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AN ILLUSTRATION OF COMPUTATIONAL METHODS FOR THE DETERMINATION OF THE PARAMETERS OF A CERTAIN RANDOM PROCESS*

by

Eugene Lukacs

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An Illustration of Computational Methods for the Determination of the Parameters of a Certain Random Process*

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National Bureau of Standards

1. <u>Introduction</u>: Statistical evaluation of the result of measurements is called for whenever repeated measurements of the same quantity are taken. However one encounters quite frequently situations where the physical quantity observed varies with time. We can then take a sequence of observations, but due to the variability in time we can not say that they are measurements of the same quantity. Moreover the situation can be such that it is impossible to take simultaneously several independent observations or to repeat the experiment under identical conditions. As an example we mention the motion of certain physical bodies which receive an initial impulse and are then moving freely subject only to random influences. The observed quantity would be in this case either the position or the velocity or the acceleration of the body. The motion of a wide variety of missiles belongs into this class. As another example we mention the decrease in thickness (or weight) of a shoe sole or a tire due to the natural wear over a period of time.

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In situations of this kind, the standard statistical techniques can not be applied in a natural manner. It seems therefore desirable to find an appropriate probabilistic model.

Although we can take observations only at discrete time points, it seems quite natural to suppose that a random variable (for instance the position of a moving body) is given at each instant of time. It is customary to try to make as many observations as possible, the temporal proximity of the observations as well as the nature of the physical phenomenon will in most cases prevent the observations from being stochastically independent. This is a situation which is rarely studied in the theory of the statistical evaluation of measurements. However these considerations suggest strongly that a stochastic process, depending on a continuous time parameter would be the suitable probabilistic model for such a sequence of observations.

A mathematically simple model is obtained if we assume that the actual sequence of observations comes from a Wiener process with a certain mean value function. Such a model contains several parameters, the variance constant of the process and the parameters introduced by the mean value function. Procedures have been developed [2] [3] [4] for the estimation of these parameters. The purpose of this paper is to give a brief survey of the estimation procedures, the numerical aspects of obtaining these estimates will be emphasized. Finally an example of a Wiener process with mean value function is constructed by means of random numbers. The main purpose of the present paper is to illustrate the available statistical techniques by means of this artificial process.

2. The Wiener process

A stochastic process $x(t)$, depending on a continuous time parameter, is said to be a Wiener process if

(i) $x(t)$ is a process with independent increments and initial value $x(0) = 0$.

(ii) the increment $x(t+\tau) - x(t)$ is normally distributed with mean zero and variance $c\tau$, (where $c > 0$).

The constant c is called the variance constant of the process.

The value $x(t)$ of the process can be written according to (i) as an increment, $x(t) = x(t) - x(0)$ so that we see from (ii) that the mean value function $Ex(t)$ of a Wiener process is identically zero. It is quite often desirable to study a process which has the essential simplicity of the Wiener process but possesses a non vanishing mean value function. We give therefore the following definition:

A stochastic process $y(t)$ is said to be a <u>Wiener process with mean value function</u> $f(t)$ <u>if</u>

(2.1) $$y(t) = x(t) + f(t).$$

Here $f(t)$ is a fixed real valued function of t such that $f(0) = 0$ and $x(t)$ is a Wiener process.

The process $y(t)$ is completely determined by its variance parameter and its mean value function $f(t)$. In order to be able to apply estimation procedures to the function $f(t)$ it is desirable to give it in such a form that it contains certain parameters. A fairly general assumption is that

(2.2) $$f(t) = k_1\phi_1(t) + k_2\phi_2(t)+\dots\ k_s\phi_s(t).$$

Here the functions $\phi_1(t)\dots\phi_s(t)$ are s arbitrary but completely specified functions while the coefficients $k_1\dots k_s$ are the parameters of the mean value function. The functions $\phi_1(t)\dots\phi_s(t)$ are only subject to certain restrictions*[3] which are satisfied in all cases of practical interest.

3. The estimation procedure.

We assume in the following that the $y(t)$ process - as described by (2.1) and (2.2) - is observed over a finite interval $[o,T]$ so that one sample curve is known over this interval.

We compute the following quantities

*These assumptions are: (a) All the functions $\phi(t)$ are twice differentiable. (b) For any set $a_1,\dots,a_s$ of real numbers, not all zero, we have $a_1\phi_1'(t)+\dots a_s\phi'_s(t) \neq 0$ for some t-set of positive measure which is contained in $[o,T]$.

$$(3.1)\qquad \Phi_{ij} = \int_0^T \emptyset'_i(t)\,\emptyset'_j(t)dt \qquad (i,j = 1,2,\ldots,s)$$

and then the quantities Φ^{ij} which are determined by the matrix relation

$$(3.2)\qquad ((\Phi^{ij})) = ((\Phi_{ij}))^{-1}$$

Next one has to compute the integrals

$$(3.3)\qquad \int_0^T \emptyset'_i(t)dy(t) = y(T)\emptyset'_i(T) - \int_0^T y(t)\emptyset''_i(t)dt \qquad (i=1,\ldots,s)$$

The estimates $\hat{k}_j$ of the parameters k_j are then

$$(3.4)\qquad \hat{k}_j = \sum_{v=1}^{s} \Phi^{jv} \int_0^T \emptyset'_v(t)dy(t) \qquad (j=1,2,\ldots,s)$$

The estimates $\hat{k}_j$ $(j=1,2,\ldots,s)$ are limits of maximum likelihood estimates computed from observations taken over a finite set of points. It can be shown that under rather general conditions the estimates $\hat{k}_j$ are unbiased estimates of k_j and that the covariance of $\hat{k}_i$ and $\hat{k}_j$ is given by $\sigma(\hat{k}_i\hat{k}_j) = c\,\Phi^{ij}$.

It should be emphasized that the estimates $\hat{k}_j$ are not maximum likelihood estimates and that the estimated mean value

$$(3.5)\qquad \hat{f}(t) = \hat{k}_1\,\emptyset_1(t) + \ldots + \hat{k}_s\,\emptyset_s(t)$$

is not a least square fit of the data. Moreover it can be shown that the mean value curve $\hat{f}(t)$ has another optimal property. We say that $\hat{f}(t)$ is a best linear estimate of $f(t)$ if

(a) $E\hat{f}(t) = f(t)$ (i.e. $\hat{f}(t)$ is an unbiased estimate)

(b) $E\int_0^T [f(t) - \hat{f}(t)]^2 dt \leq E\int_0^T [f(t) - \tilde{f}(t)]^2 dt$

where $\tilde{f}(t)$ is an estimate, $\tilde{f}(t) = \sum_{v=1}^{s} \tilde{k}_v \, \emptyset_v(t)$ such that

$$\tilde{k}_v = \int_0^T \psi_v(t)dy(t) \text{ and also } E(\tilde{k}_v) = k_v.$$

The estimate (3.5) is a best linear estimate in this sense.

We finally give an estimate for the variance constant.

We divide the interval [0, T] into N equal parts of length $\tau = \frac{T}{N}$ and denote the subdivision points by $t_i = i\tau$ ($i = 0, 1, 2, \ldots, N$). The quantity

$$(3.6) \quad \hat{c} = \frac{1}{T - s\tau} \sum_{n=1}^{N} [y(t_n) - y(t_{n-1}) - \hat{f}(t_n) + \hat{f}(t_{n-1})]^2$$

can be used as an estimate for the variance constant c. This estimate is slightly biased but its bias converges to zero as N increases.

We still have to choose the functions $\emptyset_1(t), \emptyset_2(t), \ldots, \emptyset_s(t)$. In this report we consider only the case in which the $\emptyset_j(t)$ are polynomials. However other choices are possible. For instance it might be advantageous to choose the $\emptyset_j(t)$ as trigonometric functions when phenomena with a definite periodicity are studied.

As in least squares theory it seems also here to be convenient to use orthogonal polynomials. The purpose of introducing orthogonal polynomials is in the present case the reduction of the matrix $((\Phi_{ij}))$ to a diagonal matrix. From (3.1) we see that it is then necessary to assume that the derivatives $\phi'_j(t)$ are the first s polynomials of a system orthogonal with respect to the weight function 1 over an interval $[0,T]$. It follows then that the functions $\phi_j(t)$ become the integrals of Legendre polynomials adapted to the interval $[0,1]$. As a consequence the mean value function must have initial and terminal value zero, i.e.

$$f(0) = f(T) = 0 \; . \tag{3.7}$$

Condition (3.7) will not be fulfilled in general. Moreover, it is not possible to enforce this condition by a rotation of the axis. In the situation under consideration such a rotation would change the character of the process and prevent it from being a Wiener process. This means that in general it will not be possible to use orthogonal polynomials to estimate the mean value function of a Wiener process. The use of orthogonal polynomials is restricted to certain physical situations where the quantity measured starts at a certain level which is reached again at the end of the period of observation.

In the general case (3.7) is not satisfied and it is then convenient to choose for the $\phi_j(t)$ consecutive powers of t.

We assume therefore

$$\begin{cases} \phi_j(t) = t^j & (j = 1,2,\ldots,s) \\ f(t) = k_1 t + k_2 t^2 + \ldots + k_s t^s & \end{cases} \tag{3.8}$$

We obtain therefore from (3.1), (3.3) and (3.4)

$$\Phi_{ij} = \frac{ij\, T^{i+j-1}}{i+j-1} \tag{3.9}$$

$$\left\{\begin{aligned} &\int_0^T t^{v-1}dy(t) = T^{i-1}y(T) - (i-1)\int_0^T y(t)t^{i-2}dt \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } i = 2,\ldots,s \\ &\hat{k}_j = \sum_{v=1}^{s} \Phi^{jv}\, v \int_0^T t^{v-1}dy(t) \end{aligned}\right. \tag{3.10}$$

It is easy to show that

$$\Phi^{jv} = \frac{S^{jv}}{j\,v\,T^{j+v-1}} \tag{3.11}$$

where the matrix $\|S^{jv}\|$ is the inverse of the matrix $\left\|\frac{1}{j+v-1}\right\|_{j,v=1,\ldots,s}$.

The computational work involved in this estimation procedure can be greatly reduced by providing tables of the matrices $\|S^{jv}\|$. These are inverses of finite segments of the Hilbert matrix and were tabulated for segments up to and including order 10. These tables [5] should be used whenever a polynomial of degree not exceeding 10 is the mean value function. The scope of these tables will be sufficient in most cases. It should be remarked that with the use of these tables the estimation of the mean value function is by no means more laborious than the computation of a least square fit to the same data. One could even say that the estimation of the mean value function is in some respects more convenient than a least squares fit. This is due to the fact that the table to be used for estimating the mean value curve depends only on the degree of the polynomials but is independent of the number of points observed in $[0,T]$. The tables for fitting orthogonal polynomials by least squares depend on the contrary on the degree of the polynomials as well as on the number of points.

4. Construction of an example.

We next construct an example to which we will apply the technique discussed in the preceding section. We use random numbers to construct "data" which simulate observations from a Wiener process with mean value function.

We choose the following polynomial of degree four

$$f(t) = 3400 + 310t - 2.7t^2 + 43\ 10^{-3}.t^3 - 2.6\times10^{-4}t^4 \tag{4.1}$$

This polynomial will represent the mean value function of our fictitious process. We assume that $T = 100$ and that we take observations at all integer values $0 \leq n \leq 100$. We compute therefore the values of $f(n)$, $n = 0(1)100$. In order to obtain the simulated observations we add random numbers as "errors" to the values $f(n)$. These random numbers were obtained from H. Wold's table [6]. These tables contain random normal deviates representing a normal population of zero mean and unit variance. They are arranged in colums of 50 and the sums $\sum(x)$ of each column is also given. We still must select the variance constant c. In view of the arrangement of Wold's tables it was decided to choose

$$c = \frac{50}{3} \quad (4.2)$$

Random numbers from a normal population with zero mean and variance $c = \frac{50}{3}$ were obtained by dividing the values $\sum(x)$ in Wold's table by $\sqrt{3}$. A set of 100 random numbers (denoted in the following by w_i) was derived in this manner by taking 100 columns in Wold's table. We started with the first column and used all consecutive columns with the exception of the random numbers on page 7. These were skipped in agreement with the warning given in the tables. The sum $\sum_{i=1}^{n} w_i$ ($n = 0,1,2,\ldots,100$) with $w_0 = 0$ were computed and added as "errors" to the "true values". In this manner

$$g(n) = f(n) + \sum_{i=0}^{n} w_i \qquad (i = 0,1,\ldots,100) \quad (4.3)$$

was computed. These values of g(n) are the simulated observations. We will apply in the following our estimation procedure to these simulated observations. Since we wish to use a Wiener process as our model we have to adjust the observations to make the initial value equal to zero. This can be accomplished by subtracting $g(0)$ from all observations. This leaves the increments unchanged and does therefore not affect the character of the process. We denote by

$$y(n) = g(n) - g(0). \tag{4.4}$$

Table 1 gives for each n the values of $f(n)$, $\sum_{i=1}^{n} w_i$, $g(n)$ and $y(n)$.

n	f(n)	$\sum_{i=0}^{n} w_i$	g(n)	y(n)
0	3400.00	0.00	3400.00	0.00
1	3707.34	+6.05	3713.39	313.39
2	4009.54	+7.02	4016.56	616.56
3	4306.84	+2.56	4309.40	909.40
4	4599.49	-0.41	4599.08	1199.08
5	4887.71	-3.74	4883.97	1483.97
6	5171.75	-11.26	5160.49	1760.49
7	5451.82	-14.59	5437.23	2037.23
8	5728.15	-11.66	5716.49	2316.49
9	6000.94	-13.71	5987.23	2587.23
10	6270.40	-13.70	6256.70	2856.70
11	6536.73	-12.81	6523.92	3123.92
12	6800.11	-12.26	6787.85	3387.85
13	7060.75	-13.22	7047.53	3647.53
14	7318.80	- 4.09	7314.71	3914.71
15	7574.46	- 5.67	7568.79	4168.79
16	7827.89	- 6.09	7821.80	4421.80
17	8079.24	- 2.63	8076.61	4676.61
18	8328.68	- 3.20	8325.48	4925.48
19	8576.35	- 0.32	8576.03	5176.03
20	8822.40	- 0.21	8822.19	5422.19
21	9066.96	+ 7.93	9074.89	5674.89
22	9310.16	+ 9.19	9319.35	5919.35
23	9552.12	+ 8.69	9560.81	6160.81
24	9792.97	+ 4.34	9797.31	6397.31
25	10032.81	+ 5.51	10038.32	6638.32
26	10271.75	+ 6.33	10278.08	6878.08
27	10509.89	+ 6.44	10516.33	7116.33
28	10747.33	+13.89	10761.22	7361.22
29	10984.13	+18.07	11002.20	7602.20
30	11220.40	+28.26	11248.66	7848.66
31	11456.19	+27.28	11483.47	8083.47
32	11691.59	+22.54	11714.13	8314.13
33	11926.65	+22.92	11949.57	8549.57
34	12161.42	+29.42	12190.84	8790.84
35	12395.96	+30.64	12426.60	9026.60
36	12630.31	+33.62	12663.93	9263.93
37	12864.50	+30.43	12894.93	9494.93
38	13098.56	+31.53	13130.09	9730.09
39	13332.52	+22.28	13354.80	9954.80
40	13566.40	+19.28	13585.68	10185.68

n	f(n)	$\sum_{i=0}^{n} w_i$	g(n)	y(n)
40	13566.40	19.28	13585.68	10185.68
41	13800.20	24.97	13825.17	10425.17
42	14033.94	24.01	14057.95	10757.95
43	14267.61	34.91	14302.52	10902.52
44	14501.21	39.54	14540.75	11140.75
45	14734.71	37.63	14772.34	11372.34
46	14968.11	39.20	15007.31	11607.31
47	15201.37	38.24	15239.61	11839.61
48	15434.48	41.34	15475.82	12075.82
49	15667.38	43.40	15710.77	12310.77
50	15900.00	41.94	15941.94	12541.94
51	16132.34	39.91	16172.25	12772.25
52	16364.32	46.44	16410.76	13010.76
53	16595.89	54.57	16650.46	13250.46
54	16826.96	51.10	16878.06	13478.06
55	17057.46	44.44	17101.90	13701.90
56	17287.32	47.63	17334.95	13934.95
57	17516.44	48.87	17565.31	14165.31
58	17744.73	51.55	17796.28	14396.28
59	17972.07	55.07	18027.14	14627.14
60	18198.40	55.15	18253.55	14853.55
61	18423.56	56.79	18480.35	15080.35
62	18647.46	58.08	18705.54	15305.54
63	18869.95	64.22	18934.17	15534.17
64	19090.92	65.90	19156.81	15756.81
65	19310.21	64.83	19375.04	15975.04
66	19527.70	65.97	19593.67	16193.67
67	19743.22	65.96	19809.18	16409.18
68	19956.62	65.29	20021.91	16621.91
69	20167.74	67.54	20235.28	16835.28
70	20376.40	74.56	20450.96	17050.96
71	20582.44	78.49	20660.93	17260.93
72	20785.66	78.65	20864.31	17464.31
73	20985.89	79.59	21065.48	17665.48
74	21182.92	81.32	21264.24	17864.24
75	21376.56	83.12	21459.68	18059.68
76	21566.60	84.18	21650.78	18250.78
77	21752.83	84.83	21837.66	18437.66
78	21935.02	81.67	22016.69	18616.69
79	22112.96	86.44	22199.40	18799.40
80	22286.40	82.16	22368.56	18968.56

n	f(n)	$\sum_{i=0}^{n} w_i$	g(n)	y(n)
80	22286.40	82.16	22368.56	18968.56
81	22455.12	81.37	22536.49	19136.49
82	22618.86	80.44	22699.30	19299.30
83	22777.38	84.07	22861.45	19461.45
84	22930.42	82.34	23012.76	19612.76
85	23077.71	82.09	13159.80	19759.80
86	23219.00	78.16	23297.16	19897.16
87	23353.99	73.10	23427.09	20027.09
88	23482.42	74.72	23557.14	20157.14
89	23603.98	75.86	23679.84	20279.84
90	23718.40	84.38	23802.78	20402.78
91	23825.36	86.21	23911.57	20511.57
92	23924.57	86.97	24011.54	20611.54
93	24015.70	91.86	24107.56	10707.56
94	24098.44	90.60	24189.04	20789.04
95	24172.46	81.73	24254.19	20854.19
96	24237.44	78.38	24315.82	20915.82
97	24293.03	80.21	24373.24	20973.24
98	24338.88	80.01	24418.89	21018.89
99	24374.66	83.05	24457.71	21057.71
100	24400.00	79.30	24479.30	21079.30

5. Computation of the estimated mean value curve and of the variance constant

We next compute the estimate of the mean value function using formulae (3.9), (3.10) and (3.11) in order to obtain the integral (3.10) we have to compute the integrals $\int_0^T y(t)\, t^{i-2} dt$ for $i = 2, \ldots, s$. These integrals were evaluated by means of Simpson's rule

$$(5.1) \qquad \int_{x_0}^{x_0+nh} y\, dx = \frac{h}{3}[y_0+4(y_1+y_s+\ldots+y_{n-1})+2(y_2+y_4+\ldots+y_{n-2}) + y_n]$$

where n is an even number.

The results were compared in all cases to the approximation by summation and for $i = 5$ also to the use of a seven point Lagrangian interpolation formula [1]. It was found that results obtained by the seven-point formula and Simpson's rule differ only little while straight summation would be too crude an approximation to the integral. As an example, we mention that $\int_0^T y(t)t^3 dt = 461{,}157{,}455{,}032$ by Simpson's rule, while 461,160,141,829 was obtained from the 7-point formula and 471,752,177,029 by straight summation. The integrals $\int_0^T y(t)t^{i-2}dt$ were therefore computed by (5.1), the values are given in table 2.

Table 2

i	$\int_0^T y(t)t^{i-2}\,dt$
2	1,209,256.51
3	78,807,137.03
4	5,825,790,539.54
5	461,157,455,032.19
6	38,111,464,889,820.59

We next compute from table 2 the values $a_i = iy(T)T^{2-1} - -i(i-1)\int_0^T y(t)t^{i-2}dt$ for $i = 2,\ldots,6$ while $a_1 = y(T) = y_{100}$. These are given in table 3.

Table 3

i	a_i
1	2.10793×10^4
2	1.79735×10^6
3	1.59536×10^8
4	1.44077×10^{10}
5	1.31650×10^{12}
6	1.21414×10^{14}

We finally compute form (3.10) the estimates $\hat{k}_j$ of the coefficients of the mean value function and obtain $\hat{k}_j = \sum_{v=1}^{s} \Phi^{jv} a_v$ for $j = 1,\ldots,s$. The elements Φ^{jv} of the inverse matrix were taken from the tables [5] mentioned above. The coefficients $\hat{k}_j$ were computed for mean value function of various degrees, using always the same "data" to be fitted. The results of these computations are given in table 4.

Table 4

Coefficients $\hat{k}_j$ for estimating mean value curves of degree s

j	s=3	s=4	s=5	s=6
1	257.267	308.773	307.495	300.785
2	46.962×10^{-2}	-2.6211	-2.4932	-1.4868
3	-9.34×10^{-3}	4.2166×10^{-2}	3.8332×10^{-2}	-8.638×10^{-3}
4		-2.5753×10^{-4}	-2.128×10^{-4}	7.266×10^{-4}
5			-1.79×10^{-7}	-8.63352×10^{-6}
6				2.8182×10^{-8}

Using these coefficients the values $\hat{f}_s(t)$ of the estimated mean value curve of degree s were computed for t=1 and t=10(10)100 and s = 3,4,5,6. In addition the mean value estimate $\hat{f}_4(t)$ of degree 4 was computed for t=0(1)100 and a least square fit to the simulated observations was also determined. A comparison of these curves given in table 5, while table 6 permits to compare the 4-th degree estimate with the 4-th degree least squares fit. We denote by $\hat{f}_4(t)$ the 4-th degree mean value estimate and by $\tilde{f}_4(t)$ the 4-th degree least square fit. If we write again y(t) for the (adjusted) simulated observations then we see from table 6 that

$$\sum_{t=0}^{100} [\hat{f}_4(t)-y(t)]^2 = 3351.71$$

while

$$\sum_{t=0}^{100} [\tilde{f}_4(t)-y(t)]^2 = 3166.90$$

Table 5

Comparison between the coordinates, the observations and the true values

t	"true values" $f(t)-f(0)$	"Simulated observations" $y(t)$	Least squares fit $f_4(t)$
1	307.34	313.39	303.52
10	2870.40	2856.70	2865.12
20	5422.40	5422.19	5423.79
30	7820.40	7848.66	7834.18
40	10166.40	10185.68	10195.57
50	12500.00	12541.94	12545.23
60	14798.40	14853.55	14858.43
70	16976.40	17050.96	17048.42
80	18886.40	18968.56	18966.44
90	20318.40	20402.78	20401.75
100	21000.00	21079.30	21081.58

Estimates of mean value curve of degree

t	3	4	5	6
1	257.73	306.19	305.04	299.29
10	2610.26	2865.21	2861.83	2991.90
20	5258.32	5423.14	5424.66	5442.31
30	7888.16	7834.08	7839.22	7851.25
40	10443.74	10196.58	10200.83	10191.15
50	12869.00	12547.08	12547.31	12526.12
60	15107.92	14860.69	14856.81	14846.98
70	17104.44	17050.36	17045.67	17057.57
80	18802.54	18967.36	18966.27	18983.58
90	20146.18	20401.13	20404.87	20399.69
100	21082.70	21079.30	21079.50	21079.30

Table 6

t	$y(t)$	$\hat{f}_4(t)$	$\tilde{f}_4(t)$
0	0	0	-3.11
1	313.39	306.19	303.52
2	616.56	607.39	605.13
3	909.40	903.85	901.95
4	1199.08	1195.79	1194.23
5	1483.97	1483.45	1482.21
6	1760.49	1767.05	1766.09
7	2037.23	2046.82	2046.12
8	2316.49	2322.96	2322.50
9	2587.23	2595.70	2595.43
10	2856.70	2865.21	2865.12
11	3123.92	3131.70	3131.77
12	3387.85	3395.36	3395.57
13	3647.53	3656.37	3656.69
14	3914.71	3914.90	3915.32
15	4168.79	4171.12	4171.62
16	4421.80	4425.20	4425.76
17	4676.61	4677.30	4677.90
18	4925.48	4927.56	4928.19
19	5176.03	5176.13	5176.77
20	5422.19	5423.14	5423.79
21	5674.89	5668.74	5669.38
22	5919.35	5913.05	5913.66
23	6160.81	6156.18	6156.76
24	6397.31	6398.27	6398.79
25	6638.32	6639.40	6639.86
26	6878.08	6879.67	6880.08
27	7116.33	7119.20	7119.53
28	7361.22	7358.06	7358.31
29	7602.20	7596.33	7596.50
30	7848.66	7834.08	7834.18
31	8083.47	8071.42	8071.43
32	8314.13	8308.42	8308.30
33	8549.57	8545.08	8544.85
34	8790.84	8781.48	8781.14
35	9026.60	9017.66	9017.22
36	9263.93	9253.68	9253.12
37	9494.93	9489.55	9488.88
38	9730.09	9725.32	9724.53
39	9954.80	9960.99	9960.09
40	10185.68	10196.58	10195.57

t	y(t)	$f_4(t)$	$f_4(t)$
40	10185.68	10196.58	10195.57
41	10425.17	10432.11	10430.99
42	10657.95	10667.58	10666.34
43	10902.52	10902.98	10901.63
44	11140.75	11138.30	11136.84
45	11372.34	11373.52	11371.96
46	11607.31	11608.64	11606.97
47	11839.61	11843.60	11841.84
48	12075.82	12078.39	12076.54
49	12310.77	12312.97	12311.02
50	12541.94	12547.09	12545.23
51	12772.25	12781.06	12779.14
52	13010.76	13014.66	13012.67
53	13250.46	13247.81	13245.77
54	13478.06	13480.45	13478.36
55	13701.90	13712.50	13710.36
56	13934.95	13943.86	13941.69
57	14165.31	14174.47	14172.26
58	14396.28	14404.21	14401.98
59	14627.14	14632.98	14630.74
60	14853.55	14860.69	14858.43
61	15080.35	15087.20	15084.94
62	15305.54	15312.41	15310.20
63	15534.17	15536.17	15533.93
64	15756.81	15758.37	15756.15
65	15975.04	15978.86	15976.67
66	16193.67	16197.50	16195.34
67	16409.18	16414.13	16412.01
68	16621.91	16628.59	16626.52
69	16835.28	16840.73	16838.72
70	17050.96	17050.36	17048.42
71	17260.93	17257.32	17255.45
72	17464.31	17461.42	17459.63
73	17665.48	17662.48	17660.77
74	17864.24	17860.29	17858.68
75	18059.68	18054.66	18053.14
76	18250.78	18245.38	18243.97
77	18437.66	18432.23	18430.93
78	18616.69	18614.99	18613.82
79	18799.40	18793.45	18792.40
80	18968.56	18967.36	18966.44

t	y(t)	$f_4(t)$	$f_4(t)$
80	18968.56	18967.36	18966.44
81	19136.49	19136.50	19135.71
82	19299.30	19300.60	19299.96
83	19461.45	19459.43	19458.93
84	19612.76	19612.73	19612.38
85	19759.80	19760.23	19760.03
86	19897.16	19901.66	19901.62
87	20027.09	20036.75	20036.93
88	20157.14	20165.22	20165.51
89	20279.84	20286.78	20287.23
90	20402.78	20401.13	20401.75
91	20511.57	20507.98	20508.77
92	20611.54	20607.02	20607.98
93	20707.56	20697.93	20699.06
94	20789.04	20780.40	20781.70
95	20854.19	20854.10	20855.57
96	20915.82	20918.71	20920.34
97	20973.24	20973.88	20975.68
98	21018.89	21019.26	21021.23
99	21057.71	21054.52	21056.65
100	21079.30	21079.30	21081.58

We still have to estimate the variance constant. This is done for the estimate $\hat{f}_4(t)$ by means of (3.6) and the data contained in table 6. We obtain the estimate $\hat{c} = 15.051$ as compared with the "true value" $c = 50/3$.

We also give four figures which permit to visualize the relative errors of our estimates. Figures 1 and 2 compare the estimating polynomials of degrees 3,4,5 and 6. In figure 1 the graphs of

$$\hat{R}_4(t) = \frac{\hat{f}_4(t) - f(t)}{f(t)} \quad \text{and of} \quad \tilde{R}_4(t) = \frac{\tilde{f}_4(t) - f(t)}{f(t)}$$

are given, in figure 2 we find the corresponding errors

referred to the observations, i.e. the curves $\hat{r}_4(t) = $
$$= \frac{\hat{f}_4(t) - y(t)}{y(t)} \quad \text{and} \quad \tilde{r}_4(t) = \frac{\tilde{f}_4(t) - y(t)}{y(t)} \quad .$$

In figures 3 and 4 we compare the error of the estimating polynomials of degress 3,4,5 and 6, the basis of reference is in figure 3 the curve f(t) in figure 4 the curve y(t).

6. <u>Conclusion</u>. In actual applications we will often encounter situations where more than a single sample curve of the process are given. In these cases a number of problems can arise. It might for instance be necessary to test the hypothesis whether two or more sample curves come from processes with the same variance constant or with the same mean value function. Due to the fact that increments from a Wiener process behave as if they were independent observations from a normal population many of the questions which might arise can be treated by standard statistical techniques. To go into such a discussion would however exceed the scope of this paper.

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FIG. 1

$100\ \hat{R}_4(t)$ and $100\ \tilde{R}_4(t)$

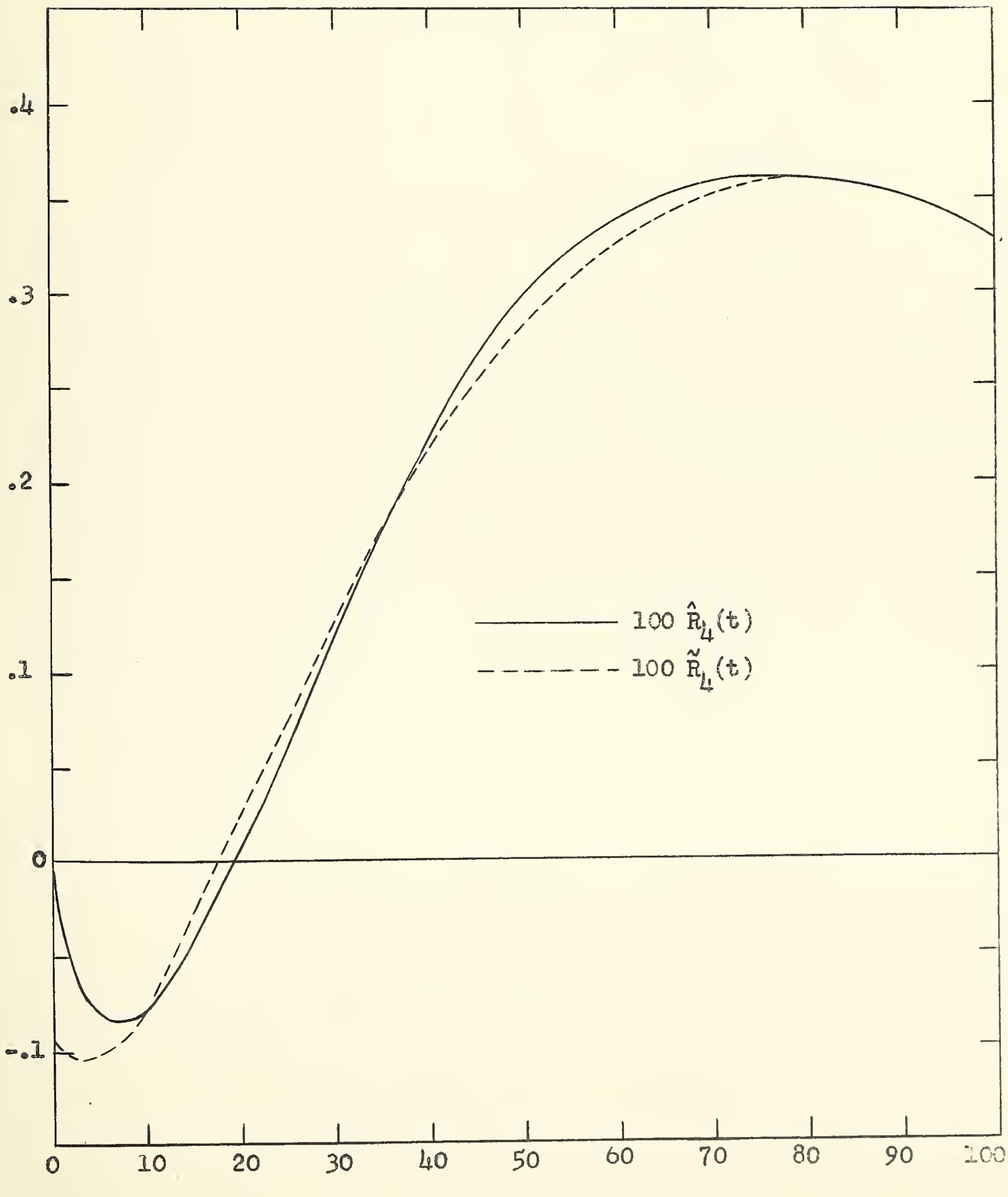


FIG. 2

100 $\hat{r}_4(t)$ and 100 $\tilde{r}_4(t)$

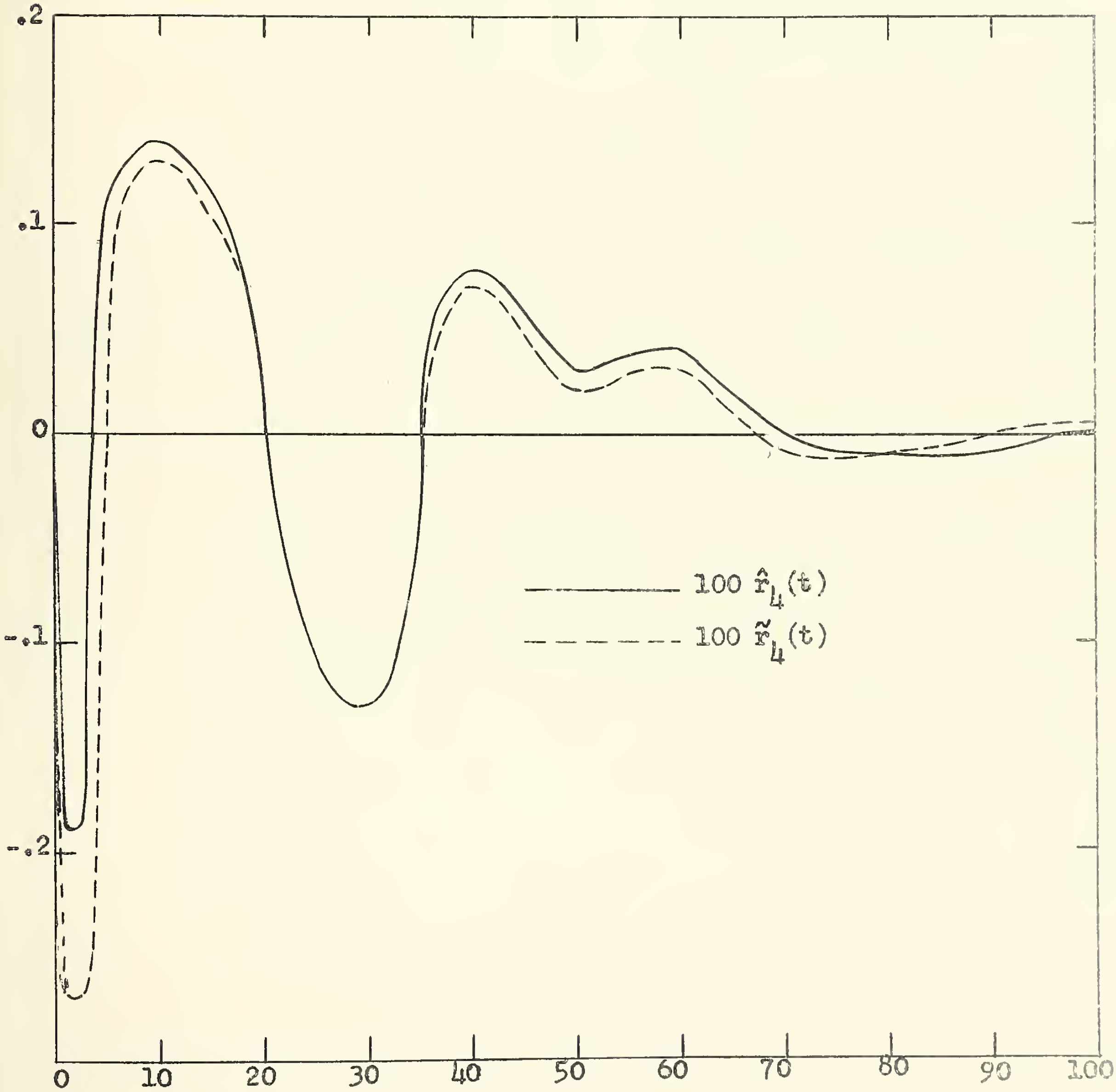


FIG. 3

$100\,\hat{R}_j(t)$

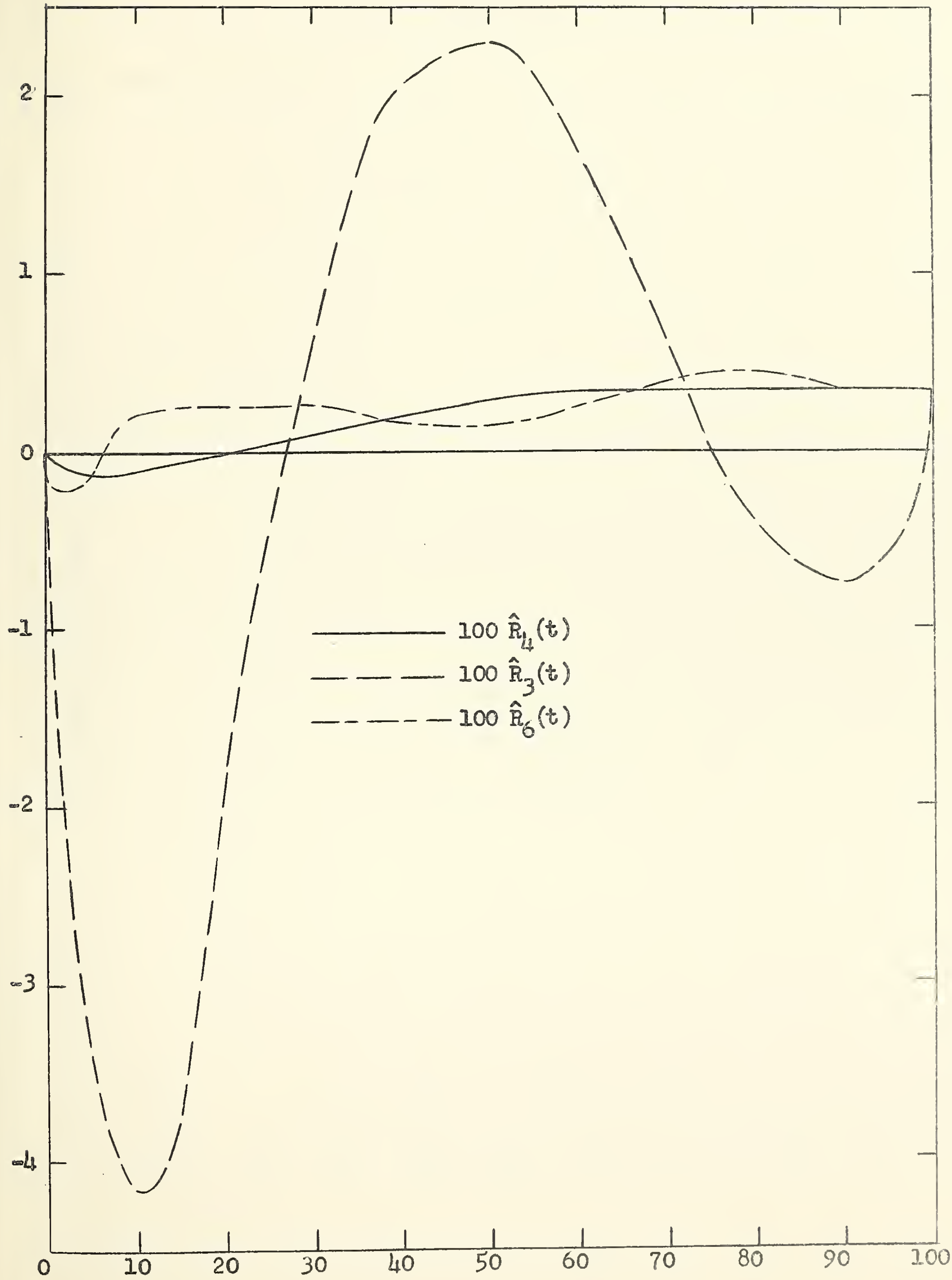
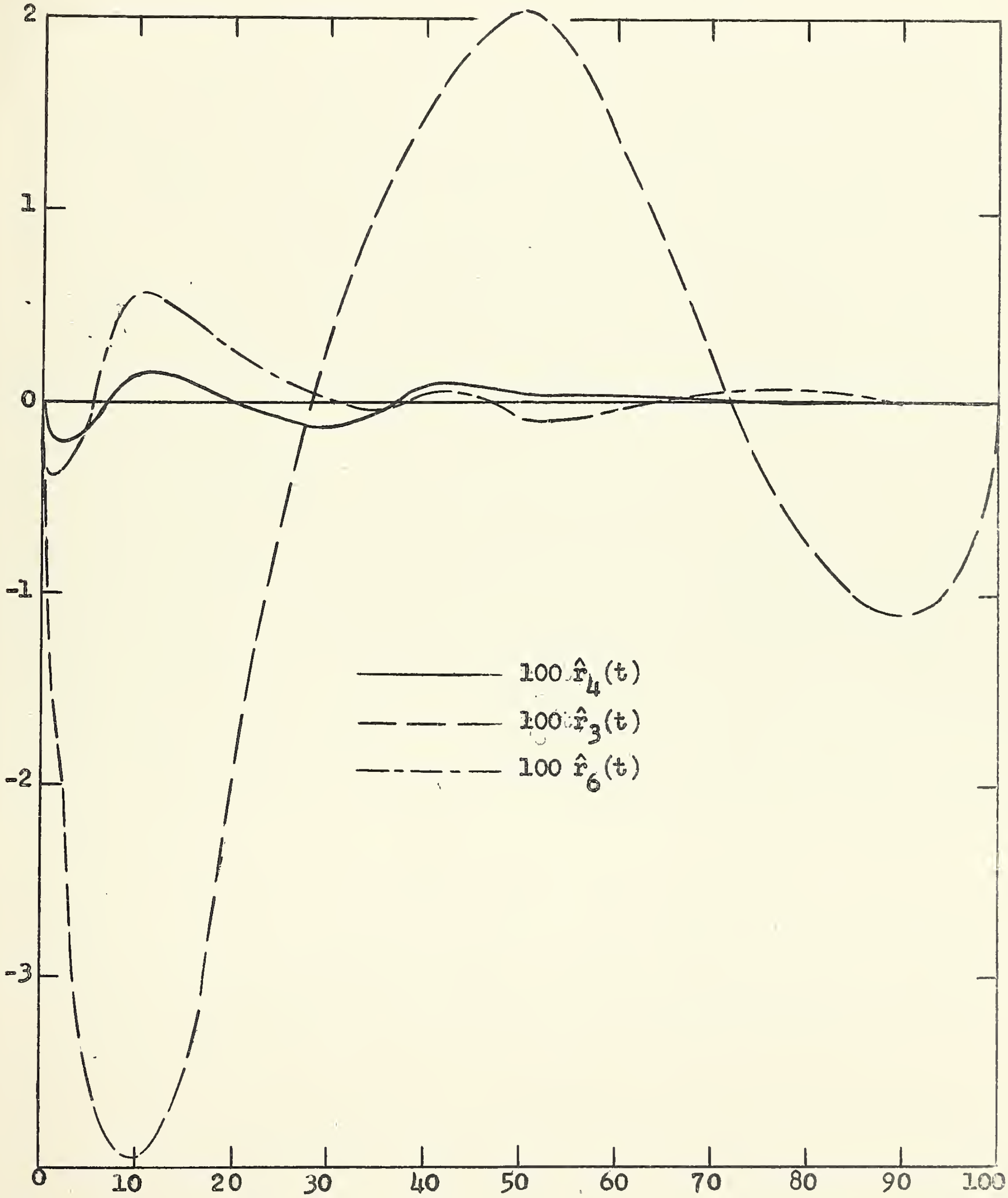


FIG. 4

$100\,\hat{r}_j(t)$



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