiler, 60 hp, oil burning	Fenn
Air conditioner 30 ton	Nameplate
Plant altitude 6,000 ft, approximately	Ward
Mean annual temperature 56° F	Well water
" " " range 0° F - 110° F	Ward

Table 4. OPERATING CONDITIONS - 10 AM September 6

Operating conditions					Source
Wall temperature, corridor				74.5° F	Thermometer in bolt hole
" " , engine room (K-1)				75° F	do
Floor " , corridor				76° F	Thermometer in drain
Air conditions by Plant instruments:					
Supply	71° DB	57 WB	41% RH		
Fresh	73°	59	42%		
Return	75°	59	37%		
Air conditions by Sling Psychrometer:					
Supply	76	58			
Fresh	75.5	58			
Return	76.5	58			
Outdoor conditions:					
Sandia - 9 AM - Wind, S.E. 12 MPH (estimate)					Sling psychrometer
	72° DB	57 WB	38% RH		
Main portal - noon:					
	79.5°	57.5	25%		
Diesel engine - generator set 400 KW Burns 24 G.P.H.:					
Operates 8 hrs per week for testing					Ward
Dehumidifier - operated total of:					
June, 1953		77 hrs			Engine Room Log Book
July		255 "			do
Aug.		107 "			do
Sept.		19 "			do
Total, 1953		458 "			
Boiler - 60 Hp - Burns 12 G.P.H. of Diesel Oil					Fenn
Operated when dehumidifier operates. Operates occasionally in winter to warm engine room					Log book and Fenn
Air conditioner - 30 ton					
Operated when engine operates to cool engine rm. Operated when dehumidifier operates to cool supply air; capacity insufficient for both					Log book and Fenn

Table 5. COMPUTATIONS

---

(1) Heat equivalent of electrical input:

$$130 \times 3,415 = 444,000 \text{ Btu hr}^{-1}$$

$$= 37 \text{ tons}$$

(2) Heat input per sq ft of rock area, gross:

$$444,000/343,000 = 1.3 \text{ Btu hr}^{-1}$$

(3) Refrigerating effect of fresh air

(Based upon mean annual temperature of 56° F):

$$5,000 \times (75 - 56) \times 1.08 = 102,600 \text{ Btu hr}^{-1}$$

(4) Heat input, net:

$$444,000 - 102,600 = 341,500 \text{ Btu hr}^{-1}$$

$$= 25.5 \text{ tons}$$

(5) Heat input per sq ft of rock area, net:

$$341,500/343,000 = 1.0 \text{ Btu hr}^{-1}$$

(6) Estimated rock temperature rise due to net heat input:

3.1° F per year

DISCUSSION

The data gathered indicate that in the Atchison site the rock surrounding the site is absorbing about 1.7 Btu per hour per square foot and that it has been absorbing heat at this rate for half a year with a small but unknown temperature rise which eventually will require use of some air conditioning means to keep the temperature within tolerable limits. Indications are that the heat acceptance of the rock in this case is greater than that occurring either at the Albuquerque site or in the experimental chamber at Mt. Weather. This may be explained by the precooling of the rock when the Atchison site was refrigerated for use as a cold storage space.

The net heat acceptance of the rock at the Albuquerque site, close to 1.0 Btu per hour per square foot, is below that observed at the Mt. Weather site and is of the order expected for rock after several years of occupancy of a site. It is to be noted that at Albuquerque the air conditioner is only operated when either the dehumidifier or the Diesel engine is operated and then for the purpose of the engine room. However, Table 5 shows that the capacity of the air conditioner exceeds the net mechanical input so that the possibility is good that this plant can be operated in future by judicious use of the ventilating air and the air conditioner without additional equipment.

Excessive water is not a problem at either Atchison or Albuquerque. In damper regions, additional air conditioning capacity, particularly dehumidifying capacity, will be required.

NATIONAL BUREAU OF STANDARDS REPORT

2895

ON A CERTAIN INTEGRAL INVOLVING BESSEL FUNCTIONS

by

H. A. Antosiewicz

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

U. S. DEPARTMENT OF COMMERCE
Sinclair Weeks, *Secretary*

NATIONAL BUREAU OF STANDARDS
A. V. Astin, *Director*

THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.

Electricity. Resistance Measurements. Inductance and Capacitance. Electrical Instruments. Magnetic Measurements. Applied Electricity. Electrochemistry.

Optics and Metrology. Photometry and Colorimetry. Optical Instruments. Photographic Technology. Length. Gage.

Heat and Power. Temperature Measurements. Thermodynamics. Cryogenics. Engines and Lubrication. Engine Fuels. Cryogenic Engineering.

Atomic and Radiation Physics. Spectroscopy. Radiometry. Mass Spectrometry. Solid State Physics. Electron Physics. Atomic Physics. Neutron Measurements. Infrared Spectroscopy. Nuclear Physics. Radioactivity. X-Rays. Betatron. Nucleonic Instrumentation. Radiological Equipment. Atomic Energy Commission Instruments Branch.

Chemistry. Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Gas Chemistry. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

Mechanics. Sound. Mechanical Instruments. Aerodynamics. Engineering Mechanics. Hydraulics. Mass. Capacity, Density, and Fluid Meters.

Organic and Fibrous Materials. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Organic Plastics. Dental Research.

Metallurgy. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion.

Mineral Products. Porcelain and Pottery. Glass. Refractories. Enameled Metals. Concreting Materials. Constitution and Microstructure. Chemistry of Mineral Products.

Building Technology. Structural Engineering. Fire Protection. Heating and Air Conditioning. Floor, Roof, and Wall Coverings. Codes and Specifications.

Applied Mathematics. Numerical Analysis. Computation. Statistical Engineering. Machine Development.

Electronics. Engineering Electronics. Electron Tubes. Electronic Computers. Electronic Instrumentation.

Radio Propagation. Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Frequency Utilization Research. Tropospheric Propagation Research. High Frequency Standards. Microwave Standards.

Ordnance Development. Electromechanical Ordnance. Ordnance Electronics. These three divisions are engaged in a broad program of research and development in advanced ordnance. Activities include basic and applied research, engineering, pilot production, field testing, and evaluation of a wide variety of ordnance matériel. Special skills and facilities of other NBS divisions also contribute to this program. The activity is sponsored by the Department of Defense.

Missile Development. Missile research and development: engineering, dynamics, intelligence, instrumentation, evaluation. Combustion in jet engines. These activities are sponsored by the Department of Defense.

- Office of Basic Instrumentation
- Office of Weights and Measures.

NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT		NBS REPORT
1102-10-1104	October 26, 1953	2895

ON A CERTAIN INTEGRAL INVOLVING BESSEL FUNCTIONS

by

H. A. Antosiewicz

National Bureau of Standards
The American University

NBS

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

ON A CERTAIN INTEGRAL INVOLVING BESSEL FUNCTIONS*

by

H. A. Antosiewicz

This note deals with the integral

$$\frac{1}{\pi}\int_0^\infty e^{-kx^2t}\ \frac{J_0(rx)Y_0(ax)-J_0(ax)Y_0(rx)}{J_0^{\ 2}(ax)+Y_0^{\ 2}(ax)}\ \frac{dx}{x}$$

which arises in the solution of the equation of heat flow in a region internally bounded by a circular cylinder with constant surface temperature [1, pp. 280-282]. Its evaluation for large values of t presents certain difficulties because the standard Abelian theorems for Laplace integrals are not applicable. Although asymptotic expansions of similar integrals can be found in the literature [1,3,4], they generally are stated without explanation of their derivation. It seems desirable, therefore, to bridge this gap by giving a rather complete discussion, which may prove useful in the evaluation of related integrals.

The problem was brought to our attention by B. Peavy of the National Bureau of Standards; we appreciate his interest in the publication of our result in this form.

1. Let

*This paper was prepared under a National Bureau of Standards contract with The American University.

$$(1) \qquad F(x) = \frac{J_0(rx)Y_0(ax)-J_0(ax)Y_0(rx)}{[J_0^{\,2}(ax)+Y_0^{\,2}(ax)].x}$$

and put

$$(2) \qquad f(t) = \frac{1}{\pi} \int_0^{\infty} e^{-kx^2t} F(x)dx$$

where it is assumed that a, k, r are positive constants.

From the definitions of the Bessel functions $J_0(z)$, $Y_0(z)$ we obtain the expressions

$$(3) \qquad J_0(rx)Y_0(ax) - J_0(ax)Y_0(rx) = \frac{2}{\pi} \ln \frac{a}{r} [1 + O(x^2)],$$

$$(4) \qquad J_0^{\,2}(ax) + Y_0^{\,2}(ax) = [1 + (\frac{2}{\pi} \ln \frac{\gamma ax}{2})^2][1 + O(x^2)]^*$$

where, as customary, $O(x^2)$ denotes a function $\varphi(x)$ such that $|\varphi(x)/x^2| < A$ for some $A > 0$ as $x \to 0$. Consequently, we find

$$(5) \qquad \begin{aligned} F(x) &= \frac{2}{\pi} \ln \frac{a}{r} [1 + (\frac{2}{\pi} \ln \frac{\gamma ax}{2})^2]^{-1} \frac{1 + O(x^2)}{1 + O(x^2)} \frac{1}{x} = \\ &= \frac{2}{\pi} \ln \frac{a}{r} ([1+(\ln \frac{\gamma ax}{2})^2]x)^{-1}[1 + G(x)] \end{aligned}$$

where the function $G(x)$ is such that $G(x) = O(x^2)$ as $x \to 0$ and $G(x) = O(e^x)$ as $x \to \infty$. Thus, $F(x)$ has an infinite discontinuity at $x = 0$, and the integral in (2) is for any(finite) value of t an improper integral. However, if

*Throughout, $\ln \gamma = C = 0.577...$ Euler's constant.

we denote by $g(x)$ the numerator in the second expression of (5) and note that $|g(x)| \leq M$ on any interval $0 \leq x \leq X$, then for any sufficiently small $\epsilon_2 > \epsilon_1 > 0$

$$
\begin{aligned}
\left|\int_{\epsilon_1}^{\epsilon_2} F(x)dx\right| &= \left|\int_{\epsilon_1}^{\epsilon_2} g(x)\left[1 + \left(\tfrac{2}{\pi}\ln \tfrac{\gamma ax}{2}\right)^2\right]^{-1} \frac{dx}{x}\right| < \\
&< M\int_{\epsilon_1}^{\epsilon_2} \left[1 + \left(\tfrac{2}{\pi}\ln \tfrac{\gamma ax}{2}\right)^2\right]^{-1} \frac{dx}{x} = \\
&= \tfrac{\pi}{2}M \left[\arctan\left(\tfrac{2}{\pi}\ln \tfrac{\gamma ax}{2}\right)\right]\Big|_{\epsilon_1}^{\epsilon_2} \leq \\
&\leq \tfrac{\pi}{2}M \arctan\left(\tfrac{2}{\pi}\ln \tfrac{\epsilon_2}{\epsilon_1}\right)
\end{aligned}
\tag{6}
$$

so that the integral in (2) certainly converges at its lower limit for any value of $t \geq 0$. The convergence at its upper limit is evident.

It is to be noted that the fact that the integral in (2) is an improper integral a fortiori precludes the possibility of using directly numerical methods for its evaluation. And this, of course, holds true regardless of the value of t for which (2) is to be calculated.

2. In analogy to the asymptotic theory of functions defined by Laplace integrals it can be shown that the important contribution to $f(t)$ for large values of t will come from the values of $F(x)$ near $x = 0$. Hence we need to consider the two integrals, obtained by substituting the expression (5) into (2), namely,

$$f(t) = \frac{2}{\pi^2} \ln \frac{a}{r} \int_0^\infty e^{-kx^2 t}[1 + (\frac{2}{\pi} \ln \frac{\gamma ax}{2})^2]^{-1} \frac{dx}{x} +$$

$$(7) \qquad + \frac{2}{\pi^2} \ln \frac{a}{r} \int_0^\infty e^{-kx^2 t}.G(x)[1 + (\frac{2}{\pi} \ln \frac{\gamma ax}{2})^2]^{-1} \frac{dx}{x} \equiv$$

$$\equiv f_1(t) + f_2(t).$$

We shall first deal with the second integral $f_2(t)$.

Putting $y = kx^2$ in $f_2(t)$ we find

$$(8) \qquad f_2(t) = \frac{2}{\pi^2} \ln \frac{a}{r} \int_0^\infty e^{-yt} G(\sqrt{y/k})[1 + (\frac{2}{\pi} \ln \frac{\gamma a}{2} \sqrt{\frac{y}{k}})^2]^{-1} \frac{dy}{y}.$$

The integral on the right is clearly a Laplace integral to which we can apply the standard Abelian theorems of the asymptotic theory of Laplace integrals [2, p. 202]. We conclude that for $t \longrightarrow \infty$

$$(9) \qquad f_2(t) \sim \frac{2}{\pi^2} \ln \frac{a}{r} B[1 + (\frac{2}{\pi} \ln \sqrt{\frac{c}{t}})^2]^{-1} (1/t)$$

where $c = [\gamma a/(2\sqrt{k})]^2$ and B is the constant implied by $G(\sqrt{y/k}) = O(y)$ as $y \longrightarrow 0$.

3. We now come to the much longer discussion of the integral I(t) in $f_1(t)$, i.e.

$$(10) \qquad I(t) = \int_0^\infty e^{-kx^2 t}[1 + (\frac{2}{\pi} \ln \frac{\gamma ax}{2})^2]^{-1} \frac{dx}{x}.$$

Putting $kx^2 = (1/c)y$ where $c = [\gamma a/(2\sqrt{k})]^2$ we find

$$(11)\quad I(t) = \tfrac{1}{2}\int_0^{\infty} e^{-\frac{t}{c}y}[1 + (\tfrac{1}{\pi}\ln y)^2]^{-1}\frac{dy}{y} =$$

$$= \frac{\pi}{2}\int_0^{\infty} e^{-\frac{t}{c}y}\, d(\arctan\frac{\ln y}{\pi}).$$

Performing an integration by parts on the last integral, we obtain

$$I(t) = \frac{\pi}{2}[e^{-\frac{t}{c}y}\arctan\frac{\ln y}{\pi}]\Big|_0^{\infty} + \frac{\pi}{2}\int_0^{\infty} e^{-\frac{t}{c}y}\arctan\frac{\ln y}{\pi}\, d(\frac{t}{c}y) =$$

$$(12)$$

$$= \frac{\pi^2}{4} + \frac{\pi}{2}\int_0^{\infty} e^{-\frac{t}{c}y}\arctan\frac{\ln y}{\pi}\, d(\frac{t}{c}y)$$

We now let $ty = cz$; noting that $\int_0^{\infty} e^{-z}dz = 1$, we then can write

$$I(t) = \frac{\pi^2}{4} + \frac{\pi}{2}\int_0^{\infty} e^{-z}\arctan[\frac{\ln(cz/t)}{\pi}]dz =$$

$$(13)\qquad = \frac{\pi}{2}\left\{\frac{\pi}{2}\int_0^{\infty} e^{-z}dz + \int_0^{\infty} e^{-z}\arctan[\frac{\ln(cz/t)}{\pi}]\right\}dz =$$

$$= \frac{\pi}{2}\int_0^{\infty} e^{-z}\left\{\frac{\pi}{2} + \arctan[\frac{\ln(cz/t)}{\pi}]\right\}dz.$$

Recalling that $\arctan(\infty) = \pi/2$ and making use of an elementary trigonometric identity, we finally obtain from (13)

(14) $$I(t) = -\frac{\pi}{2}\int_0^\infty e^{-z}\arctan\left[\pi/\left(\ln\frac{cz}{t}\right)\right] dz.*$$

In this integral we put $s = \ln\frac{t}{c}$ which yields

(15) $$I(t) \equiv \mathcal{J}(s) = -\frac{\pi}{2}\int_0^\infty e^{-z}\arctan\left[\pi s/(s\ln z - 1)\right] dz =$$

$$= \frac{\pi}{2}\int_0^\infty e^{-z}\arctan\left[\pi s/(1 - s\ln z)\right] dz.$$

It is this last integral which we shall evaluate for small s.

4. To obtain an asymptotic representation of (15) for $s \longrightarrow 0$ we shall prove that

(16) $$\lim_{s\to 0}\frac{1}{s}\int_0^\infty e^{-z}\arctan\left[\pi s/(1 - s\ln z)\right]dz = \pi.$$

We must show that given any $\epsilon > 0$, however small, there exists a $\eta(\epsilon) > 0$ such that

(17) $$\left|\frac{1}{s}\int_0^\infty e^{-z}\arctan\left[\pi s/(1 - s\ln z)\right]dz - \pi\right| \le \epsilon$$

for all $s \le \eta(\epsilon)$. Evidently, this inequality will be satisfied if for all $s \le \eta(\epsilon)$

*The use of the trigonometric identity involving arctangents was pointed out to us by P. Henrici. Our original argument was somewhat longer.

$$(18)\quad \left|\int_0^Z e^{-z}\left\{\frac{1}{s}\arctan\left[\pi s/(1-s\ln z)\right]-\pi\right\}dz\right| + \left|\int_Z^\infty e^{-z}\frac{1}{s}\arctan\left[\pi s/(1-s\ln z)\right]dz\right| + \pi\left|\int_Z^\infty e^{-z}dz\right| \le \epsilon$$

where $Z > 0$ is some constant at our disposal. Since the function

$$(19)\qquad h(t,z) = \frac{1}{t}\arctan\left[\pi t/(1-t\ln z)\right]$$

is uniformly bounded for all z and all $t \geq 0$ and $h(t,z) \longrightarrow \pi$ as $t \longrightarrow 0$ for z finite, we have

$$(20)\quad \left|\int_Z^\infty e^{-z}\frac{1}{s}\arctan\left[\pi s/(1-s\ln z)\right]dz\right| < K\int_Z^\infty e^{-z}dz = Ke^{-Z}.$$

Choose $Z > 0$ so large that $\max(Ke^{-Z}, \pi e^{-Z}) \le \frac{\epsilon}{3}$. Then all we have to prove is that there exists a $\eta(\epsilon) > 0$ such that

$$(21)\quad \left|\int_0^Z e^{-z}\left\{\frac{1}{s}\arctan\left[\pi s/(1-s\ln z)\right]-\pi\right\}dz\right| \le \frac{\epsilon}{3}$$

for all $s \le \eta(\epsilon)$. This, however, is now trivial since the boundedness of $h(t,z)$ and $h(t,z) \longrightarrow 0$ as $t \longrightarrow 0$ together imply the bounded convergence of (21) to zero as $s \longrightarrow 0$.

By virtue of (16) we can conclude from (15) that for $s \longrightarrow 0$

$$(22)\qquad \mathcal{D}(s) \sim \frac{\pi^2}{2}s$$

whence for $t \longrightarrow \infty$

(23) $$I(t) \sim \frac{\pi^2}{2} \frac{1}{\ln(t/c)}$$

Recalling that we have from (7) and (10)

(24) $$f_1(t) = (\frac{2}{\pi^2} \ln \frac{a}{r}) I(t)$$

we finally obtain

(25) $$f_1(t) \sim \frac{\ln\frac{a}{r}}{\ln t - \ln c} = \frac{\ln\frac{a}{r}}{\ln 4tk - 2\ln \gamma a}$$

If we compare this relation with that derived in (9) for $f_2(t)$, we see that for sufficiently large values of t the contribution of $f_2(t)$ to $f(t)$ can be neglected.

Thus, the behavior of $f(t)$ for large values of t is described by the asymptotic relation

(26) $$f(t) \sim \frac{\ln\frac{a}{r}}{\ln 4tk - 2\ln \gamma a}$$

5. It is possible to derive similarly higher order terms in the asymptotic relation (25) for $f_1(t)$. In fact, we can obtain in this manner the asymptotic expansion

(27) $$f_1(t) \sim \ln \frac{a}{r} \left[\frac{1}{\ln 4tk - 2\ln \gamma a} - \frac{\ln \gamma}{(\ln 4tk - 2\ln \gamma a)^2} + \ldots\right]$$

for $t \longrightarrow \infty$. In view of the asymptotic relation (9) for $f_2(t)$ this expansion can be considered as an asymptotic expansion, in the sense of Poincaré, of the function $f(t)$ for $t \longrightarrow \infty$.

In conclusion we remark that the expansion

$$(28) \quad \frac{1}{\pi}\int_0^\infty \frac{e^{-kx^2 t}}{J_0^2(ax) + Y_0^2(ax)} \frac{dx}{x} \sim \frac{\pi}{2}\left[\frac{1}{\ln 4tk - 2\ln \gamma a} - \frac{\ln \gamma}{(\ln 4tk - 2\ln \gamma a)^2} + \ldots\right]$$

can be found in the literature [1,3,4], where it is stated without derivation.

REFERENCES

1. Carslaw, H.S. and Jaeger, J.C. Conduction of Heat in Solids. Oxford University Press, 1947.

2. Doetsch, G. Theorie und Anwendung der Laplace-Transformation. (American Ed.). Dover Publications, 1943.

3. Goldstein S. Some two-dimensional Diffusion Problems with Circular Symmetry. Proc. London Math. Soc., vol. 34(1932), pp. 51-88.

4. Jaeger, J. C. Heat Flow in the Region Bounded Internally by a Circular Cylinder. Proc. Roy. Soc. Edinburgh (A), vol. 61(1942), pp. 223-228.

The American University

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

NBS