NATIONAL BUREAU OF STANDARDS REPORT

2489

ASYMPTOTIC SOLUTION OF VAN DER POL'S EQUATION

by

A. A. Dorodnicyn [Dorodnitsyn]

Translated from the Russian by Curtis D. Benster

Edited by

Wolfgang R. Wasow

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

U. S. DEPARTMENT OF COMMERCE
Sinclair Weeks, *Secretary*

NATIONAL BUREAU OF STANDARDS
A. V. Astin, *Director*

THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.

Electricity. Resistance Measurements. Inductance and Capacitance. Electrical Instruments. Magnetic Measurements. Applied Electricity. Electrochemistry.

Optics and Metrology. Photometry and Colorimetry. Optical Instruments. Photographic Technology. Length. Gage.

Heat and Power. Temperature Measurements. Thermodynamics. Cryogenics. Engines and Lubrication. Engine Fuels. Cryogenic Engineering.

Atomic and Radiation Physics. Spectroscopy. Radiometry. Mass Spectrometry. Solid State Physics. Electron Physics. Atomic Physics. Neutron Measurements. Infrared Spectroscopy. Nuclear Physics. Radioactivity. X-Rays. Betatron. Nucleonic Instrumentation. Radiological Equipment. Atomic Energy Commission Instruments Branch.

Chemistry. Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Gas Chemistry. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

Mechanics. Sound. Mechanical Instruments. Aerodynamics. Engineering Mechanics. Hydraulics. Mass. Capacity, Density, and Fluid Meters.

Organic and Fibrous Materials. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Organic Plastics. Dental Research.

Metallurgy. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion.

Mineral Products. Porcelain and Pottery. Glass. Refractories. Enameled Metals. Concreting Materials. Constitution and Microstructure. Chemistry of Mineral Products.

Building Technology. Structural Engineering. Fire Protection. Heating and Air Conditioning. Floor, Roof, and Wall Coverings. Codes and Specifications.

Applied Mathematics. Numerical Analysis. Computation. Statistical Engineering. Machine Development.

Electronics. Engineering Electronics. Electron Tubes. Electronic Computers. Electronic Instrumentation.

Radio Propagation. Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Frequency Utilization Research. Tropospheric Propagation Research. High Frequency Standards. Microwave Standards.

Ordnance Development.
Electromechanical Ordnance.
Ordnance Electronics.
These three divisions are engaged in a broad program of research and development in advanced ordnance. Activities include basic and applied research, engineering, pilot production, field testing, and evaluation of a wide variety of ordnance matériel. Special skills and facilities of other NBS divisions also contribute to this program. The activity is sponsored by the Department of Defense.

Missile Development. Missile research and development: engineering, dynamics, intelligence, instrumentation, evaluation. Combustion in jet engines. These activities are sponsored by the Department of Defense.

- Office of Basic Instrumentation
- Office of Weights and Measures.

NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT | **NBS REPORT**

1101-10-5100 April 30, 1953 2489

ASYMPTOTIC SOLUTION OF VAN DER POL'S EQUATION*

by

A. A. Dorodnicyn [Dorodnitsyn]

Translated from the Russian by Curtis D. Benster

Edited by

Wolfgang R. Wasow

National Bureau of Standards, Los Angeles

[redacted]

*This translation was sponsored in part by the Office of Naval Research, USN.

NBS

ASYMPTOTIC SOLUTION OF VAN DER POL'S EQUATION[1]

A. A. Dorodnicyn [Dorodnitsyn]

Moscow

1. Statement of the problem. In this article is considered the solution of Van der Pol's equation

(1.1) $$\frac{d^2x}{dt^2} - \nu(1 - x^2)\frac{dx}{dt} + x = 0$$

for large values of the parameter ν.

In the phase plane xp, equation (1.1) is transformed into the form

(1.2) $$pp' - \nu(1 - x^2)p + x = 0 \qquad \left(p = \frac{dx}{dt}\right)$$

where the prime sign denotes differentiation with respect to x.

The solution of this equation has the character schematically represented in Fig. 1 (for the limit cycle).

It is known that in domain I and in domain III the solution of equation (1.2) tends to the solutions of the "shortened" equations

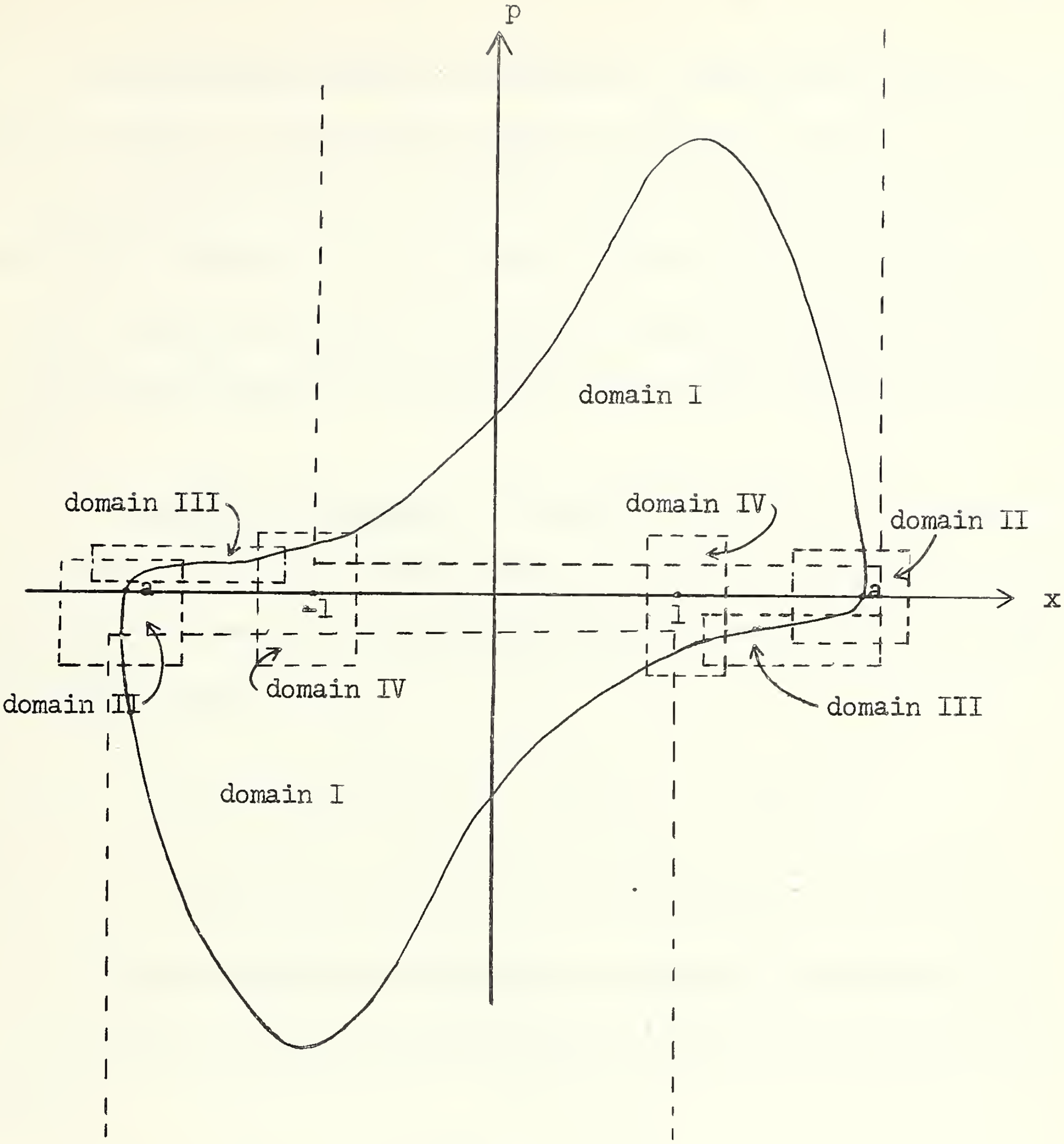
(1.3) $$pp' - \nu(1 - x^2)p = 0$$

(1.4) $$-\nu(1 - x^2)p + x = 0$$

respectively.

---

[1]Prikladnaîa matematika i mekhanika. Vol. XI, 1947, pp. 313-328.



The domains in which these two limit solutions are suitable are not, however, contiguous, and therefore these solutions cannot be linked. We do not know how to choose the constant of integration in equation (1.3) so that for the analytic continuation of the solution into domain III this solution shall pass into that which tends to the solution of the second equation, (1.4).

In the present paper we introduce two "connecting" domains II and IV, for which are established singular asymptotic solutions of equation (1.2), different from the solutions of the "shortened" equations (1.3) and (1.4). Domains I, II, III, and IV overlap each other, the possibility being thus obtained of finding the complete solution for the entire cycle accurate to quantities of any order of smallness relative to ν.

2. Solution for domain I. Denoting by a_1 and a_2 the values of x for which $p = 0$ (for the limit cycle $a_1 = a_2 = a$, where a is the amplitude of the steady-state auto-oscillations), we will define two parts of domain I thus:

$$-1 + \epsilon < x < a_1 - \epsilon \quad , \quad p > 0 \quad , \quad \epsilon > 0 \quad ;$$

$$-a_2 + \epsilon < x < 1 - \epsilon \quad , \quad p < 0 \quad , \quad \epsilon > 0 \quad .$$

It will obviously be sufficient to consider the solution in one of the parts of domain I, say the first. We seek the solution in the form

$$p = \nu \sum_{n=o}^{\infty} f_n(x) \nu^{-2n} \quad . \tag{2.1}$$

Substituting (2.1) in (1.2) and equating the coefficients of like powers of ν, we obtain the recurrent system of equations:

$$f_o' = 1 - x^2 \quad , \quad f_o f_1' = -x \quad ; \quad -f_o f_2' = f_1 f_1', \ \ldots \ ,$$

$$f_o f_{n+1}' = - \sum_{k=1}^{n} f_k f_{n+1-k}' \tag{2.2}$$

whose solution is elementary. Thus for the first two functions we have

(2.3) $$f_o(x) = c + x - \frac{1}{3}x^3$$

(2.4)
$$f_1(x) = \frac{x_1}{x_1^2 - 1}\left[\log\left(1 - \frac{x}{x_1}\right) - \frac{1}{2}\log\frac{(2x + x_1)^2 + 3(x_1^2 - 4)}{4(x_1^2 - 3)}\right]$$
$$+ \frac{x_1^2 - 2}{x_1^2 - 1}\sqrt{\frac{3}{x_1^2 - 4}}\left[\text{arc tg}\,\frac{2x + x_1}{\sqrt{3(x_1^2 - 4)}} - \text{arc tg}\,\frac{x_1}{\sqrt{3(x_1^2 - 4)}}\right] .$$

Here, in (2.4), by x_1 is denoted the real (positive) root of the equation $f_o(x) = c + x - \frac{x^3}{3} = 0$, and it is assumed that $c > 2/3$ (which holds, for example, for the limit cycle).

The functions $f_n(x)$ have a singularity in the neighborhood of the point $x = x_1$. From system (2.2) it is not hard to discover the character of these singularities, namely:

(2.5) $$f_n(x) \sim \frac{[\log(x - x_1)]^{n-1}}{(x - x_1)^{n-1}} .$$

Hence it follows that series (2.1) preserves its asymptotic character up to values of x satisfying the condition $O(x_1 - x) > O(\log \nu/\nu^2)$.

In particular, series (2.1) is an asymptotic series for $x = x_1 - O\left(\frac{1}{\nu}\right)$. Here p will be of the order of unity. We shall utilize this in what is to follow.

We exhibit the expansions of the first three functions $f_n(x)$ in the neighborhood of x_1:

$$f_o(x) = -(x_1^2 - 1)(x - x_1) - x_1(x - x_1)^2 - \frac{1}{3}(x - x_1)^3$$

$$f_1(x) = \frac{x_1}{x_1^2 - 1} \log \left(1 - \frac{x}{x_1}\right) + g - \frac{1}{(x_1^2 - 1)^2}(x - x_1)$$

$$- \frac{x_1(x_1^2 - 4)}{6(x_1^2 - 1)^3}(x - x_1)^2 + \frac{x_1^4 - 3x_1^2 - 1}{9(x_1^2 - 1)^4}(x - x_1)^3 \cdots$$

(2.6)

$$f_2(x) = - \frac{x_1^2}{(x_1^2 - 1)^3} \log \left(1 - \frac{x}{x_1}\right) \left[\frac{1}{x - x_1} + \frac{1 + g(1 + x_1^2)}{x_1^2(x_1^2 - 1)} + \cdots\right]$$

$$- \frac{(1 + g)x_1^2}{(x_1^2 - 1)^3} \frac{1}{x - x_1} - \frac{x_1(x_1^2 + 1)}{2(x_1^2 - 1)^4} \log^2 \left(1 - \frac{x}{x_1}\right) + \cdots$$

where

$$(2.7) \qquad g = \frac{x_1}{x_1^2 - 1} \left[\frac{\sqrt{3}(x_1^2 - 2)}{x_1\sqrt{x_1^2 - 4}} \text{ arc tg } \frac{x_1\sqrt{x_1^2 - 4}}{\sqrt{3}(x_1^2 - 2)} - \frac{1}{2} \log \frac{3(x_1^2 - 1)}{x_1^2 - 3}\right] .$$

We will not present here proof of the convergence of series (2.1). This proof is obtained from a consideration of solution (1.2) by the method of successive approximations:

$$p_o' = \nu(1 - x^2), \ \ldots, \quad p_{n+1}' = \nu(1 - x^2) - \frac{x}{p_n}$$

which converge in domain I. After this, estimating the difference $p_{n+1} - p_n$, we become convinced that it has the order $\nu^{-(2n+1)}$, whence indeed follows the convergence of expansion (2.1) (at least the asymptotic convergence).

3. Solution for domain II. Domain II is the neighborhood of the point $p = 0$, $x = a_1$, $x = -a_2$. For definiteness we will consider the part of domain II corresponding to the neighborhood of the point $p = 0$, $x = a_1$. We will introduce the variable $q = -\nu p$ and seek x as a function of q. Equation (1.2) is written in the form:

$$\frac{dx}{dq} = \frac{1}{\nu^2} \frac{q}{q(x^2 - 1) - x} . \tag{3.1}$$

The solution of this equation we will represent in the form:

$$x = \sum_{n=0}^{\infty} \chi_n(q) \nu^{-2n} \tag{3.2}$$

Substitution of this expression in equation (3.1) gives a recurrent system of equations for the determination of the functions $\chi_n(q)$. We obtain

$$\chi_0' \equiv 0 \quad , \quad \chi_0 \equiv a_1$$

$$[q(a_1^2 - 1) - a_1]\chi_1' = q \quad , \quad [q(a_1^2 - 1) - a_1]\chi_2' = (1 - 2a_1 q)\chi_1 \chi_1'$$

(3.3) $\quad \cdots\cdots\cdots\cdots$

$$[q(a_1^2 - 1) - a_1]\chi_{n+1}' = \sum_{k=1}^{n} \chi_k \chi_{n+1-k}' - q \sum_{\alpha,\beta,\gamma} \chi_\alpha' \chi_\beta \chi_\gamma$$

$$(\alpha + \beta + \gamma = n + 1 \quad , \quad 1 \leq \alpha \leq n \quad , \quad 0 \leq \beta \leq n \quad , \quad 0 \leq \gamma \leq n) .$$

The solution of this system is elementary. For the first two functions $\chi_n(q)$ we have

$$\chi_1(q) = \frac{1}{a_1^2 - 1}\left[q + \frac{a_1}{a_1^2 - 1}\log\left(1 - \frac{a_1^2 - 1}{a_1}q\right)\right]$$

$$\chi_2(q) = \frac{a_1}{(a_1^2 - 1)^4}\left\{(a_1^2 - 1)q\left(q + \frac{a_1^2 + 1}{a_1(a_1^2 - 1)}\right) + \left[\frac{a_1^2 + 1}{a_1^2 - 1} + 2a_1q\right.\right.$$

$$\left.\left. - 2(a_1^2 - 1)q^2\right]\frac{\log[1 - q(a_1^2 - 1)/a_1]}{1 - q(a_1^2 - 1)/a_1} + \frac{3a_1^2 + 1}{2(a_1^2 - 1)}\log^2\left(1 - \frac{a_1^2 - 1}{a_1}q\right)\right\} .$$

The functions $\chi_n(q)$ have singularities for $q \to a_1/(a_1^2 - 1)$ and for $q \to -\infty$. Let us discover the character of these singularities. From formulas (3.4) it is evident that for $q_1 \to a_1/(a_1^2 - 1)$ the function χ_1 has a singularity of the form $\log(1 - u)$, and χ_2 a singularity of the form $(1 - u)^{-1}\log(1 - u)$, where $u = q(a_1^2 - 1)/a_1$. We then easily obtain from system (3.3) that generally

$$\chi_n \sim \left[\frac{\log(1 - u)}{1 - u}\right]^{n-1} \tag{3.4}$$

Hence it follows that series (3.2) preserves its asymptotic character up to values of q satisfying the condition

$$O\left(\frac{a_1}{a_1^2 - 1} - q\right) > O\left(\frac{\log \nu}{\nu^2}\right) .$$

Analogously for large negative values of q we obtain $\chi_1 \sim q$, $\chi_2 \sim q^2$ and generally $\chi_n \sim q^n$. Thus for negative values of q series (3.2) preserves its asymptotic character up to values of q bounded by the inequality $O(q) < O(\nu^2)$. In particular, asymptotic convergence obtains for $q = -\nu (p = 1)$.

We will not adduce proof of the convergence of series (3.1) either. It is not difficult to obtain this proof from the solution of equation (3.1) by the method of successive approximations, setting

$$x_o \equiv a_1 \ , \quad \frac{dx_{n+1}}{dq} = \frac{1}{\nu^2} \frac{q}{q(x_n^2 - 1) - x_n} \ .$$

For linking the solutions obtained for domains I and II, the constant a_1 must be determined in accordance with the assigned value of the constant c in (2.3) or, what is the same thing, in accordance with a given value of x_1. Since series (2.1) converges asymptotically up to the values of x for which $p = O(1)$, and since it is for these values that series (2.3) converges, we obtain the possibility of linking solutions (2.1) and (3.2). Putting $p = 1$ $(q = -\nu)$ in (2.1) and (3.2), we obtain two equations in the two unknowns x^* and a_1:

$$(3.5) \qquad 1 = \nu \sum_{n=o}^{\infty} f_n(x^*)\nu^{-2n} \ , \quad x^* = \sum_{n=o}^{\infty} \chi_n(-\nu)\nu^{-2n}$$

The solution reduces to finding x^* from the first of equations (3.5), and next—by the x^* found—finding, from the second equation of (3.5), a_1, which figures in the expressions for the functions $\chi_n(q)$.

Substituting for the functions $f_n(x^*)$ their expressions (2.6), we will find x^* by the method of iterations. Wanting to obtain x^* with a definite accuracy, we stop the process of iterations when subsequent iterations do not alter the quantities of a given order of smallness relative to ν. Thus, for example, the functions cited in (2.6) are sufficient for the computation of x^* accurate to a quantity of the order $\log^2\nu/\nu^4$. Accurate to quantities of the order of $1/\nu^3$, performing three iterations, we obtain

$$x^* = x_1 - \frac{1}{\nu}\frac{1}{x_1^2 - 1} - \frac{\log \nu}{\nu^2}\frac{x_1}{(x_1^2 - 1)^2}$$

$$(3.6) \qquad - \frac{1}{\nu^2}\left[\frac{x_1}{(x_1^2 - 1)^2}\log x_1(x_1^2 - 1) - \frac{g}{x_1^2 - 1} + \frac{x_1}{(x_1^2 - 1)^3}\right]$$

$$- \frac{\log \nu}{\nu^3}\frac{2x_1}{(x_1^2 - 1)^4} + 0(\frac{1}{\nu^3}) \quad .$$

Proceeding in exactly the same way with the second equation of (3.5), we obtain for a_1 the expression

$$(3.7) \quad a_1 = x_1 - \frac{\log \nu}{\nu^2}\frac{2x_1}{(x_1^2 - 1)^2} - \frac{1}{\nu^2}\left[\frac{2x_1}{(x_1^2 - 1)^2}\log(x_1^2 - 1) - \frac{g}{x_1^2 - 1}\right] + 0(\frac{1}{\nu^3}) \quad .$$

4. Solution for domain III. Domain III is defined by the interval of variation of the variables

$$a_1 - \epsilon > x > 1 + \epsilon \quad , \quad p < 0 \quad , \quad \epsilon > 0 \quad ;$$

$$-a_2 + \epsilon < x < -1 - \epsilon \quad , \quad p > 0 \quad , \quad \epsilon > 0 \quad .$$

Domain III has this essential significance for relaxation oscillations, that when the oscillatory system falls into this region, it at once passes, with a high degree of accuracy, to steady-state auto-oscillations. We will dwell in detail on the obtaining of the solution in this region.

First we will find a particular solution satisfying the condition $p \to 0$ for $x \to \infty$, denoting this solution by $P(x)$. This is precisely the

solution for whose expansion the solution of the second shortened Van der Pol equation, (1.4), is the principal term. We therefore put

$$(4.1)^{1)} \qquad P(x) = -\frac{x}{\nu(x^2-1)} + \pi(x)$$

(we are considering that part of domain III for which $p < 0$). Then for the function $\pi(x)$ we obtain the equation

$$(4.2) \quad \pi'(x) - \left[\nu^2\frac{(x^2-1)^2}{x} + \frac{x^2+1}{x(x^2-1)}\right]\pi(x) = -\frac{x^2+1}{\nu(x^2-1)^2} + \frac{\nu(x^2-1)}{x}\pi\pi' .$$

Regarding the right side of equation (4.2) as the free term, we reduce equation (4.2) to an integral equation:

$$(4.3) \qquad \pi(x) = \frac{1}{\nu^3}F(x) + \nu\frac{x^2-1}{x}e^{\nu^2 k(x)}\int_{\infty}^{x} e^{-\nu^2 k(\xi)}\pi\pi'\,d\xi$$

where

$$F(x) = \nu^2\frac{x^2-1}{x}e^{\nu^2 k(x)}\int_{x}^{\infty} e^{-\nu^2 k(\xi)}\frac{\xi(\xi^2+1)}{(\xi^2-1)^3}d\xi ,$$

(4.4)

$$k(x) = \frac{1}{4}x^4 - x^2 + \log x + \frac{3}{4}$$

and $k(1) = 0$, $k(x) > 0$ for $x > 1$, and $k'(x) = (x^2-1)^2/x > 0$.

Moreover, it is easily seen that $F(x) = O(1)$. Indeed, integrating (4.4) once by parts, for which we multiply and divide the integrand by $k'(\xi)$, we obtain

---

1) Editor's note: Formulas (3.8) and (3.9) do not appear in the original paper.

$$\therefore \quad F(x) = \frac{(x^2+1)x}{(x^2-1)^4} - \frac{x^2-1}{x} e^{\nu^2 k(x)} \int_x^\infty e^{-\nu^2 k(\xi)} \frac{6\xi^5+12\xi^3+2\xi}{(\xi^2-1)^6} d\xi = O(1)$$

and, in addition, since $F(x) > 0$ and the integral term in the last expression is positive, we have

$$F(x) \leq \frac{(x^2+1)x}{(x^2-1)^4} \quad . \tag{4.5}$$

Next performing an integration by parts in equation (4.3), we shall have reduced it to a non-linear integral equation,

$$\begin{aligned} \pi(x) &= \frac{1}{\nu^3} F(x) + \frac{\nu}{2} \frac{x^2-1}{x} \pi^2(x) \\ &+ \frac{\nu^3}{2} \frac{x^2-1}{x} e^{\nu^2 k(x)} \int_\infty^x e^{-\nu^2 k(\xi)} (\xi^2-1)^2 \pi^2(\xi) \frac{d\xi}{\xi} \quad . \end{aligned} \tag{4.6}$$

Lastly, replacing the sought function by the formula

$$\pi(x) = \frac{2x}{x^2-1} \omega(x) \tag{4.7}$$

we shall have for $\omega(x)$ the integral equation

$$\omega(x) = \frac{1}{\nu^3} f(x) + \nu \omega^2(x) + \nu^3 \frac{(x^2-1)^2}{x^2} e^{\nu^2 k(x)} \int_\infty^x e^{-\nu^2 k(\xi)} \xi \omega^2(\xi) d\xi \tag{4.8}$$

where

$$f(x) = \frac{x^2-1}{2x} F(x) \quad . \tag{4.9}$$

We solve this equation by the method of successive approximations, putting

$$\omega_1(x) = \frac{1}{\nu^3}f(x)$$

(4.10) $$\cdots\cdots\cdots$$

$$\omega_{n+1}(x) = \frac{1}{\nu^3}f(x) + \nu\omega_n^2(x) + \nu^3\frac{(x^2-1)^2}{x^2}e^{\nu^2k(x)}\int_\infty^x e^{-\nu^2k(\xi)}\xi\omega_n^2(\xi)d\xi \quad .$$

Denoting max $|f(x)|$ by M and max $|\omega_n|$ by Ω_n in the interval $1 + \varepsilon < x \leq \infty$, we shall have the estimate

$$\Omega_1 = \frac{M}{\nu^3}$$

$$\cdots\cdots$$

$$\Omega_{n+1} \leq \frac{M}{\nu^3} + \nu\,\Omega_n^2 + \nu^3\Omega_n^2 \max\left|e^{\nu^2k(x)}\frac{(x^2-1)^2}{x^2}\int_\infty^x e^{-\nu^2k(\xi)}\xi\,d\xi\right| \quad .$$

Since $\xi^2/(\xi^2-1)^2$ is a monotonically decreasing function, we have

$$e^{\nu^2k(x)}\frac{(x^2-1)^2}{x^2}\int_x^\infty e^{-\nu^2k(\xi)}\xi\,d\xi$$

$$= e^{\nu^2k(x)}\frac{(x^2-1)^2}{x^2}\int_x^\infty e^{-\nu^2k(\xi)}k'(\xi)\frac{\xi^2}{(\xi^2-1)^2}d\xi < \frac{1}{\nu^2} \quad .$$

Accordingly

(4.11) $$\Omega_1 = \frac{M}{\nu^3},\ \cdots\ ,\ \Omega_{n+1} \leq \frac{M}{\nu^3} + 2\nu\Omega_n^2 \quad .$$

Let us consider the series of relations

$$Y_1 = \frac{M}{\nu^3},\ \cdots\ ,\ Y_{n+1} = \frac{M}{\nu^3} + 2\nu Y_n^2 \quad .$$

This series is formed during the solution, by the method of iterations, of the equation $Y = M/\nu^3 - 2\nu Y^2$, and converges, if this equation has real roots, to the lesser root. From the condition of the reality of the roots, we obtain that the process of iterations converges as long as $8M/\nu^2 < 1$.

Since

$$f(x) < \frac{(x^2 + 1)x}{2x(x^2 - 1)^3} < \frac{1}{8(x - 1)^3} ,$$

the condition $8M/\nu^2 < 1$ can be represented in the form $x - 1 > \nu^{-2/3}$.

On fulfillment of the condition $8M/\nu^2 < 1$, we shall have

$$Y_n < Y = \frac{1}{4\nu} - \sqrt{\frac{1}{16\nu^2} - \frac{M}{2\nu^4}} < \frac{2M}{\nu^3}$$

and consequently, so much the more will $\Omega_n < 2M/\nu^3$. From the boundedness of Ω_n promptly follows the convergence of the method of successive approximations. Indeed, from (4.13) we shall have

$$\max|\omega_{n+1} - \omega_n| < 4\nu Y \max|\omega_n - \omega_{n-1}|$$

and consequently the series

$$\omega_1 + (\omega_2 - \omega_1) + (\omega_3 - \omega_2) + \cdots$$

converges if $4\nu Y < 1$, i.e., also for $8M/\nu^2 < 1$, and thus the successive approximations converge uniformly to the solution of equation (4.8), given fulfillment of the condition $x - 1 > \nu^{-2/3}$.

Lastly, noting that $\omega_{n+1} - \omega_n = O(\nu^{-(2n+3)})$ and that f(x) and each $\omega_n(x)$ are developable in asymptotic series in ν, we conclude that the functions $\omega(x)$, which means $\pi(x)$ too, are expansible in asymptotic series in powers of $1/\nu$.

This asymptotic expansion is most simply obtained directly from differential equation (1.2), setting

$$P(x) \approx -\frac{1}{\nu} \sum_{n=o}^{\infty} P_n(x)\nu^{-2n} \tag{4.12}$$

which gives

$$P_o(x) = \frac{x}{x^2 - 1}, \quad P_1(x) = \frac{P_o P_o'}{x^2 - 1} = -\frac{x(x^2 + 1)}{(x^2 - 1)^4}$$

(4.13) $\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$$P_{n+1}(x) = \frac{1}{x^2 - 1} \sum_{k=o}^{n} P_k'(x) P_{n-k}(x) .$$

One is easily convinced that the functions $P_n(x)$ for $x \to 1$ have a singularity of the form $1/(x - 1)^{3n+1}$, and consequently series (4.12) preserves its asymptotic character under the condition $O(x - 1) > O(\nu^{-2/3})$. We note that for an approach to the boundary of convergence P(x) will have the order of $\nu^{-1/3}$.

Let us pass on to finding the solution in accordance with the assigned initial conditions.

In the preceding N° we had the possibility of obtaining the solution up to values of p falling short of attaining the line $p = -x/[\nu(x^2 - 1)]$ by a quantity of the order of $\log \nu/\nu^3$. In particular, we could obtain the solution for

$$p = p_o = -\frac{1}{\nu}\frac{a_1}{a_1^2 - 1} + \frac{1}{\nu^2}$$

where x differs from a_1 by a quantity of the order of $\log \nu/\nu$.

Thus we have to construct in domain III a solution satisfying the condition $p = p_o$ for $x = x_o$, where

$$p_o + \frac{x_o}{\nu(x_o^2 - 1)} = 0(\frac{1}{\nu^2}) > 0 \quad .$$

We will seek the solution in the form

$$p(x) = P(x) + \sigma(x) \quad ; \tag{4.14}$$

then for $\sigma(x)$ we obtain the equation

$$P(x)\sigma'(x) + [P(x) + \nu(x^2 - 1)]\sigma(x) = -\sigma(x)\sigma'(x) \tag{4.15}$$

which, analogously to what was done for $\pi(x)$, we reduce to the integral equation

$$\begin{aligned}\sigma(x) = {} & \frac{c}{\nu^2}\frac{n(x_o)}{n(x)}e^{-\nu^2 m(x)} + \frac{\nu}{2n(x)}[\sigma^2(x) - \sigma_o^2 e^{-\nu^2 m(x)}] \\ & + \frac{\nu^3}{2n(x)}e^{-\nu^2 m(x)}\int_{x_o}^{x} e^{\nu^2 m(\xi)}\frac{\xi^2 - 1}{n(\xi)}\sigma^2(\xi)\,d\xi\end{aligned} \tag{4.16}$$

where we have used the notation

$$m(x) = \int_{x_o}^{x}\frac{x^2 - 1}{\nu P(x)}dx \quad , \quad n(x) = -\nu P(x) \quad , \quad \sigma_o = \sigma(x_o) = \frac{c}{\nu^2} \tag{4.17}$$

From expansion (4.12) it follows that

$$(4.18)\qquad m(x) = O(1) > 0 \quad , \quad n(x) = O(1) > 0 \quad .$$

In exactly the same way, from the initial conditions for p one may consider $c = O(1)$ (or even less). Now put

$$(4.19)\qquad \sigma(x) = 2n(x)e^{-\nu^2 m(x)}s(x) \quad ;$$

we obtain for s(x) the equation

$$(4.20)\qquad \begin{aligned} s(x) &= \frac{1}{\nu^2}\phi(x) + \nu s^2(x)e^{-\nu^2 m(x)} \\ &+ \frac{\nu^3}{n^2(x)}\int_{x_o}^{x} e^{-\nu^2 m(\xi)} n(\xi)(\xi^2 - 1)s^2(\xi)d\xi \end{aligned}$$

where

$$\phi(x) = \frac{n(x_o)c - c^2/\nu}{2n^2(x)} \quad .$$

We will obtain the solution of this equation, again by the method of successive approximations, setting

$$(4.21)\qquad \begin{aligned} s_1(x) &= \frac{1}{\nu^2}\phi(x) \\ &\ldots\ldots\ldots \\ s_{n+1}(x) &= \frac{1}{\nu^2}\phi(x) + \nu e^{-\nu^2 m(x)} s_n^2(x) + \frac{\nu^3}{n^2(x)}\int_{x_o}^{x} e^{-\nu^2 m(\xi)} n(\xi)(\xi^2 - 1)s_n^2(\xi)d\xi \quad . \end{aligned}$$

Having denoted by Σ_n the maximum of the modulus of $s_n(x)$ in the interval $(1 + \epsilon, x_o)$, by M the maximum of the modulus of $\phi(x)$, and

taken into account the fact that n(x) is a decreasing function of x, we obtain the estimate

$$\Sigma_1 = \frac{M}{\nu^2} , \quad \Sigma_{n+1} \leq \frac{M}{\nu^2} + 2\nu\Sigma_n^2 .$$

From this, by a method completely analogous to the preceding, we shall prove the convergence of the successive approximations under the condition $8M/\nu < 1$. Here we have $|s(x)| < 2M/\nu^2$.

Turning to the expression for $\phi(x)$, one can write the last inequality in form

$$(4.22) \qquad |s(x)| < \frac{c}{\nu^2 n(x)} , \quad \text{or} \quad |\sigma(x) < \frac{2c}{\nu^2} e^{-\nu^2 m(x)} .$$

This result can be formulated as follows. Whatever be the initial values or origin of motion of the oscillatory system, on falling into domain III its motion approaches the limit cycle with an accuracy to quantities of the order of $\nu^{-2}\exp(-\nu^2 m)$, where $m > 0$ is of the order of unity.

Since the basic solution P(x) can be obtained accurate to quantities of the order of $1/\nu^n$, where n is any number, the order of smallness of the correction $\sigma(x)$ to this solution is less than any remainder term in the series for P(x) and therefore it makes no sense to seek an asymptotic expression for $\sigma(x)$.

5. <u>Solution in domain</u> IV. This domain is defined as follows:

$$1 - \epsilon < x < 1 + \epsilon , \quad p < 0 , \quad \epsilon > 0 ;$$

$$-1 - \epsilon < x < -1 + \epsilon , \quad p > 0 , \quad \epsilon > 0 .$$

We have already seen that for an approach to the boundary of domain III, which is defined by the condition $(x - 1) > O(\nu^{-2/3})$, the order of p(x) approaches $O(\nu^{-1/3})$.

We therefore introduce the following change of variables:

$$p = -\nu^{-1/3}Q(u) \quad , \quad u = \nu^{2/3}(x - 1) \tag{5.1}$$

(we are considering the first part of domain IV). In the new variables equation (1.2) will take the form

$$Q\frac{dQ}{du} - 2uQ + 1 = \nu^{-2/3}(u^2Q - u) \quad . \tag{5.2}$$

We will seek the solution in the form of the series

$$Q(u) = \sum_{n=o}^{\infty} Q_n(u)\nu^{-2/3\ n} \quad . \tag{5.3}$$

Substitution of (5.3) in equation (5.2) gives for the functions $Q_n(u)$ the recurrent system:

(a) $$Q_o\frac{dQ_o}{du} - 2uQ_o + 1 = 0$$

(b) $$Q_o\frac{dQ_1}{du} - \frac{Q_1}{Q_o} = u^2Q_o - u$$

(5.4) (c) $$Q_o\frac{dQ_2}{du} - \frac{Q_2}{Q_o} = u^2Q_1 - Q_1\frac{dQ_1}{du}$$

. . . .

$$Q_o\frac{dQ_{n+1}}{du} - \frac{Q_{n+1}}{Q_o} = u^2Q_n - \sum_{k=1}^{n} Q_k\frac{dQ_{n+1-k}}{du} \quad .$$

As regards the initial conditions for $Q_n(u)$, they are determined subsequently by the condition of the linkage of the solution with the solution in domain III.

Let us find the solution of equation (5.4, a). Setting $Q_o = \frac{du}{d\tau}$, after the substitution and integration we obtain in succession

$$(5.5) \qquad \frac{d^2u}{d\tau^2} - 2u\frac{du}{d\tau} + 1 = 0 \quad , \quad \frac{du}{d\tau} - u^2 + \tau = 0$$

(we take the constant of integration as equal to zero, in view of the arbitrariness in the choice of the variable τ). Equation (5.5) is the Riccati equation. It reduces to the equation

$$(5.6) \qquad \frac{d^2v}{d\tau^2} - \tau v = 0 \qquad \left(u = -\frac{1}{v}\frac{dv}{d\tau}\right)$$

the general solution of which is

$$(5.7) \qquad v = \sqrt{\tau}[c_1 K_{1/3}(2/3\ \tau^{3/2}) + c_2 I_{1/3}(2/3\ \tau^{3/2})] \ .$$

To link it with the solution P(x), we must require that $Q_o = \frac{du}{d\tau} \to 0$ as $u \to \infty$, i.e., from (5.5) we obtain that $u \to \infty$ as $\tau \to \infty$. This condition is satisfied only by the solution

$$(5.8) \qquad v = c_1\sqrt{\tau}K_{1/3}(2/3\ \tau^{3/2}) \ .$$

Utilizing the familiar relations for the Bessel functions

$$K'_n(x) = K_{n-1}(x) + K_n(x)n/x \quad , \quad K_{-n}(x) = K_n(x)$$

we obtain for u the solution

(5.9)[2] $$u = \sqrt{\tau} K^{2/3}(2/3\ \tau^{3/2})/K_{1/3}(2/3\ \tau^{3/2}) .$$

For negative values of τ, formula (5.9) is more conveniently represented in the form ($\tau_1 = -\tau$)

(5.10) $$u = \sqrt{\tau_1}\{J_{-2/3}(2/3\ \tau_1^{3/2}) - J_{2/3}(2/3\ \tau_1^{3/2})\}/\{J_{1/3}(2/3\ \tau_1^{3/2}) + J_{-1/3}(2/3\ \tau_1^{3/2})\} .$$

According to (5.5), the quantity $Q_o = u^2 - \tau$, and, using asymptotic expansions for the Bessel functions, the asymptotic expansion for Q(u) is easily obtained. We have

(5.11) $$u \approx \sqrt{\tau}(1 + \tau^{-3/2} - \dots) , \quad Q_o = u^2 - \tau \approx \frac{1}{2\sqrt{\tau}} + \dots \approx \frac{1}{2u} + \dots$$

The more complete asymptotic expansion is more simply obtained directly from (5.4, a). We have

(5.12) $$Q_o(u) \approx \frac{1}{2u} - \frac{1}{8u^4} + \frac{5}{32u^7} - \frac{11}{32u^{10}} + \frac{539}{512u^{13}} - \dots$$

For negative values of τ, the denominator in expression (5.10) has zeros. Denote by α the least root of the equation

$$J_{1/3}(2/3\ \tau_1^{3/2}) + J_{-1/3}(2/3\ \tau_1^{3/2}) = 0 .$$

---

2) Editor's note: For $K^{2/3}$ read $K_{2/3}$.

Then $u \to -\infty$ as $\tau_1 \to \alpha\,(\tau_1 < \alpha)$. Moreover, according to (5.6), $u = -v^{-1}\frac{dv}{d\tau}$, or $u = v^{-1}\frac{dv}{d\tau_1}$, and therefore u has for $\tau_1 = \alpha$ a simple pole with residue equal to unity, and accordingly

$$u = \frac{1}{\tau_1 - \alpha} + \text{a holomorphic function}$$

Hence $\tau_1 = \alpha + \frac{1}{u} + \cdots$, and consequently as $u \to -\infty$ we have

$$Q_o = u^2 + \alpha + \frac{1}{u} + \cdots$$

A more detailed computation gives

$$(5.13)\quad Q_o(u) = u^2 + \alpha + \frac{1}{u} - \frac{\alpha}{3} - \frac{1}{u^3} - \frac{1}{4u^4} + \frac{\alpha^2}{5u^5} + \frac{7\alpha}{18u^6} - \frac{1}{7}\alpha^2 - \frac{5}{4}\frac{1}{u^7} + \cdots .$$

Let us pass on to the determination of $Q_1(u)$. The general solution of (5.4, b) will be

$$(5.14)\quad Q_1(u) = \frac{1}{A(u)}\left[c + \int_0^u A(u)\left(u^2 - \frac{u}{Q_o}\right)du\right], \quad A(u) = \exp\left(-\int_0^u \frac{du}{Q_o^2}\right).$$

For linking with the solution P(x) we have to require that the quantity $\nu^{-2/3}Q_1(u)$ be bounded for $u = O(\nu^{\epsilon})$.

Utilizing asymptotic expansion (5.12), we easily obtain for the constant of integration c the value

$$c = -\int_0^\infty A(u)\left[u^2 - \frac{u}{Q_o}\right]du .$$

Hence finally

(5.15) $$Q_1(u) = \frac{1}{A(u)} \int_u^{\infty} A(u)\left(\frac{u}{Q_o} - u^2\right)du \quad .$$

For Q_1 we have the expansion

for $u \to \infty$

(5.16) $$Q_1(u) \approx \frac{1}{4} - \frac{1}{64u^6} + 0\left(\frac{1}{u^9}\right)$$

for $u \to -\infty$

(5.17) $$Q_1(u) = \frac{1}{3}u^3 + b_o - \frac{\alpha}{6u^2} - \left(\frac{1}{27} + \frac{b_o}{3}\right)\frac{1}{u^3} + \left(\frac{1}{450} + \frac{2}{5}b_o\right)\frac{1}{u^5} + \cdots - \frac{2}{3}\log|u|\left(1 - \frac{1}{3u^3} + \frac{2\alpha}{5u^5} - \cdots\right)$$

where

(5.18) $$b_o = \frac{1}{A(-\infty)} \int_{-\infty}^{\infty} A(u)\left[\frac{u}{Q_o} - \frac{1}{3}\frac{u^3}{Q_o^2} - \frac{2}{3}\frac{u}{u^2 + \alpha/2} + \frac{1}{3Q_o^2}\log\left(u^2 + \frac{\alpha}{2}\right)\right]du \quad .$$

The solutions for the rest of the functions will have the form

(5.19) $$Q_{n+1}(u) = \frac{1}{A(u)} \int_u^{\infty} A(u)\left[\sum_{k=1}^{n} Q_k \frac{dQ_{n+1-k}}{du} - u^2 Q_n\right]\frac{du}{Q_o}$$

where the constants of integration are determined by the condition of the boundedness of the quantities $\nu^{-2n/3} Q_n(u)$ for $u = O(\nu^{\epsilon})$.

The asymptotic expressions for the functions $Q_n(u)$ will be:

for $u \to +\infty$

$$Q_2(u) \approx -\frac{u}{8} + O\left(\frac{1}{u^5}\right) ,$$

(5.20)

$$Q_3(u) \approx \frac{u^2}{16} + O\left(\frac{1}{4}\right) , \quad \ldots , \quad Q_n(u) \approx (-1)^{n-1}\frac{u^{n-1}}{2^{n+1}}$$

for $u \to -\infty$

$$Q_2(u) = \frac{2}{9}u + b_o^{(2)} + \frac{\alpha}{3u} + O\left(\frac{\log u}{u^2}\right) ,$$

(5.21)

$$Q_3(u) = -\frac{u^2}{27} - \frac{2b_o}{81} u + \ldots .$$

Generally, the principal singularity of the functions $Q_n(u)$ for $u \to -\infty$, beginning with $n = 2$, will be $Q_n \sim u^{n-1}$.

These results show that series (5.3) preserves its asymptotic character up to values of u bounded by the condition $O(u) < O(\nu^{2/3})$, i.e., for values of x satisfying the condition $O(x - 1) < O(1)$, and thus the domains in which solutions (5.3) and (4.12) are suitable overlap. As we shall see, the same thing holds for solutions (5.3) and (2.1).

6. <u>Linking the solutions for domains I and IV.</u> Let us return to the first part of domain I. We shall have to link solution (2.1) with solution (5.3), in which by u should be understood the quantity $-\nu^{2/3}(1 + x)$ and one sets

(5.22)

$$p = + \nu^{-1/3}Q(u)$$

We note at the outset that since $p > 0$ for $x = -1$ ($u = 0$), the constant c in formula (2.3) must be greater than $2/3$; (for $c = 2/3$ we shall have $f_o(-1) = 0$). Set $c = 2/3 + \gamma$. The order of magnitude of γ can be at once determined; since $p(-1) = O(\nu^{-1/3})$, $\nu\gamma$ will also be of the order of $\nu^{-1/3}$ and consequently $\gamma = O(\nu^{-4/3})$.

Up to what negative values of x is expansion (2.1) applicable? Let us now discover this.

Consider the case when $c = 2/3$. Increasing the constant c leads only to improving the convergence. For $c = 2/3$ we have

$$f_o(x) = \frac{2}{3} + x - \frac{1}{3}x^3 = \frac{1}{3}(x + 1)^2(2 - x)$$

$$f_1(x) = -\int_o^x \frac{xdx}{f_o} = -\frac{1}{x + 1} - \frac{2}{3}\log(x + 1) + \frac{2}{3}\log\left(1 - \frac{x}{2}\right) .$$

From system of equations (2.2) it is now easy to find that in the neighborhood of $x = -1$ the principal singularity of the functions $f_n(x)$ has the form $f_n(x) \sim (x + 1)^{-(3n-2)}$, and accordingly series (2.1) preserves its asymptotic character up to values of x satisfying the condition $O(x + 1) > O(\nu^{-1/3})$ [which corresponds to the values $-u > O(1)$], and thus the domains in which expansions (5.3) and (2.1) are suitable overlap.

In particular, the asymptotic convergence of expansions (2.3) and (5.3) is guaranteed for $x = -1 + \nu^{-1/3}$), and thus the constant of integration c can be determined by equating, for $x = -1 + \nu^{-1/3}$, the values of p obtained from formulas (5.3) and (2.1).

$$\nu^{-1/3} \sum_{n=o}^{\infty} \nu^{-2n/3} Q_n(-\nu^{-1/3}) = \nu \sum_{n=o}^{\infty} \nu^{-2n} f_n(-1 + \nu^{-1/3}) \tag{6.1}$$

We will not exhibit here the quite cumbersome but elementary calculations. Using the expansions of the functions $Q_n(u)$ for large negative values of u [formulas (5.13), (5.17), (5.21)] we obtain for the left side of equation (6.1) the expression

$$\begin{aligned} p(-1 + \nu^{-1/3}) &= \nu^{1/3} - \frac{1}{3} + \alpha\nu^{-1/3} - \nu^{-2/3} + b_o\nu^{-1} \\ &+ \frac{\alpha}{3} - \frac{2}{9}\nu^{-4/3} - \ldots - \frac{2}{9}\frac{\log \nu}{\nu}\left(1 + \frac{1}{3\nu} - \ldots\right) \end{aligned} \tag{6.2}$$

On the other hand, the expansions of the functions $f_n(x)$ in the neighborhood of x = -1 give for the right side of equation (6.1) the expression:

$$\begin{aligned} p(-1 + \nu^{-1/3}) &= \nu^{1/3} - \frac{1}{3} - \nu^{-2/3} + \left(1 + \frac{2}{3}\log\frac{3}{2}\right)\nu^{-1} \\ &- \frac{2}{9}\nu^{-4/3} - \ldots + \frac{2}{9}\frac{\log \nu}{\nu}\left(1 + \frac{1}{3\nu} + \ldots\right) \\ &+ \gamma\left(\nu + \frac{1}{3} - \ldots\right) - \gamma^2\left(\frac{1}{5}\nu^{2/3} - \ldots\right) + \ldots \end{aligned} \tag{6.3}$$

(the cited terms of the expansions are sufficient for the determination of γ accurate to quantities of the order of $\nu^{-8/3}$).

Equating expressions (6.2) and (6.3), we obtain the equation for the determination of γ. Thus accurate to quantities of the order of $\nu^{-8/3}$ we obtain

$$(6.4)\quad \gamma \approx \alpha \nu^{-4/3} - \frac{4}{9}\frac{\log \nu}{\nu} + \left(b_o - 1 - \frac{2}{3}\log\frac{3}{2}\right)\nu^{-2} + O(\nu^{-8/3}) \quad .$$

7. Determination of the amplitude of the steady-state auto-oscillations. After the determination of the constant $c = 2/3 + \gamma$, the root x_1 of the equation $f_o(x_1) = 0$ is easily computed. The solution of this cubic equation can be represented in the form

$$x_1 = 2 + \frac{1}{3}\gamma - \frac{2}{27}\gamma^2 + \frac{2}{243}\gamma^3 - \cdots$$

and substitution of the value of γ gives

$$(7.1)\quad x_1 \approx 2 + \frac{\alpha}{3}\nu^{-4/3} - \frac{4}{27}\frac{\log \nu}{\nu^2} + \frac{1}{3}\left(b_o - 1 - \frac{2}{3}\log\frac{2}{3}\right)\nu^{-2} + O(\nu^{-8/3}) \quad ,$$

after which equation (3.9) allows[3)] the amplitude of the auto-oscillations to be found, the computations yielding

$$(7.2)\quad \begin{aligned} a \approx\ & 2 + \frac{\alpha}{3}\nu^{-4/3} - \frac{16}{27}\frac{\log \nu}{\nu^2} \\ & + \frac{1}{9}(3b_o - 1 + 2\log 2 - 8\log 3)\nu^{-2} + O(\nu^{-8/3}) \end{aligned}$$

8. Determination of the period of the auto-oscillations. The period of the auto-oscillations is computed by the formula

$$(8.1)\qquad T = 2\int_{-a}^{a} \frac{dx}{p(x)} \quad .$$

---

3) Editor's note: Apparently equation (3.7) is meant.

We will here limit ourselves to an indication only of the method of calculating the period, without adducing the calculations.

We subdivide the whole interval of integration into five parts, corresponding to the different domains:

1) from $-a$ to $-x_2$ over domain II (this part of the integral we will designate as T_2''); here x_2 is the value of x obtained by formula (3.2) for a value of q equal, for instance, to $(1 - \nu^{-4/3})a/(a^2 - 1)$;

2) from $-x_2$ to $-(1 + \nu^{-1/3})$ over domain III (this part of the integral we will designate by the letter T_3);

3) from $-(1 + \nu^{-1/3})$ to $-(1 - \nu^{-1/3})$ over domain IV (this part of the integral we will designate by the letter T_4);

4) from $-(1 - \nu^{-1/3})$ to x^* over domain I (this part of the integral is designated by the letter T_1); here x^* is determined by formula (3.6);

5) from x^* to $\underline{a}$ over domain II (this part of the integral is designated by the letter T_2');

The complete period T will then equal

(8.2) $$T = 2(T_1 + T_2' + T_2'' + T_3 + T_4) .$$

In each of the integrals we substitute for p the respective expansions (in domain II replacing the variable of integration by q and in domain IV by u). Using the estimates of the singularities of the functions set forth for each region, we determine without difficulty the number of terms of the developments needed to determine the period T to the assigned accuracy, after which the calculation reduces to the computation of the integrals.

We will set forth the results of the calculations accurate to quantities of the order of ν^{-1} inclusively:

$$(8.3)\quad T_1' = \int_{-1+\nu^{-1/3}}^{0} \frac{dx}{p} \approx \nu^{-2/3} + \frac{1}{9}\frac{\log \nu}{\nu} - \left(1 - \frac{1}{3}\log\frac{3}{2}\right)\nu^{-1} + O(\nu^{-4/3})$$

$$(8.4)\quad T_1'' = \int_{0}^{x^*} \frac{dx}{p} \approx \frac{1}{3}\frac{\log \nu}{\nu} + \frac{2}{3}\left(1 + \log 3 + \frac{1}{2}\log 2\right)\nu^{-1} + O(\nu^{-4/3})$$

$$(8.5)\quad T_1 = T_1' + T_1'' \approx \nu^{-2/3} + \frac{4}{9}\frac{\log \nu}{\nu} + \left(\log 3 - \frac{1}{3}\right)\nu^{-1} + O(\nu^{-4/3})$$

$$(8.6)\quad T_2' \approx \frac{1}{3}\frac{\log \nu}{\nu} + \frac{1}{3}\log\frac{3}{2}\nu^{-1} + (\nu^{-4/3})$$

$$(8.7)\quad T_2'' \approx \frac{4}{9}\frac{\log \nu}{\nu} + O(\nu^{-5/3}\log \nu)$$

$$(8.8)\quad \begin{aligned} T_3 \approx{}& \left(\frac{3}{2} - \log 2\right)\nu - \nu^{1/3} + \frac{1}{3} + \left(\frac{\alpha}{2} - \frac{1}{4}\right)\nu^{-1/3} + \frac{7}{10}\nu^{-2/3} \\ &- \frac{43}{18}\frac{\log \nu}{\nu} + \left(\frac{1}{2}b_o + \frac{1}{12} + \frac{11}{6}\log\frac{2}{3}\right)\nu^{-1} + O(\nu^{-4/3}) \end{aligned}$$

$$(8.9)\quad \begin{aligned} T_4 \approx{}& \nu^{1/3} - \frac{1}{3} + \left(\alpha + \frac{1}{4}\right)\nu^{-1/3} - \frac{17}{10}\nu^{-2/3} \\ &- \frac{1}{18}\frac{\log \nu}{\nu} + \left(\frac{1}{6} - d\right)\nu^{-1} + O(\nu^{-4/3}) \end{aligned}$$

where

$$(8.10)\quad d = \frac{1}{2}\log\alpha - b_o - \frac{1}{4} + \int_{-\infty}^{0}\left(\frac{u}{Q_o} - \frac{u}{u^2+\alpha}\right)du + \int_{0}^{\infty}\left(Q_o - \frac{1}{2}\frac{u^2}{u^3+1}\right)du \,.$$

Accordingly for the complete period we obtain

$$T \approx (3 - 2 \log 2)\nu + 3\alpha\nu^{-1/3} - \frac{22}{9}\frac{\log \nu}{\nu}$$

(8.11)

$$+ \left(3 \log 2 - \log 3 - \frac{1}{6} + b_0 - 2d\right)\nu^{-1} + O(\nu^{-4/3}) \quad .$$

We exhibit the numerical values of the coefficients of the basic formulas:

$$\alpha = 2.338107 \quad , \quad b_0 = 0.1723 \quad , \quad d = 0.4889 \quad .$$

$$p(0) \approx \frac{2}{3}\nu + 2.338107\nu^{-1/3} - \frac{4}{9}\frac{\log \nu}{\nu} - 1.0980\nu^{-1} + O(\nu^{-5/3})$$

$$a \approx 2 + 0.779369\nu^{-4/3} - \frac{16}{27}\frac{\log \nu}{\nu^2} - 0.8762\nu^{-2} + O(\nu^{-8/3})$$

$$T \approx 1.613706\nu + 7.01432\nu^{-1/3} - \frac{22}{9}\frac{\log \nu}{\nu}$$

$$+ 0.0087\nu^{-1} + O(\nu^{-4/3}) \quad .$$

We have performed here the asymptotic solution of the simplest equation of non-linear oscillations, so that it would be possible to carry through the computations to the end. But the device used here, of introducing "connecting" regions, is not limited to this particular case only, being applicable to a more general equation

$$\frac{d^2x}{dt^2} - \nu\psi\left(x, \frac{dx}{dt}\right) + \phi(x) = 0$$

under certain limitations, to be placed on the function $\psi(x, p)$. The connecting domains will be neighborhoods of the lines $\psi(x, p) = 0$.

If at a nodal point of the line $\psi(x, p) = 0$ the expansion of the function ψ in a Taylor's series commences with the term

$$a(x - x_o)p \quad ,$$

where a is a numerical coefficient, then the basic solution for domain IV will remain without modification, since, setting $p = \pm\nu^{-1/3}Q$ in the neighborhood of the nodal point $\pm(x - x_o)\nu^{2/3} = u$, we shall obtain for the principal term of the expansion the equation

$$Q_oQ_o' - auQ_o + \phi(x_o) = 0$$

reducing to (5.4, a) by the simple substitution

$$Q_o = [2\phi^2(x_o)/a]^{1/3}Q_o^* \quad , \quad u^* = [a^2/4\phi(x_o)]^{1/3}u$$

9. Example. In conclusion we will present an example of the solution of the equation for $\nu = 10$. For this case the asymptotic formulas exhibited give $p(0) = 7.540$, $a = 2.0138$, $T = 18.831$. In Fig. 2 is depicted the graph of the function $p(x)$, two terms of the expansion having been taken for each domain:

for domain I

$$p(x) \approx \nu f_o(x) + \frac{1}{\nu}f_1(x)$$

for domain II

$$x \approx a + \frac{1}{\nu^2}\chi_1(q)$$

for domain III

$$p \approx \frac{1}{\nu} P_o(x) + \frac{1}{\nu^3} P_1(x)$$

for domain IV

$$p \approx \nu^{-1/3} Q_o(u) + \frac{1}{\nu} Q_1(u) \quad .$$

The values of the functions $Q_o(u)$ and $Q_1(u)$ are set forth in Table 1.

Table 1.

Table of the Functions $Q_o(u)$ and $Q_1(u)$

u	$Q_o(u)$	$Q_1(u)$	u	$Q_o(u)$	$Q_1(u)$
-6.0	38.1747	-73.0343	0.0	1.0187	0.1869
-5.0	27.1436	-42.5848	0.2	0.8424	0.2149
-4.0	18.0985	-22.1123	0.4	0.7018	0.2310
-3.8	16.5269	-19.0382	0.6	0.5904	0.2398
-3.6	15.0342	-16.2668	0.8	0.5023	0.2445
-3.4	13.6203	-13.7819	1.0	0.4331	0.2471
-3.2	12.2848	-11.5673	1.2	0.3778	0.2485
-3.0	11.0276	-9.6070	1.4	0.3334	0.2492
-2.8	9.8484	-7.8847	1.6	0.2974	0.2495
-2.6	8.7469	-6.3844	1.8	0.2678	0.2496
-2.4	7.7225	-5.0898	2.0	0.2432	0.2497
-2.2	6.7749	-3.9849	2.2	0.2225	0.2498
-2.0	5.9032	-3.0532	2.4	0.2049	0.2499
-1.8	5.1068	-2.2790	2.6	0.1897	0.2499
-1.6	4.3845	-1.6458	2.8	0.1766	0.2500
-1.4	3.7351	-1.1379	3.0	0.1652	0.2500
-1.2	3.1568	-0.7393	3.2	0.1551	0.2500
-1.0	2.6476	-0.4344	3.4	0.1461	0.2500
-0.8	2.2047	-0.2080	3.6	0.1381	0.2500
-0.6	1.8249	-0.0457	3.8	0.1310	0.2500
-0.4	1.5041	+0.0664	4.0	0.1245	0.2500
-0.2	1.2372	0.1403			
0.0	1.0187	0.1869			

In Fig. 2 the computational points corresponding to the formulas of domains I and III are depicted as circlets, and those corresponding to the formulas for domains II and IV, as crosses.

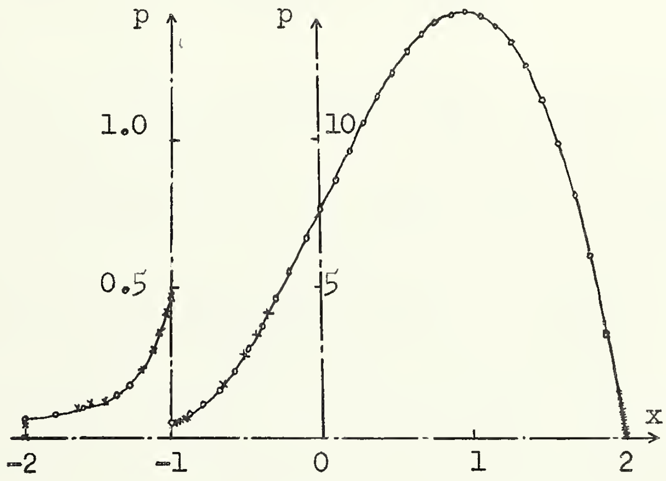


Fig. 2

The scale for p in the section from -a to -1 has been increased to ten times that of the reamining part of the curve.

Received at the editorial office
April 17, 1947

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

NBS