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Error bounds for eigenvalues of symmetric integral equations

by

Helmut Wielandt

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Helmut Wielandt

American University
Universität Tübingen, Germany

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Error bounds for eigenvalues of symmetric integral equations[1]

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1. The problem.

When the theory of integral equations was developed fifty years ago, its aim was to provide a general method for dealing with the common features of large groups of problems, rather than to furnish an effective method to solve specific problems numerically. In fact, the determination of an unknown function of one variable as the solution of an integral equation involves a function of two variables, viz. the kernel, and hence requires an amount of numerical work which could not, as a rule, effectively be performed with the equipment available at that time. Thus, the classical theory was interested in existence theorems, convergence theorems and asymptotic formulae more than in numerical aspects. The situation has been changed by the recent development of high speed calculating machines, which has created considerable interest especially in the problem of error estimates. It may be seen from BÜCKNER's report [1] that much has been done in this direction during the last years. However, interesting

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and important questions are still open. For instance, in the basic problem of calculating the eigenvalues of a real symmetric kernel no error estimate seems to be known for the oldest and most natural method which consists in performing the integration approximately by a formula of numerical quadrature. It is the purpose of this paper to fill this gap.

To state our problem precisely we have to describe the method under discussion in some detail. Let the integral equation be written in the form

$$(1) \qquad \int_0^1 K(x,\xi)y(\xi)d\xi = \kappa y(x) .$$

We assume the given kernel $K(x,\xi)$ to be real symmetric, or at least hermitian, and square-integrable:

$$(2) \qquad \overline{K(x,\xi)} = K(\xi,x), \qquad \int_0^1\int_0^1 |K(x,\xi)|^2 \, dx \, d\xi < \infty .$$

We wish to calculate the eigenvalues of $K(x,\xi)$, i.e. the constants $\kappa \neq 0$ for which (1) has a solution $y(x)$ such that

$$(3) \qquad 0 < \int_0^1 |y(\xi)|^2 \, d\xi < \infty .$$

It is known that the eigenvalues are a real bounded set which is either finite or has 0 as its only point of accumulation, even if multiplicities are counted.

The first step in the numerical procedure which we want to discuss consists in selecting some rule S for numerical quadrature. We choose a natural number n, a set of n numbers $\xi_1 < \ldots < \xi_n$ contained in the interval $0 \leqq x \leqq 1$, a set of n positive numbers p_ν with sum 1, and form the sum

(4) $$Sf = \sum_{\nu=1}^{n} p_\nu f(\xi_\nu)$$

as an approximation to the integral of the function f(x); we write

(5) $$\int_0^1 f(\xi)d\xi \approx Sf.$$

In the second step of the procedure the selected n-point-rule S is used to replace the integral equation (1) by a system of n linear algebraic equations (7) in the following way.

Suppose κ is an eigenvalue of (1) and y(x) a corresponding eigenfunction. If we specify $x = \xi_\mu (\mu = 1,\ldots,n)$ and approximate the integral by S we find from (1)

(6) $$\sum_{\nu=1}^{n} K(\xi_\mu, \xi_\nu) p_\nu\, y(\xi_\nu) \approx \kappa y(\xi_\mu) \qquad (\mu = 1,\ldots,n).$$

Introducing n unknown constants y_ν and a new eigenvalue κ^S we form the system of equations corresponding to (6):

(7) $$\sum_{\nu=1}^{n} K(\xi_\mu, \xi_\nu) p_\nu y_\nu = \kappa^S y_\mu \qquad (\mu = 1,2,\ldots,n).$$

In the third step the n eigenvalues κ^S of this algebraic problem are computed by any of the several available methods.

In the last step we assign these n values as approximations to certain of the (possibly infinitely many) eigenvalues of (1). There is a natural way of doing this. We remark that every κ^S is real; for κ^S is an eigenvalue of the n $\times$ n matrix

$$K^S = \left(K(\xi_\mu, \xi_\nu)p_\nu\right) \tag{8}$$

which is similar to the hermitian matrix $(p_\mu^{\frac{1}{2}}K(\xi_\mu, \xi_\nu)p_\nu^{\frac{1}{2}})$. Since each κ as well as each κ^S is real, and since the κ's have no other limit point than 0, it is natural to take the largest positive κ^S as an approximation to the largest positive κ, the negative κ^S largest in modulus as an approximation to the negative κ largest in modulus, and so on. Whenever a term we need is missing we shall take 0 in its place (following WEYL [11], p. 443), since almost all of the κ's are near to 0. To formalize this correspondence we define $\kappa_p^S (p = 1,2,\ldots)$ to be the p-th positive eigenvalue of K^S (counting from the top and taking regard of the multiplicities) provided a p-th positive eigenvalue exists; if not, we define $\kappa_p^S = 0$. In the same way we define κ_{-p}^S $(p=1,2,\ldots)$ to be the p-th negative eigenvalue of K^S (counting from the bottom) or 0; and similarly we denote by κ_ϱ $(\varrho = \pm 1, \pm 2,\ldots)$ the eigenvalues of K, possibly completed by zeros. The notation is such that

$$(9) \qquad \kappa_1 \geqq \kappa_2 \geqq \ldots \to 0 , \qquad \kappa_{-1} \leqq \kappa_{-2} \leqq \ldots \to 0 ,$$

$$\kappa_1^S \geqq \kappa_2^S \geqq \ldots \to 0 , \qquad \kappa_{-1}^S \leqq \kappa_{-2}^S \leqq \ldots \to 0 .$$

(In what follows we shall refer to these numbers as the eigenvalues of $K(x,\xi)$ and K^S, though this is not the usual definition as far as 0 is concerned.) We take κ_ϱ^S as an approximation to κ_ϱ, for every integer $\varrho \neq 0$, and ask for a method to calculate error bounds in the case of any given hermitian kernel $K(x,\xi)$ and any given n-point-rule S for numerical quadrature.

It appears from BÜCKNER's report [1] that so far only limit relations for $n \to \infty$ have been obtained. HILBERT, in his first proof for the existence of eigenvalues for a continuous real symmetric kernel [7], proved that for the special n-point rules S_n defined by $\xi_\nu = \nu/n$, $p_\nu = 1/n$ we have $\kappa_\varrho^{S_n} \to \kappa_\varrho$ as $n \to \infty$. BÜCKNER ([1], p. 362; [2], p. 110) extended this result to a more general class of rules. In addition he obtained, for certain rules T_n, the stronger result

$$(10) \qquad \lim_{n \to \infty} n^p(\kappa_\varrho - \kappa_\varrho^{T_n}) = 0$$

under the assumption that $K(x,\xi)$ has a continuous p-th derivative with respect to x. (see [1], p. 113; [2], p. 370). His proof of (10) (related to perturbation theory) involves in the pre-limit stage the unknown eigenfunction $y_\varrho(x)$ belonging to κ_ϱ. For this reason it does not seem to provide a suitable way to answer our question, and we prefer to develop a new method.

2. Reduction to an approximation problem for functions of two variables.

The difficulty lies in the fact that we have to compare kernels and matrices: it would be easier to deal with two kernels, or with two matrices. In fact, there is an important theorem of WEYL ([11], p. 445) which allows us to compare the eigenvalues of two hermitian kernels $K(x,\xi)$ and $G(x,\xi)$, or of two hermitian matrices K and G of the same order, without any previous knowledge concerning eigenfunctions or eigenvectors. To formulate the theorem we define, for <u>any</u> kernel or square matrix H, the nonnegative number

$$\|H\| = \max |\eta| \tag{11}$$

where η runs over all eigenvalues of H. Then WEYL's theorem implies that

$$|\kappa_\varrho - \gamma_\varrho| \leqq \|K - G\| \qquad (\varrho = \pm 1, \pm 2, \ldots) \tag{12}$$

whenever K and G are two hermitian kernels, or two hermitian matrices, with eigenvalues κ_ϱ and γ_ϱ numbered according to (9).

To overcome the discrepancy between kernels and matrices we consider those hermitian kernels $K(x,\xi)$ which have the special property, with respect to a fixed rule S for numerical quadrature, that the matrix K^S derived from $K(x,\xi)$ by S has the same eigenvalues as $K(x,\xi)$ has:

$$(13) \qquad \kappa_\varrho^S = \kappa_\varrho \qquad (\varrho = \pm 1, \pm 2, \ldots).$$

These kernels will be said to <u>allow</u> S. Postponing the question of how to construct kernels allowing S, we state

Theorem 1. <u>Let</u> $K(x,\xi)$ <u>be a hermitian kernel and</u> S <u>a rule</u> (4) <u>for numerical quadrature. Then the eigenvalues</u> κ_ϱ <u>and</u> κ_ϱ^S <u>defined by</u> (9) <u>satisfy</u>

$$(14) \qquad |\kappa_\varrho - \kappa_\varrho^S| \leqq \| K - G \| + \| K^S - G^S \| \qquad (\varrho = \pm 1, \pm 2, \ldots)$$

<u>for every hermitian kernel</u> $G(x,\xi)$ <u>allowing</u> S.

<u>Proof</u>. If the eigenvalues of $G(x,\xi)$ and G^S are denoted by γ_ϱ and γ_ϱ^S then by (13) $\gamma_\varrho = \gamma_\varrho^S$. Hence

$$|\kappa_\varrho - \kappa_\varrho^S| \leqq |\kappa_\varrho - \gamma_\varrho| + |\kappa_\varrho^S - \gamma_\varrho^S|.$$

Since $K(x,\xi)$ and $G(x,\xi)$ are hermitian, and since K^S and G^S may be transformed into hermitian matrices simultaneously by a similarity transformation which does not affect $\| K^S - G^S \|$ we have by (12)

$$|\kappa_\varrho - \gamma_\varrho| \leqq \| K - G \|, \quad |\kappa_\varrho^S - \gamma_\varrho^S| \leqq \| K^S - G^S \|$$

which proves theorem 1.

Corollary 1'. <u>Let</u> $K(x,\xi)$ <u>be a hermitian kernel</u>, S <u>a rule</u> (4), $G(x,\xi)$ <u>a hermitian kernel allowing</u> S <u>which coincides with</u> $K(x,\xi)$ <u>on the mesh points determined by</u> S, <u>that is</u>

(15) $$G(\xi_\mu, \xi_\nu) = K(\xi_\mu, \xi_\nu) \qquad (\mu, \nu = 1,2,\ldots,n).$$

Then

(16) $$|\kappa_\rho - \kappa_\rho^S| \leqq \|K - G\| \qquad (\rho = \pm 1, \pm 2, \ldots).$$

In order to facilitate the numerical application of these theorems we recall some simple estimates for the right hand sides. For arbitrary (not necessarily hermitian) kernels (or matrices) H we have[2)]

(17) $$\|H\| \leqq \sup_x \int_0^1 |H(x,\xi)|\,d\xi \qquad \left(\text{or} \leqq \sup_\mu \sum_{\nu=1}^n |h_{\mu\nu}|\right)$$

(18) $$\|H\|^2 \leqq \int_0^1\!\!\int_0^1 |H(x,\xi)|^2\,dx\,d\xi \qquad \left(\text{or} \leqq \sum_{\mu,\nu} |h_{\mu\nu}|^2\right)$$

(19) $$\|H\| \leqq \sup_{x,\xi} |H(x,\xi)|$$

(20) If $|H_1(x,\xi)| \leqq H_2(x,\xi)$ then $\|H_1\| \leqq \|H_2\|$.

If $K(x,\xi) - G(x,\xi) \leqq M$ then by (19) and (17)

$$\|K - G\| \leqq M, \quad \|K^S - G^S\| \leqq M \sum_{\nu=1}^n p_\nu = M.$$

Hence we have the convenient

Corollary 1". <u>If</u> $K(x,\xi)$ <u>and</u> $G(x,\xi)$ <u>are hermitian kernels such that</u> $G(x,\xi)$ <u>allows</u> S <u>then</u>

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2) (17) and (18) are easily derived from (11), (19) from (18). For (20) cf. JENTZSCH [9] and FROBENIUS [7], p. 516.

(21) $$|\kappa_\varrho - \kappa_\varrho^S| \leqq 2 \sup_{x,\xi} |K(x,\xi) - G(x,\xi)| \quad (\varrho = \pm 1, \pm 2, \ldots).$$

This leads us to consider the following question: How well can a given kernel $K(x,\xi)$ be approximated by hermitian kernels allowing a given rule S?

3. Construction of kernels which allow S.

Let us say that a set of functions $g_1(x),\ldots,g_m(x)$ admits a given quadrature formula S if each $g_\mu(x)$ is square-integrable over $0 \leqq x \leqq 1$ and the scalar product of every two of them can be calculated exactly by S, that is

(22) $$\int_0^1 \overline{g_\alpha(\xi)}\, g_\beta(\xi)\, d\xi = S\bar{g}_\alpha g_\beta \qquad (\alpha,\beta = 1,2,\ldots,m).$$

Every set with this property provides a method for constructing kernels which allow S in the sense of (13):

Theorem 2. Let $g_1(x),\ldots,g_m(x)$ admit S, and let $c_{\alpha\beta}$ be complex constants such that $\overline{c_{\alpha\beta}} = c_{\beta\alpha}$ for $\alpha,\beta = 1,2,\ldots,m$. Then

(23) $$\sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} c_{\alpha\beta}\, g_\alpha(x)\, \overline{g_\beta(\xi)}$$

is a hermitian kernel allowing S.

Proof. Let us denote the expression (23) by $K(x,\xi)$. Evidently $K(x,\xi)$ fulfills (2). Hence it will be sufficient to prove that the eigenvalues $\kappa \neq 0$ of $K(x,\xi)$ coincide with the eigenvalues $\kappa^S \neq 0$ of the matrix K^S defined by (8).

Let $y(x)$ be a solution of (1) and (3). Since $\kappa \neq 0$ we find from (1) and (3) that $y(x)$ has the form

$$y(x) = \sum_{\alpha=1}^{m} c_\alpha g_\alpha(x) \tag{24}$$

with certain constants c_α. Now (22) shows that (6) holds not only approximately but exactly. This means that

$$\kappa^S = \kappa, \quad y_\nu = y(\xi_\nu) \qquad (\nu = 1,2,\ldots,n) \tag{25}$$

satisfy (7). Moreover we know that some $y_\nu \neq 0$ since by (24) and (22)

$$0 < \int_0^1 |y(\xi)|^2 \, d\xi = S\,|y|^2 = \sum_{\nu=1}^{n} p_\nu |y_\nu|^2.$$

Hence κ is an eigenvalue of K^S, and the linear mapping (25) of eigenfunctions into eigenvectors preserves linear independence. As a consequence the multiplicity of κ as an eigenvalue for K^S is not less than for $K(x,\xi)$. So we know $\sum (\kappa_\varrho^S)^2 \geqq \sum \kappa_\varrho^2$. To prove theorem 2 it is sufficient to show that here equality holds. We have (from SMITHIES [10] formula 5.1.1)

$$\sum_\varrho \kappa_\varrho^2 = \int_0^1\!\!\int_0^1 K(x,\xi)\, K(\xi,x)\,dx\, d\xi$$

which can be written, by (22) and (23), in the form

$$\sum_\varrho \kappa_\varrho^2 = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} K(\xi_\mu, \xi_\nu) p_\nu \, K(\xi_\nu, \xi_\mu) p_\mu$$
$$= \operatorname{trace} (K^S)^2 = \sum_\varrho (\kappa_\varrho^S)^2.$$

This completes the proof.

Remarks. (a) The eigenvectors of K^S are explicitly given by (25) in terms of the eigenfunctions of $K(x,\xi)$.
(b) The restriction of theorem 2 to hermitian kernels can be avoided by a slight modification of the proof (calculate trace $(K^S)^p$; $p = 2,3,\ldots$).

Theorem 2 suggests the following procedure to obtain bounds for $|\kappa_\rho - \kappa_\rho^S|$: (I) Select a set $g_1(x),\ldots,g_m(x)$ which admits S. (II) Select constants $c_{\alpha\beta}$ which make (23) an approximation to $K(x,\xi)$. (III) Apply theorem 1 or its corollaries.

Postponing the discussion of I to section 5 where several rules S will be dealt with we turn to the approximation problem II.

4. Reduction to an approximation problem for functions of a single variable.

We want to reduce our problem of approximating $K(x,\xi)$ by $\sum\sum c_{\alpha\beta}\, g_\alpha(x)\overline{g_\beta(\xi)}$ to the "one-dimensional" problem of approximating functions $f(x)$ by $\sum c_\mu\, g_\mu(x)$. For the sake of simplicity let us assume that

(26) $\overline{K(x,\xi)} = K(\xi,x)$; $K(x,\xi)$ continuous in $0 \leq x,\xi \leq 1$.

Suppose we have defined an approximation operator A which assigns to every continuous function $f(x)$ a linear combination of m fixed arbitrary functions $g_\mu(x)$:

(27) $$Af(x) = \sum_{\mu=1}^{m} c_\mu g_\mu(x)$$

where the coefficients c_μ depend on f. Then it is natural to assign to the kernel $K(x,\xi)$ the new kernel

(28) $$G(x,\xi) = A_x \overline{A}_\xi K(x,\xi)$$

where the subscripts denote the variable to which the operator is to be applied, and the conjugate complex operator $\overline{A}$ is defined by

(29) $$\overline{A}f(x) = \overline{A\overline{f(x)}}.$$

Under suitable assumptions every error estimate for the approximation of f by Af yields an error estimate for the approximation of $K(x,\xi)$ by $G(x,\xi)$:

Theorem 3. _Let the operator_ (27) _be linear, i.e., for any complex_ β_i,

$$A[\beta_1 f_1(x) + \beta_2 f_2(x)] = \beta_1 \, Af_1(x) + \beta_2 \, Af_2(x);$$

and let there be a constant c _such that_[3)]

(30) $$|Af(x)| \leq c \max_\xi |f(\xi)| \qquad (0 \leq x \leq 1)$$

3) It will be seen later that this assumption is a weak point in our method of reducing the two-dimensional to the one-dimensional approximation problem. It would seem desirable to have a reduction method avoiding the constant c.

for every function $f(x)$ *continuous in* $0 \leq x \leq 1$. *Let* $K(x,\xi)$ *satisfy* (26). *Then* (28) *defines a hermitian kernel* $G(x,\xi)$ *of the form* (23) *such that for* $0 \leq x,\xi \leq 1$

(31) $$|K(x,\xi) - G(x,\xi)| \leq (1 + c)\sup_{x,\xi} |K(x,\xi) - A_x K(x,\xi)|.$$

Proof. Since $K(x,\xi)$ is continuous we can approximate $K(x,\xi)$ uniformly by a polynomial:

(32) $$\left|K(x,\xi) - \sum_{\varrho=0}^{r} \sum_{\sigma=0}^{r} a_{\varrho\sigma} x^{\varrho} \xi^{\sigma}\right| \leq \varepsilon \qquad (0 \leq x,\xi \leq 1)$$

where $\varepsilon > 0$ may be prescribed arbitrarily small; and by symmetrizing, i.e. exchanging x for ξ, we can achieve $\overline{a_{\varrho\sigma}} = a_{\sigma\varrho}$ since $K(x,\xi)$ is hermitian. Define $h_\sigma(\xi) = A_\xi \xi^\sigma$. Then by (30)

(33) $$\left|\overline{A}_\xi K(x,\xi) - \sum_{\varrho=0}^{r} \sum_{\sigma=0}^{r} a_{\varrho\sigma} x^{\varrho} \overline{h}_\sigma(\xi)\right| \leq c\varepsilon \quad (0 \leq x,\xi \leq 1)$$

which shows that $\overline{A}_\xi K(x,\xi)$ can be approximated with arbitrarily high accuracy by a polynomial in x, for every fixed ξ. Hence $\overline{A}_\xi K(x,\xi)$ is continuous in x, and we may apply A_x so that (28) really defines a function $G(x,\xi)$. Applying A_x to (33) we find

(34) $$\left|G(x,\xi) - \sum_{\varrho=0}^{r} \sum_{\sigma=0}^{r} a_{\varrho\sigma} h_\varrho(x) \overline{h}_\sigma(\xi)\right| \leq c^2\varepsilon \qquad (0 \leq x, \xi \leq 1).$$

Since $\overline{a_{\varrho\sigma}} = a_{\sigma\varrho}$, the double sum is hermitian. Hence by letting $\varepsilon \to 0$ we find that $G(x,\xi)$ is hermitian. In addition, since the $h_\varrho(x)$ are linear combinations of $g_1(x),\ldots,g_m(x)$ we find that $G(x,\xi)$ has the form (23). Finally

$$|K(x,\xi)-G(x,\xi)| \leqq |K(x,\xi) - A_x K(x,\xi)| + |A_x[K(x,\xi)-\overline{A}_\xi K(x,\xi)]|$$

$$\leqq M + c\, M$$

if

$$\sup_{x,\xi} |K(x,\xi) - A_x K(x,\xi)| = M,$$

since

$$|K(x,\xi) - \overline{A}_\xi K(x,\xi)| = |\overline{K(x,\xi)} - \overline{\overline{A}_\xi K(x,\xi)}|$$

$$= |K(\xi,x) - A_\xi \overline{K(x,\xi)}| = |K(\xi,x) - A_\xi K(\xi,x)| \leqq M.$$

Remarks. (a) From (34) we see that also $\overline{A}_\xi\, A_x\, K(x,\xi)=G(x,\xi)$. (b) If A is an integral operator with a real non-negative kernel such that for $f_o(x) \equiv 1$ we have $Af_o(x) \equiv 1$ then (30) is true with

$$c = 1. \tag{35}$$

To prove this let $f(x) = u(x) + iv(x)$. Then we have, with $M = \sup|f(x)|$,

$$\text{Re}[A\, f(x)] = A\, u(x) = A[M\, f_o(x)]$$

$$- A[M\, f_o(x) - u(x)].$$

Since $M\, f_o(x) - u(x) \geqq 0$ and A has a positive kernel we conclude that

$$\text{Re}[A\, f(x)] \leqq A[M\, f_o(x)] = M.$$

In the same manner we find for every complex number ζ with modulus 1

$$\operatorname{Re} \zeta [A\, f(x)] = \operatorname{Re} [A\, \zeta f(x)] \leqq M.$$

Hence $|A\, f(x)| \leqq M$. This means that (35) holds.

5. <u>Five special quadrature formulas</u>.

A. We begin with the simple rule for which the abscissas ξ_ν are the left end points of n subintervals of equal length and all weights p_ν are equal. We denote this rule by Leq_n:

(36) $\quad \mathrm{Leq}_n: \quad \xi_\nu = \frac{\nu - 1}{n}, \quad p_\nu = \frac{1}{n} \quad (\nu = 1, \ldots, n).$

Obviously the n piecewise constant functions

(37)
$$g_\nu(x) = \begin{cases} 1 \text{ if } \frac{\nu-1}{n} \leqq x < \frac{\nu}{n} & (\nu = 1, \ldots, n) \\ 1 \text{ if } \nu = n \text{ and } x = 1 & \\ 0 \text{ else} & \end{cases}$$

admit Leq_n in the sense of (22). (In a similar way, for every n-point-rule S a system of n piecewise constant functions admitting S can be constructed; see [1], p. 109). We form the kernel

$$G(x, \xi) = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} K(\xi_\mu, \xi_\nu) g_\mu(x)\, \overline{g_\nu(\xi)}$$

which is constant in every subsquare

$$\frac{\mu - 1}{n} \leqq x < \frac{\mu}{n}, \quad \frac{\nu - 1}{n} \leqq \xi < \frac{\nu}{n} \qquad (\mu, \nu = 1, \ldots, n).$$

$G(x,\xi)$ is of type (23) and coincides with $K(x,\xi)$ at the mesh points (ξ_μ, ξ_ν). Hence (16) holds. To estimate $\|K - G\|$ we assume that $K(x,\xi)$ satisfies a Lipschitz condition with a known constant L at each of the n^2 mesh points:

$$\left|K(x,\xi) - K(\tfrac{\mu}{n},\tfrac{\nu}{n})\right| \leqq L(|x - \tfrac{\mu}{n}| + |\xi - \tfrac{\nu}{n}|)$$

(38)

$$\text{if } \frac{\mu}{n} \leqq x < \frac{\mu+1}{n}, \quad \frac{\nu}{n} \leqq \xi < \frac{\nu+1}{n},$$

This condition is certainly satisfied if $K(x,\xi)$ has a partial derivative $K_x(x,\xi)$ such that $|K_x(x,\xi)| \leqq L$ for every x,ξ. (38) means that

$$|K(x,\xi) - G(x,\xi)| \leqq L\ P(x,\xi)$$

where

(39) $\quad P(x,\xi) = p(x) + p(\xi); \quad p(x) = x - \frac{\mu}{n}$ if $\frac{\mu}{n} \leqq x < \frac{\mu+1}{n}$. By (20) we have $\|K - G\| \leqq L\ \|P\|$. By (39) every eigenfunction of $P(x,\xi)$ has the form $c_1 p(x) + c_2$, hence the eigenvalues of $P(x,\xi)$ can be calculated by solving the quadratic equation

$$\pi^2 - 2\pi \int_0^1 p dx + [\int_0^1 p dx]^2 - \int_0^1 p^2 dx = 0.$$

The roots are

$$\pi_1 = \frac{1}{n}\left(\frac{1}{2} + \sqrt{\frac{1}{3}}\right), \quad \pi_{-1} = \frac{1}{n}\left(\frac{1}{2} - \sqrt{\frac{1}{3}}\right).$$

Hence $\|P\| = \pi_1$ by definition (11). We have proved

Theorem 4. <u>Let</u> $S = Leq_n$. <u>Let</u> $K(x,\xi)$ <u>be a hermitian kernel satisfying the Lipschitz condition</u> (38). <u>Then</u>

(41) $$\left|\kappa_\varrho - \kappa_\varrho^S\right| \leqq \frac{C\,L}{n} \qquad (\varrho = \pm 1, \pm 2, \ldots)$$

where
$$C = \frac{1}{2} + \sqrt{\frac{1}{3}} < 1.08 .$$

We remark that the constant C cannot be replaced by any smaller constant independent of $K(x,\xi)$. For in the special case $K(x,\xi) = LP(x,\xi)$ equality holds in (41) for $\varrho = 1$.

B. By the same method we can treat the rule Meq_n whose abscissas are the mid-points of n subintervals of equal length:

(42) $$Meq_n: \quad \xi_\nu = \frac{2\nu - 1}{2n} \; ; \quad p_\nu = \frac{1}{n} \; ; \quad (1 \leqq \nu \leqq n).$$

Changing (38) and (39) in an obvious way (with $p(x) = \min\limits_{\mu} |x - \xi_\mu|$) we arrive at

Theorem 5. Let $S = Meq_n$. Let the hermitian kernel $K(x,\xi)$ satisfy

(43) $$\left|K(x,\xi) - K\left(\frac{2\mu - 1}{n}, \frac{2\nu - 1}{n}\right)\right| \leqq L\left(\left|x - \frac{2\mu - 1}{n}\right| + \left|\xi - \frac{2\nu - 1}{n}\right|\right)$$
$$\text{if } \left|x - \frac{2\mu - 1}{n}\right| \leq \frac{1}{2n}, \qquad \left|\xi - \frac{2\nu - 1}{n}\right| < \frac{1}{2n} .$$

Then

(44) $$\left|\kappa_\varrho - \kappa_\varrho^S\right| \leqq \frac{CL}{n} \qquad (\varrho = \pm 1, \pm 2, \ldots)$$

where $C = \frac{1}{4} + \sqrt{\frac{1}{12}} < .54$ is the best possible constant.

C. For the trapezoidal rule

(45) $\quad \mathrm{Tra}_n: \quad \xi_\nu = \frac{\nu-1}{n-1}\,,\ p_1 = p_n = \frac{1}{2n},\ p_2 = \ldots = p_{n-1} = \frac{1}{n}$

$$(1 \leqq \nu \leqq n)$$

we find similarly, replacing (37) by step functions which jump at $(\nu - \frac{1}{2})/(n-1)$ instead of ν/n:

Theorem 6. Let $S = \mathrm{Tra}_n$. Let the hermitian kernel $K(x,\xi)$ satisfy

$$\left|K(x,\xi) - K(\tfrac{\mu}{n}, \tfrac{\nu}{n})\right| \leqq L\left(\left|x - \tfrac{\mu}{n}\right| + \left|\xi - \tfrac{\nu}{n}\right|\right)$$

if $\quad \left|x - \frac{\mu}{n}\right| < \frac{1}{2n}, \quad \left|\xi - \frac{\nu}{n}\right| < \frac{1}{2n}$.

Then

(46) $$\left|\kappa_\varrho - \kappa_\varrho^S\right| \leqq \frac{CL}{n-1} \qquad (\varrho = \pm 1, \pm 2, \ldots)$$

where $C = \frac{1}{4} + \sqrt{\frac{1}{12}} < .54$ is the best possible constant.

D. We now turn to Simpson's rule Sim_n defined by

(47) $\quad \mathrm{Sim}_n: \quad \xi_\nu = (\nu-1)h,\ h = \frac{1}{n-1} \qquad (\nu = 1,2,\ldots,n;\ n = 2m+1)$

$$p_1 = p_n = \frac{h}{3},\ p_3 = p_5 = \ldots = p_{n-2} = \frac{2h}{3},\ p_2 = p_4 = \ldots = p_{n-1} = \frac{4h}{3} .$$

In order to get good approximations we try to avoid step functions. Since Simpson's rule integrates exactly every continuous function $f(x)$ which is quadratic in each of the $m = (n-1)/2$ intervals

(48) $$\xi_{2\mu-1} \leqq x \leqq \xi_{2\mu+1} \qquad (\mu = 1,2,\ldots,m),$$

each set of <u>continuous</u> functions which are <u>linear</u> in each of the m intervals (48) will admit Sim_n. We choose any set of m+1 linearly independent functions $g_1(x),\ldots,g_{m+1}(x)$ of this type. Their linear combinations exhaust the continuous piecewise linear functions with vertices at $\xi_1, \xi_3,\ldots, \xi_n$.

In order to apply theorem 3 we define an approximation operator A by interpolation at the m+1 points $\xi_1, \xi_3,\ldots, \xi_n$:

$$(49)\quad Af(x) = \sum_{\mu=1}^{m+1} a_\mu\, g_\mu(x),\ Af(\xi_{2\mu-1}) = f(\xi_{2\mu-1}) \quad (\mu = 1,\ldots,m+1).$$

Obviously A is linear. The best constant c in (30) is $C=1$, since the piecewise linear function $Af(x)$ attains its maximum for some $\xi_{2\mu-1}$, and there coincides with $f(x)$. Let $f(x)$ have a continuous second derivative in $0 \leqq x \leqq 1$, and

$$(50)\qquad |f''(x)| \leqq M.$$

To estimate the error $|f(x) - Af(x)|$, it will be sufficient to consider the interval $\xi_1 \leqq x \leqq \xi_3$, that is $0 \leqq x \leqq 2h$. In addition, since neither our assumption (50) nor our conclusion (55) is affected by subtracting a linear function from $f(x)$, we may assume without loss of generality that

$$(51)\qquad f(0) = f(2h) = 0.$$

Then

$$(52)\qquad f(x) = \int_0^{2h} L(x,\xi)\, f''(\xi)d\xi \qquad (0 \leqq x \leqq 2h)$$

where

(53)
$$L(x,\xi) = \begin{cases} -\dfrac{\xi}{2h}(2h-x) & \text{if } 0 \leqq \xi \leqq x, \\ -\dfrac{x}{2h}(2h-\xi) & \text{if } x \leqq \xi \leqq 2h. \end{cases}$$

Hence for $0 \leqq x \leqq 2h$

(54)
$$|f(x)| \leqq M \int_0^{2h} |L(x,\xi)|\, d\xi = M \frac{x(2h-x)}{2} \leqq M \frac{h^2}{2}.$$

From (51) we have $Af(x) \equiv 0$ for $0 \leqq x \leqq 2h$, hence (54) gives

(55)
$$|f(x) - A\, f(x)| \leqq \frac{Mh^2}{2} \qquad (0 \leqq x \leqq 2h).$$

Since the intervals $[\xi_3, \xi_5], \ldots$ may be treated in the same manner, (55) holds for $0 \leqq x \leqq 1$. Now let

(56)
$$\overline{K(x,\xi)} = K(\xi,x), \qquad |K_{xx}(x,\xi)| \leqq M$$

where the derivative K_{xx} is supposed to be continuous. Then theorem 3 states that $G(x,\xi) = A_x \overline{A_\xi} K(x,\xi)$ is a hermitian kernel allowing Sim_n and approximating $K(x,\xi)$, on account of $c = 1$ and (55), with a uniform error $\leqq Mh^2$. Hence we have by (20)

(57) for $S = \mathrm{Sim}_n$:
$$|\kappa_\rho - \kappa_\rho^S| \leqq 2Mh^2 = \frac{2M}{(n-1)^2} \qquad (\rho = \pm 1, \pm 2, \ldots).$$

If we had been able to make coincide $G(x,\xi)$ with $K(x,\xi)$ at <u>all</u> mesh points then the constant 2 in (57) could have been replaced by 1. But obviously our $m+1 = (n+1)/2$ functions $g_\mu(x)$ are too few to do this; we would need n functions $g_\nu(x)$ which are linearly independent on the n points ξ_ν. So we are led to search for m more functions $g_{m+2}(x), \ldots, g_n(x)$ which complete the m+1 functions g (x) defined above [after (48)] to a system $g_1(x),\ldots,g_n(x)$ still admitting Sim_n. It may be checked by a simple calculation that the step functions

$$(58)\qquad g_{m+1+\mu}(x) = \begin{cases} 1 & \text{if } |x - \xi_{2\mu}| \leqq \frac{2h}{3} \\ 0 & \text{else} \end{cases} \qquad (\mu = 1,\ldots,m)$$

do us this favor[4]. Using these functions we may proceed as before, replacing (49) by

$$Af(x) = \sum_{\nu=1}^{n} a_\nu g_\nu(x),\ Af(\xi_\nu) = f(\xi_\nu) \qquad (\nu = 1,\ldots,n).$$

We have, say in $0 \leqq x \leqq 2h$,

$$(59)\qquad Af(x) = \begin{cases} \frac{x}{2h} f(0) + (1 - \frac{x}{2h})f(2h) & \text{if } |x-h| > \frac{2h}{3}, \\ f(h) + \frac{f(2h)-f(0)}{2h}(x-h) & \text{if } |x-h| \leqq \frac{2h}{3}. \end{cases}$$

Hence $|Af(x)| \leqq \frac{5}{3} \sup|f(x)|$, that is (30) holds with

$$(60)\qquad c = \frac{5}{3}.$$

To estimate $|f(x)-Af(x)|$ we assume, as before, (50) and (51). Then by (59)

$$f(x) - Af(x) = \begin{cases} f(x) & \text{if } (x-h) > \frac{2h}{3}, \\ f(x) - f(h) & \text{if } (x-h) \leqq \frac{2h}{3}. \end{cases}$$

4) The author is not aware of any convenient system of continuous functions $g_{m+2},\ldots,g_n$ serving our purpose. This lack of information concerning (22) seems to indicate that an investigation of numerical quadrature of <u>products of two functions</u> is desirable. Such a theory would be useful in numerical applications of Hilbert space.

Hence by (54), still in $0 \leqq x \leqq 2h$,

(61) $\left|f(x) - Af(x)\right| \leqq M \frac{x(2h-x)}{2} \leqq M \frac{5h^2}{18}$ if $|x-h| > \frac{2h}{3}$,

and

$$\left|f(x) - Af(x)\right| \leqq M \int_0^{2h} \left|L(x,\xi) - L(h,\xi)\right| d\xi \text{ if } |x-h| \leqq \frac{2h}{3}$$

where $L(x,\xi)$ is given by (53). The right hand side is found, after some computation, to be

(62) $$M \frac{(h^2+x^2)(h-x)}{2(h+x)} \leqq M \frac{5h^2}{18} \qquad \text{if } |x-h| \leqq \frac{2h}{3} .$$

From (61) and (62), we have, for every x with $0 \leqq x \leqq 2h$, and hence in the entire interval $0 \leqq x \leqq 1$

(63) $$\left|f(x) - Af(x)\right| \leqq M \frac{5h^2}{18} \qquad (h = \frac{1}{n-1})$$

instead of (55). Using (16), (19), (31) and (60) we find

Theorem 7. <u>Let</u> $S = \mathrm{Sim}_n$. <u>If</u> $K(x,\xi)$ <u>is hermitian and has a continuous second derivative such that</u> $\left|K_{xx}(x,\xi)\right| \leqq M$ <u>then</u>

(64) $$\left|\kappa_\varrho - \kappa_\varrho^S\right| \leqq \frac{CM}{(n-1)^2}$$

<u>where</u> $C = (1 + \frac{5}{3}) \frac{5}{18} < .75$.

It should be possible to replace C by a substantially smaller constant by using (17) instead of (19).

E. Our last example is Gauss' rule Gau_n. Its (irrational) abscissas and (positive) weights are uniquely determined by the property that

(65) $\quad \mathrm{Gau}_n$: $\quad \int_0^1 x^\tau \, dx = \mathrm{Gau}_n x^\tau \quad (\tau = 0,1,\ldots,2n-1).$

From (65) we see that the system $1, x, \ldots, x^{n-1}$ admits Gau_n. Hence we find from theorems 1,2,3:

Theorem 8. Let $S = \mathrm{Gau}_n$. Let there exist a linear operator A_n which assigns to every continuous function $f(x)$ $(0 \leqq x \leqq 1)$ a polynomial $A_n f(x)$ of degree $< n$ such that

(66) $$\left| A_n f(x) \right| \leqq c_n \max \left| f(\xi) \right| \qquad (0 \leqq x \leqq 1),$$

and for some $p<n$

(67) $$\left| f(x) - A_n f(x) \right| \leqq c_n^{(p)} \max \left| f^{(p)}(\xi) \right| \text{ if } f^{(p)} \text{ is continuous,}$$

with constants c_n and $c_n^{(p)}$ independent of f. Then we have for every hermitian kernel $K(x,\xi)$ which has a continuous p-th partial derivative $K^{(p)}(x,\xi)$ with respect to x,

(68) $$\left| K_\rho - K_\rho^S \right| \leqq 2(1+c_n)c_n^{(p)} \max_{x,\xi} \left| K^{(p)}(x,\xi) \right|.$$

From the extensive literature on approximation, several operators A_n are either immediately available or can be obtained from trigonometric approximation operators by substituting $x = \frac{1}{2}(1 + \cos \varphi)$, see [4].

Thus, the "trigonometric" operator defined by JACKSON ([8] p. 3) leads to the "polynomial" constants[5)]

5) These constants are better than those given by JACKSON, p. 17. They have been calculated by Dr. P. HENRICI. He found that if $|f^{(p)}(x)| \leqq 1$, $f(\frac{1}{2}) = \ldots = f^{(p-1)}(\frac{1}{2}) = 0$ and $f(\frac{1}{2} + \frac{1}{2}\cos\varphi) = g(\varphi)$ then $|g^{(p)}(\varphi)| \leqq H^p$ where $H < .64$ for $1 \leqq p \leqq 6$. It is known [4] that $H < e/2$ for $1 \leqq p < \infty$.

$$(69) \qquad c_n^{(p)} = \begin{cases} \left(\frac{1.92}{n-1}\right)^p & (p \leqq 6), \\ \left(\frac{4.1}{n-1}\right)^p & (p \geqq 7). \end{cases}$$

Since JACKSON's operator is an integral operator with a non-negative kernel, (66) holds with $c_n=1$. Hence we have

Corollary 8'. Let $S = \mathrm{Gau}_n$. Let the hermitian kernel $K(x,\xi)$ possess a continuous p-th derivative $K^{(p)}(x,\xi)$ with respect to x. Then we have for $n \geqq p$

$$(70) \qquad |K_\rho - K_\rho^S| \leqq 4 \left(\frac{d}{n-1}\right)^p \max_{x,\xi} |K^{(p)}(x,\xi)|$$

where $d = 1.92$ if $p \leqq 6$, and $d = 4.1$ if $p \geqq 7$.

FAVARD [5] has defined a trigonometric operator which leads to the "polynomial" constants

$$(71) \quad c_n^{(p)} = \begin{cases} \left(\frac{.67}{n}\right)^p & (2 \leqq p \leqq 6), \\ \left(\frac{1.36}{n}\right)^p & (p \geqq 7). \end{cases}$$

These constants are better than those given by (69). Unfortunately no constant c_n satisfying (66) seems to be known for FAVARD's operator, hence we cannot use (71) to improve (70). However, there is a special class of kernels where we can do without c_n, viz. kernels of the form

$$(72) \qquad K(x,\xi) = h_1(x+\xi) + h_2(x-\xi) \qquad (0 \leqq x, \xi \leqq 1).$$

We prove

Theorem 9: Let $S = \mathrm{Gau}_n$, and let $K(x,\xi)$ be a hermitian

<u>kernel of type</u> (72). <u>If there exist complex polynomials</u> $p_1(x)$ <u>and</u> $p_2(x)$ <u>of degrees</u> $< n$ <u>such that for some constants</u> $\varepsilon_1, \varepsilon_2$

$$(73)\quad |h_2(x)-p_2(x)| \leqq \varepsilon_2 \ (-1<x<1), |h_1(x)-p_1(x)| \leqq \varepsilon_1 \ (0<x<2)$$

<u>then we have</u>

$$(74)\qquad |\kappa_\rho - \kappa_\rho^S| \leqq 2(\varepsilon_1 + \varepsilon_2) \qquad (\rho = \pm 1, \pm 2, \dots).$$

<u>Proof</u>. Put

$$M(x,\xi) = p_1(x+\xi) + p_2(x-\xi) .$$

Then we have from (73)

$$|K(x,\xi) - M(x,\xi)| \leqq \varepsilon_1 + \varepsilon_2 \qquad (0 \leqq x,\xi \leqq 1).$$

Since $K(x,\xi)$ is hermitian the kernel

$$G(x,\xi) = \tfrac{1}{2}[M(x,\xi) + \overline{M(\xi,x)}]$$

also satisfies

$$|K(x,\xi) - G(x,\xi)| \leqq \varepsilon_1 + \varepsilon_2.$$

Moreover, $G(x,\xi)$ is hermitian and a polynomial of degree $< n$ in x (and in ξ); hence $G(x,\xi)$ allows Gau_n by theorem 2. Theorem 9 now follows from corollary 1".

We may apply FAVARD's operator to $h_1(x)$ in $0 \leqq x \leqq 2$, and to $h_2(x)$ in $-1 \leqq x \leqq 1$. Since we now have intervals of length 2 the numerators occurring in (71) have to be multiplied by 2. We obtain

Corollary 9'. <u>Let</u> $S = \mathrm{Gau}_n$. <u>Let</u> $K(x,\xi)$ <u>be a hermitian kernel of type</u> (72) <u>such that</u> h_1 <u>and</u> h_2 <u>have continuous</u> p-th <u>derivatives. Then we have for</u> $n \geqq p$

$$(75) \qquad \left| K_\rho - K_\rho^S \right| \leqq 2.\left(\frac{d}{n}\right)^p \max_{\substack{0 \leqq t \leqq 2 \\ -1 \leqq u \leqq 1}} \left[h_1^{(p)}(t) + h_2^{(p)}(u) \right]$$

<u>where</u> $d = 1.34$ <u>if</u> $2 \leqq p \leqq 6$, <u>and</u> $d = 2.72$ <u>if</u> $p \geqq 7$.

F. <u>Once more</u> Leq_n. Whenever $K(x,\xi)$ is periodic then better results may be expected from the simple rule Leq_n than from Gau_n. To show this we note that the system of n exponential functions

$$(76) \qquad g_\mu(x) = e^{2\pi i \mu x} \text{ where } \mu = \begin{cases} -m,\ldots,m & \text{if } n = 2m+1 \\ -m,\ldots,m+1 & \text{if } n = 2m+2 \end{cases}$$

admits Leq_n. (It also admits Meq_n and Tra_n which, on account of the periodicity, is not essentially different from Leq_n.) Repeating the reasoning of section 5E, we arrive at the following analogues of theorems 8, etc.:

Theorem 10. <u>Let</u> $S = \mathrm{Leq}_n$. <u>Let there exist a linear operator</u> A_n <u>which assigns to every continuous function</u> $f(x)$ <u>of period</u> 1 <u>a linear combination of the functions</u> $g_\mu(x)$ <u>defined by</u> (76) <u>such that for some constant</u> c_n

$$(77) \qquad \left| A_n f(x) \right| \leqq c_n \max \left| f(\xi) \right| \qquad (-\infty < x < \infty)$$

<u>and further, if</u> $f^{(p)}(x)$ <u>exists and is continuous, for some</u>

<u>constant</u> $c_n^{(p)}$

(78) $\left|f(x) - A_n f(x)\right| \leqq c_n^{(p)} \max \left|f^{(p)}(\xi)\right|$ $(-\infty < x < \infty)$.

<u>If</u> $K(x,\xi)$ <u>is a hermitian kernel of period</u> 1 <u>and</u> $K^{(p)}(x,\xi)$ $= (\partial/\partial x)^p K(x,\xi)$ <u>is continuous (in the entire</u> (x,ξ)<u>-plane) then</u>

(79) $$\left|\kappa_\rho - \kappa_\rho^S\right| \leqq 2(1+c_n)c_n^{(p)} \max_{x,\xi} \left|K^{(p)}(x,\xi)\right|.$$

JACKSON [8] and FAVARD [3] have given trigonometric operators A_n leading to the constants[6)]

(80) $$c_n = 1 \; ; \quad c_n^{(p)} = \left(\frac{.956}{n-2}\right)^p \qquad (p \geqq 1)$$

(81) $$c_n^{(p)} = \left(\frac{.354}{n}\right)^p \qquad (p \geqq 2)$$

Hence we have

Corollary 10'. <u>Let</u> $S = \mathrm{Leq}_n$. <u>Let</u> $K(x,\xi)$ <u>be a hermitian kernel with period</u> 1. <u>Then</u>

(82) $$\left|\kappa_\rho - \kappa_\rho^S\right| \leqq 4 \left(\frac{.956}{n-2}\right)^p \max_{x,\xi} \left|K^{(p)}(x,\xi)\right|.$$

To theorem 9 there corresponds

Theorem 11. <u>Let</u> $S = \mathrm{Leq}_n$. <u>Let</u>

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6) JACKSON [6] p. 10 gives the constant $(3/m)^p$ for approximation by trigonometrical polynomials of order $\leqq m$ in $0 \leqq x \leqq 2\pi$. In our case we have $m=(n-1)/2$ or $m=(n-2)/2$, and 3 is to be replaced by $3/2\pi < .48$ for the interval $0 \leqq x \leqq 1$. This leads to (80). Similarly (81) is obtained from FAVARD's trigonometric constants.

$$K(x,\xi) = h_1(x+\xi) + h_2(x-\xi)$$

be a hermitian kernel, with functions $h_1(x)$, $h_2(x)$ of period 1. If there exist linear combinations $e_1(x)$, $e_2(x)$ of the exponentials functions (76) such that for $-\infty < x < \infty$

(83) $$|h_\alpha(x) - e_\alpha(x)| \leqq \varepsilon_\alpha \qquad (\alpha = 1,2; \quad \varepsilon_\alpha \text{ constant})$$

then we have

(84) $$|K_\varrho - K_\varrho^S| \leqq 2(\varepsilon_1+\varepsilon_2) \qquad (\varrho = \pm 1, \pm 2, \ldots).$$

Here the approximation operators work even better than in the polynomial case since, on account of the periodicity, the duplication of the interval does no harm. Using FAVARD's constants (81) we obtain

Corollary 11'. Let $S = Leq_n$. Let $K(x,\xi)$ be a hermitian kernel of type (72) where $h_1(x)$, $h_2(x)$ have the period 1 and a continuous p-th derivative such that

$$|h_\alpha^{(p)}(x)| \leqq L_\alpha \qquad (-\infty < x < \infty; \; \alpha = 1,2).$$

Then we have for $\varrho = \pm 1, \pm 2, \ldots$

(85) $$|K_\varrho - K_\varrho^S| \leqq 2 \left(\frac{.35}{n}\right)^p (L_1 + L_2).$$

Remark. Under the assumption that $K(x,\xi)$ is analytic in a known neighborhood of $0 \leqq x \leqq 1$, $0 \leqq \xi \leqq 1$, especially convenient bounds of $|K_\varrho - K_\varrho^S|$, for an arbitrary quadrature formula S, might be obtained from the error estimates given by DAVIS for the approximation of analytic functions [3].

6. <u>Two Numerical Examples.</u>

I. Suppose we want to calculate the eigenvalues of

$$K(x,\xi) = e^{x\xi} \qquad (0 \leqq x, \xi \leqq 1)$$

using the trapezoidal n-point-rule. How large should we choose n to be sure that the error of each eigenvalue is less than .01? This question is answered by theorem 6. Using the trivial estimate $|K_x(x,\xi)| \leqq e$ we find

$$|\kappa_\rho - \kappa_\rho^{\mathrm{Tra}_n}| \leqq \frac{.54\ e}{n-1} < \frac{1.47}{n-1} \qquad (\rho = \pm 1, \pm 2, \ldots).$$

Hence $n \geqq 148$ is sufficient. If we want to answer the same question for Gauss' rule we may either estimate $K^{(p)}(x,\xi)$ and apply Corollary 8' for some suitable chosen value of p, or we may search for a good explicit polynomial approximation to $K(x,\xi)$ and then apply theorem 2 and corollary 1". The first method gives the estimates

$$|\kappa_\rho - \kappa_\rho^{\mathrm{Gau}_n}| \leqq \begin{cases} 4e\left(\dfrac{1.92}{n-1}\right)^p & (p \leqq 6,\ p \leqq n), \\[2ex] 4e\left(\dfrac{4.1}{n-1}\right)^p & (7 \leqq p \leqq n). \end{cases}$$

The smallest bound results from choosing p = 6; this leads to $n \geqq 8$. In the second method it is natural to use the Taylor expansion with center $\frac{1}{2}$. If $0 \leqq z \leqq 1$ then

$$e^z = e^{\frac{1}{2}} e^{z-\frac{1}{2}} = e^{\frac{1}{2}} \sum_{\nu=0}^{n-1} \frac{(z-\frac{1}{2})}{\nu!} + R_n, \qquad |R_n| \leqq \frac{(\frac{1}{2})^n}{n!}\, e \ .$$

Putting

$$G(x,\xi) = e^{\frac{1}{2}} \sum_{\nu=0}^{n-1} \frac{(x\xi - \frac{1}{2})^{\nu}}{\nu!}$$

we find from (21) the estimate

$$|\kappa_\varrho - \kappa_\varrho^{Gau_n}| \leq \frac{(\frac{1}{2})^{n-1}}{n!} e.$$

This shows that already $n \geq 5$ is sufficient for $|\kappa_\varrho - \kappa_\varrho^{Gau_n}| < .01$.

II. Sometimes our theorems can be applied in the inverse direction: We have some information concerning the eigenvalues of $K(x,\xi)$ and ask for similar information concerning the eigenvalues of K^S. For instance, let

$$(80) \qquad K(x,\xi) = \frac{5}{13-12\cos 2\pi(x+\xi)} \qquad (0 \leq x, \xi \leq 1).$$

Since $K(x,\xi) > 0$ and $\int K(x,\xi)d\xi = 1$ for every x, we know that $\kappa_1 = 1$, $|\kappa_\varrho| < 1$ $(\varrho \neq 1)$ (cf. JENTZSCH [7], FROBENIUS [4]). Suppose we want to perform some iteration process with the matrix K^S which is derived from $K(x,\xi)$ by one of the n-point quadrature formulas S discussed in section 5, knowing that the process will converge satisfactorily if all $|\kappa_\varrho^S| \leq 1.1$. How large should we choose n in order to secure convergence?[7)]

The kernel (80) is pretty bad for numerical integration

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7) This question was proposed to the author by Dr. J. Todd and gave rise to the present investigation. The kernel (86) appeared in a numerical procedure for conformal mapping of the interior of an ellipse with axis ratio 5 onto the unit circle.

since it has a high crest along $x + \xi = 1$. The derivatives have the upper bounds

(87)

$p =$	0	1	2	3	4	5	6
$\lvert K^{(p)}\rvert \leq$	5	49.7	1270	85800	683×10^4	444×10^6	492×10^8

Using these values (and $p = 6$ where $K^{(p)}(x,\xi)$ occurs) we find that

(88) $$|K_\varrho - K_\varrho^S| \leq .1 \qquad (\varrho = \pm 1, \pm 2, \ldots)$$

(and hence $|K_\varrho^S| \leq 1.1$) if S is one of the following rules:

(89)

(41)	(44)	(64)	(70)	(75)	(82)	(85)
Leq_{537}	Meq_{269}	Sim_{135}	Gau_{216}	Gau_{134}	Leq_{110}	Leq_{36} .

Comparing the results for Simpson's and Gauss' rule we see that our Gauss estimates (70) and (75) are far from being good (they could be greatly improved if the constant c_n for FAVARD's operator turned out to be near to 1). In (75) and (85) the fact is used, that $K(x,\xi)$ is a function of a single variable. (82) and (85) use the periodicity of $K(x,\xi)$.

Better results than by our general theorems, are, of course, obtained if the special structure of our kernel (86) is used. To do this we develop $K(x,\xi)$ into its Fourier series:

$$K(x,\xi) = \sum_{j=-\infty}^{\infty} \left(\tfrac{2}{3}\right)^{|j|} e^{2\pi i j(x+\xi)},$$

and then, say for $n = 2m+1$, choose

$$G(x,\xi) = \sum_{j=-m}^{m} (\tfrac{2}{3})^{|j|} e^{2\pi i j(x+\xi)} = \sum_{j=-m}^{m} c_{\alpha\beta}\, g_\alpha(x)\, \overline{g_\beta(\xi)}$$

with $c_{\alpha\beta} = \delta_{\alpha,-\beta} (\tfrac{2}{3})^{|\alpha|}$ and $g_\alpha(x)$ defined by (81). We have

$$\left| K(x,\xi) - G(x,\xi) \right| \leq 2 \sum_{j=m+1}^{\infty} (\tfrac{2}{3})^{j} = 6(\tfrac{2}{3})^{m+1} = 6(\tfrac{2}{3})^{\frac{n+1}{2}},$$

hence by (84)

$$\left| \kappa_\varrho - \kappa_\varrho^{Leq_n} \right| \leq 12.(\tfrac{2}{3})^{\frac{n+1}{2}} .$$

From this estimate it follows that already Leq_{25} is sufficient. Which is the smallest value of n such that $S = Leq_n$ satisfies (88)?

In the case of the kernel (86) this value of n can actually be determined. For the eigenvalues of $K(x,\xi)$ can be calculated exactly, as it is always the case when $K(x,\xi) = f(x+\xi) + g(x-\xi)$ with periodic functions f, g (the eigenfunctions have the form $ae^{2\pi irx} + be^{-2\pi irx}$). In a similar way the eigenvalues of K^{Leq_n} can be calculated. The result is (after a convenient change in the numbering of the eigenvalues):

$$\text{(90)} \qquad \kappa_\varrho = \begin{cases} q^\varrho & (\varrho = 0,1,2,\ldots), \\ -q^{|\varrho|} & (\varrho = -1,-2,\ldots), \end{cases} \qquad (q = \tfrac{2}{3})$$

$$\text{(91)} \qquad \kappa_\varrho^{Leq_n} = \begin{cases} q^\varrho \dfrac{1+q^{n-2\varrho}}{1-q^n} & (\varrho = 0,1,2,\ldots,[\tfrac{n}{2}]), \\ -q^{|\varrho|} \dfrac{1+q^{n-2|\varrho|}}{1-q^n} & (\varrho = -[\tfrac{n-1}{2}],\ldots,-1), \\ 0 & \text{else}. \end{cases}$$

From this it follows that for $n \leqq 10$

$$|K_{-5} - K_{-5}^{\mathrm{Leq}_n}| = |K_{-5}| = (\tfrac{2}{3})^5 > .1,$$

but for $n \geqq 11$ always $|K_\rho - K_\rho^{\mathrm{Leq}_n}| \leqq .1$. Hence $n = 11$ is the lowest admissible value.

Perhaps it will be felt that all of the results mentioned in the table (89) are not very satisfactory. However, one must bear in mind that no general approximation theorem which is based on uniform bounds for some derivative is likely to give good results for the special kernel (86) since this kernel has high values of $K^{(p)}(x,\xi)$ only at a small part of the unit square, though our estimates are valid for kernels which have everywhere derivatives of the order of magnitude allowed by (87). Better results for the kernel (86) could be derived from approximation theorems involving say $\max_\xi \int |K^{(p)}(x,\xi)|\, dx$ instead of $\max_{x,\xi} |K^{(p)}(x,\xi)|$. A close look at example II will show that there are at least two more directions where we may look for refinements.

From (90) and (91) we see that, for fixed n, the errors $|K_\rho - K_\rho^{\mathrm{Leq}_n}|$ depend on ρ in such a way that they are small for small $|\rho|$ (they attain their maximum value for $\rho \approx \pm n/2$). This means that our estimates which are uniform in ρ are unnecessarily bad if we are interested, as usual, only in the first eigenvalues (those of large modulus). It would be worth searching for estimates of the type

(92) $$|K_\rho - K_\rho^S| \leqq C_\rho$$

where C_ρ depends on $K(x,\xi)$, S and ρ, and is small for small $|\rho|$.

Another direction for improvements is suggested by the fact that, in our example, we are interested only in one-sided bounds (namely upper bounds for $|K_\rho^S|$). For some quadrature formulas S there exist much better upper bounds for $|K_\rho^S|$, and lower bounds for $|K_\rho|$, than those implied by our estimates of $|K_\rho - K_\rho^S|$. An example will be given in the next section.

7. <u>One-sided bounds for</u> Meq_n.

Theorem 12. <u>Let</u> $S = \mathrm{Meq}_n$ <u>be the quadrature formula defined by</u> (42). <u>Let</u> $K(x,\xi)$ <u>be a hermitian kernel with a continuous second derivative</u> $K_{xx}(x,\xi)$ <u>such that</u> $|K_{xx}(x,\xi)| \leqq L'$ $(0 \leqq x, \xi \leqq 1)$. <u>Then</u>

$$\begin{aligned} &(93a) \quad K_p^S - K_p \\ &(93b) \quad K_{-p} - K_{-p}^S \end{aligned} \Bigg\} \leqq \frac{CL'}{n^2} \qquad (p = 1,2,\ldots)$$

<u>where</u> $C = \frac{1}{8}\left(\frac{1}{3} + \sqrt{\frac{1}{5}}\right) < .098$ <u>is the best possible constant</u>. The estimate (93) is remarkable for the fact that it is quadratic in $1/n$ though the two-sided estimate (44) for Meq_n is only of the first order. Applied to our "bad" kernel (86) theorem 12 shows that $|K_\rho^S| \leqq 1.1$ for $S = \mathrm{Meq}_{49}$. This

is a very satisfactory result in view of the fact that neither the periodicity nor the special form (72) of the kernel has been used.

Proof of theorem 12. We define a piecewise continuous hermitian kernel $G(x,\xi)$ which is constant in each of the n^2 subsquares

(94) $$|x-\xi_\mu| < \frac{1}{2n},\ |\xi-\xi_\nu| < \frac{1}{2n} \qquad (\mu,\nu = 1,2,\ldots,n),$$

where $\xi_\nu = (2\nu-1)/2n$, such that $G(x,\xi)$ coincides with $K(x,\xi)$ at the mesh points determined by the rule Meq_n:

(95) $$G(x,\xi) = K(\xi_\mu,\xi_\nu) \qquad \text{if (94) holds.}$$

By theorem 2 this kernel $G(x,\xi)$ admits $S = Meq_n$, hence its eigenvalues are κ_ρ^S. To prove (93a) let p be a fixed natural number. If $\kappa_p^S = 0$ then (93a) is true since $\kappa_p \geqq 0$ by definition. Let $\kappa_p^S \neq 0$. Then the eigenfunctions $z_\rho(x)$ of $G(x,\xi)$ corresponding to the first p eigenvalues κ_ρ^S ($\rho = 1,\ldots,p$) are piecewise constant. We determine numbers a_π such that

(96) $$z(x) = \sum_{\pi=1}^{p} a_\pi z_\pi(x)$$

is normalized and orthogonal to those eigenfunctions $y_\rho(x)$ of $K(x,\xi)$ which belong to eigenvalues $\kappa_\rho > 0$ with $1 \leqq \rho \leqq p-1$. (Such numbers a_π exist since the orthogonality requirement imposes at most p-1 linear homogeneous conditions on the p unknowns a_π.) Put

$$D(x,\xi) = G(x,\xi) - K(x,\xi).$$

We prove theorem 12 by estimating the integral

$$J = \int_0^1\int_0^1 \overline{z(x)}\; D(x,\xi)\; z(\xi)\; dx\, d\xi$$

in two ways. From (96) we have

$$\int_0^1\int_0^1 \overline{z(x)}\; G(x,\xi) z(\xi) dx\, d\xi \geqq K_p^S ,$$

and from the orthogonality condition

$$\int_0^1\int_0^1 \overline{z(x)}\; K(x,\xi) z(\xi) dx\, d\xi \leqq K_p ,$$

hence

(97) $$J \geqq K_p^S - K_p .$$

On the other hand we remark that $z(x)$ is constant in each of the subintervals $(\nu-1)/n < x < \nu/n$. Hence J will not be altered if we replace $D(x,\xi)$ by any kernel $\tilde{D}(x,\xi)$ which has the same mean value as $D(x,\xi)$ in each of the subsquares (94). Let us choose for $\tilde{D}(x,\xi)$ the kernel which is obtained by "symmetrizing" $D(x,\xi)$ in each of the subsquares in the following way: With the notation (94) define

$$\tilde{D}(x,\xi) = \frac{1}{4} \sum_{\sigma}{}' \sum_{\tau}{}' D(\xi_\mu + \sigma (x-\xi_\mu),\ \xi_\nu + \tau (\xi - \xi_\nu)) \quad (\sigma,\tau = \pm 1).$$

Obviously $\tilde{D}(x,\xi)$ has the same mean value as $D(x,\xi)$ in each subsquare, and in addition we know

(98a) $\tilde{D}(\xi_\mu,\xi_\nu) = 0, \quad \tilde{D}_x(\xi_\mu,\xi) = \tilde{D}_\xi(x,\xi_\nu) = 0,$

(98b) $|\tilde{D}_{xx}(x,\xi)| \leqq L' \qquad (0 \leqq x,\xi \leqq 1).$

We integrate (98b) twice, using the initial conditions (98a); the result is

$$|\tilde{D}(x,\xi)| \leqq L'[p(x) + p(\xi)], \quad p(x) = \min_\mu \frac{(x-\xi_\mu)^2}{2} .$$

Now we can proceed as in the proof of theorem 4. We find

(99) $$\|\tilde{D}\| \leqq L' \pi_1$$

where π_1 is the largest root of equation (40) formed with the present definition of $p(x)$. Solving (40) we find $\pi_1 = C/n^2$ with the value of C given in theorem 12. Now (99) gives $J \leqq CL'/n^2$. This inequality together with (97) completes the proof of (93a). Applying (93a) to $-K(x,\xi)$ instead of $K(x,\xi)$ we obtain (93b).

8. Limitations of the method.

Though the method of numerical integration can in principle be applied to every kernel it will give, as a rule, good results only if the kernel is smooth at least in the neighborhood of the mesh points (ξ_μ,ξ_ν). For only these n^2 points are used to calculate the approximations K_ϱ^S, so they have to represent the whole of $K(x,\xi)$. In cases where $K(x,\xi)$ is not very smooth better results may be expected from Hilbert's second method which approximates, after choosing n orthonormal functions $\varphi_\nu(x)$, the eigenvalues κ_ϱ of the integral equation (1) by the eigenvalues $\kappa_\varrho^{[n]}$ of the $n \times n$ matrix

$$K^{[n]} = (k_{\mu\nu}) \qquad (\mu, \nu = 1, \ldots, n)$$

where

$$k_{\mu\nu} = \int_0^1\int_0^1 \overline{\varphi_\mu(x)}\, K(x,\xi)\, \varphi_\nu(\xi)\, dx\, d\xi .$$

Error estimates for this method (which may be considered as an analogue of the well-known Rayleigh-Ritz method for differential equations) are being published elsewhere [12]. We mention the following result of type (92): For $p = 1,2,\ldots$ we have

$$(100) \qquad 0 \leq \kappa_p - \kappa_p^{[n]} \leq \frac{1}{2\kappa_p^{[n]}} \left[\int_0^1\int_0^1 |K(x,\xi)|^2\, dx\, d\xi - \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} |k_{\mu\nu}|^2 \right].$$

BIBLIOGRAPHY

[1] H. Bückner, Die praktische Behandlung von Integralgleichungen. (Ergebnisse der angewandten Mathematik vol. 1.) Springer, Berlin, 1952.

[2] H. Bückner, Konvergenzuntersuchungen bei einem algebraischen Verfahren zur Lösung von Integralgleichungen. Math. Nachrichten vol. 3 (1950), pp. 358-372.

[3] P. Davis, Errors of numerical approximation for analytic functions. J. rat.Mech. Anal. 2 (1953), pp. 303-313.

[4] C. de la Vallée Poussin, Leçons sur l'approximation des fontions d'une variable réelle, Paris 1919, Chapitre V.

[5] J. Favard, Sur les meilleures procédés d'approximation de certaines classes de fonctions par des polynomes trigonométriques. Bull. Sci. math. (2) vol. 61 (1937), pp. 209-224, 243-256.

[6] G. Frobenius, Über Matrizen aus positiven Elementen, II. Sitzungsberichte Akad. Wiss. Berlin 1909, pp. 514-518.

[7] D. Hilbert, Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, I. Mitteilung. Nachr. Ges. Wiss. Göttingen, math.-phys. Kl., 1904, pp. 49-91.

[8] D. Jackson, The theory of approximation. (Amer. Math. Soc. Coll. Publ. vol. xi.) New York 1930.

[9] R. Jentzsch, Über Integralgleichungen mit positivem Kern. J. reine angew. Math. vol. 141 (1912), pp. 235-244.

[10] F. Smithies, The eigenvalues and singular values of integral equations. Proc. London math. Soc. (2) vol. 43 (1937), pp. 255-279.

[11] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Hohlraumstrahlung). Math. Ann. vol. 71 (1912), pp. 441-479.

[12] H. Wielandt, <u>Einschliessung von Eigenwerten hermitescher Matrizen nach dem Abschnittsverfahren</u>. Archiv der Mathematik (under press).

Universität Tübingen, Germany

and

American University, Washington, D. C.

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