NATIONAL BUREAU OF STANDARDS REPORT

2545

A PROPERTY OF THE NORMAL DISTRIBUTION RELATED TO
A THEOREM OF S. BERNSTEIN

by

Eugene Lukacs and Edgar P. King

NBS

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

U. S. DEPARTMENT OF COMMERCE
Sinclair Weeks, *Secretary*

NATIONAL BUREAU OF STANDARDS
A. V. Astin, *Director*

THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.

Electricity. Resistance Measurements. Inductance and Capacitance. Electrical Instruments. Magnetic Measurements. Applied Electricity. Electrochemistry.

Optics and Metrology. Photometry and Colorimetry. Optical Instruments. Photographic Technology. Length. Gage.

Heat and Power. Temperature Measurements. Thermodynamics. Cryogenics. Engines and Lubrication. Engine Fuels. Cryogenic Engineering.

Atomic and Radiation Physics. Spectroscopy. Radiometry. Mass Spectrometry. Solid State Physics. Electron Physics. Atomic Physics. Neutron Measurements. Infrared Spectroscopy. Nuclear Physics. Radioactivity. X-Rays. Betatron. Nucleonic Instrumentation. Radiological Equipment. Atomic Energy Commission Instruments Branch.

Chemistry. Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Gas Chemistry. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

Mechanics. Sound. Mechanical Instruments. Aerodynamics. Engineering Mechanics. Hydraulics. Mass. Capacity, Density, and Fluid Meters.

Organic and Fibrous Materials. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Organic Plastics. Dental Research.

Metallurgy. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion.

Mineral Products. Porcelain and Pottery. Glass. Refractories. Enameled Metals. Concreting Materials. Constitution and Microstructure. Chemistry of Mineral Products.

Building Technology. Structural Engineering. Fire Protection. Heating and Air Conditioning. Floor, Roof, and Wall Coverings. Codes and Specifications.

Applied Mathematics. Numerical Analysis. Computation. Statistical Engineering. Machine Development.

Electronics. Engineering Electronics. Electron Tubes. Electronic Computers. Electronic Instrumentation.

Radio Propagation. Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Frequency Utilization Research. Tropospheric Propagation Research. High Frequency Standards. Microwave Standards.

Ordnance Development. **Electromechanical Ordnance.** **Ordnance Electronics.** These three divisions are engaged in a broad program of research and development in advanced ordnance. Activities include basic and applied research, engineering, pilot production, field testing, and evaluation of a wide variety of ordnance matériel. Special skills and facilities of other NBS divisions also contribute to this program. The activity is sponsored by the Department of Defense.

Missile Development. Missile research and development: engineering, dynamics, intelligence, instrumentation, evaluation. Combustion in jet engines. These activities are sponsored by the Department of Defense.

- Office of Basic Instrumentation
- Office of Weights and Measures.

NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT		NBS REPORT
1103-10-1107	4 June 1953	2545

A PROPERTY OF THE NORMAL DISTRIBUTION RELATED TO
A THEOREM OF S. BERNSTEIN

by

Eugene Lukacs and Edgar P. King

PREPRINT

A PROPERTY OF THE NORMAL DISTRIBUTION RELATED TO A THEOREM OF S. BERNSTEIN

by

Eugene Lukacs and Edgar P. King*

1. Summary. The following theorem is proved.

Let $X_1, X_2, \cdots, X_n$ be n independently (but not necessarily identically) distributed random variables, and assume that the n^{th} moment of each X_i $(i = 1,2,\ldots,n)$ exists. The necessary and sufficient conditions for the existence of two statistically independent linear forms $Y_1 = \sum_{s=1}^{n} a_s X_s$ and $Y_2 = \sum_{s=1}^{n} b_s X_s$ are:

(A) Each random variable which has a nonzero coefficient in both forms is normally distributed.

(B) $\sum_{s=1}^{n} a_s b_s \sigma_s^2 = 0$.

Here σ_s^2 denotes the variance of X_s $(s = 1,2,\ldots,n)$.

For $n=2$ and $a_1=b_1=a_2=1$, $b_2=-1$ this reduces to a theorem of S. Bernstein [1] (see also [3]) which was also proved by M. Kac [4] in measure theoretic terms. Another particular case of the theorem is stated without proof in a recent paper by Yu. V. Linnik [5].

2. Introduction. We consider two linear forms

$$Y_1 = \sum_{s=1}^{n} a_s X_s \; ; \quad Y_2 = \sum_{s=1}^{n} b_s X_s \tag{1}$$

in the n independently distributed random variables $X_1, X_2, \cdots, X_n$.

*National Bureau of Standards, Washington 25, D. C.

We arrange the variables so that the first p ($X_1, X_2, \cdots, X_p$) have **nonzero** coefficients in both forms and the remaining (n-p) have zero coefficients in one form or the other. Clearly $0 \leq p \leq n$. When p=0, Y_1 and Y_2 are trivially independent; when p=1, Y_1 and Y_2 cannot be independent. For $p \geq 2$, it is clear that the statistical independence of the original linear forms (1) is completely equivalent to the independence of the forms $Z_1 = \sum_{s=1}^{p} a_s X_s$ and $Z_2 = \sum_{s=1}^{p} b_s X_s$. This means that when $p < n$ the distributions of the random variables $X_{p+1}, \cdots, X_n$ do not affect the independence of Y_1 and Y_2. This is why the theorem contains only a statement about the distributions of those random variables with **nonzero** coefficients in both forms.

If for some pairs of corresponding coefficients, say the first r ($1<r<p$), the relation

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_r}{b_r} = C \tag{2}$$

holds, then we can rewrite Z_1 and Z_2 as

$$Z_1 = C(b_1X_1 + \cdots + b_rX_r) + a_{r+1}X_{r+1} + \cdots + a_pX_p \text{ , and}$$
$$Z_2 = b_1X_1 + \cdots + b_rX_r + b_{r+1}X_{r+1} + \cdots + b_pX_p \text{ .}$$

Introducing the new variable $\widetilde{X}_1 = b_1X_1 + \cdots + b_rX_r$, we see that the independence of Y_1 and Y_2 is equivalent to the independence of the forms $\widetilde{Z}_1 = C\widetilde{X}_1 + a_{r+1}X_{r+1} + \cdots + a_pX_p$ and $\widetilde{Z}_2 = \widetilde{X}_1 + b_{r+1}X_{r+1} + \cdots + b_pX_p$. If the theorem holds for the forms $\widetilde{Z}_1$ and $\widetilde{Z}_2$, Cramér's theorem [2] shows that the normality of $\widetilde{X}_1$ implies the normality of the random variables $X_1, X_2, \cdots, X_r$. We proceed in the same manner if there are several groups of random variables for which a relation

of type (2) holds. Hence our problem reduces to the study of the independence of two linear forms whose coefficient matrix contains no vanishing minor or order 2.

Finally it is clear that the independence of Y_1 and Y_2 is equivalent to the independence of the forms $\tilde{Y}_1 = \sum_{s=1}^{n} a_s(X_s - E[X_s])$ and $\tilde{Y}_2 = \sum_{s=1}^{n} b_s(X_s - E[X_s])$. Therefore we shall assume without loss of generality that the following conditions are satisfied:

(i) $a_s b_s \neq 0 \quad (s = 1,2,\ldots,n)$

(ii) $a_s b_t - a_t b_s \neq 0$ for all $s \neq t \quad (s,t = 1,2,\ldots,n)$

(iii) $E[X_s] = 0 \quad (s = 1,2,\ldots,n)$.

3. <u>The functional equation for the characteristic functions.</u>

Denote the distribution function of the random variable X_s $(s=1,\ldots,n)$ by $F_s(x)$ and the corresponding characteristic function by $f_s(t)$. Let $h(u,v)$ be the c.f. of the joint distribution of Y_1 and Y_2 and write $h_1(u) = h(u,0)$, $h_2(v) = h(0,v)$. Clearly $h_1(u)$ and $h_2(v)$ are the c.f.'s of the distributions of Y_1 and Y_2, respectively.

We prove first that our conditions are necessary; that is, we assume that Y_1 and Y_2 are statistically independent. In terms of characteristic functions this means

(3) $$h(u,v) = h_1(u)h_2(v) \quad .$$

Further, because $X_1,\ldots,X_n$ are independent, we have

(4) $$h_1(u) = \prod_{s=1}^{n} f_s(a_s u) \quad ,$$

(5) $$h_2(v) = \prod_{s=1}^{n} f_s(b_s v) \quad ,$$

(6) $$h(u,v) = \prod_{s=1}^{n} f_s(a_s u + b_s v) \quad .$$

Finally, substituting (4), (5), and (6) in (3) we obtain the following functional equation in the characteristic functions:

$$(7) \qquad \prod_{s=1}^{n} f_s(a_s u + b_s v) = \prod_{s=1}^{n} f_s(a_s u) f_s(b_s v) \quad .$$

4. <u>The differential equations for the cumulant generating functions</u>. The general procedure for determining the explicit form of the characteristic functions $f_s(t)$ will be to differentiate the logarithm of (7) r times $(r = 1,2,\ldots,n)$ with respect to u, set u=0, and solve the resulting n differential equations for $\ln f_s(t)$ $(s = 1,\ldots,n)$.

We first note that $f_s(o) = 1$ $(s = 1,\ldots,n)$ and that $f_s(t)$ is a continuous function of t. Therefore there exists a neighborhood of the origin in which all the factors occuring in (7) are different from zero. This neighborhood could of course be the entire plane. In the following derivation we restrict the values of u and v to this neighborhood; then we may take the logarithm of both sides of (7) and obtain

$$(9) \qquad \sum_{s=1}^{n} \phi_s(a_s u + b_s v) = \sum_{s=1}^{n} \phi_s(a_s u) + \sum_{s=1}^{n} \phi_s(b_s v) \ ,$$

$$\text{where } \phi_s(x) = \ln f_s(x) \quad .$$

Differentiating (9) r times with respect to u and setting u=0 yields

$$(10) \qquad \sum_{s=1}^{n} \left[\frac{\partial^r}{\partial u^r} \phi_s(a_s u + b_s v) \right]_{u=0} = \sum_{s=1}^{n} \left[\frac{d^r}{du^r} \phi_s(a_s u) \right]_{u=0} \quad .$$

Letting $z_s = a_s u$, we find that the typical term on the left side of (10) becomes

$$\left[\frac{\partial^r}{\partial u^r}\phi_s(a_s u+b_s v)\right]_{u=o} = a_s^r\left[\frac{\partial^r}{\partial z_s^r}\phi_s(z_s+b_s v)\right]_{z_s=0} . \tag{11}$$

Employing the substitution $\psi_s(v) = \phi(b_s v)$, (11) becomes

$$\left[\frac{\partial^r}{\partial u^r}\phi_s(a_s u+b_s v)\right]_{u=0} = \left(\frac{a_s}{b_s}\right)^r \frac{d^r}{dv^r}\psi_s(v) . \tag{12}$$

Similarly the typical term on the right side of (10) becomes

$$\left[\frac{d^r}{du^r}\phi_s(a_s u)\right]_{u=0} = a_s^r\left[\frac{d^r}{dz^r}\phi_s(z_s)\right]_{z_s=0} = (ia_s)^r\kappa_r^{(s)} \tag{13}$$

where $\kappa_r^{(s)}$ is the r^{th} order cumulant of X_s. Substituting (12) and (13) in (10) we obtain

$$\sum_{s=1}^{n}\xi_s^r\frac{d^r}{dv^r}\psi_s(v) = \sum_{s=1}^{n}(ia_s)^r\kappa_r^{(s)} \qquad (r = 1,2,\ldots,n) \tag{14}$$

$$\text{where } \xi_s = \frac{a_s}{b_s} .$$

Differentiating (14) (n-r) times yields the system of differential equations

$$\left\{\begin{aligned} &\sum_{s=1}^{n}\xi_s^r\frac{d^n}{dv^n}\psi_s(v) = 0 && (r = 1,2,\ldots,n-1)\\ &\sum_{s=1}^{n}\xi_s^n\frac{d^n}{dv^n}\psi_s(v) = \sum_{s=1}^{n}(ia_s)^n\kappa_n^{(s)} . \end{aligned}\right. \tag{15}$$

We have to determine all the distribution functions whose characteristic functions satisfy this system of differential equations and the initial conditions

$$\left\{\begin{aligned} &\left[\frac{d^r}{dv^r}\psi_s(v)\right]_{v=0} = (ib_s)^r\kappa_r^{(s)} \qquad r = 1,2,\ldots,n-1\\ &\psi_s(0) = 1 . \end{aligned}\right. \tag{15a}$$

We now define

$$D_n = \begin{vmatrix} \xi_1 & \cdots & \xi_n \\ \xi_1^2 & \cdots & \xi_n^2 \\ \cdots & \cdots & \cdots \\ \xi_1^n & \cdots & \xi_n^n \end{vmatrix}$$

and denote by $D_{s,n}$ the cofactor of the element in the s^{th} column and the n^{th} row of D_n. Considering (15) as a system of n linear equations in the quantities $\frac{d^n}{dv^n}\psi_s(v)$, we obtain the solutions

(16) $$\frac{d^n}{dv^n}\psi_s(v) = \frac{D_{s,n}}{D_n}\sum_{s=1}^{n}(ia_s)^n \kappa_n^{(s)} = i^n C_{s,n}\ , \text{ say} .$$

Integrating (16) n times and employing the initial conditions (15a) yields

$$\psi_s(v) = \sum_{j=1}^{n-1}\frac{(ib_s)^j}{j!}\kappa_j^{(s)} v^j + \frac{C_{s,n}}{n!}(iv)^n \quad .$$

Since $f_s(b_s v) = \exp[\phi_s(b_s v)] = \exp[\psi_s(v)]$ we have

(17) $$f_s(b_s v) = \exp\left[\sum_{j=1}^{n-1}\frac{\kappa_j^{(s)}}{j!}(ib_s v)^j + \frac{C_{s,n}}{b_s^n n!}(ib_s v)^n\right] .$$

In case any of the functions $f_s(t)$ become zero for some real t, this solution is valid only in a certain neighborhood of the origin. We next show by an indirect proof that none of the functions $f_s(t)$ $(s = 1,\ldots,n)$ has a real zero; from this we can conclude that (17) is valid for all real t.

Let us therefore assume that one or more of the c.f.'s $f_s(t)$ have zeros. In this case at least one of the functions $f_s(b_s v)$ will have a zero. Denote by v_r^o the zero closest to the

origin and by $f_r(t)$ a function for which $f_r(b_r v_r^o) = 0$. For $|v| < |v_r^o|$ we have $f_s(b_s v) \neq 0$ $(s = 1,\dots,n)$ and formula (17) is valid. Let v be a real number such that $|v| < |v_r^o|$; then we have by (17)

$$f_r(b_r v) = \exp\left[\sum_{j=1}^{n-1} \frac{\kappa_j^{(n)}}{j!}(ib_r v)^j + \frac{C_{r,n}}{b_r^n n!}(ib_r v)^n\right] . \tag{18}$$

But $f_r(t)$ is a continuous function. Hence $\lim_{v \to v_r^o} f_r(b_r v) =$ $= f_r(b_r v_r^o) = 0$ by assumption. However, from (18) it is clear that

$$\lim_{v \to v_r^o} f_r(b_r v) = \exp\left[\sum_{j=1}^{n-1} \frac{\kappa_j^{(n)}}{j!}(ib_r v_r^o)^j + \frac{C_{r,n}}{b_r^n n!}(ib_r v_r^o)^n\right]$$

which is always different from zero. This is a contradiction, and hence formula (17) is valid for all values of v. Writing $t = b_s v$ we finally obtain

$$f_s(t) = \exp\left[\sum_{j=1}^{n-1} \frac{\kappa_j^{(s)}}{j!}(it)^j + \frac{C_{s,n}}{b_s^n n!}(it)^n\right] . \tag{19}$$

5. Proof of the theorem. We have determined all the solutions of the system (15) satisfying the initial conditions (15a). In order to find the distribution functions whose characteristic functions satisfy this system we must select those functions (19) which are characteristic functions. This is easily done by means of the following result due to Marcinkiewicz [6].

Theorem of Marcinkiewicz.

No function of the form $e^{a_0+a_1z+\dots+a_rz^r}$ $(r>2)$ can be a characteristic function.

Hence the degree of the polynomial in (19) cannot exceed 2. In

case n>2 we must have

$$\kappa_j^{(s)} = 0 \qquad j = 3,4,\ldots,n-1; \; s = 1,2,\ldots,n \quad (n>3)$$

$$C_{s,n} = 0 \qquad (n>2); \quad s = 1,2,\ldots,n \; .$$

Because the factor $\frac{D_{s,n}}{D_n}$ is $C_{s,n}$ cannot vanish, these relations reduce to

$$(20) \qquad \kappa_j^{(s)} = 0 \qquad j = 3,\ldots,n-1 \qquad n>3$$

$$\sum_{s=1}^{n} a_s^n \kappa_n^{(s)} = 0 \qquad n>2 \; .$$

There is no restriction if n = 2. In view of (iii) $\kappa_1^{(s)} = 0$ also, and (19) becomes

$$(21) \qquad f_s(t) = \exp\left[-\frac{1}{2}\sigma_s^2 t^2\right] \qquad \text{for } n > 2 \; .$$

This shows that each X_s $(s = 1,\ldots,n)$ must be normally distributed, which is condition (A) of the theorem. All cumulants of order r>2 vanish for a normal distribution, hence equations (20) impose no additional restrictions. In case n = 2 we have

$$(22) \qquad f_s(t) = \exp\left[e^{-k/2}\, t^2\right] \qquad \text{for } n = 2 \; ,$$

where k is determined from (16) and (19). The independence of Y_1 and Y_2 implies that they are uncorrelated which yields condition (b) and completes the first part of the proof.

It is easy to establish that conditions (A) and (B) are also sufficient. Assuming that (A) and (B) hold, it follows that Y_1 and Y_2 are uncorrelated and normally distributed. Hence Y_1 and Y_2 must be independent.

For $n = 2$ and $a_1 = a_2 = b_1 = 1$, $b_2 = -1$ we obtain from (22)

$$f_s(t) = \exp\left[- \left(\frac{\sigma_1^2 + \sigma_2^2}{2} \right) t^2 \right] \qquad s = 1,2 \quad .$$

This shows that $\sigma_1^2 = \sigma_2^2$ and establishes Bernstein's theorem.

REFERENCES

[1] S. Bernstein: Sur une Proprieté Caractéristique de la Loi de Gauss, Transactions of the Leningrad Polytechnic Institute (1941).

[2] H. Cramér: Über Eine Eigenschaft Der Normalen Verteilungs-funktion, Math. Ztscht. 41, 405-414 (1936).

[3] M. Fréchet: Généralisations de la Loi de Probabilité de Laplace, Annales de L'Institut Henri Poincare, 13, 1-29 (1951).

[4] M. Kac: A Characterization of the Normal Distribution. American Journal of Mathematics, 61, 726-728 (1939)

[5] Yu. V. Linnik: Remarks Concerning the Classical Derivation of Maxwell's Law, Doklady Ak. Nauk SSSR, 85, 1251-1254 (1952).

[6] J. Marcinkiewicz; Sur une Proprieté de la Loi de Gauss, Math. Ztschr. 44, 612-618 (1938) .

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

NBS