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NATIONAL BUREAU OF STANDARDS REPORT

2392

Selected Topics in the Theory of Asymptotic Expansions

By

H. A. Antosiewicz
Philip Davis
Fritz Oberhettinger

The American University

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NBS PROJECT | **NBS REPORT**

1102-10-1104 March 30, 1953 2392

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FOREWORD

A seminar on the theory of asymptotic expansions was held by The American University with the cooperation of members of the staff of the National Bureau of Standards during the fall semester of 1952-53. This volume consists essentially of notes on these lectures. While it contains such diversified topics as the titles of the respective parts indicate, it nevertheless attempts to bring together under one cover material which is of considerable interest in pure and applied mathematics.

Part I.

SOME GENERAL THEOREMS AND METHODS

by

Fritz Oberhettinger

1) <u>Introduction</u>. In 1739 Euler investigated the series

(1.1) $$y=f(x)=x^{-1}(1-1!x^{-1}+2!x^{-2}-3!x^{-3}+\dots) = \sum_{0}^{\infty}(-1)^n n!x^{-n-1} \quad x>0,$$

which is divergent for any finite value of x. Obviously (1) satisfies formally the differential equation

(1.2) $$y' - y + \frac{1}{x} = 0$$

Putting $y = v(x)e^x$ we obtain $v'(x) = -\frac{e^{-x}}{x}$ and a solution of (1.2) which vanishes for $x \to \infty$ is

(1.3) $$y = f(x) = e^x \int_x^\infty \frac{e^{-t}}{t}\,dt = -e^x \,\mathrm{Ei}(-x),$$

where Ei(-x) is the so-called exponential integral.

Euler considered (1.1) as a series expansion of the function defined in (1.3) or (1.3) as the sum of the everywhere divergent series (1.1). In order to get a better insight into the connection between the formulas (1.1) and (1.3) let us consider the integral

(1.4) $$f(z) = \int_0^\infty (z+t)^{-1}e^{-t}dt = z^{-1}\int_0^\infty (1+\frac{t}{z})^{-1}e^{-t}dt,$$

This work was performed under a National Bureau of Standards contract with American University.

which represents an analytic function in the z plane cut along the negative real axis from 0 to $-\infty$. Instead of (1.4) we may write

(1.4a) $$f(z) = e^{z} \int_{z}^{\infty} \frac{e^{-v}}{v}\, dv = -e^{z}\, \mathrm{Ei}(-z)$$

where the path of integration shall not pass the cut. In (1.4) we expand $(1 + \frac{t}{z})^{-1}$ into the finite series

$$(1 + \frac{t}{z})^{-1} = 1 - \frac{t}{z} + \frac{t^2}{z^2} + \ldots + (-1)^n \frac{t^n}{z^n} + (-1)^{n+1} \frac{t}{z^{n+1}} \frac{1}{1 + \frac{t}{z}}$$

and obtain integrating term by term

(1.5) $$f(z) = z^{-1}[1 - 1!\, z^{-1} + 2!\, z^{-2} + \ldots + (-1)^n\, n!\, z^{-n} + R_n(z)],$$

using the well known formula

(1.6) $$\int_0^{\infty} e^{-t\sigma}\, t^{\alpha}\, dt = \sigma^{-\alpha}\, \Gamma(\alpha+1),\ \mathrm{Re}\,\alpha > -1,\ \mathrm{Re}\,\sigma > 0.$$

Obviously

(1.7) $$R_n(z) = (-1)^{n+1} z^{-n-1} \int_0^{\infty} e^{-t}\, t^{n+1}\, (1 + \frac{t}{z})^{-1}\, dt.$$

For a point $z = |z|\, e^{i\vartheta}$, $(-\pi < \vartheta < \pi)$ in the cut plane we have on the path of integration $0 < t < \infty$

$$\left|1 + \frac{t}{z}\right|^{-1} = (1 + 2t\, |z|^{-1} \cos\vartheta + t^2\, |z|^{-2})^{-\frac{1}{2}} = g(t).$$

For a fixed $|z|$ and ϑ, this expression has a maximum at $t = -|z| \cos\vartheta$ and the value of the maximum is

$$[g(t)]_{max} = |\sin\vartheta| .$$

Therefore on the path of integration $0 \leqq t < \infty$ there exist the inequalities

$$\left|1 + \frac{t}{z}\right|^{-1} \leqq 1 \text{ , if } \cos\vartheta \geqq 0, \text{ or } \frac{-\pi}{2} \leqq \vartheta \leqq \frac{\pi}{2}$$

$$\left|1 + \frac{t}{z}\right|^{-1} \leqq \left|\sin\vartheta\right|^{-1}, \text{ if } \cos\vartheta \leqq 0$$

Hence, we obtain the following estimate for the absolute value of the remainder $R_n(z)$ in (1.5)

(1.8a) $$|R_n(z)| < |z|^{-n-1} \int_0^\infty e^{-t} t^{n+1} dt = |z|^{-n-1} (n+1)! \text{ , } \cos\vartheta \geqq 0.$$

(1.8b) $$|R_n(z)| < |z|^{-n-1} |\operatorname{cosec}\vartheta|(n+1)!, \cos\vartheta \leqq 0.$$

In case z is real and positive ($z=x, \vartheta = 0$) we simply get from (1.7)

(1.9) $$R_n(x) < (-1)^{n+1} x^{-n-1} (n+1)!$$

(1.8a) shows that for Re $z \geqq 0$, ($\cos\vartheta \geqq 0$) the absolute value of the remainder $R_n(z)$ is smaller than the absolute value of the first neglected term in (1.5). In addition to this we see from (1.9) that for a real and positive z, remainder and first neglected term have the same sign. However, we can only infer from (1.8b) that for a complex z with Re $z < 0$ and $I_m z \neq 0$ the absolute value of the remainder in (1.5) is of the same magnitude as the absolute value of the first neglected term. Hence the series (1.1) can be used for the numerical calculation of the integral (1.3) in spite of its divergence with limited accuracy but with an accuracy which is often close enough to be sufficient for numerical purposes. Moreover the larger the variable is, the more readily does the series yield the results just mentioned.

If, for instance we want to use (1.1) for the numerical computation of the integral (1.3) for a value x = 6, say the terms in the series (1.1) become

$$1, -\frac{1}{6}, \frac{1}{18}, -\frac{1}{36}, \frac{1}{54}, -\frac{5}{324}, \frac{5}{324}, -\frac{35}{1944}, \ldots$$

and tend to infinity with increasing number. It is therefore evident, that if the series is terminated after the fifth or sixth term, the difference between the function $-x\, e^x\, Ei(-x)$ and the n-th partial sum of this series will be least for n=5 or n=6, namely smaller than $\neq \frac{5}{324}$ respectively. It follows from (1.8) that for a fixed n the remainder $R_n(z)$ tends to zero with increasing argument z. Therefore the sum of the first n terms of the divergent series (1.1) will represent the value of the integral (1.3) with any desired degree of accuracy if only x is sufficiently large. Divergent series with this property are called asymptotic series and (1.5) considered as an infinite series is said to represent the integral (1.4) asymptotically for large values of the argument or to be an asymptotic expansion of (1.4) for large values of z and to indicate this we write

$$(1.10) \qquad \int_0^\infty (t+z)^{-1}\, e^{-t}\, dt \sim z^{-1}(1 - 1!\, z^{-1} + 2!\, z^{-2} \ldots)$$

for large values of z in $-\pi < \arg z < \pi$.

If in (1.10) z is replaced $\frac{1}{z}$ we get

$$\int_0^\infty (1+zt)^{-1} e^{-t} dt \quad 1 - 1!z + 2!\, z^2 - \ldots \tag{1.11}$$

and we call this for all $z \neq 0$ divergent series on the r.h.s. of (1.11) an asymptotic expansion of the integral on the l.h.s. of (1.11) for small values of z in $-\pi < \arg z < \pi$.

A generalization of (1.4) is

$$F(z) = \int_0^\infty (z+t)^{-1} f(t) dt = z^{-1} \int_0^\infty (1 + \frac{t}{z})^{-1} f(t) dt. \tag{1.12}$$

Again $z = |z|\, e^{i\vartheta}$ is a point in the cut complex z plane $(-\pi < \vartheta < \pi)$ and the same analysis as performed before applies to this case too and (1.12) may be represented in the form

$$F(z) = z^{-1} \sum_{n=0}^{m} a_n z^{-n} + R_m(z) \tag{1.13}$$

Here

$$a_n = (-1)^n \int_0^\infty f(t) t^n dt \tag{1.14}$$

$$R_m(z) = (-1)^{m+1} z^{-m-1} \int_0^\infty (1 + \frac{t}{z})^{-1} t^{m+1} f(t) dt \tag{1.15}$$

with

$$|R_m(z)| < |z|^{-m-1} \int_0^\infty t^{m+1} |f(t)| dt, \quad \cos\vartheta \geq 0 \tag{1.16}$$

$$|R_m(z)| < |z|^{-m-1} |\mathrm{cosec}\, \vartheta| \int_0^\infty t^{m+1} |f(t)| dt, \quad \cos\vartheta \leq 0 \tag{1.17}$$

provided the integrals on the r.h.s. of (1.14) to (1.17) exist.

Again, if $z = x$ is real and positive and if furthermore $f(t) > 0$ in $(0, \infty)$, then from (1.15)

$$R_m(x) < (-1)^{m+1} x^{-m-1} \int_0^\infty t^{m+1} f(t) dt = a_{m+1}\, x^{-m-1}. \tag{1.18}$$

Thus the absolute value of the remainder is less than the absolute value of the first neglected term and has the same sign. Again we may write

$$(1.19) \qquad \int_0^\infty (t+z)^{-1} f(t)dt \sim z^{-1}(a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots)$$

for large values of z in $-\pi < \arg z < \pi$, and

$$(1.20) \qquad \int_0^\infty (1+zt)^{-1} f(t)dt \sim a_0 + a_1 z + a_2 z^2 + \dots$$

for small values of z in $-\pi < \arg z < \pi$ and the a_n are given by (1.14), provided the integrals exist.

It may be mentioned that the problem to determine the function $f(t)$ in (1.14) when the a_n are given or in other words by given asymptotic expansion on the r.h.s. of (1.20) to find the function on the l.h.s. of (1.20) which it represents asymptotically, leads to the so called Stieltjes momentum problem.

The question under which condition the system (1.14) by given $a_n (n = 0,1,\dots,\infty)$ has a unique solution $f(t)$ has been investigated by Stieltjes.

The first systematic investigations of divergent series of the type considered before were made almost simultaneously by Th. J. Stieltjes (Annales scientifigues de l'ecole normale superieure, 3(1886) 201-258 and H. Poincare (Acta Math. 8, 1886, 295-344).

Stieltjes calls the series semi convergent. Poincare on the other hand speaks of asymptotic series.

2. Euler's and Plana's summation formulas. Series of the asymptotic type were produced for the first time by Euler's summation formula (Knopp, K. Theory and application of infinite series, London, 1928, p. 528).

Let f(x) be a real function of the real variable x, continuous with all its derivatives in $x \geq 0$. Then,

$$\sum_{l=0}^{n} f(l) = \int_0^n f(x)dx + \tfrac{1}{2}(f_n + f_o) + \tag{2.1}$$

$$+ \frac{B_2}{2!}(f_n{}' - f_o{}') + \frac{B_4}{4!}(f_n{}''' - f_o{}''') + \dots +$$

$$+ \frac{B_{2k}}{(2k)!}(f_n^{(2k-1)} - f_o^{(2k-1)} + R_k$$

$$R_k = \int_0^n f^{(2k+1)}(x)\ P_{2k+1}(x). \tag{2.2}$$

n, k are positive integers. $f_n^{(2k-1)}$ and $f_o^{(2k-1)}$ respectively means the (2k-1)th derivative of the function f(x) at the point x = n and x = 0 respectively. $P_{2k+1}(x)$ is the (2k+1)th Bernoulli polynomial which for k = 1,2,3,... in $0 \leq x \leq 1$ can be represented by

$$P_{2k+1}(x) = 2(-1)^{k+1}(2k+1)! \sum_{m=1}^{\infty} (2\pi m)^{-2k-1} \sin(2\pi mx) \tag{2.3}$$

The Bernoulli numbers B_l are defined by

$$t(e^t-1)^{-1} = \sum_{l=0}^{\infty} B_l\ t^l/l! \tag{2.4}$$

The series $\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}[f_n{}^{(2k-1)} - f_o{}^{(2k-1)}]$

will turn out divergent for almost all the functions f(x) which occur in applications, no matter what n in (2.1) may be. Thus (2.1) suggests a summation process for a certain type of divergent series. Now it can be shown (Knopp, K. Theory and application of infinite series, London, 1928, p. 539) that if $f(x)$ is of constant sign for $x > 0$ and together with all its derivatives tends monotonously to zero with $x \to \infty$ the remainder R_k in (2.1) can be written as

$$(2.5) \qquad R_k = \vartheta \frac{B_{2k+2}}{(2k+2)!} [f_n^{(2k+1)} - f_0^{(2k+1)}] \qquad 0 < \vartheta < 1,$$

In other words the remainder has the same sign and is numerically less than the first neglected term.

<u>Example</u>: <u>Asymptotic expansion of the function</u> $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$

We apply (2.1) to the function $f(x) = (y+x)^{-1}$, $(y > 0)$. Hence $f^{(n)}(x) = (-1)^n \, n! \, (y+x)^{-n-1}$

Obviously $f(x)$ is of constant sign for $x > 0$ and together with all its derivatives tend monotonously to 0 with $x \to \infty$, so that the remainder in (2.1) is of the form (2.5).

Euler's summation formula gives

$$y^{-1} + (y+1)^{-1} + (y+2)^{-1} + \dots + (y+n)^{-1} = \int_0^n (y+x)^{-1}\,dx +$$
$$+ \tfrac{1}{2}[y^{-1} + (y+n)^{-1}] + \frac{B_2}{2!}[y^{-2} - (y+n)^{-2}] + \frac{B_4}{4}[y^{-4} - (y+n)^{-4}]$$
$$+ \dots + \frac{B_{2k}}{2k}[y^{-2k} - (y+n)^{-2k}] + R_k$$

Hence

$$\log n - y^{-1} - (y+1)^{-1} \dots - (y+n)^{-1} = \log\left(\frac{n}{y+n}\right) + \log y -$$
$$- \tfrac{1}{2}[y^{-1} + (y+n)^{-1}] - \frac{B_2}{2}[y^{-2} - (y+n)^{-2}] \dots - \frac{B_{2k}}{2k}[y^{-2k} - (y+n)^{-2k}]$$
$$- R_k.$$

The l.h.s. tends to the function $\psi(y)$ when n tends to infinity and we get

$$(2.6)\quad \lim_{n\to\infty}\left[\log n - y^{-1} - (y+1)^{-1} \ldots - (y+n)^{-1}\right] = \psi(y) =$$

$$= \log y - \tfrac{1}{2}\, y^{-1} - \sum_{e=1}^{k} \frac{B_{2l}}{2l} y^{-2l} - R_k$$

with

$$(2.7)\quad R_k = \vartheta \frac{B_{2k+2}}{2K+2}\, y^{-2k-2}, \qquad 0 < \vartheta < 1$$

The series on the right hand side of (2.6) considered as an infinite series is divergent for any finite positive y, but represents the function $\psi(y)$ asymptotically for large positive y so that we can write

$$(2.8)\quad \psi(y) - \log y \sim -\tfrac{1}{2}\, y^{-1} - \frac{B_2}{2}\, y^{-2} - \frac{B_4}{4}\, y^{-4} \ldots$$

for large positive y.

A formula similar to Euler's summation formula (2.1) in case that f(w) is an analytic function of the complex variable w has been established by Plana (Lindelof, E. Le calcul des residus, New York, 1937, p. 63; Ford, W.B. Divergent series, New York, 1916, p. 12). Using this formula, the result (2.8) can be shown to be true for complex values of y for $-\pi < \arg y < \pi$. Or

$$(2.9)\quad \psi(z) - \log z \sim -\tfrac{1}{2}\, z^{-1} - \frac{B_2}{2}\, z^{-2} - \frac{B_4}{4}\, z^{-4} \ldots$$

for large z in $-\pi < \arg z < \pi$.

3. <u>Definition of asymptotic series</u>.

According to the definition as given by Poincare (1886) a function $f(z)$ is said to possess an asymptotic expansion for a large modulus of z and for a certain range of values of arg z, if for such values of arg z

$$(3.1) \qquad \lim_{z \to \infty} z^n \left[f(z) - a_0 - a_1 z^{-1} \ldots - a_n z^{-n}\right] \to 0$$

when $z \to \infty$ along a radius for every fixed $n = 0,1,2,\ldots$ in $\vartheta_1 < \arg z < \vartheta_2$, and we write

$$(3.2) \qquad f(z) \sim a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots \qquad \vartheta_1 < \arg z < \vartheta_2$$

The question whether a function $f(z)$ possesses an asymptotic expansion and what the values of the coefficients are is immediately settled in theory by the fact that the successive limiting values for $z \to \infty$ along a radius

$$(3.3) \qquad \begin{array}{ll} f(z) \to a_0 \,, & f(z) - a_0 \to a_1 \\ f(z) - a_0 - a_1 z^{-1} \to a_2, & f(z) - a_0 - a_1 z^{-1} - a_2 z^{-2} \to a_3 \end{array}$$

must exist. This shows that any function can have only one asymptotic expansion. On the other hand for

$$(3.4) \qquad f(z) = e^{-z} \quad , \quad -\frac{\pi}{2} < \arg z < \frac{\pi}{2}$$

all the a_n are zero since $z^m e^{-z} \to 0$ for every integer $m > 0$ when $z \to \infty$ in $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$.

Thus

$$(3.5) \qquad e^{-z} \sim 0 + 0\, z^{-1} + 0\, z^{-2} + \ldots, \; -\frac{\pi}{2} < \arg z < \frac{\pi}{2},$$

A result, which shows that different functions may have the same asymptotic expansion. Thus if $f(z)$ has an asymptotic expansion in $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$, $f(z) + a\, e^{-bz}$ ($b > 0$) has the same asymptotic expansion in $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$. If we write (3.2) in the form

$$(3.6) \qquad f(z) = a_o + a_1\, z^{-1} + a_2\, z^{-2} + \ldots + a_n\, z^{-n} + w_n(z)\, z^{-n}$$

with $w_n(z) \rightarrow 0$ if $z \rightarrow oo$ in $\vartheta_1 < \arg z < \vartheta_2$, the following theorems are readily established. Let be

$$(3.7) \qquad f_1(z) = a_o + a_1\, z^{-1} + \ldots + a_n\, z^{-n} + w_n^{(1)}(z)\, z^{-n}$$

$$(3.8) \qquad f_2(z) = b_o + b_1\, z^{-1} + \ldots + b_n\, z^{-n} + w_n^{(2)}(z)\, z^{-n}$$

with $w_n^{(1)}(z) \rightarrow 0$ and $w_n^{(2)}(z) \rightarrow 0$ when $z \rightarrow oo$ in $\emptyset_1 < \arg z < \emptyset_2$.

a) If $f_1(z) = f_2(z)$, then $a_n = b_n$ (uniqueness theorem)

b) If $F(z) = \quad f_1(z)+f_2(z) = c_o+c_1\, z^{-1}+\ldots+ c_n z^{-n}+w_n(z)z^{-n}$

then $c_n = a_n+b_n$ (sum theorem)

c) If $F(z)=f_1(z)f_2(z)=c_o+c_1z^{-1}+\ldots+c_nz^{-n}+w_n(z)z^{-n}$

then $c_n = \sum_{o}^{n} a_l\, b_{n-l}$ (product theorem)

d) If $F(z) = \dfrac{f_1(z)}{f_2(z)} = c_o+c_1\, z^{-1}+\ldots+ c_n\, z^{-n} + w_n(z)\, z^{-n}$

then $b_oc_o = a_o$, $b_1c_o + b_oc_1 = a_1$, $b_2c_o+b_1c_1 + b_oc_2 = a_2$

(quotient theorem)

e) If $f(z) = a_2z^{-2}+a_3z^{-3} +\ldots+a_nz^{-n} + w_n(z)\, z^{-n}$

then

$$\int_{0}^{\infty} f(z)dz = -a_2 z^{-1} - \frac{a_3}{2} z^{-2} \ldots \frac{a_n z^{-n+1}}{n-1} + \varepsilon_n(z)\, z^{-n+1} ,$$

the path of integration being taken along the asymptotic radius vector (integration theorem).

If

(f) $\quad f(z)=a_1 z^{-1}+a_2 z^{-2} +\ldots+a_n z^{-n} + w_n(z)z^{-n}$

then

$$f'(z)= -a_1 z^{-1}-2a_2 z^{-3}-\ldots- n\, a_n z^{-n-1} - \varepsilon_n(z)z^{-n-1}$$

provided $f'(z)$ can be developed asymptotically (differentiation theorem). This provision can be omitted in the complex case (see p.70).
In a) to f) $w_n(z)$ and $\varepsilon_n(z) \to 0$ when $z \to \infty$.

<u>Example to theorem f</u>

From

$$f(x) = e^{-x} \sin(e^{x}) \sim 0 + 0\, x^{-1} + 0\, x^{-2} + \ldots$$

for large positive x does not follow

$$f'(x) = -e^{-x}\sin(e^{x}) + \cos(e^{x}) \sim 0 + 0\, x^{-1} + 0\, x^{-2} + \ldots$$

because $f'(x) = -e^{-x} \sin(e^{x}) + \cos(e^{x})$ is oscillatory for real positive x.

Poincare's definition (3.1) of an asymptotic series may be slightly generalized in the following manner.

1) Let be $\emptyset(z)$ a function defined for $|z| > \rho$ and $\vartheta_1 < \arg z < \vartheta_2$. Furthermore let be

$$\sum_{n=0}^{\infty} a_n z^{-\alpha_n}$$

an infinite, generally divergent series with arbitrary real α_n such that

$$\alpha_0 < \alpha_1 < \alpha_2 < \ldots \to +\infty \ .$$

This series is called an asymptotic expansion of $\emptyset(z)$ at $z = \infty$

$$\emptyset(z) \sim \sum_{n=0}^{\infty} a_n z^{-\alpha_n} \tag{3.9}$$

if

$$z^{\alpha_n} [\emptyset(z) - \sum_{l=0}^{n} a_l z^{-\alpha_l}] \to 0 \text{ or}$$

$$z^{\alpha_{n+1}} [\emptyset(z) - \sum_{l=0}^{n} a_l z^{-\alpha_l}] \to a_{n+1}$$

when $|z| \to \infty$ in $\vartheta_1 < \arg z < \vartheta_2$. Corresponding.

2) Let be $\emptyset(z)$ a function defined for $|z| < \varrho$ and $\vartheta_1 < \arg z < \vartheta_2$. Furthermore let be

$$\sum_{n=0}^{\infty} a_n z^{\alpha_n}$$

an infinite generally divergent series with arbitrary real α_n such that

$$\alpha_0 < \alpha_1 < \alpha_2 < \ldots \to +\infty \ .$$

This series is called an asymptotic expansion of $\emptyset(z)$ at $z=0$

$$\emptyset(z) \sim \sum_{n=0}^{\infty} a_n z^{\alpha_n} , \tag{3.10}$$

if

$$z^{-\alpha_n}[\emptyset(z) - \sum_{l=0}^{n} a_l z^{\alpha_l}] \to 0 \text{ , or}$$

$$z^{-\alpha_{n+1}}[\emptyset(z) - \sum_{l=0}^{n} a_l z^{\alpha_l}] \to a_{n+1}$$

when $|z| \to 0$ in $\vartheta_1 < \arg z < \vartheta_2$.

Borel's associated series

Borel associates with the divergent series

$$a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n} + \ldots$$

a quantity S which shall be equal to the sum s of the series if it happens to be convergent and which shall have a definite meaning if the series be divergent. The series associated with (3.2) is

$$\Phi(t) = a_0 + a_1 + \frac{a_2}{2!} t^2 + \ldots \quad . \tag{3.11}$$

And Borel defines as the "sum" S of (3.2)

$$S = \int_0^\infty e^{-t} \Phi(\tfrac{t}{z}) dt \tag{3.12}$$

It can be shown that the integral (3.12) admits an expansion of the form (3.2). For reasons of simplicity let us assume that z be real and positive equal to x. Furthermore let $\Phi(t)$ be continuous for $0 \leqq t < \infty$ and let be $\Phi(t)$ such that

$$\lim_{t \to \infty} e^{-lt} \frac{d^n \Phi(t)}{dt^n} = 0 , \quad n = 0,1,2,\ldots$$

and l be an assigned positive number. Using this relation and (3.11) we get after n+1 partial integrations of (3.12) provided $x > 1$,

$$S(x) = a_0 + a_1 x^{-1} + \ldots + a_n x^{-n} + x^{-n-1} \int_0^\infty e^{-t} \Phi^{(n+1)}(\tfrac{t}{x}) dt$$

But $|\Phi^{(n+1)}(t)| < A e^{lt}$ in $0 \leqq t < \infty$

Therefore

$$x^{-n-1} \left| \int_0^\infty e^{-t} \Phi^{(n+1)}(\tfrac{t}{z}) dt \right| < A x^{-n} (x-1)^{-1}$$

for $x > 1$. Hence

$$\lim_{x \to \infty} x^n [S(x) - a_0 - a_1 x^{-1} \dots a_n x^{-n}] = 0$$

and this proves our statement.

4) <u>Asymptotic behavior of a sequence of numbers (Darboux's method)</u>

The asymptotic behavior of a sequence of numbers $a_n (n=0,1,2,\dots)$ with a finite lim sup. $\left| a_n \right|^{\frac{1}{n}}$ for large n can be determined by means of a method given by Darboux. (Journal de Math. (3) 4, 1878, 5-56; 377-416). With these a_n we can define a function f(z) by

(4.1) $$f(z) = \sum_{n=0}^{\infty} a_n z^n ,$$

which represents an analytic function within a circle C with the radius

(4.2) $$r = [\limsup \left| a_n \right|^{\frac{1}{n}}]^{-1}$$

From (4.1) we have by Cauchy's theorem

(4.3) $$2\pi i \; a_n = \oint z^{-n-1} f(z) dz .$$

The path of integration be a circle C_1 with a radius $\rho < r$ around $z = 0$. But we may take $\rho = r$ if f(z) is such, that its boundary values on the circle of convergence define a continuous function along C (boundary function of f(z)). Thus, with $r = re^{i\phi}$

(4.4) $$2\pi \; r^n \; a_n = \int_0^{2\pi} f(re^{i\phi}) e^{-in\phi} d\phi$$

The r.h.s. of (4.4) represents the Fourier coefficients of a continuous function and these tend to zero if n tends to infinity. This leads to

(4.5) $$r^n \; a_n \to 0 \quad \text{for } n \to \infty .$$

The case when the boundary function of $f(z)$ is not continuous may be treated as follows: We determine a series $\phi(z) = \sum_0^\infty b_n z^n$ (comparison function) with the same radius of convergence r such that the boundary function of

$$(4.6) \qquad F(z) = f(z) - \phi(z) = \sum_0^\infty (a_n - b_n) z^n$$

is continuous. Instead of (4.5) we obtain now

$$(4.7) \qquad r^n(a_n - b_n) \longrightarrow 0 \quad \text{for } n \longrightarrow \infty .$$

The approximation (4.7) can be improved if the boundary function of $F(z)$ has a continuous derivative of the order k.

We apply then the preceeding results to the k-th derivative of the boundary function of $f(z)$ whose Fourier coefficients are asymptotically equal to $n^k a_n$ and obtain

$$(4.8) \qquad n^k r^n (a_n - b_n) \rightarrow 0 \quad \text{for } n \longrightarrow \infty .$$

<u>Example</u>

As an example for Darboux's method we use its simplest form (4.7) to determine the asymptotic behavior of Legendre's polynomials $P_n(\cos \phi)$ for large n and fixed ϕ ($0 \leq \phi \leq \pi$). The Legendre polynomials are defined by

$$(4.9) \quad f(z) = (1-2z\cos\phi + z^2)^{-\frac{1}{2}} = [(1-ze^{i\phi})(1-ze^{-i\phi})]^{-\frac{1}{2}}$$
$$= \sum_{n=0}^\infty P_n(\cos\phi)\, z^n .$$

Obviously $f(z)$ has a pole of the order $\frac{1}{2}$ for $z = e^{i\phi}$ and $z = e^{-i\phi}$. The radius of convergence of the series in (4.9) is therefore $r = 1$. The character of $f(z)$ as given by (4.9) suggests to choose a companion function $\emptyset(z)$ the expression

$$(4.10)\quad \emptyset(z) = \sum_{n=0}^{\infty} b_n z^n = [(1-ze^{i\emptyset})(1-e^{-i2\emptyset})]^{-\frac{1}{2}}+[(1-ze^{i\emptyset})(1-e^{i2\emptyset})]^{-\frac{1}{2}}$$

Applying the binomial theorem we find easily

$$(4.11)\quad b_n = c_n\,[e^{in\emptyset}(1-e^{-i2\emptyset})^{-\frac{1}{2}} + e^{-in\emptyset}(1-e^{i2\emptyset})^{-\frac{1}{2}}]$$

where

$$(4.12)\quad c_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} = \frac{2^{-2n}\,(2n)!}{(n!)^2} = \frac{\pi^{-\frac{1}{2}}\,\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}$$

Therefore

$$(4.13)\quad b_n = (\tfrac{1}{2}\pi \sin\emptyset)^{-\frac{1}{2}}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\cos[(n+\tfrac{1}{2})\emptyset - \tfrac{\pi}{4}]$$

Thus using (4.7) we obtain for large n

$$(4.14)\quad P_n(\cos\emptyset) \rightarrow (\tfrac{1}{2}\pi \sin\emptyset)^{-\frac{1}{2}}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\cos[(n+\tfrac{1}{2})\emptyset - \tfrac{\pi}{4}]$$

By Stirling's theorem we can write

$$\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \sim n^{-\frac{1}{2}} \qquad \text{for large } n$$

This gives Laplace's formula

$$(4.15)\quad P_n(\cos\emptyset) \sim (\tfrac{1}{2}\pi n \sin\emptyset)^{-\frac{1}{2}}\cos[(n+\tfrac{1}{2})\emptyset - \tfrac{\pi}{4}]$$

for large n and $0 \lneqq \emptyset \lneqq \pi$.

5. Asymptotic behavior of a function which is represented as a Laplace transform

An often successful procedure to obtain an asymptotic expansion of a given function $f(z)$ is to represent this function in the form of an integral transform

$$(5.1)\quad f(z) = \int_{-\infty}^{\infty} \emptyset(x)\,K(x,z)\,dx.$$

$K(x,z)$ is the kernel of the transform and the function $\emptyset(x)$ may vanish over parts of the interval $(-\infty, \infty)$. The basic problem is to investigate if it is possible for an assigned kernel function $K(x,z)$ to describe the behavior of $f(z)$ for large z, say, by the behavior of $\emptyset(x)$ for some isolated distinguished points. (G. Doetsch, Journat f. Math., 167, 1932, 274-293).

We restrict ourselves here to the case that (5.1) may be a Laplace transform i.e. we choose

$$K(x,z) = e^{-xz}$$

and $\emptyset(x)$ equal to zero in $-\infty < x < 0$ and equal to $F(x)$ in $0 \leqq x < \infty$. Therefore we represent $f(z)$ by

$$f(z) = \int_0^\infty F(x)\, e^{-xz}\, dx. \tag{5.2}$$

In this case we will show that the behavior for large z is determined by the behavior of $F(x)$ for real positive x near the origin. In this connection we establish 3 theorems, a, b, c.

<u>Theorem a</u>

$$f(z) = \int_0^\infty e^{-xz}\, F(x)dx$$

be convergent in a half place $\text{Re } z > \sigma$. Then if

$$F(x) = 0 \quad \text{for} \quad 0 < x < a\,,$$

$$f(z) = O(e^{-a\text{Re } z}) \quad \text{if} \quad z \to \infty \text{ in } |\arg z| < \frac{\pi}{2}$$

<u>Proof</u>

$$f(z) = \int_a^\infty e^{-zx}\, F(x)dx = e^{-az} \int_0^\infty e^{-zt}\, F(a+t)dt$$

But by a well known theorem of the theory of the Laplace transform (Doetsch, G. Theorie und Anwendung der Laplace Transtormation, Berlin, 1937, p. 49) the integral tends to 0 as z tends to infinity in $|\arg z| < \frac{\pi}{2}$ or

$$\int_0^\infty e^{-zt} F(a+t)dt = o(1) \quad \text{if} \quad z \to \infty$$ and this establishes theorem a.

<u>Theorem b</u>.

$$f(z) = \int_0^\infty e^{-xz} F(x)\,dx$$

be convergent in a half plane $\operatorname{Re} z > \sigma$. Then, if

$$F(x) \sim Bx^{\beta}, \quad (\operatorname{Re}\beta > -1) \quad \text{if } t \to 0,$$

$$f(z) \sim B \int_0^\infty x^{\beta} e^{-xz}\,dx = B\,\Gamma(\beta+1) z^{-\beta-1}$$

for $z \to \infty$ in $|\arg z| < \frac{\pi}{2}$, by (1.6).

More precisely, when

$$F(x) = B x^{\beta} + \varepsilon(x) x^{\beta} \quad \text{with } \varepsilon(x) \to 0 \text{ when } x \to 0$$

then

$$f(z) - B\,\Gamma(\beta+1)\, z^{-\beta-1} = \eta(z)\, z^{-\beta-1}$$

with $\eta(z) \to 0$ when $z \to \infty$ in $|\arg z| < \frac{\pi}{2}$.

<u>Proof</u>

$$f(z) = \int_0^\infty [B x^{\beta} + \varepsilon(x) x^{\beta}]\, e^{-xz}\,dx =$$

$$= B\int_0^\infty x^{\beta} e^{-xz}\,dx + \int_0^T \varepsilon(x)\, x^{\beta} e^{-xz}\,dx + \int_T^\infty \varepsilon(x) x^{\beta} e^{-xz} dx$$

and T is chosen such, that $|\varepsilon(x)| \leqq \eta(T)$ for $0 < t < T$.

The last integral is $o(e^{-T \operatorname{Re} z})$ for large z by theorem a and we get using (1.6)

$$|f(z) - B\,\Gamma(\beta+1) z^{-\beta-1}| \leqq \eta \int_0^T e^{-x \operatorname{Re} z} x^{\operatorname{Re}\beta}\,dx + o(e^{-T \operatorname{Re} z})$$

$$\leqq \eta\,(1+\operatorname{Re}\beta)(\operatorname{Re} z)^{-1-\operatorname{Re}\beta} + o(e^{-T \operatorname{Re} z})$$

using (1.6) again. But with

$$z = |z|\, e^{i\psi} \text{ we have}$$

$$(\mathrm{Re}\ z)^{-1-\mathrm{Re}\beta} = |z^{-1-\beta}|(\cos\psi)^{-1-\mathrm{Re}\beta}\, e^{-\psi\, \mathrm{Im}\,\beta}$$

and

$$|f(z) - B\,\Gamma(\beta+1)z^{-1-\beta}| \leq A\eta|z^{-1-\beta}| + o(e^{-T\ \mathrm{Re}\ z})$$

If for large z we choose $T = O((\mathrm{Re}\ z)^{-\frac{1}{2}})$, say, then $T\ \mathrm{Re}\ z = O((\mathrm{Re}\ z)^{\frac{1}{2}})$ and $\eta \to 0$ when $z \to \infty$ in $|\arg z| < \frac{\pi}{2}$. This proves theorem b.

<u>Theorem c.</u>

Let F(x) for small positive x be asymptotically represented by

$$F(x) \sim \sum_{n=0}^{\infty} c_n\, x^{\alpha_n}, \quad \mathrm{Re}\alpha_0 < \mathrm{Re}\alpha_1 < \dots + \infty,$$
$$\mathrm{Re}\alpha_0 > -1$$

i.e. by (3.10).

$$G(x) = F(x) - \sum_{l=0}^{n} c_l\, x^{\alpha_l} = c_{n+1}x^{\alpha_{n+1}} + \varepsilon(x)\, x^{\alpha_{n+1}}$$

with $\varepsilon(x) \to 0$ when $x \to 0$.

We apply theorem b to G(x) and get

$$\int_0^{\infty} G(z)\, e^{-xz}\, dx = \int_0^{\infty} F(x)\, e^{-xz}\, dx - \sum_{0}^{n} c_l\, \Gamma(1+\alpha_l)z^{-1-\alpha_l}$$
$$= c_{n+1}\,\Gamma(1+\alpha_{n+1})z^{-1-\alpha_{n+1}} + \eta(z)z^{-1-\alpha_{n+1}}$$

with $\eta(z) \to 0$ when $z \to \infty$ in $|\arg z| < \frac{\pi}{2}$, or

$$f(z) = \int_0^{\infty} F(x)\, e^{-xz}\, dx \sim \sum_{l=0}^{\infty} c_l\, \Gamma(1+\alpha_l)z^{-1-\alpha_l}$$
$$|\arg z| < \frac{\pi}{2}.$$

Example for theorem c

Let be $f(z) = J_v^2(\sqrt{2az}) + Y_v^2(\sqrt{2az})$

where J_v and Y_v mean Bessel's and Neumann's cylinder function respectively. By means of a well known integral formula we can represent f(z) in the form (5.2) namely

$$(5.3)\quad J_v^2(\sqrt{2az}) + Y_v^2(\sqrt{2az}) = 2\pi^{-2}\int_0^\infty e^{-zx}x^{-1}e^{ax^{-1}}K_v(ax^{-1})\,dx$$

i.e. $F(x) = x^{-1}e^{ax^{-1}}K_v(ax^{-1})$

In order to obtain an asymptotic expansion for f(z) for large z it is according to theorem c only necessary to find an expansion that represents F(x) asymptotically for small positive x. But in this case this is simply the asymptotic expansion of the modified Hankel function for large positive argument which is known. We have then (Watson, G. N. Bessel functions, Cambridge 1944, p.207) for small positive x

$$F(x) \sim \left(\frac{\pi}{2a}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(v+m+\frac{1}{2})}{m!\,\Gamma(v-m+\frac{1}{2})}\,(2a)^{-m}x^{m-\frac{1}{2}}$$

and we get applying theorem c

$$J_v^2(\sqrt{2\ az}) + Y_v^2(\sqrt{2\ az}) \sim$$

$$\sim 2^{\frac{1}{2}}a^{-\frac{1}{2}}\pi^{-3/2}\sum_{m=0}^{\infty}\frac{\Gamma(v+m+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{m!\,\Gamma(v-m+\frac{1}{2})}\;z^{-m-\frac{1}{2}}$$

and this asymptotic expansion is valid for large z and $|\arg z| < \frac{\pi}{2}$.

6. Asymptotic behavior of a function which is represented as the inverse of a Laplace transform (Haar's method).

A method to determine the asymptotic behavior of a function f(x) for large x corresponding to Darboux's method for sequences of numbers (see 4) has been worked out by Haar (Math. Annalen, 96, 1927, 69-107). Before outlining this method we quote two well known theorems of the theory of the Laplace transform.

Lemma a

$\int_{-\infty}^{\infty} e^{ix\eta} g(\eta)\, d\eta$ tends to zero if x tends to infinity, provided $g(\eta)$ is such that the integral is uniformly convergent with with respect to the upper and to the lower limit for all $x \geqq X$.

(Doetsch, Theorie und Auwendung der Laplace Transformation, Berlin, 1937, p. 50). This theorem is an extension of Riemann's Lemma for Fourier coefficients. A function $g(\eta)$ with this property is called of the Fourier character for $\eta = \pm\infty$ for sufficiently large x.

Lemma b.

f(x) be a function of the real variable, defined for $x > 0$ and continuous for $x > X$. (X fixed $\geqq 0$). The Laplace Transform of f(x)

$$\text{(6.1)} \qquad \varnothing(z) = \int_0^{\infty} e^{-zx} f(x)dx$$

be convergent for a real $z = \sigma_0$ and such that $\varnothing(\sigma + i\eta)$ for a $\sigma > \sigma_0$ is of the Fourier character for $\eta = \pm\infty$ and sufficiently large t, i.e.

$$\text{(6.2)} \qquad \int_{-\infty}^{\infty} \varnothing(\sigma + i\eta)\, e^{ix\eta}\, d\eta$$

is uniformly convergent for all $x > X$ with respect to both limits.

Then

$$(6.3) \quad f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{xz} \phi(z)dz = \frac{1}{2\pi} e^{x\sigma} \int_{-\infty}^{\infty} e^{ix\eta} \phi(\sigma + i\eta)d\eta$$

(Doetsch, Theorie und Auwendung der Laplace Transformation, Berlin, 1937, p. 107.)

(6.3) is the inversion formula of the Laplace transform (6.1) If the Laplace transform $\phi(z)$ of $f(x)$ satisfies all the conditions mentioned in Lemma b, we obtain immediately from (6.3) for large x, using Lemma a.

$$(6.4) \qquad \lim_{x \to \infty} e^{-x\sigma} f(x) = 0$$

or

$$(6.5) \qquad f(x) = o(e^{x}) \quad \text{for} \quad x \longrightarrow \infty .$$

Equation (6.4) corresponds to (4.5) in Darboux's theory. The approximation (6.4) or (6.5) will be best for the smallest possible value of σ for which the conditions in Lemma b are satisfied. In the most favorable case σ can be chosen to be the abscissa of convergence of the integral in (6.1) or some number in an arbitrarily small neighborhood.

We will now formulate additional conditions which will enable us to replace the path of integration in (6.3) consisting of the line $\xi = \sigma$ in the half plane of convergence of the integral in (6.1) by a line $\xi = a$ in the half plane of regularity of the function $\phi(z)$. For this can be larger than the half plane of convergence of the integral in (6.1). (Doetsch: Handbuch der Laplace Transformation, Basel, 1950, vol. 1, p. 152. Widder: The Laplace Transform, Princeton 1941, p. 58).

We state now the theorem:

(I) $f(x)$ be defined for $x > 0$ and continuous for $x \geqq X$. (X fixed $\geqq 0$). The Laplace transform of $f(x)$, $\emptyset(z) = \int_0^\infty e^{-zx} f(x)dx$ be convergent for a real $z = \sigma_0$. The integral $\int_{-\infty}^{\infty} e^{ix\eta}\emptyset(\sigma + i\eta)\, d\eta$ be for a $\sigma > \sigma_0$ uniformly convergent for $x \geqq X$.

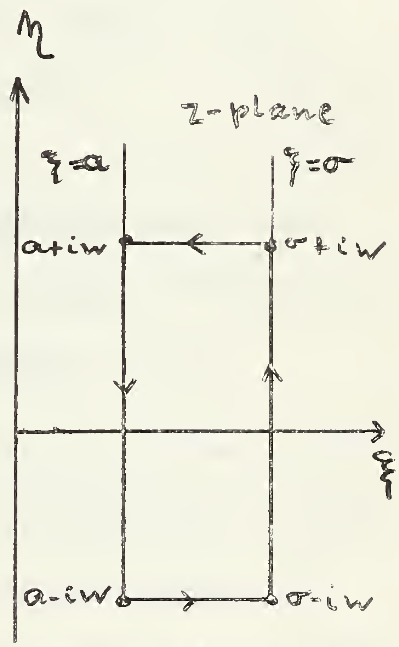
(II) $\emptyset(z)$ considered as a regular function be analytic in the half plane $\mathrm{Re}\, z > a$ ($a \leqq \sigma$) and have for $\mathrm{Re}\, z \to a$ boundary values (denoted by $\varphi(a + i\eta)$) such that the function $\varphi(z)$ which is completed by these boundary values is continuous for $\mathrm{Re}\, z \geqq a$ in the two dimensional sense. The integral $\int_{-\infty}^{\infty} e^{ix\eta}\emptyset(a+i\eta)d\eta$ be uniformly convergent for $x \geqq X$.

(III) The integrals

$$\int_{a+iw}^{\sigma+iw} e^{xz}\,\emptyset(z)dz \quad \text{and} \quad \int_{a-iw}^{\sigma-iw} e^{xz}\,\emptyset(z)dz$$

shall tend to zero if $w \to \infty$ for $x \geqq X$. Then

(6.6) $\qquad f(x) = o(e^{ax}) \qquad$ for $\quad x \to \infty$



Proof

$\emptyset(z)$ is regular inside the rectangle with the corners $a+iw$, $a-iw$, $\sigma - iw$, $\sigma + iw$ and continuous the boundary included.

Then the integral $\frac{1}{2\pi i}\int e^{xz}dz$ taken along the boundary of the rectangle is zero according to a modification of Cauchy's theorem (Heilbronn: Math. Zeitschrift, 37, 1933, 37-38). If we let w tend to infinity we obtain because of the conditions (III) and Lemma b, (6.7) $$f(x) = \frac{1}{2\pi} e^{ax} \int_{-\infty}^{\infty} e^{ix\eta} \phi(a + i\eta) d\eta.$$

Using (2) and Lemma a the result (6.6) is established. The approximation (6.6) can be improved if we assume

(IV) The conditions (I), (II), (III) are satisfied. The boundary function $\phi(a + i\eta)$ has n derivatives with respect to η. $\phi(a + i\eta)$, $\phi'(a + i\eta)$ and $\phi^{(n-1)}(a + i\eta)$ tend to zero if $\eta \longrightarrow \pm \infty$. $\phi^{(n)}(a + i\eta)$ is integrable in any finite interval and

$$\int_{-\infty}^{\infty} e^{ix\eta} \phi^{(n)} (a + i\eta)\, d\eta^{*}$$

converge uniformly for $x \geqq X$.

We have then instead of (6.7)

(6.8) $$f(x) = o(x^{-n} e^{ax}) \text{ for } x \to \infty .$$

<u>Proof</u>.

Under the assumptions mentioned above we get from (6.7) integrating by parts

$$f(x) = \frac{1}{2\pi} e^{ax} (-\frac{1}{ix})^n \int_{-\infty}^{\infty} e^{ix\eta} \phi^{(n)} (a + i\eta) d\eta.$$

This establishes (6.8).

<u>Simple singularities of the Laplace transform</u>

We assume now that the Laplace transform $\phi(z) = \int_0^{\infty} e^{-zx} f(x)dx$ of $f(x)$ is regular for Re $z > a$ but has a singularity at Rez=a.

*The fact that $\phi^{(n)}$ is of the Fourier character and $\phi^{(n-1)} \to 0$ for $\eta \to \pm \infty$ implies the Fourier character of $\phi^{(n-1)}$ to ϕ.

If it is possible to find a function $F(x)$ (comparison function) whose Laplace transform $\Phi(z) = \int_0^\infty e^{-xz} F(x)dx$, has the same singularity at Re $z = a$ (and no other), and if the difference $\phi(z) - \Phi(z)$ satisfies (IV), then

(6.9) $$\lim_{x \to \infty} x^n e^{-ax} [f(x) - F(x)] = 0.$$

We will restrict ourselves to functions $f(x)$ whose Laplace transforms $\phi(z)$ have singularities of algebraic and logarithmic character and state the theorem (V).

(A) $f(x)$ be defined for $x > 0$ and continuous for $x \geqq X$ (X fixed $\geqq 0$). The Laplace transform $\phi(z) = \int_0^\infty e^{-xz} f(x)dx$ be convergent for a real $z = \sigma_0$. The integral $\int_{-\infty}^{\infty} e^{ix\eta} \phi(\sigma + i\eta)d\eta$ be for a $\sigma > \sigma_0$ uniformly convergent for $x \geqq X$.

(B) $\phi(z)$ considered as a regular function be analytic in the open half plane Re $z > a$ ($a \leqq \sigma$). In the closed half plane Re $z \geqq a$, $\phi(z)$ shall admit one of the representations

1) $\phi_1(z) = c(z-z_0)^{-\rho} + \psi_1(z)$ Re $\rho > 0$

2) $\phi_2(z) = c(z-z_0)^{-\rho} + \psi_2(z)$ Re $\rho \leqq 0$, $\rho \neq 0, -1, -2, \ldots$

3) $\phi_3(z) = c(z-z_0)^n \log(z-z_0) + \psi_3(z)$ $n = 0, 1, 2, \ldots$

4) $\phi_4(z) = c(z-z_0)^{-\rho} \log(z-z_0) + \psi_4(z)$ $\rho \neq 0, -1, -2, \ldots$

corresponding to one of the functions $f_v(x)$, ($v = 1, 2, 3, 4$). z_0 is a point on the line Re $z = a$ and c is a constant. $\psi_v(z)$ be a continuous function in Re $z \geqq a$ (regular for Re $z > a$). The first n derivatives of the boundary function $\psi_v(a+i\eta)$ shall exist and $\psi_v^{(n)}(a+i\eta)$ be integrable in any finite interval.

(C) The integrals $\int_{a+iw}^{\sigma+iw} e^{xz}\,\phi(z)$ and $\int_{a-iw}^{\sigma-iw} e^{xz}\phi(z)dz$ shall tend to zero if $w \to \infty$ for $x \geqq X$.

(D) According to the assumptions (B) with respect to $\phi(z)$ it is obvious that the boundary function $\phi(a+i\eta)$ exists with the exception of the point $\eta = \operatorname{Im} z_o$ and is in times differentiable. The functions $\phi(a+i\eta),\ldots,\phi^{(n-1)}(a+i\eta)$ shall tend to zero if $\eta \to \pm\infty$ and the integral

$$\int_{-\infty}^{\eta_1} e^{ix\eta}\phi^{(n)}(a+i\eta)d\eta \quad \text{and} \quad \int_{\eta_2}^{\infty} e^{ix\eta}\,\phi^{(n)}(a+i\eta)d\eta$$

with fixed η_1, η_2 ($\eta_1 < \operatorname{Im} z_o < \eta_2$ for any $n = 0,1,2,\ldots$ shall be uniformly convergent for $x \geqq X$.

Under these assumptions we have the following asymptotic expressions for $f(x)$. In case 1) and 2)

$$\lim_{x\to\infty} x^n e^{-ax}\left[f(x) - \frac{c}{\Gamma(\varrho)}\, e^{z_o x}\, x^{\varrho-1}\right] = 0 \tag{6.10}$$

In case 3)

$$\lim_{x\to\infty} x^n e^{-ax}\left[f(x) + c(-1)^n n!\; e^{z_o x}\, x^{-n-1}\right] = 0 \tag{6.11}$$

In case 4)

$$\lim_{x\to\infty} x^n e^{-ax}\left\{f(x) + \frac{c}{\Gamma(\varrho)}\, e^{z_o x}\, x^{\varrho-1}\left[\log x - \frac{\Gamma'(\varrho)}{\Gamma(\varrho)}\right]\right\} = 0$$

These results can be generalized if $\phi(z)$ has a finite number of poles on $\operatorname{Re} z = a$. Is for instance for $\operatorname{Re} z \geqq a$

$$\phi(z) = \sum_{l=1}^{n} c_l\,(z - z_l)^{-\varrho_l} + \psi(z)$$

with $z_l = a + \eta_l$ and $\varrho_l \neq 0, -1, -2,\ldots$ we obtain

$$\lim_{x\to\infty} x^n e^{-ax}\left[f(x) - \sum_{l=1}^{n} \frac{c_l}{\Gamma(\varrho_l)}\, e^{z_l x}\, x^{\varrho_l-1}\right] = 0 \tag{6.10a}$$

Proof

In order to apply (6.9) we have at first to find functions $F_v(x)$ (camparison functions) whose Laplace transforms $\Phi_v(z) = \int_0^\infty e^{-xz} F_v(x)dx$ are of the form 1) to 4). These functions are respectively

1a) $F_1(x) = \frac{c}{\Gamma(\rho)} e^{z_0 x} x^{\rho-1}$, Re $\rho > 0$

2a) $F_2(x) = \begin{cases} 0 & \text{for } 0 \leqq x < 1 \\ \frac{c}{\Gamma(\rho)} e^{z_0 x} x^{\rho-1} & \text{for } x \geqq 1 \end{cases}$

Re $\rho \leqq 0, \rho \neq 0, -1, -2, \ldots$

3a) $F_3(x) = \begin{cases} 0 & \text{for } 0 \leqq x < 1 \\ -c(-1)^n n!\, e^{z_0 x} x^{-n-1} & \text{for } x \geqq 1 \end{cases}$

$n = 0, 1, 2, \ldots$

4a) $F_4(x) = \begin{cases} 0 & \text{for } 0 < x < 1 \\ \frac{-c}{\Gamma(\rho)} e^{z_0 x} x^{\rho-1}[\log x - \frac{\Gamma'(\rho)}{\Gamma(\rho)}] & \text{for } x \geqq 1 \end{cases}$

$\rho \neq 0, -1, -2, \ldots$

It can easily be shown that the Laplace transforms $\Phi_v(z)$ of these functions are respectively

1b) $\Phi_1(z) = c(z-z_0)^{-\rho}$ Re $\rho > 0$

2b) $\Phi_2(z) = c(z-z_0)^{-\rho} + Y_2(z)$

3b) $\Phi_3(x) = c(z-z_0)^n \log(z-z_0) + Y_3(x)$

4b) $\Phi_4(x) = c(z-z_0)^{-}\ \log(z-z_0) + Y_4(z)$

where $Y_v(z)$, ($v = 1,2,3,4$) is an integral function of z.

We apply now (IV) to the difference function $g_v(x)=f_v(x)-F_v(x)$, $(v = 1,2,3,4)$. The Laplace transform of this function is

$$\varphi_v(z) - \Phi_v(z) = \psi_v(z) - Y_v(z)$$

Then it is obvious that the conditions in (V) are a consequence of those in (IV) provided, that

a') $$\int_{-\infty}^{\infty} e^{ix\eta}\,\Phi_v(\sigma + i\eta)\,d\eta$$

converges uniformly for $x \geqq X$ for any $\sigma > a$.

b') $$\int_{a+iw}^{\sigma+iw} e^{xz}\,\Phi_v(z)dz \quad \text{and} \quad \int_{a-iw}^{\sigma-iw} e^{xz}\,\Phi_v(z)dz$$

tend to zero if $w \to \infty$ for $x \geqq X$.

c') $$\Phi_v^{(n)}(a + i\eta) \to 0 \quad \text{for } \eta \to \pm\infty \text{ and}$$

$$\int_{-\infty}^{\eta_1} e^{ix\eta}\,\Phi_v^{(n)}(a + i\eta)d\eta, \quad \int_{\eta_2}^{\infty} e^{ix\eta}\Phi_v^{(n)}(a+i\eta)d\eta$$

are uniformly convergent for $x \geqq X$ and any $n = 0,1,2,\ldots$

$(\eta_1 < J_m\, z_o < \eta_2)$.

It has now to be shown that the $\Phi_v(z)$ in 1b) to 4b) satisfy the conditions above. We prove this here only for the case $v=1$, i.e. we establish (6.10) under the assumptions in (V). For the remaining cases we refer to the original paper by Haar.

Since here

$$\Phi_1(z) = c(z-z_o)^{-\rho}, \qquad \operatorname{Re}\rho > 0$$

it is obvious that b') and the first part of c'), is satisfied. To prove the validity of a') and the second part of c'), we note that for a fixed $\operatorname{Re} z = \xi$ and $z_o = a + i\eta_o$

$$\Phi_1(\xi + i\eta) = c\,[\,\xi - a + i(\eta - \eta_o)]^{-\rho}$$

$$\Phi_1'(\xi + i\eta) = -i\rho c[\xi - a + i(\eta - \eta_0)]^{-\rho-1}$$

$$\Phi_1^{(n)}(\xi + i\eta) = (-i)^n c\, \rho\, (\rho+1)\ldots(\rho+n-1)[\xi - a + i(\eta - \eta_0)]^{-\rho-n}$$

We make now use of the following <u>lemma c</u>.

If $g'(\xi + i\eta)$ is absolutely integrable at $\eta = \pm\infty$ and $\lim g(\xi + i\eta) \to 0$ for $\eta \to \pm\infty$, then

$$\int_{-\infty}^{\infty} g(\xi + i\eta)\, e^{ix\eta}\, d\eta \text{ is uniformly convergent}$$

for $x > X$.

<u>Proof</u>

$$\left|\int_{\alpha}^{\beta} e^{-xi\eta} g(\xi + i\eta)\, d\eta\right| =$$

$$= \left|\frac{e^{ix\beta} g(\xi + i\beta) - e^{ix\alpha} g(\xi + i\alpha)}{ix} - \frac{1}{x}\int_{\alpha}^{\beta} e^{ix\eta} g'(\xi + i\eta)\, d\eta\right|$$

$$\leq \frac{1}{x}\left\{|g(\xi + i\beta)| + |g(\xi + i\alpha)| + \int_{\alpha}^{\beta} |g'(\xi + i\eta)|\, d\eta\right\}$$

and this can be made arbitrarily small for $x > X$ if α and β are sufficiently large. The application of this lemma proves a') and the second part of b').

<u>Asymptotic behavior of a function defined by a power series</u>.

The methods outlined before can often be used to determine the asymptotic behavior of a function which is defined by a power series. Let be

(6.13) $\quad f(y) = a_0 + a_1 y + a_2 y^2 + \ldots = \quad a_n y^n$

are integral functions of the complex variable y and let the series of the Laplace transforms of (6.13) for a real $y = x$,

$$(6.14) \qquad \phi_0(z) = \sum_0^\infty a_n \int_0^\infty e^{-xz} x^n \, dx = \sum_0^\infty a_n n! \, z^{-n-1}$$

be convergent for $|z| > R$. Then it is evident that $\varphi_0(z)$ is a function regular in the half plane Re $z > a$, where a is the abscissa of this singular point z_0 of $\phi_0(z)$ on or within the circle of convergence $|z| = R$ whose real part is a maximum. Furthermore $\phi_0(z)$ and all its derivatives (the latter at least of the second order) vanish if z tends to infinity. If now $\phi_0(z)$ is such that for Re $z \geqq a$ it admits one of the representations given in (B) we see by lemma c that all conditions in (V) are satisfied and the asymptotic behavior of $f(y)$ defined in (6.13) for large positive $y = x$ can be described by one of the formulas (6.10) to (6.12) corresponding to the character of the singular point $z = z_0$ with Re $z_0 = a$. To determine the asymptotic behavior of $f(y)$ for any complex y for large $|y|$ we put $y = x\, e^{i\vartheta}$, $(x > a)$ and investigate by the methods just described the asymptotic behavior of

$$(6.15) \qquad f(y) = \sum_0^\infty a_n \, e^{in\vartheta} x^n$$

for large positive x. We obtain now instead of (6.14)

$$(6.16) \qquad \phi_\vartheta(z) = \sum_0^\infty a_n \, e^{in\vartheta} n! \, z^{-n-1} = e^{-i\vartheta} \phi_0(ze^{-i\vartheta})$$

<u>Example</u>. As an example we consider the asymptotic behavior of an integral function given by

$$(6.17) \qquad Y(y,\alpha,s) = \sum_{n=0}^\infty \frac{y^n}{n!\,(\alpha+n)^s} = f(y)$$

$\alpha \neq 0,-1,-2,\ldots$ and s arbitrarily complex. This function has been studied by Barnes (Phil. Trans. Roy. Soc. (A) 206, 1906, 249-297). We obtain from (6.14)

$$(6.18) \qquad \phi_0(z) = \sum_0^\infty z^{-n-1}(\alpha+n)^{-s} = z^{-1}\sum_0^\infty z^{-n}(\alpha+n)^{-s}.$$

If we denote by

$$(6.19) \qquad \Phi(z,\alpha,s) = \sum_0^\infty z^n(\alpha+n)^{-s}, \qquad |z| < 1$$

we see that

$$(6.20) \qquad \phi_0(z) = z^{-1}\,\Phi(z^{-1},\alpha,s).$$

The function $\Phi(z,\alpha,s)$ defined for $|z| < 1$ by the series in (6.19) has been investigated for the first time by Lerch (Acta Math. 11, 1887, 19-24) and is called Lerch's zeta function. Barnes (Proc. London Math. Soc. 4, 1906, 284-316) has proved that apart from the essential singular point $z = \infty$ $\Phi(z,\alpha,s)$ has but one singularity at $z = 1$. Hardy (Proc. London Math. Soc. 3, 1905, 381-389) gave the representation

$$(6.21) \qquad \Phi(z,\alpha,s) = \Gamma(1-s)z^{-\alpha}(\log\tfrac{1}{z})^{s-1} + \beta(z)$$

where $\beta(z)$ is regular for Re $z \geqq 1$.

Thus

$$(6.22) \qquad \phi_0(z) = \Gamma(1-s)z^{\alpha-1}(\log\tfrac{1}{z})^{s-1} + \psi(z)$$

where $\psi(z)$ regular for Re $z \geqq 1$.

Hence by (6.16)

$$(6.23) \qquad \phi_\nu(z) = e^{-i\alpha\vartheta}z^{\alpha-1}\Gamma(1-s)(\log z - i\vartheta)^{s-1} + \psi^*(z)$$

The singular points of $\phi_\vartheta(z)$ are $z = 0$ and $z = e^{i\vartheta}$. We consider only the case $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$. Then the singular point z_0 whose real part is a maximum is the point $z_0 = e^{i\vartheta}$ and we will show that $\phi_\vartheta(z)$ admits for Re $z \geqq a = \cos\vartheta$ one of the representations 1) or 2) of (B) i.e. $\phi_\vartheta(z)$ has at $z = z_0\ e^{i\vartheta}$ a singularity of algebraic character.

We write (6.23)

(6.24) $$\phi_\vartheta(z) = e^{-i\alpha\vartheta}\Gamma(1-s)(z-e^{i\vartheta})^{s-1}z^{\alpha-1}\left(\frac{\log z-i\vartheta}{z-e^{i\vartheta}}\right)^{s-1} + \psi^*(z)$$

We expand according to Taylor's theorem $z^{\alpha-1}\left(\frac{\log z-i\vartheta}{z-e^{i\vartheta}}\right)^{s-1} =$

$$\sum_0^\infty c_n(z-e^{i\vartheta})^n ,$$

where

$$c_n = \frac{1}{n!}\frac{d^n}{dz^n}\left[z^{\alpha-1}\left(\frac{\log z-i\vartheta}{z-e^{i\vartheta}}\right)^{s-1}\right]_{z=e^{i\vartheta}}$$

Hence

(6.25) $$\phi_\vartheta(z) = \Gamma(1-s)e^{-is\vartheta}\sum_{n=0}^\infty a_n e^{-in\vartheta}(z-e^{i\vartheta})^{n+s-1} + \psi^*(z)$$

with

(6.26) $$a_n = \frac{1}{n!}\frac{d^n}{dx^n}\left[x^{\alpha-1}\left(\frac{\log x}{x-1}\right)^{s-1}\right]_{x=1}$$

or instead of (6.25)

(6.27) $$\phi_\vartheta(z) = \Gamma(1-s)e^{-is\vartheta}\sum_{n=0}^N a_n e^{-in\vartheta}(z-e^{i\vartheta})^{n+s-1} + \psi^{**}(z)$$

with

(6.28) $$\psi^{**}(z) = \psi^*(z) + \Gamma(1-s)e^{-is\vartheta}\sum_{n=N+1}^\infty a_n e^{-in\vartheta}(z-e^{i\vartheta})^{n+s-1}.$$

Since

$$\frac{d}{dz^k}(z-e^{i\vartheta})^{N+s-1} = A(z-e^{i\vartheta})^{N+s-k-1}$$

it follows from (6.28) that if M is an integer such that $M < \operatorname{Re} s < M+1$, the (M+N)th derivative of $\varphi^{**}(z)$ is continuous. (6.10a) gives then

$$\lim_{x\to\infty} x^{N+M} e^{-x\cos\vartheta}[f(x e^{i\vartheta}) - \\ - \Gamma(1-s)e^{-is\vartheta}\sum_{n=0}^{M}\frac{a_n e^{-in\vartheta}}{\Gamma(1-n-s)} e^{x e^{i\vartheta}} x^{-n-s}]$$

or with $x = y e^{-i\vartheta}$

$$(6.29)\qquad \lim_{y\to\infty} y^{N+M} e^{-\operatorname{Re} y}\left[\sum_{n=0}^{\infty}\frac{y^n}{n!(n+\alpha)^s} - \right. \\ \left. - \Gamma(1-s)\, e^y\, y^{-s}\sum_{n=0}^{M}\frac{a_n y^{-n}}{\Gamma(1-n-s)}\right] = 0$$

$$-\tfrac{1}{2}\pi < \arg y < \tfrac{1}{2}\pi$$

with the a_n given in (6.26).

The asymptotic expansion of the function f(y) defined by (6.17) for $\frac{1}{2}\pi < |\arg y| < \pi$ can not be obtained by the method developed before. It follows in this case from (6.18) and (6.16) that the singular point of $\emptyset_{\vartheta}(z)$ whose real part is a maximum is the point z = 0 which is an essential singular point of $\emptyset_{\vartheta}(z)$. But our method has so far been restricted to functions $\emptyset(z)$ with arithmetic and logarithmic singularities.

Part II

Asymptotic Solutions of Linear Differential Equations with a Parameter

by

H. A. Antosiewicz.

1. <u>Introduction</u>. These lectures represent a survey of the theory of asymptotic solutions of ordinary linear differential equations containing a parameter. We shall restrict our discussion to equations of the second order

$$y'' + a_1(x)\, y' + (\, k\, a_2(x) + a_3(x))\, y = 0 \tag{1.1}$$

because these equations are of importance in many problems of applied mathematics. They arise in the solution of linear second order partial differential equations by Bernoulli's method of separation of variables; mathematically, they are the type of equations which include as special cases the differential equations of such standard functions as the trigonometric functions, Bessel functions, Legendre functions, the functions of Mathieu, and many others.

We were guided in our presentation by an exposition of the subject by Langer [19]. We shall not strive for greatest generality here because of evident limitation of space. Whenever proofs of theorems are only outlined or details of too technical a nature omitted, these can be found in the references listed at the end.

2. <u>The basic equation</u>. It is known that the equation (1.1) can be transformed by a change of dependent variable into an equation of like form in which the coefficient of y' is identically zero. We shall assume that such a transformation has been carried out, and henceforth take as our basic equation

$$y'' + (k^2a^2(x) - b(x))y = 0 \;. \tag{2.1}$$

For convenience we write $k^2a^2(x)$ rather than $ka(x)$ although this notation shall not imply that this term is necessarily positive.

Throughout the equation (2.1) is subject to the following hypotheses:

(H_1) x is real and $x \in [x_1, x_2]$; ($x_1 = -\infty$ and $x_2 = \infty$ is permitted);

(H_2) $a^2(x)$ and $b(x)$ are real, continuous and differentiable to whatever order is required on $[x_1, x_2]$;

(H_3) k is real and positive.

It is clear that these hypotheses guarantee the applicability of the standard existence and uniqueness theorems so that for any given $x_0 \in [x_1, x_2]$ and any constants y_0, y_0' there exists a unique solution $y(x)$ of (2.1) which satisfies the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$.

3. <u>The case of</u> $a^2(x)$ <u>bounded from zero</u>. Let us assume for the present that $a^2(x)$ is of fixed sign on the interval $[x_1, x_2]$. Since $a^2(x)$ is real, we can then define $a(x)$ such that $a(x)/|a(x)| = 1$ or i according as $a^2(x) \gtrless 0$.

We begin by making the substitutions

(3.1) $t = \varrho^{-1}\int^{x}|a(x)|dx, \quad u = |a(x)|^{1/2}y, \quad i\lambda = k\varrho\left\{\frac{a(x)}{|a(x)|}\right\}$

which reduce (2.1) to the equation

$$\frac{d^2u}{dt^2} - (\lambda^2 + b_1(t))u = 0 \tag{3.2}$$

where

$$b_1 = \varrho^2|a|^{-2}\left[b + (a''/2a) - (3a'^2/4a^2)\right] \tag{3.3}$$

and the variable t ranges over an interval $[t_1, t_2]$. Note that when $|\lambda|$ is very large, then λ^2 dominates the term $\lambda^2 + b_1(t)$ in (3.2) and thus minimizes the influence of $b_1(t)$. Since the equation $\ddot{u} - \lambda^2 u = 0$ has the two solutions $u = \exp(\pm\lambda t)$, we attempt to solve (3.2) by assuming a solution of the form

$$U_n(t,\lambda) \equiv e^{\lambda t}\, v_n(t,\lambda) \tag{3.4}$$

where

$$v_n(t,\lambda) = 1 + \frac{\alpha_1}{\lambda} + \ldots + \frac{\alpha_n}{\lambda^n}. \tag{3.5}$$

On differentiating (3.4) twice, we find that $U_n(t,\lambda)$ satisfies the equation

$$\ddot{U}_n - (\lambda^2 + b_1(t) + b_2(t)/v_n(t,\lambda)\; U_n = 0 \tag{3.6}$$

where

$$b_2(t) = (2\dot{\alpha}_1 - b_1) + \sum_{j=2}^{n}\lambda^{1-j}(2\dot{\alpha}_j - \alpha_{j-1}b_1 + \ddot{\alpha}_{j-1}) + \lambda^{-n}(\ddot{\alpha}_n - \alpha_n b_1). \tag{3.7}$$

Suppose we determine the functions $\alpha_j(t)$ so that they satisfy the equations

$$\dot{\alpha}_1 = \frac{1}{2}b_1, \quad \dot{\alpha}_j = \frac{1}{2}(\alpha_{j-1}b_1 - \ddot{\alpha}_{j-1}), \quad j = 2, 3, \ldots n; \tag{3.8}$$

this clearly depends upon the integrability of $b_1(t)$ over

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It is clear that these hypotheses guarantee the applicability of the standard existence and uniqueness theorems so that for any given $x_0 \in [x_1, x_2]$ and any constants y_0, y_0' there exists a unique solution $y(x)$ of (2.1) which satisfies the initial conditions $y(x_0)=y_0$, $y'(x_0)=y_0'$.

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We begin by making the substitutions

(3.1) $t = \varrho^{-1}\int^{x}|a(x)|dx\,, \quad u = |a(x)|^{1/2} y\,, \quad i\lambda = k\varrho\left\{\frac{a(x)}{|a(x)|}\right\}$

which reduce (2.1) to the equation

(3.2) $$\frac{d^2u}{dt^2} - (\lambda^2 + b_1(t))u = 0$$

where

(3.3) $$b_1 = \varrho^2|a|^{-2}\left[b + (a''/2a) - (3a'^2/4a^2)\right]$$

and the variable t ranges over an interval $[t_1, t_2]$. Note that when $|\lambda|$ is very large, then λ^2 dominates the term $\lambda^2 + b_1(t)$ in (3.2) and thus minimizes the influence of $b_1(t)$. Since the equation $\ddot{u} - \lambda^2 u=0$ has the two solutions $u = \exp(\pm\lambda t)$, we attempt to solve (3.2) by assuming a solution of the form

(3.4) $$U_n(t,\lambda) \equiv e^{\lambda t}\, v_n(t,\lambda)$$

where

(3.5) $$v_n(t,\lambda) = 1 + \frac{\alpha_1}{\lambda} + \ldots + \frac{\alpha_n}{\lambda^n}\,.$$

On differentiating (3.4) twice, we find that $U_n(t,\lambda)$ satisfies the equation

(3.6) $$\ddot{U}_n - (\lambda^2 + b_1(t) + b_2(t)/v_n(t,\lambda)\; U_n = 0$$

where

(3.7) $$b_2(t) = (2\dot{\alpha}_1 - b_1) + \sum_{j=2}^{n} \lambda^{1-j}(2\dot{\alpha}_j - \alpha_{j-1}b_1 + \ddot{\alpha}_{j-1}) + \lambda^{-n}(\ddot{\alpha}_n - \alpha_n b_1)\,.$$

Suppose we determine the functions $\alpha_j(t)$ so that they satisfy the equations

(3.8) $$\dot{\alpha}_1 = \frac{1}{2}b_1\,, \quad \dot{\alpha}_j = \frac{1}{2}(\alpha_{j-1}b_1 - \ddot{\alpha}_{j-1})\,, \quad j = 2, 3, \ldots n;$$

this clearly depends upon the integrability of $b_1(t)$ over

$[t_1, t_2]$. Then the equation (3.6) reduces to

(3.9) $\ddot{U}_n - (\lambda^2 + b_1(t) + (\ddot{\alpha}_n - \alpha_n b_1)/\lambda^n v_n(t,\lambda))U_n = 0.$

This equation evidently resembles equation (3.2), the more so the larger $|\lambda|$ and n. We are therefore lead to presume that $U_n(t,\lambda)$ might represent an asymptotic expression of a solution of (3.2); and since similar considerations apply to $U_n(t,-\lambda)$ the same might hold for $U_n(t,-\lambda)$. This can indeed be shown [10;11].

<u>Theorem (Horn)</u>: <u>If</u> $a^2(x)$ <u>is bounded away from zero on</u> $[x_1, x_2]$ <u>and the integral</u>

$$\int_{x_1}^{x_2} \left| \frac{b}{a} + \frac{a''}{2a^2} - \frac{3a'^2}{4a^3} \right| dx$$

<u>converges</u>, <u>then the equation</u> (2.1) <u>possesses a pair of solutions</u>

$$y(x) = |a(x)|^{-\frac{1}{2}} \exp\left(\pm ik \int^x |a(x)|dx\right) \left\{1 + \frac{\beta_1(x)}{\rho k} + \ldots + \frac{\beta_n(x)}{(\rho k)^n} + \frac{\epsilon(x,k)}{(\rho k)^n}\right\}$$

<u>where</u> $\epsilon(x,k) \to 0$ <u>as</u> $\rho k \to \infty$.

It follows that if $a^2(x) < 0$ on $[x_1, x_2]$, then there exists a pair of solutions of (2.1) of the form

(3.10) $y_{1,2}(x) = |a(x)|^{-\frac{1}{2}} \exp\left(\pm k \int^x |a(x)|dx\right) \left\{1 + O(1/\rho k)\right\};$

and if $a^2(x) > 0$ on $[x_1, x_2]$, there exists solutions of the form

(3.11) $y_\gamma(x) = [a(x)]^{-\frac{1}{2}} \left\{\cos\left(k \int^x a(x)dx + \gamma\right) + O(1/\rho k)\right\}.$

Observe that the leading terms in these expressions satisfy the differential equation

(3.12) $Y'' + (k^2a^2 + (a''/2a) - (3a'^2/4a^2))\, Y = 0$

which has no singularity on $[x_1, x_2]$ since $a(x) \neq 0$ by hypothesis.

4. <u>The case of a simple zero of</u> $a^2(x)$. Suppose now that $a^2(x)$ has a simple zero at some point $x = x_0$; we may assume without loss of generality that $(x-x_0)a^2(x) > 0$ for $x \neq x_0$. Let us consider solutions of (2.1) to the right of x_0, say over an interval $[x_1, x_2]$ where $x_1 > x_0$. For sufficiently large ρk these solutions are expressible in the form (3.11), i.e. they are of oscillatory type.

Recall that we were lead to the expressions (3.10) and (3.11) by an investigation of the equation (3.2) for large values of $|\lambda|$, and that (3.2) was obtained from (2.1) by the transformation of variables (3.1) in which the constant ρ as well as the lower limit of the integral were left unspecified. Let us now choose these constants as follows:

$$t = \rho^{-1}\int_{x_1}^{x} |a(x)|\,dx\ , \quad \rho = \int_{x_0}^{x_1} |a(x)|\,dx\ . \tag{4.1}$$

Then

$$|\lambda| = k\int_{x_0}^{x_1} |a(x)|\,dx \tag{4.2}$$

and since $|\lambda|$ must be large, we see that the point x_1 may be taken closer and closer to x_0 if k is taken larger and larger. This clearly establishes a relation between the size of k and the distance of x_1 from x_0 which must be satisfied in order that a solution of (2.1) may be expressed over $[x_1, x_2]$ in the form (3.11).

Note that the function

$$\xi = k \int_{x_0}^{x} a(x)\,dx \tag{4.3}$$

increases numerically with $|x-x_0|$; thus the equation $|\xi| = C$ defines two points $c_1(k) < x_0$, $c_2(k) > x_0$ which depend upon k and tend to x_0 as $k \to \infty$. It follows that, given any $C > 0$, the expressions (3.10) and (3.11) are valid only in those intervals to the left and to the right of x_0, respectively, which are separated from one another by the interval $[c_1, c_2]$. In any such interval to the left of c_1 the solutions of (2.1) are of exponential type while in any interval to the right of c_2 they are of oscillatory type, the transition from one type to the other taking place in $[c_1, c_2]$. However, no information as to the transition itself is available from the expressions (3.10) and (3.11): they become infinite at $x = x_0$. Moreover, the equation (3.12) satisfied by their leading terms has a singularity at $x = x_0$ in contrast to equation (2.1) which remains non-singular throughout $[c_1, c_2]$.

5. Example. As an illustration of the previous considerations let us investigate the differential equation

$$y'' + k^2 xy = 0 \tag{5.1}$$

which is satisfied by the functions

$$y_{\pm}(x) = x^{\frac{1}{2}} J_{\pm\frac{1}{3}}(\xi), \quad \xi = \frac{2}{3} k x^{\frac{3}{2}}. \tag{5.2}$$

Its general solution (save for an arbitrary constant factor) may be written in the form

$$(5.3)\quad y_\alpha(x) = [(2/3)k]^{\frac{1}{6}}(2\pi/3)^{\frac{1}{2}}\xi^{\frac{1}{3}}\left\{\cos(\alpha+\tfrac{\pi}{3})J_{\frac{1}{3}}(\xi) + \cos(\alpha-\tfrac{\pi}{3})J_{-\frac{1}{3}}(\xi)\right\}$$

where α is any constant such that $-\frac{\pi}{2} < \alpha \leq \frac{\pi}{2}$. It is clear from (5.2) that when x is real and positive so is ξ while $\arg\xi = 3\pi/2$ for x negative, i.e. $|\xi| = i\xi$.

For large values of ξ the asymptotic expressions for $J_{\pm\frac{1}{3}}(\xi)$ are known explicitly; they are of the form

$$(5.4)\qquad J_{\pm\frac{1}{3}}(\xi) \sim (2\pi\xi)^{-\frac{1}{2}}\left\{A_{\pm 1}e^{i\xi}P(\xi^{-1}) + A_{\pm 2}e^{-i\xi}P(-\xi^{-1})\right\}$$

where P denotes a formal power series with constant coefficients [32]. If the constants A_1, A_2 are determined so that (5.4) is valid for positive large values of ξ, they will in general not be correct for negative values of ξ. In other words, as $\arg\xi$ varies from zero to 2π, the constants A_1, A_2 must be changed in order that (5.4) remain valid. This fact is known after its discoverer as Stokes' phenomenon [25].

To obtain the asymptotic forms of $y_\alpha(x)$ we use (5.4) with the appropriate constants and note that by (5.2) $|\xi| = C$ yields $c_{1,2} = (3C/2k)^{3/2}$. We find for $x < c_1 < 0$

$$(5.5)\quad y_\alpha(x) = |x|^{-\frac{1}{4}}\left\{\sin\alpha\, e^{|\xi|}\left\{1+O(|\xi|^{-1})\right\} + \tfrac{1}{2}\cos\alpha\, e^{-|\xi|}\left\{1+O(|\xi|^{-1})\right\}\right\}$$

and for $x > c_2 > 0$

$$(5.6)\quad y_\alpha(x) = x^{-\frac{1}{4}}\left\{\cos(\xi - \tfrac{\pi}{4} + \alpha) + O(\xi^{-1})\right\}.$$

Clearly, all solutions with the exception of $y_0(x)$ become infinite as $x \to -\infty$, i. e. $|\xi| \to +\infty$, and $y_0(x)$ approaches zero.

If there exists a value ξ (c^*) such that

$$(5.7) \qquad \tan\alpha = \tfrac{1}{2}\exp(-2\,|\,\xi\,(c^*)\,|)$$

thus for $x = c^* < c_1$, the explicit terms in (5.5) are the same, and for $x < c^*$ the first term in (5.5) will become dominant while for $x > c^*$ the second term will become dominant. The existence of c^* evidently depends upon whether (for a given α) the equation (5.7) can be solved for $\xi(c^*)$, i.e. whether α is sufficiently small, say $\alpha < \alpha_0$. Suppose we agree to identify c^* with c_1 if $\alpha \geq \alpha_0$ and with $-\infty$ if $\alpha = 0$. Then we may put the asymptotic expression (5.5) and (5.6) into the following form:

$$(5.8)\; y\ (x) = \begin{cases} |x|^{-\frac{1}{4}}\sin\alpha\, e^{|\xi|}\left\{1+O(|\xi|^{-1})\right\} & x < c^* \\ \frac{1}{2}|x|^{-\frac{1}{4}}\cos\alpha\, e^{-|\xi|}\left\{1+O(|\xi|^{-1})\right\} & c^* < x \leq c_1\ (<0) \\ x^{-\frac{1}{4}}\left\{\cos(\xi-\frac{\pi}{4}+\alpha)+O(\xi^{-1})\right\} & x \geq c_2\ (>0). \end{cases}$$

These formulas will be useful in our later discussions.

6. <u>The method of Jeffreys</u>. We shall now outline a method due to Jeffreys [12] which permits a representation of the solutions of (2.1) over intervals which include a simple zero of the coefficient $a^2(x)$.

Let us assume $b(x) \equiv 0$ in (2.1), i.e. let us consider the equation

$$(6.1) \qquad y'' + k^2a^2\,(x)y = 0$$

where we suppose further that $a^2(x)$ has a simple zero at the origin and $x\,a^2(x) > 0$ for $x \neq 0$, and that the value of the derivative of $a^2(x)$ at $x = 0$ is unity. (This normalization merely introduces a constant factor which can be taken into the parameter k.) In addition, we assume that in a neighborhood of the origin the function $a^2(x)$ may be approximated by a linear function. These assumptions imply that for sufficiently small values of x, say x in $[x_1, x_2]$, the equation (6.1) may be approximated by the equation

$$y'' + k^2 xy = 0. \tag{6.2}$$

Let us note also that

$$\xi = k\int_0^x a(x)dx \doteq \frac{2}{3}kx^{\frac{3}{2}} \tag{6.3}$$

for x in $[\bar{x}_1, \bar{x}_2]$.

We now recall that for sufficiently large negative values of x the solutions of (6.1) may be expressed according to (3.10) in the form

$$\begin{aligned} y_1 &\sim |a|^{-\frac{1}{2}}\exp(-|\xi|), \\ y_2 &\sim |a|^{-\frac{1}{2}}\exp(|\xi|); \end{aligned} \tag{6.4}$$

to be precise, these representations are valid only for $x \leq c_1 (< 0)$. If we are permitted to replace $a^2(x)$ and ξ by their approximate values, i.e.

$$\begin{aligned} |a|^{-\frac{1}{2}}\exp(-|\xi|) &\doteq |x|^{-\frac{1}{4}}\exp(-\tfrac{2}{3}k|x|^{\frac{3}{2}}) \\ |a|^{-\frac{1}{2}}\exp(|\xi|) &\doteq |x|^{-\frac{1}{4}}\exp(\tfrac{2}{3}k|x|^{\frac{3}{2}}) \end{aligned} \tag{6.5}$$

then we may identify the right hand sides of these expressions according to (5.8) we readily find

$$(6.6)\qquad \begin{aligned} |x|^{-\frac{1}{4}} \exp\left(-\tfrac{2}{3} k |x|^{\frac{3}{2}}\right) &\sim 2y_0(x) \\ |x|^{-\frac{1}{4}} \exp\left(\tfrac{2}{3} k |x|^{\frac{3}{2}}\right) &\sim 2y_{\frac{\pi}{6}}(x). \end{aligned}$$

On the other hand, we know from previous considerations that, on traversing the origin, the right hand side of (6.6) becomes

$$(6.7)\qquad \begin{aligned} 2y_0(x) &\sim 2x^{-\frac{1}{4}} \cos\left(\tfrac{2}{3} kx^{\frac{3}{2}} - \tfrac{\pi}{4}\right), \\ 2y_{\frac{\pi}{6}}(x) &\sim 2x^{-\frac{1}{4}} \cos\left(\tfrac{2}{3} kx^{\frac{3}{2}} - \tfrac{\pi}{12}\right), \end{aligned}$$

and these expressions are valid for $x > c_2(> 0)>$ Finally, if we are permitted to replace again x and ξ by $a^2(x)$ and $k \int^x a(x)dx$, respectively, then we find from (5.8)

$$(6.8)\qquad \begin{aligned} 2x^{-\frac{1}{4}} \cos\left(\tfrac{2}{3} kx^{\frac{3}{2}} - \tfrac{\pi}{4}\right) &\doteq 2|\alpha|^{-\frac{1}{2}} \cos\left(\xi - \tfrac{\pi}{4}\right) \sim 2y_{\frac{\pi}{4}}(x) \\ 2x^{-\frac{1}{4}} \cos\left(\tfrac{2}{3} kx^{\frac{3}{2}} - \tfrac{\pi}{12}\right) &= 2|\alpha|^{-\frac{1}{2}} \cos\left(\xi - \tfrac{\pi}{12}\right) \sim 2y_{\frac{\pi}{12}}(x). \end{aligned}$$

These formulas, namely (6.4),...(6.8), clearly establish the connection between the respective solutions of (6.1) on either side of the origin. According to Jeffreys we denote these connections in the following way:

$$(6.9)\qquad |\alpha|^{-\frac{1}{2}} \exp(-|\xi|) \longleftrightarrow 2|a|^{-\frac{1}{2}} \cos\left(\xi - \tfrac{\pi}{4}\right)$$

$$(6.10)\qquad |\alpha|^{-\frac{1}{2}} \exp(|\xi|) \longleftrightarrow 2|a|^{-\frac{1}{2}} \cos\left(\xi - \tfrac{\pi}{12}\right)$$

where the double arrow indicates that the functions it connects are asymptotic representations of the same functions under different circumstances. As Langer [19] has pointed out certain misconceptions may arise from this notation since the left hand side of (6.9) implies the right hand side but the converse is not necessarily true. The same statement holds for (6.10) where the right hand side implies the left hand side but not conversely.

It is clear that the expressions (6.5) are valid only if $\bar{x}_1 < c_1$, and expressions(6.8) are valid only if $\bar{x}_2 > c_2$. It is therefore a requirement of the method that the interval $[c_1, c_2]$ be included in $[\bar{x}_1, \bar{x}_2]$ for otherwise we are not justified in making these transfers. This, however, can always be accomplished by making k sufficiently large since $c_{1,2} \to 0$ as $k \to \infty$.

We remark that Kramers also established the formula (6.9); together with Ittmann he obtained the further relation

$$(6.11) \qquad |a|^{-\frac{1}{2}}\exp(|\xi|) \longleftrightarrow |a|^{-\frac{1}{2}}\cos\left(\xi+\frac{\pi}{4}\right)$$

which evidently results from (5.8) by taking $\alpha = \pi/2$ (13;14]. Among physicists all these relations are commonly referred to as part of the Wentzel-Kramers-Brillouin method [33]. Goldstein extended Jeffrey's method to the case where $a^2(x)$ may vanish of any integral order [6]. An entirely different procedure based upon the idea of encircling the zero of $a^2(x)$ in the complex plane, was given by Zwaan. [2;19;34].

7. <u>The method of Langer</u>. We now come to the discussion of a method due to Langer which is perhaps the most powerful and most elegant procedure for obtaining asymptotic represen-

tations of solutions of (2.1) for large values of the parameter k. [15; 16;17;18;19;20]. However ingenious the method of Jeffreys, it is essentially based upon connecting appropriately the solutions of (2.1) on either side of a zero of $a^2(x)$ by the use of Horn's classical representations (3.10) and (3.11). Langer starts from the same basic idea, namely that approximately identical differential equations should have approximately identical solutions. He introduces, however, a "related" equation which approximates the given equation throughout the entire interval including the zero of $a^2(x)$; moreover this related equation is explicitly solvable in terms of Bessel functions.

Let us begin with the following preliminary observation. The functions $v = \xi^{\mu} J_{\pm\mu}(\xi)$ where $\xi = k\int_{x_0}^{x} a(x)dx$, satisfy the differential equation

(7.1) $$\frac{d^2v}{d\xi^2} + \frac{1-2\mu}{\xi}\frac{dv}{d\xi} + v = 0$$

or, in terms of the variable x

(7.2) $$v'' + \left[(1-2\mu)\frac{ka}{\xi} - \frac{a'}{a}\right] v' + k^2a^2v = 0$$

If we make the substitution

(7.3) $$u = A(x)v, \; A(x) = (\xi/k)^{\frac{1}{2}-\mu}\alpha^{-\frac{1}{2}}$$

we can reduce (7.2) to the equation

(7.4) $$u'' + \left(k^2a^2(x) - \frac{A''(x)}{A(x)}\right) u = 0$$

whose solutions are the functions

(7.5) $$u = A(x)\,\xi^{\mu}\,J_{\pm\mu}(\xi).$$

The equation (7.4) is of the type (2.1). In fact, since

(7.6) $$-\frac{A''(x)}{A(x)} = \frac{a''}{2a} - \frac{3a'^2}{4a^2} + \left(\frac{1}{4} - \mu^2\right)\frac{a^2}{\left(\int_{x_0}^{x} a\,dx\right)^2}$$

this equation resembles very much the equation (3.12), namely

(7.7) $$Y'' + \left(k^2a^2 + \frac{a''}{2a} - \frac{3a'^2}{4a^2}\right)Y = 0$$

which is satisfied by the leading terms of the classical asympototic representations (3.10) and (3.11). The only difference in their coefficients is the term

(7.8) $$\left(\frac{1}{4} - \mu^2\right)\frac{a^2}{\left(\int_{x_0}^{x} a\,dx\right)^2}.$$

If x_0 is a zero of $a^2(x)$, then (7.7) clearly has a pole at $x = x_0$ which causes the breakdown of Horn's procedure. The coefficient of (7.4) also has a pole at $x = x_0$ for any arbitrary μ, and so nothing seems to have been gained. However, for one particular value of μ this pole can be removed because of the presence of the term (7.8). Suppose that $a^2(x)$ vanishes at $x = x_0$ of the order ν; then if we choose $\mu = 1/(\nu+2)$, it is easy to see that (7.6) remains bounded at $x = x_0$. Thus, for this particular choice of μ, the equation (7.4) approximates the basic equation (2.1) over the entire interval including the point x_0. The difference between these two equations is independent of the parameter k and is therefore dominated by the term $k^2a^2(x)$ for sufficiently large k. This is in contrast to Jeffrey's approximations where, except for the point $x = x_0$, the difference increases with k of the order k^2. Note also that if $a^2(x)$ does not vanish within the interval under consideration, then $\mu = \frac{1}{2}$ and so (7.4) reduces to (7.7); thus Langer's method contains as a special case the classical representation theorem of Horn.

Let us now consider the basic equation (2.1)

$$y'' + (k^2a^2(x) - b(x))y = 0 \tag{7.9}$$

under the further hypotheses that $a^2(x)$ has a simple zero at the origin and that $x\, a^2(x) > 0$ for $x \neq 0$. We will write (7.9) in the form

$$y'' + \left(k^2a^2(x) - \frac{A''(x)}{A(x)}\right) y = p(x)y \tag{7.10}$$

where

$$p(x) = b(x) - \frac{A''(x)}{A(x)} \tag{7.11}$$

Now the left hand side of (7.10) is identical with the related equation (7.4), and if we regard (7.10) formally as a non-homogeneous equation we may thus solve (7.10) in the familiar form

$$y(x) = u(x) + \frac{1}{W} \int_c^x K(x,t)y(t)dt \tag{7.12}$$

where $K(x,t) = u_0(x)u_1(t) - u_1(x)u_0(t)$ and W is the Wronskian determinant of $u_0(x), u_1(x)$. It is clear from (7.11) that at $x = c$ the values of $y(x)$ and $y'(x)$ are identical with those of $u(x)$ and $u'(x)$, respectively.

On repeatedly substituting the entire right hand side of (7.11) for y(t) in the integrand, we obtain the expansion

$$(7.13)\qquad y(x) = u(x) + \frac{1}{W}\int_c^x K(x,t)u(t)\,dt + \frac{1}{W^2}\int_c^x\int_c^t K(x,t)K(t,s)u(t)\,ds\,dt + \ldots$$

which has to be shown to converge and to represent y(x) for sufficiently large k.

Recall that under our assumptions $\mu = \frac{1}{3}$ and so the related equation (7.4) has the solutions

$$(7.14)\qquad u_\alpha(x) = A(x)k^{\frac{1}{6}}(2\pi/3)^{\frac{1}{2}}\xi^{\frac{1}{3}}\left[\cos\left(\alpha+\tfrac{\pi}{3}\right)J_{\frac{1}{3}}(\xi) + \cos\left(\alpha-\tfrac{\pi}{3}\right)J_{-\frac{1}{3}}(\xi)\right].$$

For $\alpha = 0$ and $\alpha = \frac{\pi}{2}$ we obtain a pair of linearly independent solutions which we can identify as $u_0(x)$ and $u_1(x)$ in (7.11); their Wronskian has the value k. Furthermore, their asymptotic expressions are given by (5.8), i.e.

$$(7.15)\qquad \begin{aligned} u_0(x) &\sim \tfrac{1}{2}|a|^{-\frac{1}{2}}\exp(-|\xi|) \sim a^{-\frac{1}{2}}\cos\left(\xi-\tfrac{\pi}{4}\right) \\ u_1(x) &\sim |a|^{-\frac{1}{2}}\exp(|\xi|) \sim a^{-\frac{1}{2}}\cos\left(\xi+\tfrac{\pi}{4}\right) \end{aligned}$$

for x negative or positive.

We now sketch the proof of convergence of the series (7.1) The first term of (7.1) can be split into two integrals

$$(7.16)\qquad I_1 = k^{-1}u_0(x)\int_c^x p(t)u_1(t)u(t)\,dt,\quad I_2 = k^{-1}u_1(x)\int_c^x p(t)u_0(t)u(t)\,dt.$$

If the interval of integration lies inside the interval $[c_1, c_2]$ as defined in 5, then the integration over it contributes an amount which tends to zero as $k \to \infty$. This can be concluded from the fact that the interval is in length of order $k^{-\frac{2}{3}}$ while the product of any two solutions (7.1) has the factor

$k^{\frac{1}{3}}$. Thus we may suppose the integration carried out over those intervals only where $x < c_1$ or $x > c_2$.

Let us suppose first that the graph of $u(x)$ is rising on $c < x \leq c_1$. Then, according to (5.8), we have

$$(7.17) \qquad u(x) \sim \begin{cases} \frac{1}{2}|a|^{-\frac{1}{2}} \cos\alpha \, e^{-|\xi|} & c < x \leq c_1 \\ a^{-\frac{1}{2}} \cos\left(\xi - \frac{\pi}{4} + \alpha\right) & x \geq c_2 \end{cases}$$

where by (5.7)

$$(7.18) \qquad \tan\alpha = \tfrac{1}{2}\exp(-2|\xi(c)|).$$

It is now easy to see that $I_1 = O(1)$ for $|\xi| > C$. As regards I_2, we note that its integrand is of the form

$$\frac{p(t)}{a(t)}\, e^{-2i\,\xi(t)}\, O(1);$$

since $|\exp(-2i\,\xi(t))| < |\exp(-2i\,\xi(x))|$ for all $t \leq x$, we find $I_2 = e^{-2i\,\xi(x)}O(1)$. Hence we can conclude that

$$(7.19) \qquad I_1 + I_2 = \begin{cases} k^{-5/6}O(1) & |\xi| \leq C \\ k^{-5/6}a^{-\frac{1}{2}}e^{-i\xi}\,O(1) & |\xi| > C \end{cases}$$

A similar argument can be applied to the other terms in the expansion (7.13); it yields estimates in terms of successively higher powers of k^{-1} which are sufficient to prove convergence of the series (7.13).

<u>Theorem 1 (Langer)</u>. <u>Let</u> $u(x)$ <u>be a solution of the related equation</u> (7.4) <u>whose graph is rising at a point</u> $c < c_1$. <u>Then the solution</u> $y(x)$ <u>of</u> (7.9) <u>given by</u> (7.13) <u>can be expressed in the form</u>

(7.20)

$$y(x) = \begin{cases} \frac{1}{2} \, | a(x) |^{-\frac{1}{2}} \cos\alpha \, e^{-|\xi|} \{1 + O(\xi^{-1}) + O(k^{-1})\} & c \leq x \leq c_1 \\ u(x) + k^{-5/6} O(1) & c_1 \leq x \leq c_2 \\ a^{-\frac{1}{2}}(x) \{\cos(\xi - \frac{\pi}{4} + \alpha) + O(\xi^{-1}) + O(k^{-1})\} & c_2 \leq x \end{cases}$$

where the constant α is the same as in (7.17).

The case where the graph of $u(x)$ is falling at $x = c_1$ can be dealt with in similar manner. According to our convention as regards c^*, we have $\alpha > \alpha_0$, where $\tan \alpha_0 = \frac{1}{2} \exp(-2C)$, and thus by (5.8)

$$(7.21) \qquad u(x) \sim \begin{cases} |a|^{-\frac{1}{2}} \sin\alpha \; e^{|\xi|} & x \leq c_1 \\ a^{-\frac{1}{2}} \cos(\xi - \frac{\pi}{4} + \alpha) & x \geq c_2 \end{cases}$$

If c is any constant greater than c_1, then the integrals I_1 and I_2 can be shown to be of the form

$$(7.22) \qquad I_1 = e^{2i\,\xi(x)} O(1), \quad I_2 = O(1) \qquad |\xi| > C,$$

and similar estimates can be found for the subsequent terms in (7.1[illegible]).

Theorem 2(Langer). Let $u(x)$ be a solution of the related equation (7.4) whose graph is falling at the point $c_1 < c$. Then the solution $y(x)$ of (7.9) given by (7.13), can be expressed in the form

$$(7.23) \quad y(x) = \begin{cases} |a|^{-\frac{1}{2}} \sin\alpha \, e^{|\xi|} \{1 + O(\xi^{-1}) + O(k^{-1})\} & x \leq c_1 \\ u(x) + k^{-5/6} O(1) & c_1 \leq x \leq c_2 \\ \text{[illegible]} \{\cos(\xi - \frac{\pi}{4} + \alpha) + O(\xi^{-1}) + O(k^{-1})\} & c_2 \leq x \end{cases}$$

where α is the same constant as in (7.21).

We have outlined Langer's method for a particularly simple case. In general, it can be applied to any equation of type (2.1) where $a^2(x)$ is a real or complex function representable in the form $(x-x_0)^{\nu} a_1^2(x)$ in some region of the complex plane, and k is a real or complex parameter. Langer himself applied this method to an investigation of Bessel functions for large complex arguments and large complex orders [16] and to a study of Mathieu's equation for complex arguments and at least one large parameter [18].

8. Extensions of Langer's method. In the foregoing discussion it was assumed throughout that in the equation

$$y'' + (k^2a^2(x) - b(x))y = 0 \tag{8.1}$$

the coefficient b(x) is independent of the parameter k. This is an unnecessary restriction so long as $a^2(x)$ has a simple zero, for in this case the theory requires only slight modifications to cover the case where b(x) is a bounded function of k.

Goldstein was the first to investigate equations of the more general type

$$y'' - (k^2a^2(x) + k\,\alpha_1(x) + b(x,k))y = 0 \tag{8.2}$$

under the assumptions that x and k are real and $a^2(x)$ vanishes of the second order at a point $x = x_0$ and $b(x,k) \equiv 0$. His method of attack is essentially based upon Jeffrey's idea of fitting together the classical representations on either side of the point x_0[8]. A similar treatment was also given by Voss

[30]. Langer succeeded in extending his method to equations of the form (8.2) in which x as well as k may be complex [17]. We shall not state these results here; a brief sketch of them for the special case of real parameters can be found in [19].

In recent years Cherry has successfully attempted to improve Langer's method with a view toward obtaining expansions better suited for numerical calculations [3;4]. Langer's results have only limited utility because the expansions generally involve functions far too complex to be handled numerically. As a consequence, the first terms in the expansions are only of use in numerical work. For example, Langer has shown that

$$(8.3)\quad J_\nu(\nu\sqrt{1-x^2}) = \frac{1}{\Gamma(\nu+1)}\left(\frac{\nu}{e}\right)^\nu\sqrt{\frac{2}{\pi x}}\,L\{\nu(\operatorname{artanh} x - x)\}\cdot\left\{1+\sum_{i=1}^{\infty}\frac{E_i(x,\nu)}{\nu^i}\right\}$$

where $L(x) = \sqrt{x}\,K_{1/3}(x)$ and the series $\sum E_i\nu^{-i}$ converges uniformly in $0 \leq x \leq 1$ provided $\nu > \nu_0$. We shall outline Cherry's method for the case of these Bessel functions [3].

9. <u>The method of Cherry</u>. Let us consider the function $y = \sqrt{x}\,J_\nu(k\sqrt{1-x^2})$ as a function of $t = \operatorname{artanh} x - x$; the introduction of the variable t is suggested by Langer's formula (8.3) in which $J_\nu(\nu\sqrt{1-x^2})$ is expressed in terms of the function $L(\nu t)$. The value of the constant parameter k will be specified later. On differentiating y twice, we find for y the differential equation

$$(9.1)\quad \frac{d^2y}{dt^2} + y\left\{-\nu^2 + (x^{-2}-1)\left(\frac{5}{4}x^{-4} - \frac{1}{4}x^{-2} + k^2 - \nu^2\right)\right\} = 0.$$

Since for sufficiently small values of x we have

$$(9.2)\qquad \begin{aligned} t &= \frac{1}{3}x^3\left(1+\frac{3}{5}x^2+\ldots\right)\\ x &= (3t)^{1/3}\left(1-\frac{1}{5}(3t)^{2/3}+\ldots\right)\end{aligned}$$

the coefficient of y in (9.1) can be written in the form

$$(9.3)\qquad -\nu^2+\frac{5}{36}t^{-2}+\left(k^2-\nu^2-\frac{1}{35}\right)(3t)^{-2/3}+P(t^{2/3})$$

where P denotes a power series whose coefficients can be calculated from (9.1) and (9.2). Thus the equation (9.1) may be approximated for small values of t by the equation

$$(9.4)\qquad \frac{d^2u}{dt^2}+u\left(-\nu^2+\frac{5}{36t^2}\right)=0;$$

and the best agreement (near $t = 0$) will be obtained by taking $k^2=\nu^2+\frac{1}{35}$. We will show that the solutions of (9.1) can be expressed in terms of solutions of (9.4).

The general solution of (9.4) is known to be

$$(9.5)\qquad u(t) = A\,L(\nu t) + B\,L(\nu t\, e^{-\pi i})$$

where $L(z)=\sqrt{z}\,K_{\frac{1}{3}}(z)$. From the properties of the function L(z) we easily deduce the following inequalities

$$(9.6)\qquad |L(z)|\le A_1|e^{-z}|,\ |L(ze^{-\pi i})|\le A_2|e^{z}|,\ \left|\frac{L(ze^{-\pi i})}{L(z)}\right|\le A_3|e^{2z}|$$

which hold for arbitrary $|z|$ so long as $-\frac{1}{2}\pi+\epsilon<\arg z<\frac{3}{2}\pi-\epsilon$.

We now write (9.1) in the form

$$(9.7)\qquad \frac{d^2y}{dt^2}+y\left(-\nu^2+\frac{5}{36t^2}\right)=y\,f(t)$$

where

$$f(t) = \frac{5}{36t^2} - \left(\frac{1}{x^2} - 1\right)\left(\frac{5}{4} x^{-4} - \frac{1}{4} x^{-2} + k^2 - \nu^2\right) \tag{9.8}$$

and solve this equation by iteration; we suppose throughout that $0 < t < \infty$ and ν real and positive. Taking $y_0 = L(\nu t)$, we obtain

$$y_1(t) = L(\nu t) + \frac{1}{\nu\pi} L(\nu t) \int_t^\infty \left\{ L(\nu x e^{-\pi i}) L(\nu x) - L^2(\nu x) \frac{L(\nu t e^{-\pi i})}{L(\nu t)} \right\} f(x)\, dx \tag{9.9}$$

and thus $y_1(t) \sim L(\nu t)$ for t large. It is easy to see that the integral on the right of (9.9) is bounded by $M \int_0^\infty |f(x)|\, dx$, and so we find for $0 < t < \infty$

$$\left| \frac{y_1(t)}{L(\nu t)} - 1 \right| < \frac{C}{\nu} \tag{9.10}$$

where C is a certain positive constant. It follows in similar manner that

$$\left| \frac{y_n(t)}{L(\nu t)} - \frac{y_{n-1}(t)}{L(\nu t)} \right| < \left(\frac{C}{\nu}\right)^n . \tag{9.11}$$

The sequence of functions $y_n(t)$ converges to a solution of (9.7) such that

$$y(t) = L(\nu t) \left\{ 1 + O(\nu^{-1}) \right\} \tag{9.12}$$

uniformly for $0 < t < \infty$ and $\nu \geq C + \epsilon$.

We must now identify this solution. Since by (9.8) $f(t) = O(t^{-2})$ for large t, we can improve (9.10) and (9.11), namely

$$\left| \frac{y_1(t)}{L(\nu t)} - 1 \right| \leq \frac{C'}{t\nu}, \qquad \left| \frac{y_n(t)}{L(\nu t)} - \frac{y_{n-1}(t)}{L(\nu t)} \right| \leq \left(\frac{C'}{t\nu}\right)^n; \tag{9.13}$$

hence we have

$$(9.14) \quad y(t) = L(\nu t)\left\{1+O_\nu(t^{-1})\right\} = \sqrt{\frac{\pi}{2}}\, e^{-\nu t}\left\{1 + O_\nu(t^{-1})\right\}.$$

On the other hand, we know that the general solution of (9.1), and so of (9.7), is of the form

$$(9.15) \qquad y(t) = A\sqrt{x}\, J_\nu\left\{k\sqrt{1-x^2}\right\} + B\sqrt{x}\, Y_\nu\left\{k\sqrt{1-x^2}\right\}.$$

Since, by virtue of $t = \operatorname{arctanh} x - x$, we have

$$(9.16) \qquad \sqrt{\frac{2}{1-x}} = e^{1+t}\left\{1 + O(e^{-2t})\right\},$$

we find from (9.15)

$$y(t) = \frac{A}{\Gamma(\nu+1)}\left(k\sqrt{\frac{1-x}{2}}\right)^\nu\left\{1+O_\nu(1-x)\right\} + B.O_\nu(1-x)^{-\frac{\nu}{2}}$$

$$(9.17) \qquad = \frac{A}{\Gamma(\nu+1)}\left(\frac{k}{e}\right)^\nu e^{-\nu t}\left\{1+O_\nu(e^{-2t})\right\} + B.O_\nu(e^{\nu t}).$$

By letting $t \to \infty$ we can prove from (9.17) that $B = 0$ and determine A, so that we finally obtain

$$(9.18) \qquad y(t) = \Gamma(\nu+1)\left(\frac{k}{e}\right)^{-\nu}\sqrt{\frac{\pi x}{2}}\, J_\nu\left\{k\sqrt{1-x^2}\right\}.$$

On substituting for $y(t)$ the series determined by iteration we can deduce, for $k = \nu$, Langer's formula (8.3); writing down the first term only, we have

$$(9.19) \qquad J_\nu\left\{\nu\sqrt{1-x^2}\right\} = \frac{L(\nu t)}{\pi\sqrt{\nu x}}\left\{1 + O(\nu^{-1})\right\}.$$

We now try to improve this result by transforming the original equation (9.1) into a form which resembles (9.4) more closely. We begin with the following preliminary observations. The differential equation

$$(9.20) \qquad \frac{d^2w}{dz^2} + w\left(-\nu^2 + c + \frac{5}{36z^2}\right) = 0$$

in which c is a disposable constant, can be reduced by the substitution

$$(9.21) \qquad z = g(t), \quad w = u\sqrt{g'(t)}$$

to the differential equation

$$(9.22) \qquad \frac{d^2u}{dt^2} + u\left\{(-\nu^2+c)g'^2 + \frac{5g'^2}{36z^2} + \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2\right\} = 0.$$

Let us consider the difference between the coefficient of u in (9.22) and that of y in (9.1),

$$(9.23) \quad f = (-\nu^2+c)g'^2 + \frac{5g'^2}{36z^2} + \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + \nu^2 - \left(\frac{1}{x^2}-1\right)\left(\frac{5}{4x^4} - \frac{1}{4x^2} + k^2 - \nu^2\right)$$

and let us attempt to determine g(t) so that this difference is suitably reduced. We prescribe $g'(0) = 1$ and put

$$(9.24) \qquad g'(t) = 1 + h_1(t)\nu^{-2} + h_2(t)\nu^{-4} + \ldots$$

$$(9.25) \qquad z = g(t) = \int_0^t g'(t)dt = t + g_1(t)\nu^{-2} + g_2(t)\nu^{-4} + \ldots;$$

we also let

$$(9.26) \qquad c = c_0 + c_1\nu^{-2} + c_2\nu^{-4} + \ldots$$

$$k^2 - \nu^2 = k_0 + k_1\nu^{-2} + k_2\nu^{-4} + \ldots$$

Then, and substituting these formal expressions into (9.23) and expanding f in powers of ν^{-2} we can reduce f by equating to zero as many of the coefficients as we please. This reduction yields for the coefficients of ν^0 and ν^{-2} the following set of equations

$$(9.27)\quad -2h_1(t)+c_0+\frac{5}{36t^2}-\left(\frac{1}{x^2}-1\right)\left(\frac{5}{4x^4}-\frac{1}{4x^2}+k_0\right)=0$$

$$(9.28)\quad -2h_2(t)+c_1-h_1^2(t)+2c_0h_1+\frac{5}{18t^2}\left(h_1-\frac{g'}{t}\right)+\frac{1}{2}h_1''-\left(\frac{1}{x^2}-1\right)k_1=0.$$

The terms in x can be developed in powers of $t^{2/3}$ as in (9.3) and so (9.27) becomes

$$(9.29)\quad -2h_1+c_0-\frac{k_0-\frac{1}{35}}{(3t)^{2/3}}+\left(k_0+\frac{8}{525}\right)+\frac{36}{11.25.49}(3t)^{\frac{2}{3}}+\dots=0;$$

since $h_1(t)$ must vanish for $t = 0$, we have to take

$$(9.30)\qquad k_0=\frac{1}{35}\text{ and }c_0=-\left(k_0+\frac{8}{525}\right)=-\frac{23}{525}.$$

With the values of c_0 and k_0, $h_1(t)$ is uniquely determined by (9.27); in fact, according to (9.29), we have

$$(9.31)\qquad h_1(t)=\frac{18}{11.25.49}(3t)^{2/3}+\dots$$

By integration, we find

$$(9.32)\qquad g_1(t)=\int_0^t h_1(t)dt=\frac{18}{11.125.49}(3t)^{5/3}+\dots$$

Now $h_2(t)$ can be determined in similar manner from (9.28). Here, the salient fact is that because of the form of $h_1(t)$ and $g_1(t)$ the constants k_1 and c_1 can be determined so as to annul the terms in $t^{-2/3}$ and t^0; hence $h_2(t)$ and $g_2(t)$ have a local development similar to that of $h_1(t)$ and $g_1(t)$. In this manner we can determine as many of the functions $h_r(t)$

and $g_r(t)$ as we please and thereby reduce the function f in (9.23) to any desired degree.

It can be shown that the functions $h_r(t), \frac{g_r(t)}{t}$ are bounded in any finite t-interval. It follows that if the series (9.24) -(9.26) are terminated after k_{n-1}, c_{n-1}, $h_n(t)$, $g_n(t)$ have been determined, then the difference $|z-t|$ can be made arbitrarily small for ν sufficiently large. The expansion of f(t) in powers of ν^{-2} converges for $|\nu| > \nu_0$; moreover, $f(t)=O(\nu^{-2n}t^{-2/3})$.

We now use (9.22) as approximating equation and solve (9.1) by iteration as before; to simplify matters, we can reduce first (9.1) by the inverse of the transformation $z = g(t)$ to an equation in z and then compare the solutions of the resulting equation with those of (9.20). Here it is more convenient to express the final result in terms of Airy-functions by introducing

(9.33) $Z=(\frac{3}{2}z)^{2/3}$, $M=\mu^{2/3}=(\nu^2-c)^{1/3}$, $L(\mu z) = \pi\sqrt{2}(MZ)^{1/4}\mathrm{Ai}(MZ)$.

It can then be shown by arguments similar to those used before that

(9.34) $J_\nu(k\sqrt{1-x^2}) = \sqrt{\frac{2}{\mu x g'(t)}}(MZ)^{1/4}\mathrm{Ai}(MZ)\{1+O(\nu^{-2n-1})\}$

where Z ranges over the upper half of the complex Z plane from which the part of the sector $0 < \arg Z < \frac{2}{3}\pi$ that corresponds to $\mathrm{Im}\, z > \frac{1}{2}\pi$ is removed; and this formula is uniformly valid without restriction upon x, except for slight modifications near the zeros of $L(\mu z)$, provided ν is real and positive.

The formula (9.34) was tested numerically for accuracy [3]; it was found that in the case of n=2 it gives, for instance, at least 8 decimals for $\nu \geq 9$, nearly 8 for $\nu \geq 5$, and more than 5 for $\nu \geq 2$. Thus it is always more accurate than any other standard formulas (like the two Debye "A-formulas" [32]).

Cherry's method can be extended to differential equations of the form

$$y'' + (k^2 a(x) + b(x,k^{-2}))y = 0 \tag{9.35}$$

where a(x) has a simple zero at the origin and k is a large parameter and the function b(x,y) is regular at x = 0, y = 0 [4]. However, we shall not present these generalizations here.

The theory of asymptotic solution of systems of linear differential equations containing a parameter has very recently been discussed by Turrittin [29]. His method of attack is essentially part of the "WKB" procedure we briefly mentioned earlier.

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Part III

Existence and Uniqueness Theorems for Asymptotic Expansions

Philip Davis

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Existence and Uniqueness Theorems for Asymptotic Expansions

by

Philip Davis

In the present part, we shall discuss some function theoretic aspects of asymptotic expansions. These are closely related to the theory of interpolation and approximation for analytic functions, for infinitely differentiable functions, and to the theory of quasi-analytic families of functions.

1. Asymptotic Series in the Sense of Poincaré

Let D be a fixed region in the complex $z = x + iy$ plane. We distinguish a point of the plane which is a limit point of D. For simplicity, we may take the distinguished point as $z = 0$. For a given $f(z)$ which is defined in D, we consider the following limits:

(1.1)
$$\begin{aligned} a_0 &= \lim_{z \to 0} f(z) \\ a_1 &= \lim_{z \to 0} z^{-1}\left[f(z) - a_0\right] \\ &\vdots \\ a_n &= \lim_{z \to 0} z^{-n}\left[f(z)-a_0-a_1(z)-\cdots-a_{n-1}z^{n-1}\right] \\ &\vdots \end{aligned}$$

If all the limits in (1.1) exist and yield the same value a_n independently of the manner in which z goes to 0 in D, then we shall say that f possesses the asymptotic expansion

$\sum_{n=0}^{\infty} a_n z^n$ at $z = 0$, and we shall indicate this by writing

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^n \tag{1.2}$$

The association (1.2) is a purely formal one, and nothing is said either as to the convergence of the right hand member of (1.2) or as to the way the series represents the function. We restrict our attention to functions which are regular analytic in the interior of D, and designate by $A = A(D)$ the subclass of functions which are analytic in D and possess an asymptotic expansion at $z = 0$. Let us note the following elementary consequences of the definition. Given an $f(z)$ defined in D. Some or all of the limits (1.1) may fail to exist. In such a case f will not possess an asymptotic expansion at $z = 0$. If, however, f possesses an expansion then that expansion is unique. The class A of functions possessing asymptotic expansions is a linear class. Each limit a_n in (1.1), though defined in terms of the preceding limits, may be regarded as a linear functional over A:

$$a_n = a_n(f) \equiv L_n(f); \quad (n = 0,1,\ldots),\ f \in A \tag{1.3}$$

Thus, for $f_1, f_2 \in A$ and for arbitrary complex constants α_1 and α_2 we have $\alpha_1 f_1 + \alpha_2 f_2 \in A$ and

$$L_n(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 L_n(f_1) + \alpha_2 L_n(f_2), (n=0,1,\ldots). \tag{1.4}$$

The asymptotic series (1.2) may be written in the form

(1.5) $$f(z) \sim \sum_{n=0}^{\infty} L_n(f)\, z^n$$

exhibiting the role of f in the formation of the series. If the distinguished point $z = 0$ is interior to D, then the class A coincides with the class of all functions which are regular in D. For, if f is regular, it possesses a Taylor expansion

(1.6) $$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \quad ; \quad |z| < \rho,\ \rho > 0$$

From (1.1) we have

(1.7) $$\begin{aligned} a_0 &= \lim_{z \to 0} f(z) = f(0) \\ a_1 &= \lim_{z \to 0} z^{-1} \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = f'(0) \\ &\vdots \\ a_n &= \lim_{z \to 0} z^{-n} \Big[\sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \Big] = \frac{f^{(n)}(0)}{n!} \\ &\vdots \end{aligned}$$

The asymptotic series for a function at a point of regularity therefore coincides with its Taylor series. In this case, the functionals L_n may be identified as derivatives.

(1.8) $$L_n(f) = f^{(n)}(0)/n! \quad (n = 0,1,\ldots).$$

On the other hand, suppose that f is regular in D-(0). Then the existence of the first limit of (1.1) already implies by Riemann's theorem on removable singularities that f must be regular in D. If f is regular at $z = 0$, then its asymptotic series converges in a neighborhood of $z = 0$. The converse here is not true; if $z = 0$ is a singular point of a function, its asymptotic series may either converge or diverge. A function f possesses a convergent asymptotic expansion if and only if it may be written in the form

$$f = g + h \tag{1.9}$$

where g is regular at $z = 0$ and h possesses an asymptotic expansion which is identically zero. The existence of non-trivial functions h which have this property (and are therefore singular at $z = 0$) will be shown in §3 for certain domains D. For these reasons it is profitable to deal only with functions possessing singularities at $z = 0$ and to restrict the distinguished point to be a non-isolated boundary point of D. If a singular function possesses an asymptotic expansion, the singularity must be of a transcendental nature, for it is easily seen that if f had a pole, an algebraic branch point or a logarithmic singularity at $z = 0$, all the limits in (1.1) would not exist. No extensive classification of singularities which admit asymptotic expansions appears to have been made.

The identification of L_n in (1.8) cannot in general be made. Whether or not this is true depends to a certain extent upon the domain D. For asymptotic expansions along the real line all the limits (1.1) may exist without implying any differentiability properties whatever on the function. On the other hand, if D contains a positive angle at $z = 0$, then this identification may be made.

Theorem *: Let D contain a sector S: $|\arg z| \leq \theta$, $|z| \leq \rho$; $\rho, \theta > 0$. If $f(z)$ is regular in D and possesses an asymptotic expansion (1.2) in D, then $a_n = \lim_{z \to 0} f^{(n)}(z)/n!$ where the limit may be carried out in any S_1: $|\arg z| < \theta_1$, $\theta_1 < \theta$. Furthermore all successive derivatives of f possess asymptotic expansions valid in S_1 which may be obtained by term by term differentiation of (1.2).

Proof: Let $\theta_1 < \theta' < \theta$, and designate by T_h the triangle bounded by $\arg z = \pm \theta'$, $x = h$. The limits (1.1) imply that

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n + \epsilon(z) z^n \tag{1.10}$$

where $\epsilon(z)$ is regular in T_h and continuous on its boundary, including $z = 0$, and where $\lim_{z \to 0} \epsilon(z) = 0$. Therefore,

$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \ldots + na_n z^{n-1} + nz^{n-1} \epsilon(z) + z^n \epsilon'(z) \tag{1.11}$$

Now, by Cauchy's integral formula

$$\epsilon'(z) = \frac{1}{2\pi i} \int_{T_h} \frac{\epsilon(t)}{(t-z)^2}\, dt \tag{1.12}$$

Hence for $|\arg z| \leq \theta_1$,

$$\left| z^n \epsilon'(z) \right| \leq \frac{\max_{t \in T_h} |\epsilon(t)|}{2\pi}\, \ell(T_h)\, \frac{|z|^n}{|z|^2 \sin(\theta' - \theta_1)} \tag{1.13}$$

where $\ell(T\ )$ is the length of T . For $n \geq 2$, we have

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*Ritt(11)

$\overline{\lim\limits_{z\to 0}}\left|z^n \epsilon'(z)\right| \le \dfrac{\ell(T_h)}{2\pi\sin(\theta'-\theta_1)} \max\limits_{t\in T_h}\left|\epsilon(t)\right|$. But h is arbitrary, and this last factor goes to 0 as $h \to 0$. Hence, $\lim\limits_{z\to 0}\left|z^n\epsilon'(z)\right| = 0$, $(n = 2, 3, \ldots)$. Thus we see from (1.11) that $\lim\limits_{z\to 0} f'(z)$ exists and equals a_1 and that $f'(z)$ has the asymptotic expansion $\sum\limits_{n=1}^{\infty} n a_n z^{n-1}$ in $|\arg z| \le \theta_1$. We may now repeat this argument for the successive derivatives.

A converse of this result is also possible under certain circumstances.

Theorem: *Let* $f(x)$ *be real valued and infinitely differentiable in* $0 \le x \le a$ *and let* $\lim\limits_{x\to 0+} f^{(n)}(x)$ *exist* $(n=0,1,\ldots)$, *or, in the complex case, let* $f(z)$ *be regular in a sector* S: $|\arg z| \le \theta$, $|z| \le p$ *and* $\lim\limits_{z\to 0} f^{(n)}(z)$ *exist. Then* f *possesses an asymptotic expansion on the line (or in the sector) and the values of the limits* (1.1) *are given by* $a_n = L_n(f) = \lim\limits_{z\to 0} f^{(n)}(z)/n!$.

Proof: Let n be a fixed positive integer. For an arbitrary a and z in S we have the formal Taylor identity

$$(1.14)\quad f(z) = f(a) + (z-a)f'(a) + \ldots + \frac{(z-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{1}{(n-1)!}\int_a^z (z-t)^{n-1} f^{(n)}(t)\,dt, \text{ or,}$$

$$(1.14')\quad f(z) = f(a) + (z-a)f'(a) + \ldots + \frac{(z-a)^{n-1}}{(n-1)!} + \frac{z^n}{(n-1)!}\int_{a/z}^{1} (1-u)^{n-1} f^{(n)}(uz)\,du.$$

If we allow $a \to 0$ in S and designate $\lim\limits_{z\to 0} f^{(n)}(z)$ by $f^{(n)}(0)$,

$$(1.15)\quad f(z)=f(0)+f'(0)z+\ldots+\frac{f^{(n-1)}(0)}{(n-1)!}z^{n-1}$$

$$+\frac{z^n}{(n-1)!}\int_0^1 (1-u)^{n-1} f^{(n)}(uz)\,du$$

From the first limit in (1.1) we have $a_0 = \lim_{z \to 0} f(z) = f(0)$. We proceed by induction and assume we have established that $a_k=\lim_{z \to 0} \frac{f^{(k)}(z)}{k!} = \frac{f^{(k)}(0)}{k!}$ for $k=0,1,\ldots,n-1$. Then,

$$(1.16)\quad a_n=\lim_{z\to 0} z^{-n}[f(z)-a_0-a_1 z-\ldots-a_{n-1} z^{n-1}]$$

$$= \lim_{z \to 0} \frac{1}{(n-1)!}\int_0^1 (1-u)^{n-1} f^{(n)}(uz)\,du$$

Now $a_n - \frac{f^{(n)}(0)}{n!} = \lim_{z \to 0} \frac{1}{(n-1)!}\int_0^1 (1-u)^{n-1}[f^{(n)}(uz)-f^{(n)}(0)]du = 0.$

The result follows by the continuity of $f^{(n)}(z)$.

Example 1. We cite here an example discussed in a previous part. Let

$$(1.17)\quad f(z) = \int_0^\infty \frac{e^{-w}}{1+zw}\,dw, \quad |\arg z| \le \alpha < \pi$$

A formal computation which may be justified yields

$$(1.18)\quad f^{(n)}(z) = (-1)^n n! \int_0^\infty \frac{e^{-w}\, w^n}{(1+zw)^{n+1}}\,dw$$

and

$$(1.19)\quad \lim_{z \to 0} f^{(n)}(z) = (-1)^n n! \int_0^\infty e^{-w}\, w^n dw = (-1)^n (n!)^2.$$

Hence

$$(1.20)\quad f(z) \sim \sum_{n=0}^{\infty} (-1)^n\, n!\, z^n.$$

Example 2. Consider the function $F(z)$ defined by the power series

$$(1.21)\quad F(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{\sqrt{n}}}$$

We have $\lim_{n \to \infty} \left(\frac{1}{2^{\sqrt{n}}}\right)^{1/n} = \lim_{n \to \infty} \left(\frac{1}{2^{1/\sqrt{n}}}\right) = 1$ so that $|z| = 1$ is the circle of convergence of (1.21). Now,

$$(1.22) \quad \frac{F^{(k)}(z)}{k!} = \sum_{n=k}^{\infty} \frac{n(n-1)\ldots(n-k+1)z^n}{k!\; 2^{\sqrt{n}}} \quad (k=0,1,\ldots)$$

and all the series (1.22) converge for $z = 1$. Hence, by Abel's theorem*,

$$(1.23) \quad \lim_{z \to 1} \frac{F^{(k)}(z)}{k!} = \sum_{n=k}^{\infty} \binom{n}{k} 2^{-\sqrt{n}} \quad (k = 0,1,\ldots)$$

the limit holding as $z \to 1$ between two chords with vertex at $z = 1$. Thus, we may write

$$(1.24) \quad F(z) \sim \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \binom{n}{k} 2^{-\sqrt{n}} \right) (z-1)^k .$$

2. <u>Some Generalizations of Asymptotic Series</u>

As has been pointed out, not every function possesses an asymptotic expansion. Thus, for instance, $\sqrt{z}$ does not possess one at $z = 0$. In order to widen the class A of functions with such expansions, we may allow the basis elements in the expansions themselves to have singularities at the distinguished point. This can be carried out as follows: Let $z = 0$ be a boundary point of B and by $\phi_0(z), \phi_1(z), \ldots, \phi_n(z), \ldots$ denote a sequence of functions which are regular in B and continuous in the closure of B. Moreover let $\phi_n(0) = 0$ $(n=1,2,\ldots)$, but vanish nowhere else in B and let

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*Cf., e.g., Titschmarsh (12) p. 229

(2.1) $$\lim_{z \to 0} \phi_{n+1}(z)/\phi_n(z) = 0 \quad (n=0,1,\ldots).$$

We shall say that $\sum_{n=0}^{\infty} c_n \phi_n(z)$ is an asymptotic series in a generalized sense for $f(z)$ at $z=0$ if

(2.2) $$c_0 = \lim_{z \to 0} f(z)/\phi_0(z)$$

$$c_1 = \lim_{z \to 0} [f(z)-c_0 \phi_0(z)]/\phi_1(z)$$
$$\vdots$$
$$c_n = \lim_{z \to 0} [f(z)-c_0\phi_0(z)-\ldots-c_{n-1}\phi_{n-1}(z)]/\phi_n(z)$$
$$\vdots$$

If we again designate by $A = A(D; \{\phi_n\})$ the class of functions which are regular in D and possess the above limits, we see that A is a linear class and that the n^{th} limit in (2.2) is a linear functional defined over A: $c_n = L_n(f)$ $(n=0,1,\ldots)$. L_n may, under certain circumstances, be identified as the limit as $z \to 0$ of an appropriate differential operator.

<u>Example</u>: Let $\lambda_0 < \lambda_1 < \ldots$ and $\phi_n(z)=z^{\lambda_n}$. In this case the asymptotic series are of the form $f(z) \sim \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ and the theory would be related to the theory of Dirichlet series.

Series defined by (2.2) may be written alternatively as

(2.3) $$f(z) \sim \sum_{n=0}^{\infty} L_n(f)\, \phi_n(z); \; f \in A.$$

The principal formal property of (2.3) is that of <u>biorthogonality</u>. This means that

(2.4) $$L_m(\phi_n) = \delta_{mn} \qquad (m,n=0,1,\ldots) .$$

These relations are readily established from (2.2) and the vanishing of ϕ_{n+1}/ϕ_n at $z=0$. Asymptotic series in the sense of

Poincaré, or in the extended sense of (2.2) are therefore particular examples of <u>biorthogonal series</u>. The general concept of a biorthogonal series may be explained as follows: Let A be a linear class of functions and $\{L_n\}$ a sequence of linear functionals defined on A. Let $\{\phi_n\}$ be a sequence of functions of A for which the biorthogonality relationship (2.4) holds. A formal series $f \sim \sum_{n=0}^{\infty} L_n(f)\, \phi_n$, $f \in A$, is known as a biorthogonal series for f. Fourier series, Taylor series, interpolation series are additional familiar examples of biorthogonal series.

For biorthogonal series generated by given sequences $\{L_n\}$ and $\{\phi_n\}$, the following major problems may be raised.

A. <u>The Interpolation Problem</u>: What are necessary and sufficient conditions on a given sequence of constants β_n in order that the equations

$$L_n(f) = \beta_n \qquad (n = 0,1,\ldots) \tag{2.5}$$

possess a solution f in A. In particular, is there a solution for an arbitrary sequence β_n.

B. <u>The Uniqueness Problem</u>: Is the set $\{L_n\}$ <u>complete</u> for A. That is, do the conditions

$$L_n(f) = 0 \qquad (n=0,1,\ldots);\ f \in A \tag{2.6}$$

imply that $f \equiv 0$. If the set $\{L_n\}$ is complete for A, then the solution to an interpolation problem is unique if it exists.

C. <u>The Problem of Uniqueness Classes</u>: If $\{L_n\}$ is not complete for A, can we distinguish non-trivial subclasses $A^* \subset A$

such that $\{L_n\}$ is complete for A*.

D. <u>The Problem of Representation</u>: If $\{L_n\}$ is complete for A (or for an A*) in what sense does the formal series $f \sim \sum_{n=0}^{\infty} L_n(f)\phi_n$ represent f, or, how can we reconstruct f from a knowledge only of the sequence of values $L_n(f)$ (n=0,1,...). If we operate within a uniqueness class, then such a reconstruction is, in principle, always possible. If we do not operate within a uniqueness class, we can only hope to obtain f to within a constant multiple of a zero function, i.e., to within a solution of $L_n(f) = 0$ (n=0,1,...).

The main object of Part III is to discuss aspects of these problems for asymptotic series in the sense of Poincaré. For this case, interpolation is the construction of functions with preassigned asymptotic series. The uniqueness problem is that of constructing non identically vanishing functions with zero asymptotic expansions. The problem of uniqueness classes and of representation is that of finding supplementary conditions under which expansions are unique and of finding summability theorems.

3. <u>Functions Possessing Preassigned Asymptotic Expansions</u>

It is easily shown that for a wide class of domains D the sequence $\{L_n\}$ is incomplete for A(D).

<u>Theorem</u>: <u>Let the region</u> D <u>be contained in the sector</u> $0 < |z| \leq \rho \leq \infty$, $|\arg z| \leq \alpha\pi$, $\alpha < 1$. <u>Then</u> $\{L_n\}$ <u>is incomplete for</u> A(D).

<u>Proof</u>: This is established by exhibiting a non-identically vanishing function f which is in A(D) and for which $L_n(f) = 0$ $(n = 0,1,\ldots)$. Consider the function

(3.1) $$f(z) = e^{-\sigma z^{-k}}$$

where

(3.2) $$k > \frac{1}{2\alpha} .$$

We have $\frac{1}{r^n}|f| = \frac{1}{r^n} e^{-r^{-k}\cos k\theta}$. In the sector $|\theta| < \alpha\pi$ $|k\theta| < \frac{\pi}{2}$, so that $\cos k\theta \geq \delta > 0$. Thus, $\lim_{r\to 0} \frac{1}{r^n}|f| = 0$, $n=0,1,\ldots$ From (1.1) this implies that $L_n(f) = 0$, $n = 0,1,\ldots$.

Other examples of such functions are easily found. Indeed, for any $g(z) \in A$ and for any f with the above property, $f(z)g(z)$ will again have this property. Furthermore it is seen that this proof holds for regions D with the property that there exists an $r > 0$ such that the circle $|z| = r$ intersects D in a common region which is included in a sector. The example (3.1) does not hold in any region which winds around $z= 0$ infinitely many times; indeed this function does not possess any asymptotic expansion in such a region. We turn now to the interpolation problem for the sequence $\{L_n\}$.

<u>Theorem</u>: <u>Let</u> β_n <u>be a completely arbitrary sequence of complex numbers</u>. <u>Then there exist functions</u> f(z) <u>which are regular in a region</u> D <u>for which</u> z=0 <u>is a boundary point and such that</u>

(3.3) $$f(z) \sim \sum_{n=0}^{\infty} \beta_n z^n$$

This theorem is frequently given another formulation as follows: Let β_n be a completely arbitrary sequence, then there are functions f(z) which are regular in a region D* for which z=0 is a boundary point and such that

$$(3.4)\qquad \lim_{z \to 0} f^{(n)}(z) = \beta_n \qquad (n=0,1,\ldots).$$

By the results of §1, these formulations are equivalent if D and D* contain sectors. In general, the origin will be a singularity of the function in question. This will surely be the case if $\limsup_{n \to \infty} |\beta_n|^{1/n} = \infty$. This theorem has been proven under conditions of varying generality by E. Borel (3), J. F. Ritt (10), T. Carleman(5), A. Besikowitsch(1), P. Franklin(6), G. Pólya(9), and by numerous others. In view of the great interest in this problem and the diversity of methods employed, we shall sketch several proofs here.

First Proof. This is due to Carleman and may be described as a method of condensation of singularities. We shall construct a function which is analytic in the closed half circle D: $|z| \leq 1$, $\mathrm{Re}(z) \geq 0$, is of the form

$$(3.5)\qquad f(z) = c_0 + \sum_{n=1}^{\infty} \frac{c_n \epsilon_n z^n}{z + \epsilon_n}$$

and satisfies (3.3). Let $n > 1$ be fixed. Then given an $\epsilon > 0$ we have

$$(3.6)\qquad \left|\frac{\epsilon z^n}{z+\epsilon}\right| \leq \epsilon \left|\frac{z}{z+\epsilon}\right| < \epsilon, \quad z \in D.$$

Note further the identities

$$(3.7)\quad \frac{1}{k!}\frac{d^k}{dz^k}\left[\frac{\epsilon z^n}{z+\epsilon}\right]_{z=0} = \begin{cases} 0 & 0 \le k < n \\ 1 & k = n \end{cases}$$

Let α_n be an arbitrary sequence of positive numbers such that $\sum_{k=1}^{\infty} \alpha_k < \infty$. We select $c_0 = \beta_0$, $c_1 = \beta_1$ and $\epsilon_1 > 0$ such that $\left|\frac{c_1 \epsilon_1 z}{z+\epsilon_1}\right| \le \alpha_1$ in D. We proceed now by induction. Having selected constants $c_0, c_1, \ldots, c_{n-1}$ and positive constants $\epsilon_1, \epsilon_2, \ldots \epsilon_{n-1}$, we determine c_n by the requirement that

$$(3.8)\quad \frac{1}{n!}\frac{d^n}{dz^n}\left[c_0 + \frac{c_1\epsilon_1 z}{z+\epsilon_1} + \ldots + \frac{c_{n-1}\epsilon_{n-1} z^{n-1}}{z+\epsilon_{n-1}}\right]\Bigg|_{z=0} + c_n = \beta_n.$$

This is possible since the bracketed sum is regular at z=0. Having determined c_n, we now determine an ϵ_n such that

$$(3.9)\quad \left|\frac{c_n\epsilon_n z^n}{z+\epsilon_n}\right| \le \alpha_n, \quad z \in D.$$

This is possible in view of (3.6). The series (3.5) is dominated by $\sum_{k=1}^{\infty} \alpha_k < \infty$ in D and hence converges uniformly and absolutely to a function which is analytic in the interior of and continuous on the boundary of D. It remains to prove that

$$\lim_{z \to 0}\,[f(z) - \beta_0 - \beta_1 z - \ldots - \beta_{n-1} z^{n-1}]\, z^{-n} = \beta_n \quad (n=1,2,\ldots).$$

Now,

$$(3.10)\quad f(z) = c_0 + \frac{c_1\epsilon_1 z}{z+\epsilon_1} + \ldots + \frac{c_n\epsilon_n z^n}{z+\epsilon_n} + z^{n+1} R_n(z)$$

$$\equiv P_n(z) + z^{n+1} R_n(z)$$

where $R_n(z)$ is bounded in D. Hence it suffices to prove that

$$(3.11)\quad \lim_{z \to 0}\,[P_n(z) - \beta_0 - \beta_1 z - \ldots - \beta_{n-1} z^{n-1}]z^{-n} = \beta_n.$$

Now, in view of (3.7) and (3.8) we have

$$\frac{1}{k!}\frac{d^k}{dz^k}P_n(z)\Big|_{z=0} = \beta_k \qquad (k=0,1,\ldots,n) \tag{3.12}$$

Hence by the second theorem of § 1, (3.11) follows.

<u>Second Proof</u>: This is due to Pólya. The approach here is probably the most natural one, but it leads to the necessity of solving a system of infinitely many linear equations in infinitely many unknowns. We have seen that it is not always possible to construct a power series with preassigned derivatives at a point interior to its region of convergence. But this is possible when the distinguished point lies on the circle of convergence. We take the distinguished point to be z=1, and suppose that we have a power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots \tag{3.13}$$

convergent for $|z| < 1$. Suppose further that the following sequence of series derived from (3.13) by successive differentiations also converge to the value indicated

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 + \ldots &= \beta_0 \\ a_1 + 2a_2 + 3a_3 + \ldots &= \beta_1 \\ a_2 + 3a_3 + \ldots &= \beta_2 \\ a_3 + \ldots &= \beta_3 \\ &\vdots \end{aligned} \tag{3.14}$$

By Abels limit theorem* we can assert that as $z \to 1$ in an angle,

*Cf. e.g., Titchmarsh (12) p. 229.

$\lim_{z \to 1} f^{(n)}(z)/n! = \beta_n$ $(n=0,1,\ldots)$. We can therefore write $f(z) \sim \sum_{n=0}^{\infty} \beta_n (z-1)^n$. Thus, the problem will be solved if given an arbitrary set of values β_n we can produce a solution of the system (3.14). The first equation of (3.14) implies that $a_n \to 0$ and in such a case f would surely be regular in $|z| < 1$. The solution of the system (3.14) depends upon the following sufficient condition which Pólya has given for the solution of such systems. <u>Let there be given an infinite system of linear equations in the unknowns</u> x_i:

$$\sum_{i=0}^{\infty} a_{ni}\, x_i = \beta_n \qquad (n=0,1,\ldots) \tag{3.15}$$

<u>in which the coefficients satisfy</u>

$$\lim_{k \to \infty} a_{j-1,\,k}/a_{j,k} = 0 \quad (j=0,1,\ldots). \tag{3.16}$$

$$D_{n_k} = \begin{vmatrix} a_{ok} \ldots a_{o\;k+n-1} & \\ & \\ a_{n-1,k} & a_{n-1,k+n-1} \end{vmatrix} \neq 0 \qquad \begin{matrix} (n=1,2,\ldots) \\ \\ (k=0,1,\ldots) \end{matrix} \tag{3.17}$$

<u>The constants</u> β_n <u>may be completely arbitrary</u>. <u>Then there exists a solution of the system</u> (3.15) <u>which is such that all the series in</u> (3.15) <u>converge absolutely</u>.

The proof of this theorem involves only elementary notions of convergence, but will be omitted here. During the course of its proof a solution x_i is actually constructed by a stepwise process. In the system (3.14) we have $a_{j,k} = k!/j!(k-j)!$ so that

$$\lim_{k \to \infty} a_{j-1,k}/a_{j,k} = j/_{k-j+1} = 0 \;, \tag{3.18}$$

while $D_{n,k} = 1$. The above conditions are therefore fulfilled and (3.14) possesses an absolutely convergent solution for all sequences β_n.

Third Proof: We construct here a function which is regular in the entire z plane cut along the negative real axis and which possesses a preassigned asymptotic expansion at $z = 0$, in this region. This method is due to Ritt(10) and extended by Franklin(6).

Let $a_n = n!\,\beta_n$, and select quantities b_n such that

$$(3.19) \qquad 0 < b_n < 1 \quad \text{and} \quad b_n < k/|a_n| \qquad (n=0,1,\ldots)$$

for some fixed $k > 0$. If we set

$$(3.20) \qquad f(z) = \sum_{n=0}^{\infty} \beta_n z^n (1 - e^{-b_n z^{-1/3}})$$

where $z^{1/3}$ designates that branch which is real for $x > 0$, we can show that $\lim_{z \to 0} f^{(n)}(z)/n! = \beta_n$ $(n=0,1,2,\ldots)$ in the cut plane. The proof is, briefly, as follows. If $Re(t) < 0$ then $|1 - e^t| < 2|t|$, for if $|t| \leq 1$ then $|1 - e^t| = |t + \frac{t^2}{2!} + \ldots|$ $\leq |t(1 + \frac{t}{2!} + \frac{t^2}{3!} + \ldots)| \leq |t|\,|e-1| \leq 2|t|$. While if $|t| > 1$, $Re(t) < 0$, $|1 - e^t| \leq 1 + |e^t| \leq 2 \leq 2|t|$. Now, $Re(z^{-1/3}) = \rho^{-1/3} \cos \frac{\theta}{3} > 0$ in $-\pi \leq \theta \leq \pi$. Hence, $|1 - e^{-b_n z^{-1/3}}| < 2\,|b_n z^{-1/3}|$. The series (3.20) is therefore dominated by

$$(3.21) \quad F(z) = \sum_{n=0}^{\infty} \frac{2k}{n!} |z^n|\,|z^{-1/3}| = 2k\, e^{|z|}\,|z^{-1/3}|.$$

It follows that the series (47) converges uniformly and absolutely in any region $0 < h < |z| < R$ lying in the slit plane. Since h and R are arbitrary $f(z)$ is analytic throughout the slit plane. We have moreover,

$$(3.22) \quad f(z) - a_0 = \sum_{n=1}^{\infty} \beta_n z^n (1 - e^{-b_n z^{-1/3}}) - \beta_0 e^{-b_0 z^{-1/3}}$$

Now,

$$(3.23) \quad \left| \sum_{n=1}^{\infty} \beta_n z^n (1 - e^{-b_n z^{-1/3}}) \right| \leq 2k(e^{|z|} - 1) \left| z^{-1/3} \right| = o(1), \quad (z \to 0)$$

while $\left| e^{-b_0 z^{-1/3}} \right| = e^{-b_0 \rho^{-1/3} \cos \theta/3} = o(1)$, $z \to 0$. Therefore $\lim_{z \to 0} f(z) = a_0$ in the slit plane. Differentiating (3.20) formally there is obtained

$$(3.24) \quad f'(z) = \sum_{n=1}^{\infty} n \beta_n z^{n-1} (1 - e^{-b_n z^{-1/3}}) - \sum_{n=0}^{\infty} \frac{\beta_n b_n z^n}{3z^{4/3}} e^{-b_n z^{-1/3}}.$$

The second series is dominated by an exponential series while the first may be treated as above. The series (3.24) represents the derivative of f. As before, we can show $\lim_{z \to 0} f'(z) = \beta_1$. Repeated differentiations yield additional series which can be handled similarly.

Franklin has proved a very general existence theorem as follows:

<u>Theorem: Let</u> P_n <u>be an infinite set of points such that no</u> P_k <u>is a limit point of the set. Let</u> C_n <u>be a set of cuts in the complex plane joining</u> P_n to $z = \infty$. <u>For each</u> n <u>let there be</u>

<u>given an arbitrary sequence of complex numbers</u> β_{nk} $(k=0,1,\ldots)$. <u>Then there exist functions</u> $f(z)$ <u>which are regular in the slit plane and for which</u>

$$(3.25) \qquad \lim_{z \to P_n} f^{(k)}(z)/k! = \beta_{nk} \quad (k,n=0,1,\ldots).$$

In the language of asymptotic series, this means that we can find functions which have arbitrarily prescribed asymptotic behavior at the points of an infinite isolated set. Further generality can be achieved by replacing $z^{1/3}$ in the last proof by $z^{\frac{1}{2m+2}}$ which will enable us to deal with regions with a finite branch point at $z=0$. Finally, functions with preassigned asymptotic expansions can be constructed when we adopt the generalized definition of §2.

4. <u>The Birkhoff-Besikovitch Problem of Approximation</u>

This may be described as a problem of approximation with an infinity of auxiliary conditions. Let D and G be bounded regions with $\bar{G} \subset D$ and the origin in $D-\bar{G}$. Let there be given a function $g(z)$ which is analytic in D. It is desired to approximate $g(z)$ in G by a function h which possesses a prescribed asymptotic expansion at $z=0$. We may use the construction of the third proof of §3 to obtain a solution. From (3.21) we have $|f(z)| < 2k\, e^{|z|}|z^{-1/3}|$. Since k was completely arbitrary, we may in any bounded region G for which the origin is an exterior point, find a function f whose modulus is arbitrarily small in G and such that the quantities $\lim_{z \to 0} f^{(n)}(z)$ $(n=0,1,\ldots)$ are preassigned. Let $\sum_{n=0}^{\infty} a_n z^n$ be the required asymptotic expansion at $z=0$ and let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in a neighborhood of $z=0$. Construct a function $f(z)$ with an asymptotic expansion $\sum_{n=0}^{\infty} (a_n - b_n) z^n$ at $z=0$ which is of modulus $\leq \epsilon$ in G. Then the function

$$h(z) = g(z) + f(z) \tag{4.1}$$

will have the asymptotic expansion $\sum_{n=0}^{\infty} a_n z^n$ at $z=0$, while

$$\left| h(z) - g(z) \right| \leq \epsilon \text{ in } G \tag{4.2}$$

By making use of the generalizations mentioned in § 3, it is possible to prove the following.

<u>Theorem</u>: <u>Let</u> D <u>and</u> G <u>be bounded regions with</u> $\bar{G} \subset D$ <u>and let</u> $z_1, \ldots, z_n$ <u>lie on the boundary of</u> D. <u>If</u> g <u>is analytic in</u> D, <u>it is possible to find a function</u> h <u>with preassigned asymptotic expansions at</u> $z_1, \ldots, z_n$ <u>and such that</u> $h^{(k)}(z)$ <u>approximates</u> $g^{(k)}(z)$ <u>arbitrarily closely in</u> G, $0 \leq k \leq K < \infty$.

This theorem was first stated in Birkhoff(2) for functions of class C^{∞} and for two points. The first proof was given by A. Besikovitch(1). The general theorem above is due to P. Franklin (6). Nor does this exhaust the number of auxiliary conditions which can be imposed. We can, e.g., insist further that h actually take on the values $g(\alpha_i)$ at a finite number N+1 of points α_i in G. We shall show this for K=0. Let h* approximate g to within ϵ in G and have the prescribed asymptotic expansions at z_i. Let $\phi(z)$ be a function which has zero asymptotic expansions at z_i and such that $0 < \frac{1}{2} \leq |\phi(z)| \leq 1$ in G. Such a function ϕ may also be constructed by the above theorem. Let $p_N(z)$ be the polynomial of degree N for which

$$p_N(\alpha_i) = \frac{g(\alpha_i) - h^*(\alpha_i)}{\phi(\alpha_i)} \equiv \beta_i \quad (i=1,2,\ldots,N+1). \tag{4.3}$$

Consider now

$$h(z) = h^*(z) + \phi(z)\, p_N(z) \tag{4.4}$$

The function $\phi(z)p_N(z)$ has zero asymptotic expansions at each point z_i hence h has the required preassigned expansions at z_i. Furthermore, $h(\alpha_i) = g(\alpha_i)$. Now,

$$\begin{aligned} |h(z)-g(z)| &\leq |h^*(z)-g(z)| + |\phi(z)|\,|p_N(z)| \\ &< \epsilon + |p_N(z)|. \end{aligned} \tag{4.5}$$

From (4.3), $|\beta_i| \leq 2\epsilon$, while from Lagrange's formula

$$p_N(z) = \sum_{i=1}^{N+1} \beta_i \frac{w(z)}{w'(\alpha_i)} \; ; \quad w(z) = \prod_{i=1}^{N+1} (z-\alpha_i) \tag{4.6}$$

so that $\max\limits_{z \in G} |p_N(z)| \leq 2\epsilon M$ where $M = \max\limits_{z \in G} \sum\limits_{i=1}^{N+1} \left|\frac{w(z)}{w'(\alpha_i)}\right|$

Thus

$$\max_{z \in G} |h(z) - g(z)| \leq \epsilon(1 + 2M), \tag{4.7}$$

establishing the stated result.

It appears, then, that in the absence of additional hypotheses (i.e., utilizing only the coefficients a_n), an asymptotic series may be summed to arbitrary values on a finite set of points.

5. Uniqueness Theorems for Asymptotic Series

We have observed that the set of functionals $\{L_n\}$ given by (1.1) is frequently not complete for the class A(D). We may therefore raise the question whether non-trivial subclasses A* of A may be found for which $\{L_n\}$ is complete. Such a class will be called a uniqueness class for asymptotic expansions. Within a uniqueness class, the interpolation problem of § 3 need no longer possess a solution, but if it does, the solution will be unique. Uniqueness classes can be obtained by the requirement that all asymptotic expansions of functions in the class deviate in the large from their respective functions by not too great an amount. Let $f(z) \in A$ and $f(z) \sim \sum_{n=0}^{\infty} a_n z^n$ at $z=0$. Write

$$R_n(z) = f(z) - a_0 - a_1 z - \dots - a_{n-1} z^{n-1} \tag{5.1}$$

and consider the ratios

$$f_n(z) = z^{-n} R_n(z) \quad (n=1,2,\dots) \tag{5.2}$$

The functions $f_n(z)$ are analytic in D and are bounded as $z \to 0$. Let $\{m_n\}$ be an arbitrary sequence of positive numbers and consider the subset of functions $f(z) \in A$ which satisfy in addition

$$|f_n(z)| \leq Mm_n; \; z \in D, \; (n=0,1,\dots) \tag{5.3}$$

for some $M > 0$. Designate this subset by $A^*=A^*(m_n)$, exhibiting the dependence of A* on the preassigned sequence $\{m_n\}$. If $\{m_n\}$ does not become infinite too rapidly, then $A^*(m_n)$ will be a uniqueness class. This will be the case, e.g., if we select m_n so that

$$\liminf_{n \to \infty} (m_n)^{1/n} = r < \infty \quad . \tag{5.4}$$

To show this, let $f, g \in A^*(m_n)$ and possess identical expansions $f(z) \sim \sum_{n=0}^{\infty} a_n z^n$, $g(z) \sim \sum_{n=0}^{\infty} a_n z^n$. Now,

$$(5.5)\quad |z^{-n}|\left|f(z) - \sum_{i=0}^{n-1} a_i z^i\right| \le M_1 m_n, \quad |z^{-n}|\left|g(z) - \sum_{i=0}^{n-1} a_i z^i\right| \le M_2 m_n$$

so that

$$(5.6)\quad |z^{-n_k}|\,|f(z)-g(z)| \le M_3 m_{n_k} \le M_4(r+\epsilon)^{n_k} \qquad (k=1,2,\dots).$$

Hence for $|z| \le t < (r+\epsilon)^{-1}$ we have $|f(z)-g(z)| \le M_4(t(r+\epsilon))^{n_k}$. Allowing $n_k \to \infty$, we find that $f-g\equiv 0$ for $|z| < t$, $z \in D$, so that by analytic continuation, $f\equiv g$ throughout D.

On the other hand there are sequences $\{m_n\}$ for which $A^*(m_n)$ is not a uniqueness class. For let D^+ be a region such that every point of $\bar{D}$ with the exception of $z=0$ is interior to D^+. Select an $F(z)$ which is a regular in D^+, possesses an identically zero asymptotic expansion, but is itself not identically zero. Such an F can be found for a wide class of regions D^+. Now from the quantities $m_n = \max_{z\in D} |f_n(z)|$ formed form (5.2) using F. The class $A^*(m_n)$ cannot be a uniqueness class. A fundamental problem, therefore, is <u>to give necessary and sufficient conditions on the sequence $\{m_n\}$ in order that the class</u> $A^*(m_n)$ <u>be a uniqueness class for asymptotic expansions</u>. This problem may be formulated for arbitrary regions D, and for norms other than (5.3), e.g., for $\iint_D |f_n(z)|^2\, dx\,dy < Mm_n$.

If the growth of $\{m_n\}$ is restricted too severely, then every function of $A^*(m_n)$ will be regular at $z=0$ so that the class will be a trivial one. This will occur if the lim inf in (5.4) is replaced by lim sup. To show this, observe that from (5.1) we have

(5.7) $$R_n(z) = R_{n+1}(z) + a_n z^n$$

so that

(5.8) $$f_n(z) = a_n + z\, f_{n+1}(z)$$

Hence

(5.9) $$\left| a_n \right| < m_n + \left| z \right| m_{n+1}$$

and by letting $z \to 0$ we obtain the general inequality

(5.10) $$\left|a_n\right| < m_n$$

With $\limsup_{n \to \infty} m_n^{1/n} = r < \infty$, we have then $a_n < K(r+\epsilon)^n$. The function $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is regular for $|z| < r^{-1}$. If C designates the circle $|z| \le \eta < r^{-1}$, then as we have seen $A^*(m_n)$ is a uniqueness class for expansions in $D \cap C$, hence $f \equiv g$ in $D \cap C$ and is therefore regular in $|z| < r^{-1}$.

An important uniqueness class was exhibited by G.N. Watson (13) who showed that m_n may be increased to

(5.11) $$m_n = O\left(r^n\, \Gamma(n(1-\epsilon) + 1)\right)$$

for an appropriate D. This was later improved by F. Nevanlinna (8), who showed that we may take

(5.12) $$m_n = O\left(r^n\, \Gamma(n + 1)\right).$$

We shall give Nevanlinna's proof. D is an infinite sector, and for convenience we take $z=\infty$ as the distinguished point. We must first prove a lemma of Phragmén-Lindelöf type.

Lemma: Let $f(z)$ be analytic in the angle $-\pi/2k < \arg z < \pi/2k$ and satisfy the inequality

(5.13) $$|f(z)| \leq C e^{-\sigma r^k} ; \quad C, \sigma > 0, \; r = |z|$$

there. Then $f(z) \equiv 0$.

Proof. By analytic continuation it suffices to prove that f vanishes on the positive x axis. Select a $T > \sigma$ and consider the function

(5.14) $$F(z) = e^{T z^k} f(z).$$

We have $|F(z)| = e^{Tr^k \cos k\phi} |f(z)|$ so that by (5.13)

(5.15) $$|F(z)| \leq C e^{-(\sigma - T \cos k\phi) r^k}.$$

Select an α, $k < \alpha$, such that $\sigma = T \cos (k\pi/2\alpha)$. On the sides of the angle $-\pi/2\alpha < \arg z < \pi/2\alpha$ we have

(5.16) $$|F(z)| \leq C$$

while at every point in and on this angle we have

(5.17) $$|F(z)| \leq C e^{(T-\sigma) r^k}.$$

Selecting a k' such that $k < k' < \alpha$, we have, uniformly for $-\pi/2\alpha \leq \arg z \leq \pi/2\alpha$,

(5.18) $$\lim_{r \to \infty} e^{-r^{k'}} F(z) = 0.$$

By a standard Phragmén-Lindelöf theorem*, we have $|F(z)| \leq C$ interior to $-\pi/2\alpha < \arg z < \pi/2\alpha$. Therefore $|f(z)| \leq C |e^{-Tz^k}|$.

*See, e.g., Titchmarsh (12) p. 177.

For a point x on the real axis we have $|f(x)| < C\, e^{-Tx^k}$. We may now select T arbitrarily large and obtain $f(x) \equiv 0$.

<u>Theorem</u>: <u>If</u> $m_n = O(\sigma^n \Gamma(\frac{n}{k} + 1))$ $(n \to \infty)$, <u>then</u> $A^*(m_n)$ <u>is a uniqueness class for asymptotic expansions at</u> $z = \infty$ <u>in</u> $-\pi/2k' \leq \arg z \leq \pi/2k'$, $0 < k' \leq k$.

<u>Proof</u>: As we have observed previously, it suffices to show that if $f(z)$ is analytic in this region and if $|z^n f(z)| \leq M \sigma^n \Gamma(\frac{n}{k} + 1)$ $(n=0,1,\ldots)$ then $f(z) \equiv 0$. In the sector we have $|f(z)| \leq M(\frac{\sigma}{r})^n \Gamma(\frac{n}{k} + 1) \leq M_1 (\frac{\sigma}{r})^n (\frac{n}{k})^{n/k} e^{-\frac{n+1}{k}} \sqrt{n}$. Now for $\sigma (\frac{n}{k})^{1/k} \leq |z| \leq \sigma(\frac{n+1}{k})^{1/k}$, which, for fixed z can be satisfied by selecting n sufficiently large, we have $\sigma^{-n}(\frac{n}{k})^{n/k} \leq h^n$, $(\frac{r}{\sigma})^k \leq \frac{n+1}{k}$ so that $|f(z)| \leq M_1\, e^{-(\frac{r}{\sigma})^k} \sqrt{k(\frac{r}{\sigma})^k} \leq M_2\, e^{-\frac{(1-\epsilon)}{\sigma^k} r^k}$. But by the preceding lemma, this implies that $f(z) \equiv 0$.

Specializing this theorem to the case $k = 1$, we see that if

$$m_n = O\,(\sigma^n \Gamma(n+1)) \tag{5.19}$$

then $A^*(m_n)$ is a uniqueness class for asymptotic expansions in angles at least as large as $-\pi/2 \leq \arg z \leq \pi/2$. Within a uniqueness class, we can theoretically sum an asymptotic series and arrive at that function which gave rise to the series. That is, since the functionals $\{L_n\}$ are complete for a uniqueness class $A^*(m_n)$, then we can reconstruct any $f \in A^*(m_n)$ from a knowledge only of the sequence of asymptotic coefficients $L_n(f)$. This comes under Problem D of §2, the problem of representation.

One result in this direction is that due to G.N.Watson.

This tells us that if m_n satisfy (5.19) and if the basic sector is at least as large as $-\pi/2 - \lambda \leq \arg z \leq \pi/2 + \lambda$, $0 < \lambda < \pi/2$, then asymptotic expansions are summable to their function by a Borel method in $-\lambda < -\delta \leq \arg z \leq \delta < \lambda$ and for $|z|$ sufficiently large.

Theorem: Let D designate a region of the form

(5.20) $$-\pi/2 - \lambda \leq \arg z \leq \pi/2 + \lambda, \quad |z| \geq k > 0, \; 0 < \lambda < \pi/2.$$

Let $f(z)$ be analytic in D and possess an asymptotic expansion

(5.21) $$f(z) = a_0 + a_1 z^{-1} + \ldots + a_{n-1} z^{-(n-1)} + R_n(z)$$

where

(5.22) $$\left|z^n R_n(z)\right| = O\left(\sigma^n \Gamma(n+1)\right), \sigma > 0$$

uniformly for all z in D. Then the function

(5.23) $$a(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}$$

is regular in the circle $|t| < 1/\sigma$ and can be continued analytically to any sector

(5.24) $$-\lambda < -\delta \leq \arg t \leq \delta < \lambda$$

by means of the representation

(5.25) $$a(t) = \frac{1}{2\pi i} \int_L f\left(\frac{u}{t}\right) \frac{e^u}{u} \, du.$$

Here L is a contour consisting of two infinite radii $\arg u = \pm\nu < \lambda$, $|u| \geq \ell > k/\sigma$ joined by the circular arc $|u| = \ell$, $|\arg u| \leq \nu$, and traced positively from the fourth to the first quadrants.

Finally,

$$(5.26) \quad f(z) = \int_0^{\infty} e^{-w}\, a\left(\frac{w}{z}\right) dw, \quad \text{for } |z| > k, \ |\arg z| \leq \delta.$$

This theorem which may be regarded as an inversion theorem for Laplace transforms is established by making formal substitutions and justifying this by appropriate majorizations utilizing (5.22). For exact details the reader is referred to Hardy (7) pp. 192-194.

6. <u>The Carleman Theory</u>

For the case of the unit circle with $z = 1$ as the distinguished point, T. Carleman (5) has given necessary and sufficient conditions on the sequence $\{m_n\}$ in order that $A^*(m_n)$ be a uniqueness class for asymptotic expansions. Carleman's method is of great ingenuity, involving the solution of an interesting minimum problem for analytic functions. We shall consider the proof in detail.

<u>Theorem</u>: $A^*(m_n)$ <u>is a uniqueness class for asymptotic expansions at</u> $z = 1$ <u>in</u> $|z| < 1$ <u>if and only if</u>

$$(6.1) \qquad \int_1^{\infty} \log \left[\sum_{k=0}^{\infty} \frac{r^{2k}}{m_k^{\,2}} \right] \frac{dr}{r^2} = \infty .$$

The equality sign in (6.1) is to be given the following meaning: either the series appearing behind the integral has a finite radius of convergence, or its radius of convergence is infinite and the integral itself diverges. A finite radius of convergence implies that $\liminf_{n \to \infty} (m_n)^{1/n} = \rho < \infty$, and we have already seen that in such a case $A^*(m_n)$ is a uniqueness class.

The interesting possibility is therefore the second one.

We make the following preliminary observation: $A^*(m_n)$ is not a uniqueness class if and only if there exists an $f \not\equiv 0$, regular in $|z| < 1$ for which

$$\left| \frac{f(z)}{(1-z)^n} \right| < m_n \quad (n=0,1,\ldots), \quad |z| < 1. \tag{6.2}$$

For, if $A^*(m_n)$ is not a uniqueness class, there will exist two functions $g(z)$, $h(z) \in A^*(m_n)$ possessing the same asymptotic expansion $\sum_{n=0}^{\infty} a_n(1-z)^n$ satisfying

$$\left| \frac{g(z) - \sum_{k=0}^{n-1} a_k (1-z)^k}{(1-z)^n} \right| \leq m_n ; \qquad \left| \frac{h(z) - \sum_{k=0}^{n-1} a_k (1-z)^k}{(1-z)^n} \right| \leq m_n \tag{6.3}$$

$(n=0,1,\ldots)$. Therefore, (6.2) must hold with $f = \frac{1}{2}(g-h)$. Conversely, let $f(z)$ be regular in $|z| < 1$ and satisfy (6.2). These conditions imply that

$$\lim_{z \to 1} \left| \frac{f(z)}{(1-z)^n} \right| = 0 \qquad (n=0,1,\ldots). \tag{6.4}$$

so that from (1.1) we have $f(z) \sim 0 + 0\cdot(1-z) + 0\cdot(1-z)^2 + \ldots$

Thus, f is in $A^*(m_n)$ and possesses a zero asymptotic expansion.

For an arbitrary $f(z)$ regular in $|z| < 1$ we introduce an integral

$$I_n(f;\sigma) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{1}{m_k^2 \sigma^{2k}} \int_{|z|=1} \left| \frac{f(z)}{(1-z)^k} \right|^2 ds \tag{6.5}$$

which is of fundamental importance to the proof of Carleman's Theorem, and inquire as to its minimum value under the normalization $f(0) = 1$.

Lemma 1. As f(z) runs through the set of functions regular in $|z| \le 1$ and satisfying the normalization $f(0) = 1$, the integral $I_n(f;\sigma)$ possesses a minimum value $I_n^*(\sigma)$ which is given by

$$(6.6)\quad \log I_n^*(\sigma) = \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty} \log\left(\sum_{k=0}^{n} \frac{r^{2k}}{m_k^2\sigma^{2k}}\right) \frac{dr}{r^2\sqrt{1-\frac{1}{4r^2}}}.$$

Proof: Define a function $w_n(z)$ by means of

$$(6.7)\quad w_n(z) = \frac{(1-z)^n(1-z^{-1})^n}{m_0^2} + \frac{(1-z)^{n-1}(1-z^{-1})^{n-1}}{m_1^2} + \dots + \frac{1}{m_n^2}$$

Then,

$$(6.8)\quad z^n w_n(z) = \frac{(1-z)^n(z-1)^n}{m_0^2} + \frac{z(1-z)^{n-1}(z-1)^{n-1}}{m_1^2} + \frac{z^2(1-z)^{n-2}(z-1)^{n-2}}{m_2^2} + \dots + \frac{z^n}{m_n^2}$$

On $|z| = 1$, we have $z = (\bar{z})^{-1}$, so that

$$(6.9)\quad w_n(z)\Big|_{|z|=1} = \frac{|(1-z)^n|^2}{m_0^2} + \frac{|(1-z)^{n-1}|^2}{m_1^2} + \dots + \frac{1}{m_n^2} \equiv \Omega_n(z)$$

Now $z^n w_n(z)$ is a polynomial of degree $2n$. From (6.7), $z=0$ is not a root, while from (6.9) there are no roots on $|z| = 1$. From (6.7) if $z = \rho$ is a root, then $z = \rho^{-1}$ is also a root. We designate the roots which lie in the unit circle by $\rho_1, \rho_2, \dots, \rho_n$. Then,

$$(6.10)\quad z^n w_n(z) = C(z-\rho_1)\dots(z-\rho_n)(z-\rho_1^{-1})\dots(z-\rho_n^{-1})$$

and $C = (-1)^n/m_o^{\,2}$ so that

$$(6.11) \qquad w_n(z) = (m_o^{\,2}\rho_1\cdots\rho_n)^{-1}\prod_{k=1}^{n}(1-\rho_k z^{-1})(1-\rho_k z).$$

It follows that

$$(6.12) \quad w_n(z)\Big|_{|z|=1} = (m_o^{\,2}\rho_1\cdots\rho_n)^{-1}\prod_{k=1}^{n}(1-\rho_k\bar{z})(1-\rho_k z).$$

Since the coefficients of the polynomial $z^n w_n(z)$ are real, the quantities $\rho_1,\ldots,\rho_n$ are conjugate two by two (if n is odd, one ρ_i will be real) so that $\prod_{k=1}^{n}(1-\rho_k z) = \prod_{k=1}^{n}(1-\bar{\rho}_k z)$. Thus,

$$(6.13) \quad w_n(z)\Big|_{|z|=1} = (m_o^{\,2}\rho_1\cdots\rho_n)^{-1}\prod_{k=1}^{n}|(1-\rho_k z)|^2 .$$

From (6.13), (6.9), and (6.5),

$$(6.14)\, I_n(f;1) = (m_o^2\rho_1\cdots\rho_n)^{-1}\frac{1}{2\pi}\int_{|z|=1}\left|\frac{\prod_{k=1}^{n}(1-\rho_k z)}{(1-z)^n}f(z)\right|^2 ds$$

If we set $g(z) = \dfrac{\prod_{k=1}^{n}(1-\rho_k z)}{(1-z)^n}\, f(z)$, then g is regular in $|z| < 1$,

$g(0) = 1$ if $f(0) = 1$, and we must have $\int_{|z|=1}|g|^2\, ds < \infty$. If $g(z) = 1 + \sum_{n=1}^{\infty} a_k z^k$, then

$$\frac{1}{2\pi}\int_{|z|=1}|g|^2\, ds = 1 + \sum_{k=1}^{\infty}|a_k|^2$$

The minimum value of this integral therefore occurs with $a_k = 0 (k=1,2,\ldots)$, that is, with $g(z) \equiv 1$, and has the value 1. The function which minimizes (6.14) is therefore $f(z) = \dfrac{(1-z)^n}{\prod_{k=1}^{n}(1-\rho_k z)}$.

It is regular in $|z| \leq 1$. Furthermore

$$I_n^*(1) = (m_0^2 \rho_1 \cdots \rho_n)^{-1}. \tag{6.15}$$

We now apply Jensen's formula.* This states that if a function $h(z)$, $h(0) \neq 0$, is analytic in $|z| < R$, and if $\gamma_1 \leq \gamma_2 \leq \cdots$ are the moduli of its zeros in $|z| < R$, then for $r_n < \gamma < \gamma_{n+1}$,

$$\log \frac{r^n |h(0)|}{r_1 \cdots r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| h(\gamma e^{i\theta}) \right| d\theta. \tag{6.16}$$

In (6.16) select $h(z) = z^n w_n(z)$, $\gamma = 1$, then we obtain from (6.16), (6.13), (6.9),

$$\log I_n^*(1) = \frac{1}{2\pi} \int_0^{2\pi} \log \Omega_n(e^{i\theta}) d\theta. \tag{6.17}$$

From (6.9), $\Omega_n(e^{i\theta}) = (2\sin \frac{\theta}{2})^n \sum_{k=0}^{n} [m_k^2 (2\sin \frac{\theta}{2})^{2k}]^{-1}$

so that

$$\log I_n^*(1) = \frac{n}{\pi} \int_0^{2\pi} \log\left(2\sin\frac{\theta}{2}\right) d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log\left(\sum_{k=0}^{n} \left[m_k^2 \left(2\sin\frac{\theta}{2}\right)^{2k} \right]^{-1} \right) d\theta \tag{6.18}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log\left(\sum_{k=0}^{n} \left[m_k^2 \left(2\sin\frac{\theta}{2}\right)^{2k} \right]^{-1} \right) d\theta$$

in view of the vanishing of the first integral. Setting $2\sin \frac{\theta}{2} = r^{-1}$, and $m_k = \sigma^k m_k'$, we obtain (6.6).

<u>Lemma 2</u>. <u>The sequence</u> $I_n^*(\sigma)$ <u>is bounded as</u> $n \to \infty$ <u>if and only if</u>

*See, e.g., Titchmarsh (12) p. 125.

(6.19) $$\int_1^\infty \log \Big(\sum_{k=0}^{\infty} \frac{r^{2k}}{m_k^2} \Big) \frac{dr}{r^2} < \infty .$$

Proof: By Lemma 1, $I_n^*(\sigma)$ is bounded if and only if the quantities (6.6) are bounded. Since $I_n^*(\sigma)$ are non-decreasing as $n \to \infty$, $I_n^*(\sigma)$ are bounded if and only if

$$\int_{1/2}^{\infty} \log \Big(\sum_{k=0}^{\infty} \frac{r^{2k}}{m_k^2 \sigma^{2k}} \Big) \frac{dr}{r^2\sqrt{1-\frac{1}{4r^2}}} < \infty .$$

Setting $r' = r/\sigma$, we see that this statement is equivalent to

$$\int_{1/2\sigma}^{\infty} \log \Big(\sum_{k=0}^{\infty} \frac{r^{2k}}{m_k^2} \Big) \frac{dr}{r^2\sqrt{1-\frac{1}{4r^2}}} < \infty .$$

This again, is equivalent to (6.19) because of the inequalities

$$\frac{1}{\sqrt{1-\frac{1}{4\sigma^2}}} \geq \frac{1}{\sqrt{1-\frac{1}{4r^2\sigma^2}}} \geq 1, \text{ for } 1 \leq r < \infty,$$ and the convergence of $\displaystyle\int_{\frac{1}{2\sigma}} \frac{dr}{\sqrt{1-\frac{1}{4r^2\sigma^2}}}$.

After these preliminary lemmas, we turn to the proof of Carleman's theorem. It suffices to show that there exists an f regular in $|z| < 1$, $f(z) \neq 0$, such that

(6.20) $$\left| \frac{f(z)}{(1-z)^n} \right| \leq m_n \quad (n=0,1,\ldots), \quad |z| < 1.$$

if and only if (6.19) holds. Let $f(z)$ be regular in $|z| < 1$ and satisfy (6.20). If f has a zero of the p^{th} order at $z = 0$, $0 \leq p < \infty$, write $f(z) = z^p h(z)$. Then, $h(z)/h(0)\Big|_{z=0} = 1$.

Now,

(6.21) $$I_n(h(z)/h(0);\sigma) = \frac{1}{2\pi |h(0)|^2} \sum_{k=0}^{n} \frac{1}{m_k^2 \sigma^{2k}} \int_{|z|=1} \left| \frac{h(z)}{(1-z)^k} \right|^2 ds$$

so that by (6.20)

$$(6.22)\quad I_n\left(h(z)/h(0);\sigma\right) \leq \frac{1}{|h(0)|^2}\sum_{k=0}^{n}\sigma^{-2k} \leq \frac{\sigma^2}{|h(0)|^2(\sigma^2-1)},$$

for $\sigma > 1$. But $I_n^* \leq I_n$ and therefore $I_n^*(\sigma)$ is bounded as $n \to \infty$. Applying Lemma 2, we conclude that (6.19) holds.

Conversely, assume that (6.19) holds. Then we have,

$$(6.23)\qquad I_n^*(1) \leq M \quad (n=0,1,\ldots).$$

Let $\varphi_n(z)$ designate the minimizing function arising in Lemma 1 with $\sigma = 1$, and given by $\varphi_n(z) = \dfrac{(1-z)^n}{\prod_{k=1}(1-\rho_k z)}$. We have,

$$(6.24)\quad \frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{m_k^2}\int_{|z|=1}\left|\frac{\varphi_n(z)}{(1-z)^k}\right|^2 ds \leq M \quad (n=0,1,\ldots)$$

In particular, by taking the 0^{th} term of the left hand member of (6.24)

$$(6.25)\qquad \frac{1}{2\pi m_0^2}\int_{|z|=1}\left|\varphi_n(z)\right|^2 ds \leq M \quad (n=0,1,\ldots).$$

From (6.25) it follows that $\{\varphi_n(z)\}$ form a normal family* in $|z| < 1$. There therefore exists a subsequence $\varphi_{n_p}(z)$ converging uniformly in every closed subregion of $|z| < 1$ to an analytic function $F(z)$. Now $\varphi_{n_p}(0) = 1$, so that $F(0) = 1$. From (6.24),

*See e.g., Titchmarsh (12) p. 169.

$$(6.26) \qquad \frac{1}{2\pi m_k^{\,2}} \int_{|z|=1} \left| \frac{\varphi_n(z)}{(1-z)^k} \right|^2 ds \le M \qquad (k=0,1,\ldots,n)$$

so that

$$(6.27) \qquad \int_{|z|=r} \left| \frac{\varphi_n(z)}{(1-z)^k} \right|^2 ds \le 2\pi\, M\, m_k^{\,2},\ 0 \le r < 1 \quad (k=0,1,\ldots,n).$$

By selecting $n=n_p$ in (6.27) and letting $p \to \infty$, we obtain

$$(6.28) \qquad \int_{|z|=r} \left| \frac{F(z)}{(1-z)^k} \right|^2 ds \le 2\pi\, M\, m_k^{\,2},\ 0 \le r < 1 \quad (k=0,1,\ldots)$$

We wish now to show that from $F(z)$ and (6.28), we can construct an $f(z)$ regular in $|z| < 1$ for which (6.20) holds. At first, we shall insert a well known inequality for analytic functions. Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be regular in $|z| < 1$. Then, $\int_{|z|=r} |g(z)|^2 ds$ $= \int_{|z|=r} \sum a_n z^n \sum \bar{a}_n \bar{z}_n \, ds = 2\pi \sum_{n=0}^{\infty} r^{n+1} |a_n|^2$, $0 \le r < 1$. On the other hand, applying the Schwarz inequality to $\sum_{n=0}^{\infty} a_n z^n$, $|g(z)|^2 \le \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \right) \dfrac{1}{1 - \frac{|z|^2}{r^2}}$, so that

$$(6.29) \qquad |g(z)|^2 < \frac{1}{2\pi} \frac{r}{r^2 - |z|^2} \int_{|z|=r} |g(z)|^2 ds, \quad |z| < r.$$

Finally, let

$$(6.30) \qquad f(z) = a(1-z)\, F(1-b(1-z)), \quad (0 < b < 1)$$

and introduce

$$(6.31) \qquad t = 1 - b(1-z).$$

Since $|z| < 1$ implies $|t| < 1$, $f(z)$ is regular in $|z| < 1$.

Now,

$$\frac{f(z)}{(1-z)^k} = ab^{k-1} \ (1-t) \ \frac{F(t)}{(1-t)^k} \tag{6.31}$$

Applying (6.29) to (6.31) with $g = F(t)/(1-t)^k$, and using (6.28) we obtain

$$\left| \frac{f(z)}{(1-z)^k \ ab^{k-1}(1-t)} \right|^2 \leq \frac{r \ M \ m_k^{\ 2}}{r^2 - |t|^2} \ , \quad |t| < h. \tag{6.32}$$

Allowing $h \to 1$,

$$\left| \frac{f(z)}{(1-z)^k} \right| \leq ab^{k-1} \ m_k \sqrt{M} \ \frac{|1-t|}{(1-|t|^2)^{\frac{1}{2}}} \ , \quad |z| < 1. \tag{6.33}$$

Now,

$$\frac{|1-t|^2}{(1-|t|^2)} = \frac{b^2(1-z-\bar{z}+|z|^2)}{2b-b(z+\bar{z})-b^2(1-|z|^2)} = \frac{b^2(1-z-\bar{z}+|z|^2)}{(b-b^2)(2-z-\bar{z})-b^2(1-|z|^2)}$$

so that for $|z| < 1$ we have

$$\frac{|1-t|^2}{(1-|t|^2)} \leq \frac{b^2(2-z-\bar{z})}{(b-b^2)(2-z-\bar{z})} = \frac{b}{1-b} \tag{6.34}$$

Hence, from (6.33),

$$\left| \frac{f(z)}{(1-z)^k} \right| \leq \frac{a \ b^{k-\frac{1}{2}} \sqrt{M}}{\sqrt{1-b}} \ m_k. \quad (k=0,1,\ldots) \ . \tag{6.35}$$

Since $0 < b < 1$ and a was entirely arbitrary, we may now select it so that the inequalities (6.20) are satisfied.

This completes the proof of Carleman's Theorem. Carleman has provided additional characterizations of sequences $\{m_n\}$ for which (6.1) holds, and has given a method for the construction of the unique f (within an $A^*(m_n)$) from a knowledge only of its asymptotic coefficients.

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The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

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