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ON A CERTAIN INTEGRAL INVOLVING BESSEL FUNCTIONS

by

H. A. Antosiewicz

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This note deals with the integral

$$\frac{1}{\pi}\int_0^\infty e^{-kx^2t}\,\frac{J_0(rx)Y_0(ax)-J_0(ax)Y_0(rx)}{J_0^2(ax)+Y_0^2(ax)}\,\frac{dx}{x}$$

which arises in the solution of the equation of heat flow in a region internally bounded by a circular cylinder with constant surface temperature [1, pp. 280-282]. Its evaluation for large values of t presents certain difficulties because the standard Abelian theorems for Laplace integrals are not applicable. Although asymptotic expansions of similar integrals can be found in the literature [1,3,4], they generally are stated without explanation of their derivation. It seems desirable, therefore, to bridge this gap by giving a rather complete discussion, which may prove useful in the evaluation of related integrals.

The problem was brought to our attention by B. Peavy of the National Bureau of Standards; we appreciate his interest in the publication of our result in this form.

1. Let

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*This paper was prepared under a National Bureau of Standards contract with The American University.

(1) $$F(x) = \frac{J_0(rx)Y_0(ax)-J_0(ax)Y_0(rx)}{[J_0^{\,2}(ax)+Y_0^{\,2}(ax)].x}$$

and put

(2) $$f(t) = \frac{1}{\pi}\int_0^{\infty} e^{-kx^2t}F(x)dx$$

where it is assumed that a, k, r are positive constants.

From the definitions of the Bessel functions $J_0(z)$, $Y_0(z)$ we obtain the expressions

(3) $$J_0(rx)Y_0(ax) - J_0(ax)Y_0(rx) = \frac{2}{\pi}\ \ln\frac{a}{r}\,[1 + O(x^2)],$$

(4) $$J_0^{\,2}(ax) + Y_0^{\,2}(ax) = [1 + (\frac{2}{\pi}\ \ln\frac{\gamma ax}{2})^2][1 + O(x^2)]*$$

where, as customary, $O(x^2)$ denotes a function $\varphi(x)$ such that $|\varphi(x)/x^2| < A$ for some $A > 0$ as $x \longrightarrow 0$. Consequently, we find

(5) $$F(x) = \frac{2}{\pi}\ \ln\frac{a}{r}\,[1 + (\frac{2}{\pi}\ln\frac{\gamma ax}{2})^2]^{-1}\,\frac{1 + O(x^2)}{1 + O(x^2)}\,\frac{1}{x} =$$

$$= \frac{2}{\pi}\ \ln\frac{a}{r}\ ([1+(\ln\frac{\gamma ax}{2})^2]x)^{-1}[1 + G(x)]$$

where the function $G(x)$ is such that $G(x) = O(x^2)$ as $x \longrightarrow 0$ and $G(x) = O(e^x)$ as $x \longrightarrow \infty$. Thus, $F(x)$ has an infinite discontinuity at $x = 0$, and the integral in (2) is for any(finite) value of t an improper integral. However, if

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*Throughout, $\ln\gamma = C = 0.577...$ Euler's constant.

we denote by g(x) the numerator in the second expression of (5) and note that $|g(x)| \leq M$ on any interval $0 \leq x \leq X$, then for any sufficiently small $\epsilon_2 > \epsilon_1 > 0$

$$\left|\int_{\epsilon_1}^{\epsilon_2} F(x)dx\right| = \left|\int_{\epsilon_1}^{\epsilon_2} g(x)\left[1 + \left(\frac{2}{\pi}\ln\frac{\gamma ax}{2}\right)^2\right]^{-1}\frac{dx}{x}\right| <$$

$$(6) \qquad < M\int_{\epsilon_1}^{\epsilon_2}\left[1 + \left(\frac{2}{\pi}\ln\frac{\gamma ax}{2}\right)^2\right]^{-1}\frac{dx}{x} =$$

$$= \frac{\pi}{2}M\left[\arctan\left(\frac{2}{\pi}\ln\frac{\gamma ax}{2}\right)\right]\Bigg|_{\epsilon_1}^{\epsilon_2} \leq$$

$$\leq \frac{\pi}{2}M \arctan\left(\frac{2}{\pi}\ln\frac{\epsilon_2}{\epsilon_1}\right)$$

so that the integral in (2) certainly converges at its lower limit for any value of $t \geq 0$. The convergence at its upper limit is evident.

It is to be noted that the fact that the integral in (2) is an improper integral a fortiori precludes the possibility of using directly numerical methods for its evaluation. And this, of course, holds true regardless of the value of t for which (2) is to be calculated.

2. In analogy to the asymptotic theory of functions defined by Laplace integrals it can be shown that the important contribution to f(t) for large values of t will come from the values of F(x) near x = 0. Hence we need to consider the two integrals, obtained by substituting the expression (5) into (2), namely,

$$f(t) = \frac{2}{\pi^2} \ln \frac{a}{r} \int_0^\infty e^{-kx^2 t}\left[1 + \left(\frac{2}{\pi} \ln \frac{\gamma ax}{2}\right)^2\right]^{-1} \frac{dx}{x} +$$

$$(7) \qquad + \frac{2}{\pi^2} \ln \frac{a}{r} \int_0^\infty e^{-kx^2 t}.G(x)\left[1 + \left(\frac{2}{\pi} \ln \frac{\gamma ax}{2}\right)^2\right]^{-1} \frac{dx}{x} \equiv$$

$$\equiv f_1(t) + f_2(t).$$

We shall first deal with the second integral $f_2(t)$.

Putting $y = kx^2$ in $f_2(t)$ we find

$$(8) \quad f_2(t) = \frac{2}{\pi^2} \ln \frac{a}{r} \int_0^\infty e^{-yt} G(\sqrt{y/k})\left[1 + \left(\frac{2}{\pi} \ln \frac{\gamma a}{2} \sqrt{\frac{y}{k}}\right)^2\right]^{-1} \frac{dy}{y}.$$

The integral on the right is clearly a Laplace integral to which we can apply the standard Abelian theorems of the asymptotic theory of Laplace integrals [2, p. 202]. We conclude that for $t \longrightarrow \infty$

$$(9) \quad f_2(t) \sim \frac{2}{\pi^2} \ln \frac{a}{r} B\left[1 + \left(\frac{2}{\pi} \ln \sqrt{\frac{c}{t}}\right)^2\right]^{-1} (1/t)$$

where $c = [\gamma a/(2\sqrt{k})]^2$ and B is the constant implied by $G(\sqrt{y/k}) = O(y)$ as $y \longrightarrow 0$.

3. We now come to the much longer discussion of the integral $I(t)$ in $f_1(t)$, i.e.

$$(10) \qquad I(t) = \int_0^\infty e^{-kx^2 t}\left[1 + \left(\frac{2}{\pi} \ln \frac{\gamma ax}{2}\right)^2\right]^{-1} \frac{dx}{x}.$$

Putting $kx^2 = (1/c)y$ where $c = [\gamma a/(2\sqrt{k})]^2$ we find

$$(11)\quad I(t) = \tfrac{1}{2}\int_0^\infty e^{-\frac{t}{c}y}[1 + (\tfrac{1}{\pi}\ln y)^2]^{-1}\,\frac{dy}{y} =$$

$$= \frac{\pi}{2}\int_0^\infty e^{-\frac{t}{c}y}\, d(\arctan \frac{\ln y}{\pi}).$$

Performing an integration by parts on the last integral, we obtain

$$I(t) = \frac{\pi}{2}[e^{-\frac{t}{c}y}\arctan\frac{\ln y}{\pi}]\Big|_0^\infty + \frac{\pi}{2}\int_0^\infty e^{-\frac{t}{c}y}\arctan\frac{\ln y}{\pi}\, d(\frac{t}{c}y) =$$

(12)

$$= \frac{\pi^2}{4} + \frac{\pi}{2}\int_0^\infty e^{-\frac{t}{c}y}\arctan\frac{\ln y}{\pi}\, d(\frac{t}{c}y)$$

We now let $ty = cz$; noting that $\int_0^\infty e^{-z}dz = 1$, we then can write

$$I(t) = \frac{\pi^2}{4} + \frac{\pi}{2}\int_0^\infty e^{-z}\arctan[\frac{\ln(cz/t)}{\pi}]dz =$$

$$(13)\quad = \frac{\pi}{2}\left\{\frac{\pi}{2}\int_0^\infty e^{-z}dz + \int_0^\infty e^{-z}\arctan[\frac{\ln(cz/t)}{\pi}]\right\}dz =$$

$$= \frac{\pi}{2}\int_0^\infty e^{-z}\left\{\frac{\pi}{2} + \arctan[\frac{\ln(cz/t)}{\pi}]\right\}dz.$$

Recalling that $\arctan(\infty) = \pi/2$ and making use of an elementary trigonometric identity, we finally obtain from (13)

(14) $$I(t) = -\frac{\pi}{2}\int_0^\infty e^{-z}\arctan\,[\pi/(\ell n\,\tfrac{cz}{t})]\,dz.^*$$

In this integral we put $s = \ell n\,\frac{t}{c}$ which yields

(15) $$I(t) \equiv \mathcal{J}(s) = -\frac{\pi}{2}\int_0^\infty e^{-z}\arctan\,[\pi s/(s\,\ell nz - 1)]\,dz =$$

$$= \frac{\pi}{2}\int_0^\infty e^{-z}\arctan\,[\pi s/(1 - s\,\ell nz)]\,dz.$$

It is this last integral which we shall evaluate for small s.

4. To obtain an asymptotic representation of (15) for $s \longrightarrow 0$ we shall prove that

(16) $$\lim_{s\to 0}\frac{1}{s}\int_0^\infty e^{-z}\arctan\,[\pi s/(1 - s\,\ell nz)]dz = \pi.$$

We must show that given any $\epsilon > 0$, however small, there exists a $\eta(\epsilon) > 0$ such that

(17) $$\left|\frac{1}{s}\int_0^\infty e^{-z}\arctan\,[\pi s/(1 - s\,\ell nz)]dz - \pi\right| \le \epsilon$$

for all $s \le \eta(\epsilon)$. Evidently, this inequality will be satisfied if for all $s \le \eta(\epsilon)$

*The use of the trigonometric identity involving arctangents was pointed out to us by P. Henrici. Our original argument was somewhat longer.

$$(18)\quad \left|\int_0^Z e^{-z}\left\{\frac{1}{s}\arctan\left[\pi s/(1 - s\ln z)\right] - \pi\right\}dz\right| +$$
$$+\left|\int_Z^\infty e^{-z}\,\frac{1}{s}\arctan\left[\pi s/(1 - s\ln z)\right]dz\right| + \pi\left|\int_Z^\infty e^{-z}dz\right| \leq \epsilon$$

where $Z > 0$ is some constant at our disposal. Since the function

$$(19)\qquad h(t,z) = \frac{1}{t}\arctan\left[\pi t/(1 - t\ln z)\right]$$

is uniformly bounded for all z and all $t \geq 0$ and $h(t,z) \longrightarrow \pi$ as $t \longrightarrow 0$ for z finite, we have

$$(20)\quad \left|\int_Z^\infty e^{-z}\,\frac{1}{s}\arctan\left[\pi s/(1 - s\ln z)\right]dz\right| < K\int_Z^\infty e^{-z}dz = Ke^{-Z}.$$

Choose $Z > 0$ so large that $\max\left(Ke^{-Z}, \pi e^{-Z}\right) \leq \frac{\epsilon}{3}$. Then all we have to prove is that there exists a $\eta(\epsilon) > 0$ such that

$$(21)\qquad \left|\int_0^Z e^{-z}\left\{\frac{1}{s}\arctan\left[\pi s/(1 - s\ln z)\right] - \pi\right\}dz\right| \leq \frac{\epsilon}{3}$$

for all $s \leq \eta(\epsilon)$. This, however, is now trivial since the boundedness of $h(t,z)$ and $h(t,z) \longrightarrow 0$ as $t \longrightarrow 0$ together imply the bounded convergence of (21) to zero as $s \longrightarrow 0$.

By virtue of (16) we can conclude from (15) that for $s \longrightarrow 0$

$$(22)\qquad \mathcal{J}(s) \sim \frac{\pi^2}{2}\,s$$

whence for $t \to \infty$

$$(23) \qquad I(t) \sim \frac{\pi^2}{2} \frac{1}{\ln(t/c)}$$

Recalling that we have from (7) and (10)

$$(24) \qquad f_1(t) = \left(\frac{2}{\pi^2} \ln \frac{a}{r}\right) I(t)$$

we finally obtain

$$(25) \qquad f_1(t) \sim \frac{\ln\frac{a}{r}}{\ln t - \ln c} = \frac{\ln\frac{a}{r}}{\ln 4tk - 2\ln\gamma a}$$

If we compare this relation with that derived in (9) for $f_2(t)$, we see that for sufficiently large values of t the contribution of $f_2(t)$ to f(t) can be neglected.

Thus, the behavior of f(t) for large values of t is described by the asymptotic relation

$$(26) \qquad f(t) \sim \frac{\ln\frac{a}{r}}{\ln 4tk - 2\ln\gamma a}$$

5. It is possible to derive similarly higher order terms in the asymptotic relation (25) for $f_1(t)$. In fact, we can obtain in this manner the asymptotic expansion

$$(27) \qquad f_1(t) \sim \ln\frac{a}{r}\left[\frac{1}{\ln 4tk - 2\ln\gamma a} - \frac{\ln\gamma}{(\ln 4tk - 2\ln\gamma a)^2} + \dots\right]$$

for $t \to \infty$. In view of the asymptotic relation (9) for $f_2(t)$ this expansion can be considered as an asymptotic expansion, in the sense of Poincaré, of the function $f(t)$ for $t \longrightarrow \infty$.

In conclusion we remark that the expansion

$$(28) \quad \frac{1}{\pi} \int_0^{\infty} \frac{e^{-kx^2 t}}{J_0^2(ax) + Y_0^2(ax)} \frac{dx}{x} \sim \frac{\pi}{2} \left[\frac{1}{\ln 4tk - 2 \ln \gamma a} - \frac{\ln \gamma}{(\ln 4tk - 2 \ln \gamma a)^2} + \ldots \right]$$

can be found in the literature [1,3,4], where it is stated without derivation.

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