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2332

ON THE ACCURACY OF THE NUMERICAL SOLUTION OF THE DIRICHLET PROBLEM BY FINITE DIFFERENCES

by

J. L. Walsh and David Young

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1. Introduction

Although finite difference methods afford a powerful tool for obtaining numerical solutions of partial differential equations, little is known about the accuracy. It is the purpose of this paper to derive numerical bounds for the error, in certain closed regions, of the difference analogue of the Dirichlet problem. We shall be concerned only with the difference between the exact solution of the difference equation and the solution of the Dirichlet problem. The error bounds which we obtain involve quantities which can actually be computed such as the mesh size, and the oscillation and modulus of continuity of the given function on the boundary. So far

1. This paper was presented to the American Mathematical Society in September 1951; an abstract was published in Bull. Amer. Math. Soc. 57, 478(1951). The paper was prepared in part while Professor Walsh was engaged as a consultant on a National Bureau of Standards contract with the University of California at Los Angeles. The research was also supported by the Office of Naval Research under Contracts N5ori-07634 and N5ori-76, Project 22, with Harvard University, and by the Army Office of Ordnance Research Project TB2-0001 (407) with the University of Maryland.

2. Harvard University; Consultant, National Bureau of Standards contract with the University of California at Los Angeles.

3. University of Maryland.

as method is concerned the chief novelty is the use of the difference analogues of harmonic measure and the Schwarz Alternating Process.

Gerschgorin [4][4] derived error bounds for boundary value problems associated with elliptic partial differential equations. These, and similar bounds derived by Collatz [2], and Mikeladze [10], involve bounds, in the closed region, of certain partial derivatives of the solution of the differential equation. However, the solution of the differential equation itself is not known, to say nothing of its derivatives, (although approximate values for the derivatives may sometimes be found by examining the corresponding difference quotients). Also, it may happen that although the derivatives in question are not bounded in the closed region, the solution of the difference equation may still converge to the solution of the differential equation.

Rosenbloom [13], presented an error bound for the Dirichlet problem which is closely related to Gerschgorin's but which utilizes special properties of harmonic functions. By use of well-known inequalities giving bounds for the partial derivatives of a harmonic function at an interior point in terms of the oscillation on the boundary and the distance from the boundary, he obtains upper bounds for the derivatives in closed subregions. Then, by solving the difference equation on subregions and, as the mesh size approaches zero, letting these

4. Numbers in brackets [ ], refer to the bibliography at the end of the paper.

subregions approach the given region, Rosenbloom obtains an error bound involving $\omega^*(\delta)$, the modulus of continuity in the closed region of the solution of the Dirichlet problem. However, Rosenbloom does not discuss the question of finding $\omega^*(\delta)$ in terms of $\omega(\delta)$, the known modulus of continuity of the given function on the boundary. Furthermore in practical numerical work one would not wish to change the boundary of the network as the mesh size is decreased.

In § 3 we use the explicit solution of the difference analogue of the Dirichlet problem for the rectangle, obtained by Le Roux [9] and by Phillips and Wiener [12], to derive an error bound involving bounds for the derivatives of the given function on the boundary. In § 4 and § 5 harmonic measure and its finite difference analogue, discrete harmonic measure, are used to obtain bounds for the moduli of continuity of harmonic and discrete harmonic functions respectively in certain regions in terms of $\omega(\delta)$. Phillips and Wiener [12] showed the existence of such bounds whereas we obtain suitable bounds in a precise numerical form. Then, in § 6, these upper bounds are used to yield a uniform error estimate for the closed rectangle in terms of the oscillation on the boundary and $\omega(\delta)$. In § 7 the extension to regions made up of two or more overlapping rectangles is discussed.

§ 2. Discrete Harmonic Functions

Let h and k be arbitrary positive numbers, and let $L[h,k]$ denote the set of points (x,y) such that both x/h and y/h are integers. Let Ω be a simply connected closed region with interior R such that the boundary S of Ω consists of straight lines each of which is parallel to a coordinate axis and contains a point of $L[h,k]$. Let Ω_L denote the subset of points of $L[h,k]$ contained in Ω. Two points (x_1,y_1) and (x_2,y_2) of $L[h,k]$ are <u>adjacent</u> if

$$(2.1) \qquad [(x_1-x_2)/h]^2 + [(y_1-y_2)/k]^2 = 1.$$

A point of Ω_L is an <u>interior</u> point of Ω_L if the four adjacent points belong to Ω_L. All other points of Ω_L are boundary points. We let R_L and S_L denote respectively the set of interior and boundary points of Ω_L. Evidently, we have $R_L \subseteq R$ and $S_L \subseteq S$.

A function U(x,y) defined on Ω_L is said to be <u>discrete harmonic</u>[5], (d.h.) in R_L if it satisfies the difference equation

$$(2.2) \qquad \delta^2[U(x,y)] = [2\sigma^2/(1+\sigma^2)][U(x+h,y)+U(x-h,y)-2U(x,y)]$$
$$+ [2/(1+\sigma^2)][U(x,y+k) + U(x,y-k)-2U(x,y)] = 0,$$

where

$$(2.3) \qquad \sigma = k/h.$$

The finite difference analogue of the Dirichlet problem is the following problem: given a function f(x,y) defined on S, to find a function U(x,y) defined on Ω_L, d.h. in R_L and coin-

5. Heilbronn [5] introduced the term <u>discrete harmonic function</u>, and studied the properties of these functions.

ciding with $f(x,y)$ on S_L. The existence and uniqueness of a solution of this problem for bounded regions is easy to prove; see for instance Gerschgorin [4]. The convergence to the solution of the Dirichlet problem has been proved using non-constructive methods by Le Roux [9], Phillips and Wiener [12], and others.

§ 3. Error Estimate for the Rectangle under Differentiability Assumptions

Let Ω be bounded by the lines $x=0$, $x=a$, $y=0$, and $y=b$, where $a=Ah$, $b=Bh$ and A and B are positive integers. Let $f(x,y)$ be defined and continuous on S and let $u(x,y)$ and $U(x,y)$ denote respectively the solution of the Dirichlet problem and its finite difference analogue with boundary values determined by $f(x,y)$. In this section we shall derive an upper bound for the error $U(x,y)-u(x,y)$ in the region Ω_L, under certain assumptions about the derivatives of $f(x,y)$. If $f_i(x,y)$ is defined and continuous on S, we denote generically by $u_i(x,y)$ and $U_i(x,y)$ the solutions respectively of the Dirichlet problem and its finite difference analogue with boundary values $f_i(x,y)$.

First, one can verify directly that the function $u_1(x,y)$ defined by

$$
\begin{aligned}
(3.1) \qquad u_1(x,y) = {} & f(0,0) + [f(a,0) - f(0,0)] \ (x/a) \\
& + [f(0,b) - f(0,0)] \ (y/b) \\
& + [g(a,b) + g(0,0) - g(a,0) - g(0,b)] \ (x/a)(y/b),
\end{aligned}
$$

which is linear on every line parallel to either coordinate axis, is both harmonic in R and d.h. in R_L. Thus if $f_1(x,y)$ is defined on S and equal to $u_1(x,y)$ on S, then $u_1(x,y) = U_1(x,y)$ in Ω_L.

Next, if $f_2(x,y) = f(x,y) - f_1(x,y)$ on S, then we have by linearity of Laplace's equation and (3.1).

$$(3.2) \qquad U(x,y) - u(x,y) = [U_2(x,y)+U_1(x,y)]-[u_2(x,y)+u_1(x,y)]$$
$$= U_2(x,y) - u_2(x,y).$$

Evidently both $U_2(x,y)$ and $u_2(x,y)$ vanish at all the corners of Ω.

We study arbitrary boundary values $f(x,y)$ by studying in turn four functions each of which has the boundary values $f_2(x,y)$ on one side of Ω and values identically zero on the other three sides. By the symmetry of the situation it is sufficient to study in detail only one of the latter functions. Let us set $f_3(x,y) = f_2(x,0)$ when $y=0$ and $f_3(x,y) = 0$ elsewhere on S. Since $f_2(x,y)$ is continuous and vanishes at the corners of Ω, $f_3(x,y)$ is continuous on S.

It can be verified directly[6] that if the Fourier series of $f_3(x,0)$ converges to $f_3(x,0)$, then we have

$$(3.3) \qquad u_3(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \frac{\sinh[(n\pi b/a)(1-y/b)]}{\sinh(n\pi b/a)}$$

where

6. See for instance [8] pages 95-96.

(3.4) $$A_n = (2/a)\int_0^a f_3(t,0)\sin(n\pi t/a)dt,$$

and that[7]

(3.5) $$U_3(x,y) = \sum_{n=1}^{A} A_n^* \sin(n\pi x/a)\,\frac{\sinh[m\pi b/a)(1-y/b)]}{\sinh(m\pi b/a)}$$

where

(3.6) $$A_n^* = (2h/a)\sum_{j=1}^{A-1}\sin(n\pi jh/a)f_3(jh,0)$$

and where m and n satisfy the relation

(3.7) $$\sinh(m\pi k/2a) = \sigma\sin(n\pi h/2a).$$

Now let $g(x) \equiv f_3(x,0)$. We now prove the following theorem:

Theorem 3.1. Let $g(x)$ satisfy the following conditions:

(a) $g(x)$ and its first $(s-2)$ derivatives are continuous, $0 < x < a$

(b) $g^{(s-1)}(x)$ fails to exist or fails to be continuous for at most a finite number of points in the interval $0 < x < a$, and $g^{(s-1)}(x)$ is uniformly continuous in each open interval in which it exists and is continuous

(c) $g^{(s)}(x)$ exists, except possibly for a finite number of points, and is bounded, $0 < x < a$.

Then uniformly for all $(x,y)\in\Omega_L$ we have

(3.8) $$\left|U_3(x,y) - u_3(x,y)\right| \leqslant (2+C)\,D_2\,(h/a),$$

if $s = 2$, and

(3.9) $$\left|U_3(x,y) - u_3(x,y)\right| \leqslant (1+Ce)\,D_3(h/a)^2$$

if $s = 3$, where

7. See for instance Le Roux [9] or Phillips and Wiener [12].

$$(3.10)\quad C = (1/24)\pi^2(1+\sigma^2)\coth(\alpha\pi b/a)/\alpha e$$

$$(3.11)\quad \begin{cases} D_2 = 2(a/\pi)^2[2J_1M_1/a + M_2] \\ D_3 = 2(a/\pi)^3[2(J_2+1)M_2/a+M_3] \end{cases}$$

$$(3.12)\quad \alpha = (2/\pi\sigma)\sinh^{-1}(\sigma)$$

and J_i, $i=1,2$, denotes the number of points in the open interval $0 < x < a$ at which $g^{(i)}(x)$ does not exist or is not continuous. Here M_i, $i=1,2,3$, denotes the least upper bound of the modulus of $g^{(i)}(x)$.

Proof.

Lemma 3.1. If m and n satisfy (3.7), then

$$(3.13)\quad 0 \leqslant n-m \leqslant (1/24)(1+\sigma^2)\pi^2n^3(h/a)^2, (n=1,2,\ldots,A-1).$$

Proof. Let $p = m\pi k/2a$, $q = n\pi h/2a$. Then (3.7) takes the form

$$(3.14)\quad \sinh p = \sigma \sin q.$$

We now study the function $p(q)$ defined by (3.14) in the interval $0 \leqslant q \leqslant \pi/2$. One can verify directly that $p(0)=0$, $p'(0)=\sigma$, and

$$p''(q) = -\frac{\sigma(1+\sigma^2)\sin q}{(1+\sigma^2\sin^2 q)^{3/2}} \leqslant 0$$

$$p'''(q) = -\frac{\sigma(1+\sigma^2)(1-2\sigma^2\sin^2 q)\cos q}{(1+\sigma^2\sin^2 q)^{5/2}}$$

where primes denote differentiation with respect to q. By the Extended Mean Value Theorem, since $p''(q) \leqslant 0$, we have $p \leqslant \sigma q$. Therefore $m \leqslant n$. On the other hand, we also have

$$p'''(q) \leqslant \frac{\sigma(1+\sigma^2)\,|1-2\sigma^2\sin^2 q|}{(1+\sigma^2\sin^2 q)^{5/2}} \leqslant \sigma(1+\sigma^2)$$

since $|(1-2\sigma^2\sin^2 q)(1+\sigma^2\sin^2 q)^{-5/2}| \leqslant 1$. Again using the Extended Mean Value Theorem we get

$$p(q) - \sigma q = (1/6)q^3 p'''(\xi) \qquad (0 < \xi < q)$$

and

$$|p(q) - \sigma q| \leq (1/6)q^3 \sigma (1+\sigma^2)$$

and the lemma follows.

Lemma 3.2. *If m and n satisfy (3.7) and if α is given by (3.12), then*

(3.15) $$m \geq \alpha n, \quad (n = 1,2,\ldots,A).$$

Proof. Defining p and q by (3.14) we have

$$\frac{d}{dq}(p(q)/q=[p'(q)q-p(q)]/q^2.$$

By the Mean Value Theorem we have

$$p(q) = q\, p'(\xi) \qquad (0 < \xi < q).$$

Since, as shown in the proof of Lemma 1, $p''(q) \leq 0$, we have $p'(q) \leq p'(\xi)$ and $p'(q)q-p(q) \leq 0$. Therefore the ratio $p(q)/q$ is a non-increasing function of q, $(0 \leq q \leq \pi/2)$. Its minimum value in the interval $0 \leq q \leq \pi/2$ is assumed when $q = \pi/2$. Therefore

$$p(q)/q \geq p(\pi/2)/(\pi/2) = (2/\pi)\sinh^{-1}(\sigma),$$

and the lemma follows.

Now for convenience let us define the function

(3.16) $$\Gamma_m(y) = \frac{\sinh[(m\pi b/a)(1-y/b)]}{\sinh(m\pi b/a)}$$

We shall study its behavior as a function of both m and y, where m is assumed to be a continuous variable.

Lemma 3.3. *If $m \geq 0$, then for $0 \leq y \leq b$ we have*

$$0 \geq \frac{d}{dm}\Gamma_m(y) \geq -(\pi\, y/a)\coth(m\pi\, b/a)\exp[-m\pi\, y/a)].$$

Proof. Differentiating (3.16) with respect to m we obtain

$$\frac{d}{dm}\Gamma_m(y) = (\pi\, b/a)\Big\{(1-y/b)\cosh[(m\pi\, b/a)(1-y/b)]\sinh(m\pi b/a)$$

$$-\cosh(m\pi\, b/a)\sinh[(m\pi\, b/a)(1-y/b)]\Big\}\sinh^{-2}(m\pi\, b/a)$$

$$= (\pi\, b/a)\Gamma_m(y)\Big\{(1-y/b)\coth[(m\pi\, b/a)(1-y/b)]-\coth(m\pi\, b/a)\Big\}.$$

We note that for $K > 0$, $x > 0$,

$$\frac{d}{dx}(x\coth(Kx))=\coth(Kx)-Kx\,\mathrm{csch}^2(Kx) = \frac{1}{2}\,\frac{\sinh(2Kx)-Kx}{\sinh^2(Kx)}\ .$$

But since $\sinh(2Kx) \geq 2Kx$, the last expression is non-negative; therefore by the Mean Value Theorem we have

$$(1-y/b)\coth[(m\pi\, b/a)(1-y/b)] \leq \coth(m\pi\, b/a)\ , \qquad (0\leq y\leq b)$$

and

$$\frac{d}{dm}\Gamma_m(y) \leq 0.$$

On the other hand

$$\frac{d}{dm}\Gamma_m(y)=(\pi\, b/a)\Big\{(1-y/b)\,\frac{\cosh[(m\pi\, b/a)(1-y/b)]}{\sinh(m\pi\, b/a)}$$

$$-\coth(m\pi\, b/a)\Gamma_m(y)\Big\}$$

$$=(\pi\, b/a)\coth(m\pi\, b/a)\Big\{(1-y/b)\,\frac{\cosh[(m\pi\, b/a)(1-y/b)]}{\cosh(m\pi\, b/a)}$$

$$-\Gamma_m(y)\Big\}$$

$$= (\pi\, b/a)\coth(m\pi\, b/a)\Big\{(1-y/b)\,\frac{\sinh(m\pi\, y/a)}{\sinh(m\pi\, b/a)\cosh(m\pi\, b/a)}$$

$$-(y/b)\Gamma_m(y)\Big\}.$$

For $0 \leq y \leq b$ the terms in the brackets are respectively non-negative and non-positive, whence

$$\frac{d}{dm}\Gamma_m(y) \geq -(\pi\, y/a)\coth(m\pi\, b/a)\Gamma_m(y).$$

Finally, we note that

$$\Gamma_m(y) = \exp(-m\pi\, y/a)\, \frac{1-\exp[-2(m\pi\, b/a)(1-y/b)]}{1-\exp[-2m\pi\, b/a)}$$

$$\leq \exp(-m\pi\, y/a)$$

provided[8] $0 \leq y \leq b$. The lemma now follows.

<u>Lemma 3.4</u>. <u>If</u> m <u>and</u> n <u>satisfy</u> (3.7), <u>then</u>

$$(3.17)\quad \left|\left[\frac{d}{dm}\Gamma_m(y)\right]_{m=m_1}\right| \leq [\coth(\alpha\pi\, b/a)/\alpha e]n^{-1}$$

$$(1 \leq n \leq A), (m \leq m_1 \leq n),$$

<u>where</u> α <u>is given by</u> (3.12).

<u>Proof</u>. It is easily verified that if $0 \leq t$, $0 \leq K$, then $|t\, e^{-Kt}| \leq (Ke)^{-1}$.

Therefore

$$y \exp(-m\pi\, y/a) \leq [(m\pi/a)e]^{-1}.$$

The lemma follows from Lemmas 3.2 and 3.3 and from the fact that $\coth(x)$ is a decreasing function of x, for $x \geq 0$.

<u>Lemma 3.5</u>. <u>If</u> m <u>and</u> n <u>satisfy</u> (3.7), <u>then</u>

$$|\Gamma_m(y) - \Gamma_n(y)| \leq C(h/a)^2 n^2, (0 \leq y \leq b), (1 \leq n \leq A),$$

<u>where</u> C <u>and</u> α <u>are given by</u> (3.10) <u>and</u> (3.12), <u>respectively</u>.

<u>Proof</u>. By the Mean Value Theorem we have

$$|\Gamma_m(y) - \Gamma_n(y)| \leq \max_{m \leq m_1 \leq n} \left|\left[\frac{d}{dm}\Gamma_m(y)\right]_{m=m_1}\right|\, |m-n|.$$

---

8. This inequality was proved by Phillips and Wiener [12].

The lemma follows from (3.13) and (3.17).

<u>Lemma 3.6.</u> <u>If</u> A_n <u>is defined by</u> (3.4), <u>then</u>

$$|A_n| \leq D_s/n^s \qquad (n = 2,3) \tag{3.18}$$

<u>where</u> D_s <u>is defined by</u> (3.11).

<u>Proof.</u> The proof involves the use of repeated integration by parts of the integral expression for A_n, and is similar to that given by Jackson [8], pages 13-14, for the case s = 2. We omit the details.

We now define the function $f_4(x,y)$:

$$f_4(x,y) = 0 \text{ unless } y = 0, \text{ and}$$
$$f_4(x,0) = \sum_{n=1}^{A} A_n \sin(n\pi\, x/a).$$

For $s \cancel{\geq 2} = 2,3$ and $0 \leq x \leq a$, we have

$$\left| f_4(x,0)-g(x) \right| = \left| \sum_{n=A+1}^{\infty} A_n \sin(n\pi\, x/a) \right| \leq D_s \sum_{n=A+1}^{\infty} a^{-s}$$

$$\leq D_s \int_A^{\infty} t^{-s}\,dt = D_s A^{1-s}/(s-1)$$

or

$$\left| f_4(x,0) - g(x) \right| \leq [D_s/(s-1)]\,(h/a)^{s-1} \tag{3.19}$$

On the other hand if we replace

$f_3(x,0)$ by $f_4(x,0)$ in (3.4) and (3.6) we get

$$A_n{}^* = A_n \qquad (n = 1,2,\ldots,A-1).$$

Therefore for $(x,y) \in L[h,k]$, we have, since $\sin(A\pi\, x/a) = 0$,

$$U_4(x,y)-u_4(x,y) = \sum_{n=1}^{A-1} A_n \sin(n\pi\, x/a)[\Gamma_m(y)-\Gamma_n(y)] \tag{3.20}$$

Corrigendum

to accompany

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(A) The following paragraphs should be inserted on page 13, line 9:

Evidently (3.22) could be used to obtain an error bound for the case s=3, and this was done in the original manuscript of the present paper. Since then, however, a paper by Wasow, [15], has appeared which contains a stronger result for the case s=3 than the original form of Theorem 3.1. A combination of his methods and the original ones now yields an improvement (the present Theorem 3.1) of Wasow's result in this case since we allow $g^{(3)}(x)$ and $g^{(2)}(x)$ to fail to exist for a finite number of points.

Thus, following [15], we obtain from Lemmas 3.1, 3.2, 3.3, and the Mean Value Theorem

$$\left|\Gamma_n(y) - \Gamma_n(y)\right| \leqslant C\alpha e(h/a)^2(\pi y/a)n^3\exp(-\alpha n\pi y/a).$$

By (3.20) and Lemma 3.6 we have

$$(3.24)\quad \left|U_4(x,y)-u_4(x,y)\right| \leqslant C\, D_3\alpha e(h/a)^2(\pi y/a)\sum_{n=1}^{\infty}\exp(-\alpha n\pi y/a)$$
$$\leqslant Ce\, D_3(h/a)^2,$$

since for $y>0$, $\sum_{n=1}^{\infty}\exp(-\alpha n\pi y/a)\leqslant[\exp(\alpha\pi y/a)-1]^{-1}\leqslant(\alpha\pi y/a)^{-1}$.

and

$$(3.21)\quad \left|U_4(x,y)-u_4(x,y)\right| \leq \sum_{n=1}^{A-1} |A_n| \left|\Gamma_m(y) - \Gamma_n(y)\right|$$

$$\leq CD_s(h/a)^2 \sum_{n=1}^{A-1} n^{2-s}$$

by Lemma 3.5 and (3.18). It can be shown that

$$\text{(3.22)} \; \cancel{(3.23)}\quad \sum_{n=1}^{A-1} n^{2-s} \leq 1 + \int_1^A t^{2-s}dt$$

$$\leq \begin{cases} A \quad = (h/a)^{-1} & , (s=2) \\ 1+\log A = 1-\log(h/2) & , (s=3). \\ \cancel{1+(s-3)^{-1} \leq 2} & , \cancel{(s\geq 4).} \end{cases}$$

Therefore, if s=2

$$\text{(3.23)} \; \cancel{(3.22)}\quad \left|U_4(x,y) - u_4(x,y)\right| \leq \cancel{E}\, CD_2(h/a).$$

(A) ~~where E is defined in (3.9).~~ Since d.h. functions possess mean value properties they assume their maximum and minimum values on the boundary as do harmonic functions. Since the functions $[u_3(x,y)-u_4(x,y)]$ and $[U_3(x,y)-U_4(x,y)]$ are harmonic and d.h. respectively we have

$$\left|u_3(x,y)-u_4(x,y)\right| \leq \max_{0\leq x\leq a} \left|g(x)-f_4(x,0)\right|,$$

$$\left|U_3(x,y)-U_4(x,y)\right| \leq \max_{0\leq x\leq a} \left|g(x)-f_4(x,0)\right|.$$

Therefore in Ω_L

$$\left|U_3(x,y)-u_3(x,y)\right| \leq \left|U_3(x,y)-U_4(x,y)\right| + \left|U_4(x,y)-u_4(x,y)\right|$$

$$+ \left|u_4(x,y)-u_3(x,y)\right| \leq \left|U_4(x,y)-u_4(x,y)\right|$$

$$+ 2 \max_{0\leq x\leq a} \left|g(x)-f_4(x,0)\right|.$$

By (3.19) (3.23) ~~(3.21)~~ and (3.24), ~~(3.23)~~, the theorem follows. Q.E.D.

§4. Harmonic Measure

Let Ω denote a simply connected region with interior R and boundary S. Let S' denote a subset of S consisting of a finite number of connected subsets of S. We define the <u>harmonic measure</u>, (h.m.), $H[(x,y), S', \Omega]$ as the unique function which is harmonic and bounded in R, is continuous in Ω except perhaps at a finite number of points of S, and equals unity on S' and zero on S-S'. The properties of harmonic measure have been studied in considerable detail, see for instance Nevanlinna [11] Chapter III.

By analogy we define <u>discrete harmonic measure</u>, (d.h.m.), for regions of the type described in §2, as follows: $H_L[(x,y), S', \Omega_L]$ is a function d.h. and bounded in R_L, equal to unity on $S_L \cap S'$ and to zero on $(s-S') \cap S_L$.

For bounded regions the existence and uniqueness of h.m. is well known, see for instance [11]. The existence and uniqueness for the half plane can be proved by the use of conformal mapping.

The existence and uniqueness of d.h.m. for bounded regions follows from the existence and uniqueness of the solution of the difference analogue of the Dirichlet problem. Later we shall prove existence and uniqueness of d.h.m. for a half plane and for certain other unbounded regions.

We list some elementary properties of d.h.m. which are analogues of well known properties of h.m.

(4.1) $\quad 0 \leq H_L[(x,y),\ S',\Omega_L] \leq 1 \quad$ for all $S' \subseteq S$.

(4.2) If $S' \subseteq S'' \subseteq S$, then

$$H_L[(x,y),S',\Omega_L] \leq H_L[(x,y),S'',\Omega_L].$$

(4.3) If S' is included in the boundary of both Ω_L and Ω_L^* where $\Omega_L \subseteq \Omega_L^*$ and if $(x,y) \in \Omega_L$, then

$$H_L[(x,y),\ S',\Omega_L] \leq H_L[(x,y),S',\Omega_L*].$$

The first property follows at once from the fact that the maximum and minimum values of d.h. functions are assumed on the boundary. The second follows from this fact and from the fact that the expression $\left\{H_L[(x,y),S'',\Omega_L] - H_L[(x,y),S',\Omega_L]\right\}$ is non-negative on S_L. The third follows since by the maximum and minimum principles $H_L[(x,y),\ S',\Omega_L^*\ ] \geq 0$ and hence $\left\{H_L\ [(x,y),\ S',\Omega_L^*\ ] - H_L[(x,y),\ S',\Omega_L]\right\} \geq 0$ for any point (x,y) on the boundary of Ω_L. This is the so-called principle of <u>gebietseweiterung</u>.

We shall use h.m. and d.h.m. for two purposes. In this section and in §5 lower bounds will be derived to enable us to obtain upper bounds for the modulus of continuity of harmonic and d. h. functions in a closed region in terms of their moduli of continuity on the boundary. In § 7, upper bounds will be derived which will enable us to use the Schwarz Alternating Process and its difference analogue for overlapping rectangles.

Let Ω denote the rectangle of § 3 and let S' denote the side contained in the line $y = 0$.

Theorem 4.1. If $(x-a/2)^2 + y^2 \leq \delta^2$ then

$$H[(x,y),S',\Omega] \geq 1-\nu(b/a)(\delta/a) \tag{4.4}$$

$$H_L[(x,y),S',\Omega_L] \geq 1-\nu(b/a)(\delta/a) \tag{4.5}$$

where

$$\nu(b/a) = [4 + \pi^2\coth^2(\pi\, b/a)]^{1/2}. \tag{4.6}$$

Proof. Let $u(x,y)$, $U(x,y)$ be harmonic and d.h. functions respectively in R vanishing on S-S' and equal to $\sin(\pi\, x/a)$ on S'. By (3.3)-(3.7) we have

$$u(x,y) = \sin(\pi\, x/a)\,\frac{\sinh[(\pi\, b/a)(1-y/b)]}{\sinh(\pi\, b/a)}$$

and

$$U(x,y) = \sin(\pi\, x/a)\,\frac{\sinh[(m\pi\, b/a)(1-y/b)]}{\sinh(m\pi\, b/a)}$$

where

$$\sinh(m\pi\, k/2a) = \sigma\sin(\pi\, h/2a).$$

By Lemma 3.1 we have $m \leq 1$. Also by Lemma 3.3 we have $\frac{d}{dm}\Gamma_m(y) \leq 0$. Therefore $\Gamma_m(y) \geq \Gamma_1(y)$ and we get

$$U(x,y) \geq u(x,y). \tag{4.7}$$

Now, for $0 \leq \theta \leq \pi/2$ we have[9] $\sin\theta \geq (2/\pi)\theta$ and hence for $0 \leq x \leq a/2$ we have $\sin(\pi\, x/a) \geq 2x/a = 1 - (2/a)(a/2-x)$. But since $\sin(\pi/2)(a/2 + \rho) = \sin(\pi/2)(2-\rho)$ for all ρ it follows that

9. This is Jordan's inequality; see for instance Copson [3] page 136.

$$\sin(\pi x/a) \geq 1 - (2/a)\ |a/2-x|\ , \ (0 \leq x \leq a).$$

Also

$$\left|\frac{d}{dy}\sinh[(\pi b/a)(1-y/b)]\right| = \left|(\pi/a)\cosh[(\pi b/a)(1-y/b)]\right|$$

$$\leq (\pi/a)\cosh(\pi b/a)$$

and by the Mean Value Theorem

$$\sinh[(\pi b/a)(1-y/b)] \geq \sinh(\pi b/a) - (\pi y/a)\cosh(\pi b/a).$$

Therefore

$$1-u(x,y) \leq 1-\left\{[1-(2/a)\ |a/2-x|\ ][1-(\pi y/a)\coth(\pi b/a)]\right\}$$

$$\leq (2/a)\ |a/2-x| + (\pi y/a)\coth(\pi b/a).$$

By the Schwarz Inequality the last expression does not exceed

$$(1/a)[(a/2-x)^2 + y^2]^{1/2}\, \nu\,(b/a) \leq (\delta/a)\,\nu\,(b/a)$$

provided $(x-a/2)^2 + y^2 \leq \delta^2$.

We now observe that

$$H[(x,y),S',\Omega] \geq u(x,y)$$

$$H_L[(x,y),\ S',\Omega_L] \geq U(x,y).$$

The theorem now follows from (4.7).

Now let Ω_L denote the subset of $L[h,k]$ such that $y \geq 0$ and let R_L and S_L denote the interior and boundary of Ω_L respectively.

<u>Theorem 4.2.</u> <u>For the region Ω_L, d.h.m. exists and is unique</u>.

<u>Proof</u>. Consider the sequence $\left\{H_L{}^{(n)}(x,y)\right\}$ where

$$H_L^{(n)}(x,y) = H_L[(x,y), S' \wedge S_L^{(n)}, \Omega_L^{(n)}]$$

and where $\Omega_L^{(n)}$ denotes the rectangle with vertices $(nh,0)$, $(-nh,0)$, (nh,nk) and $(-nh,nk)$. Evidently $H_L^{(n)}(x,y)$ exists since d.h.m. exists for bounded regions. By (4.3) it follows that the sequence $\{H_L^{(n)}(x,y)\}$ is non-decreasing and bounded above by unity. Therefore a unique limit exists which we denote by $H_L(x,y)$.

Since $H_L^{(n)}(x,y)$ is d.h. it satisfies (2.2). Taking limits of both sides of (2.2) we find that $H_L(x,y)$ satisfies (2.2) and is therefore d.h. Evidently $H_L(x,y) \leq 1$, $H_L(x,y) = 1$ on $S' \wedge S_L$ and $H_L(x,y) = 0$ on $S' - S_L$. Hence $H_L(x,y)$ has all the properties required for d.h.m.

In order to establish uniqueness we prove

<u>Lemma 4.1</u>. <u>If</u> $U(x,y)$ <u>is</u> d.h. <u>and bounded in</u> R_L <u>and vanishes on</u> S_L <u>then</u> $U(x,y)$ <u>vanishes in</u> R_L.

<u>Proof</u>. By hypothesis there exists a constant P such that for $(x,y) \in \Omega_L$ we have $|U(x,y)| \leq P$.

Given any $(x,y) \in \Omega_L$ and any $\epsilon > 0$, let a and b be chosen so that $(a,b) \in L[h,k]$ and

$$a \geq P(x^2 + y^2)^{1/2} \, \nu \, (1/2)/2 \, \epsilon$$

$$b \geq a.$$

Let Ω^* denote the rectangle with vertices at (a,b), $(-a,b)$, $(a,0)$ and $(-a,0)$ and let I_a denote the interval $(-a < x < a)$. Since $|U(x,y)| \leq P$ on S^* and vanishes on I_a, we have

$$|U(x,y)| \leq P\, H_L[(x,y),\, S^* - I_a, \Omega^*_L].$$

By Theorem 4.1 we have, replacing a by 2a,

$$H_L[(x,y), Ia, \Omega_L^*] \geq 1 - \nu\,(b/2a)(\delta/2a)$$

provided $x^2 + y^2 \leq \delta^2$. But by hypothesis we have

$$(x^2 + y^2)^{1/2} \leq 2\,\epsilon\, a/P\,\nu\,(1/2).$$

Substituting we get

$$H_L[(x,y), I_a, \Omega_L^*] \geq 1 - \nu\,(b/2a)\,\epsilon\,/P\,\nu\,(1/2).$$

Since $b \geq a$ we have $\nu\,(b/2a) \leq \nu\,(1/2)$; hence

$$H_L[(x,y), I_a, \Omega_L^*] \geq 1 - \epsilon/P,$$

and

$$|U(x,y)| \leq \epsilon .$$

This proves the lemma.

The uniqueness can now be proved by assuming two d.h.m.'s and showing that their difference vanishes identically.

The proof of Theorem 4.2 is complete.

We note that the existence of a unique bounded solution of the difference analogue of the Dirichlet problem for Ω_L with bounded boundary values is almost immediate. If the boundary values are determined by g(x), then the limit of the absolutely and uniformly convergent series

$$\sum_{\mu=-\infty}^{\infty} H_L[(x,y), \mu h, \Omega_L]\; g(\mu h)$$

is d.h. in R_L, bounded in Ω_L and equals g(x) on S_L. Thus a solution exists. The uniqueness follows at once from Lemma 4.1.

We next consider the region $x \geq 0$, $y \geq 0$. To find d.h.m. for subsets of the line $y = 0$ we perform a sign-changing reflection about the line $x = 0$ and use Theorem 4.2. Since the d.h.m. is zero on the line $x = 0$, this can be done in such a way that the new function will be d.h. in R_L. Similarly to find d.h.m. for subsets of the line $y = 0$ we reflect in the line $x = 0$.

By similar methods the existence and uniqueness of the d.h.m. can be established for other regions such as the semi-infinite strip $0 \leq x \leq a$, $y \geq 0$.

§ 5. Modulus of Continuity

The modulus of continuity of a function $f(x,y)$ in a closed region Ω is defined by

$$\omega(\delta) = \text{LUB} \left| f(x_1,y_1) - f(x_2,y_2) \right|$$

where (x_1,y_1), $(x_2,y_2) \in \Omega$ and $(x_1-x_2)^2 + (y_1-y_2)^2 \leq \delta^2$. Evidently if δ is not less than the diameter of Ω then $\omega(\delta)$ equals the oscillation of $f(x,y)$ in Ω.

For harmonic functions we prove the following theorem:

Theorem 5.1. Let Ω denote a bounded simply connected closed region with interior R and whose boundary S is a closed Jordan curve with the following property: there exist constants $r_0 > 0$ and $\Theta \geq 0$ such that for any point P of S there exists a circular sector with vertex at P, with radius r_0 and included angle Θ containing no point of R. Let $u(x,y)$ be

<u>harmonic in</u> R, <u>continuous in</u> Ω <u>with modulus of continuity</u> $\omega(\delta)$ <u>on</u> S. <u>If for some positive number</u> D <u>we have</u>

$$\delta < \operatorname{Max}\left\{r_0, D, D(r_0/D)^{(\pi+\psi)/\pi}\right.$$

<u>then the modulus of continuity of</u> u(x,y) <u>in</u> Ω <u>satisfies the inequality</u>

(5.1) $$\omega^*(\delta) \leq \omega\left[D(\delta/D)^{\pi/(\pi+\psi)}\right] + (4M/\pi)(\delta/D)^{\pi/(\pi+\psi)}$$

where

$$\psi = \operatorname{Max}(2\pi - \theta, \pi)$$

<u>and</u>

(5.2) $$M = \operatorname*{Max}_{(x,y)\in S}[u(x,y)] - \operatorname*{Min}_{(x,y)\in S}[u(x,y)]$$

<u>is the oscillation of</u> u(x,y) <u>on</u> S.

<u>Proof</u>.

<u>Lemma 5.1</u>. <u>Let</u> I_r <u>denote the interval</u> $-r \leq x \leq r$, $y = 0$ <u>and let</u> C_ρ <u>denote the region</u> $x^2 + y^2 \leq \rho^2$, $y \geq 0$. <u>If</u> $(x,y) \in C_\rho$, <u>then</u> $H[(x,y), I_r, y \geq 0] \geq 1 - 2\rho/\pi r$.

<u>Proof</u>. It is easy to show that if ϕ is the angle at (−r,0) from I_r to the circle C through the points (−r,0), (r,0) and (x,y), then

$$H[(x,y), I_r, y \geq 0] = 1 - \phi/\pi.$$

Now if

$$\rho/r = (1 - \cos\phi)/\sin\phi$$

then C_ρ is contained in the region bounded by C and the line y = 0. Hence, by the minimum principle for harmonic functions we have for $(x,y) \in C_\rho$

$$H[(x,y),\ I_r, y \geq 0] \geq 1-(2/\pi)\tan^{-1}(\rho/r) \geq 1-2\rho/\pi r$$

and the lemma is proved.

<u>Lemma 5.2.</u> <u>Let</u> $C_{r,\psi}$ <u>denote the region</u> $x^2+y^2 \leq r^2$, $0 \leq \tan^{-1}(y/x) \leq \psi \leq 2\pi$ <u>and let</u> $B_{r,\psi}$ <u>denote the bounding radii of</u> $C_{r,\psi}$. (<u>If</u> $\psi = 2\pi$ <u>then</u> $B_{r,\psi}$ <u>denotes the line</u> $0 \leq x \leq r$, $y = 0$). <u>If</u> $\rho \leq r$ <u>and if</u> $(x,y) \in C_{\rho,\psi}$, <u>then</u>

$$H[(x,y), B_{r,\psi}, C_{r,\psi}] \geq 1-(4/\pi)(\rho/r)^{\pi/\psi}.$$

<u>Proof.</u> We first consider the case $\psi = \pi$. If $x^2+y^2=r^2$ and $y > 0$, then as in Lemma 1 we have

$$H[(x,y),\ I_r,\ y \geq 0] = 1/2.$$

Hence, as we verify at once,

$$H[(x,y),\ I_r, C_r] = 2H\,[(x,y), I_r,\ y \geq 0] - 1.$$

If $(x,y) \in C_{\rho,\pi}$, then by Lemma 5.1 we have

$$H[(x,y), I_r, C_r] \geq 1-(4/\pi)(\rho/r).$$

The lemma can now be verified for the general case by mapping $C_{r,\pi}$ onto $C_{r,\psi}$ by means of the conformal transformation

$$(w/r) = (z/r)^{\psi/\pi}$$

We omit the details.

Now by the maximum principle for harmonic functions, the maximum value of $|U(x_1,y_1)-U(x_2,y_2)|$ for $(x_1-x_2)^2+(y_1-y_2)^2 \leq \delta^2$ occurs when either (x_1,y_1) or (x_2,y_2) belongs to S. If $(x_1,y_1) \in S$ then for all r such that $0 < r \leq r_0$ there exists a circular sector $C_{r,\psi}$ containing at least one point of R with vertex at (x_1, y_1) with radius $r \geq \delta$ and angle ψ, such that $B_{r,\psi}$ the union of the bounding radii is disjoint from R. Since the theorem is trivially true if $(x_2, y_2) \in S$, we assume $(x_2, y_2) \in R$. Let Ω_1 denote the closure of the connected component of (x_2, y_2) for the region $\Omega_1 \cap C_{r,\psi}$. Evidently $\Omega_1 \subseteq C_{r,\psi}$. Since for $(x,y) \in \Omega_1 \cap R$, we have

$$H[(x,y), B_{r,\psi}, C_{r,\psi}] \leq 1 = H[(x,y), \Omega_1 \cap R, \Omega_1]$$

and for (x,y) contained in Ω_1 and on the arc of $C_{r,\psi}$, we have

$$H[(x,y), B_{r,\psi}, C_{r,\psi}] = 0,$$

then for all $(x,y) \in \Omega_1$ it follows that

$$H[(x,y), B_{r,\psi}, C_{r,\psi}] \leq H[(x,y), \Omega_1 \cap R, \Omega_1].$$

But by Lemma 5.2 we have

$$H[(x_2,y_2), B_{r,\psi}, C_{r,\psi}] \geq 1-(4/\pi)(\delta/r)^{\pi/\psi}$$

since the lemma is obviously true if we rotate the sector.

Therefore, we have

$$\begin{aligned} |u(x_2,y_2)-u(x_1,y_1)| &\leq \omega(r)H[(x_2,y_2), C_{r,\psi} \cap S, \Omega] \\ &\quad + MH[(x_2,y_2), S-C_{r,\psi}, \Omega] \\ &\leq \omega(r) + (4M/\pi)(\delta/r)^{\pi/\psi} \end{aligned}$$

Now by the assumptions on δ we have

$$D(\delta/D)^{\pi/(\pi+\psi)} \leq D(r_0/D) = r_0.$$

Hence, we can choose $r=D(\delta/D)^{\pi/(\pi+\psi)} \leq r_0$ and the theorem follows.

We now consider the case where Ω is an arbitrary simply connected region. We first prove the following general theorem, which is essentially a formulation for harmonic functions which is equivalent to Carathéodory's form of the theorem of Milloux for analytic functions:

Theorem 5.2. Let G be a Jordan subregion of $|z| < 1$ whose boundary consists of a Jordan arc α_1 which passes through $z = 0$ plus an arc α_2 of Γ: $|z| = 1$. Let $u(z)$ be harmonic and bounded in G, continuous in the corresponding closed region except at the end points of α_1 and α_2, equal to unity in the interior points of α_1 and to zero in the interior points of α_2. Then in every point z of G we have

$$(5.3) \qquad u(z) \geq 1-(4/\pi)\tan^{-1} |z|^{1/2}.$$

Proof. Let $v(z)$ be conjugate to $u(z)$ in G, and let $f(z) \equiv \exp[-u(z)-i\, v(z)]$. Except perhaps at the end points of the arcs, on α_1 we have $|f(z)| = e^{-1}$, and on α_2 we have $|f(z)| = 1$; on these open arcs $|f(z)|$ is continuous in the two-dimensional sense. Then by the form of Milloux's theorem presented by Carathéodory, [1], § 358, we have for $z \in G$

$$\log |f(z)| = -u(z) \leq -1 + (4/\pi)\tan^{-1} |z|^{1/2}$$

as we were to prove.

Theorem 5.3. Let Ω denote a bounded simply-connected closed region with interior R and boundary S. Let $u(x,y)$ be harmonic in R, continuous in Ω and having modulus of continuity $\omega(\delta)$ on S. If D is any positive constant, then the modulus of continuity

<u>of</u> $u(x,y)$ <u>in Ω satisfies the inequality</u>

$$\omega^*(\delta) \leq \omega[D(\delta/D)^{1/3}] + (4M/\pi)(\delta/D)^{1/3}$$

<u>for all</u> $\delta \leq D$.

<u>Proof</u>. As in the proof of the previous theorem it is sufficient to show that if C_r is a circle with center (x_1,y_1) on S and radius r and if $(x_2,y_2) \in \Omega$ and $(x_1-x_2)^2 + (y_1-y_2)^2 \leq \delta^2 \leq r^2$, then

$$(5.4) \qquad H[(x_2,y_2), C_r \cap S, \Omega] \geq 1-(4/r)(\delta/r)^{1/2}.$$

The theorem would then follow if we let $r=D(\delta/D)^{1/3}$ since $\delta \leq D$ and hence $D(\delta/D)^{1/3} \geq \delta$.

Now, if Ω is a Jordan region, (5.4) follows at once from Theorem 5.2 since $\tan^{-1} |z|^{1/2} \leq |z|^{1/2}$ for $|z| \leq 1$. We indicate the modifications necessary to include the case where Ω is an arbitrary simply connected region.

Let O be a boundary point of Ω, let C_r be the circle $|z|= r$, and let z_0 be a point interior to Ω and to C_r. If no point of Ω lies exterior to C_r, then a function $v(z)$ harmonic in R and continuous in Ω equal to unity on S is identically unity in Ω so we have $v(z_0) = 1$. We proceed to study the contrary case. If points of R lie exterior to C_r, such points can be joined to z_0 by a Jordan arc lying wholly in R, so at least one arc of C_r lies in R. Let the totality of mutually disjoint arcs of C_r in R be $A_1, A_2, \ldots$. Denote by R_0 the subregion of R interior to C_r containing z_o and by R_1 the sum of R_o its reflection (inverse) in C_r, and the arcs A_k which form part of the boundary of R_o; if an arc of C_r is part of the boundary of R, it does not belong to R_1. We modify R_1 by adjoining to R_1 the interior of each Jordan curve that can be drawn in R_1 and to which O and z_o are exterior. Denote this new region by R_2. Then R_2 contains z_o, is simply connected, and has O as either an exterior or a boundary point. The subregion R_3 of R_2 interior to C_r is also simply connected, and part of its boundary is an open arc A_o of C_r of which every point is accessible; we choose A_o as the largest such arc, so that every boundary point of R_3 not on A_o is either a boundary point of R interior to C_r or is a point of C_r not contained in an arc of C_r consisting wholly of accessible boundary points of R_3. By a conformal map of R_3 onto the interior of a circle γ (in which A_o necessarily corresponds to an arc of γ) it is clear that $H(z,A_o,R_3)$ exists and is unique; this function takes the boundary value unity at every point of A_o and the boundary value zero at every boundary point of R_3 not on the closure of A_o. Carathéodory's proof of Milloux's theorem is valid for the region R_3, the distance from z_o to the boundary of R_3 is not greater than $|z_o|$, so (5.4) is valid for an arbitrary simply-connected region, and the theorem follows.

The modulus of continuity of a function $F(x,y)$ on any subset Ω_L of $L[h,k]$ is defined by

$$\omega_L(\delta) = \mathrm{LUB}\,|F(x_1,y_1) - F(x_2,y_2)|$$

where (x_1,y_1), $(x_2,y_2) \in \Omega_L$ and where $(x_1-x_2)^2+(y_1-y_2)^2 \leq \delta^2$.

For d.h. functions in a rectangle we prove the following theorem:

Theorem 5.4. Let Ω *denote the rectangle* $0 \leq x \leq a$, $0 \leq y \leq b$, *where* $(a,b) \in L[h,k]$. *Let* Ω_L *denote the subset of* $L[h,k]$ *contained in* Ω. *If* $U(x,y)$ is d.h. *in* R_L, *the interior of* Ω_L, *and has modulus of continuity* $\omega(\delta)$ *on* S_L, *the boundary of* Ω_L, *then for* $\delta \leq r = (ab)^{1/2}$ *the modulus of continuity of* $U(x,y)$ *in* Ω_L *satisfies the inequality*

$$(5.5)\quad \omega_L^*(\delta) \leq \omega_L[2^{1/2}\{(r\delta)^{1/2}+h+k\}]+(M/2)\nu(1/2)(\delta/r)^{1/2}$$

where M *and* $\nu(1/2)$ *are defined by* (5.2) *and* (4.6), *respectively.*

Proof. As in the proof of Theorem 5.1 the maximum value of $|U(x_1,y_1)-U(x_2,y_2)|$ for $(x_1-x_2)^2+(y_1-y_2)^2 \leq \delta^2$ is assumed when either (x_1,y_1) or (x_2,y_2) belongs to S_L. Let us assume that $(x_1,y_1) \in S_L$. Now if (x_2,y_2) also belongs to S_L the theorem is trivial since $\delta \leq r$. We therefore assume $(x_2,y_2) \in R_L$.

Let p denote a straight line containing (x_1,y_1) and including one of the sides of the rectangle. Let C denote the closed interior of a semi-circle with center at (x_1,y_1), with radius $\delta_1 = 2^{1/2}\{(\delta r)^{1/2}+h+k\}$, with bounding diameter included in p and containing at least one point of R_L. Since $\delta \leq r$ we have $\delta_1 \geq 2^{1/2}\delta$. Evidently there exists a rectangle T included in C with one side contained in p and with vertices contained in $L[h,k]$ such that the sides perpendicular to p are at least half as long as those parallel to p and such that the latter sides have length at least $2(\delta r)^{1/2}$.

By (4.2) and (4.3) we have

$$H_L[(x,y), C \cap S_L, \Omega_L] \geq H_L[(x,y), T \cap S_L, \Omega_L]$$

$$\geq H_L[(x,y), T \cap S_L, \Omega_L \cap T].$$

Now, for $(x,y) \in S_L \cap T$ we have

$$H_L[(x,y), T \cap p, T] \leq 1 = H_L[(x,y), T \cap S_L, \Omega_L \cap T].$$

Moreover, since the intersection of q and the boundary of $T \cap \Omega_L$ is contained in $T \cap S_L$ we have

$$H_L[(x,y), T \cap p, T] = 0$$

for all points of the boundary of $\Omega_L \cap T$ not contained in $S_L \cap T$. Therefore, for all points of the boundary of $T \cap \Omega_L$ we have

$$H_L[(x,y), T \cap S_L, \Omega_L \cap T] \geq H_L[(x,y), T \cap p, T].$$

It follows that for all $(x,y) \in T \cap \Omega_L$ we have

$$H_L[(x,y), T \cap S_L, \Omega_L \cap T] \geq H_L[(x,y), T \cap p, T].$$

Now since $(x_2,y_2) \in T \cap \Omega_L$ we have by Theorem 4.1

$$H_L[(x_2,y_2), T \cap p, T] \geq 1-(1/2)\,\nu\,(1/2)\ (\delta/r)^{1/2}.$$

Hence

$$H_L[(x_2,y_2), S_L-(S_L \cap C), \Omega_L] \leq (1/2)\,\nu(1/2)(\delta/r)^{1/2}.$$

Therefore

$$|U(x_1,y_1)-U(x_2,y_2)| \leq \omega_L[2^{1/2}\{(r\delta)^{1/2} + h + k\}]$$

$$+(M/2)\,\nu\,(1/2)(\delta/r)^{1/2}$$

and the theorem follows.

In the above proof if $h = k$, one could have obtained a rectangle with the desired properties by letting C have radius $2^{1/2}[(r\delta)^{1/2}+ h]$. Therefore we have

Corollary. If $h = k$, then

$$\omega_L^*(\delta) \leq \omega_L[2^{1/2}\{(r\delta)^{1/2}+h\}]+(M/2)\nu(1/2)(\delta/r)^{1/2}.$$

§ 6. Error Estimate for the Rectangle in Terms of the Modulus of Continuity on the Boundary

In this section we obtain an error bound for the rectangle of § 3 under the assumption that the function $f(x,y)$, which determines the boundary values, has modulus of continuity $\omega(\delta)$ on S. The function $f_3(x,y)$ defined in § 3 is also continuous and we denote by $\omega_3(\delta)$ its modulus of continuity for $0 \leq x \leq a$, $y = 0$.

Let $U_3(x,y)$ and $u_3(x,y)$ be given by (3.3) and (3.6) respectively. We define the function $f_5(x)$ by the partial sum

$$f_5(x) = \sum_{n=1}^{A} d_{A,n}A_n \sin(n\pi x/a) \tag{6.1}$$

where the A_n are the Fourier coefficients for $f_3(x,0) = g(x)$ as defined by (3.4) and where the coefficients $d_{A,n}$ are summation coefficients defined by Jackson [7] page 9. Let $u_5(x,y)$ and $U_5(x,y)$ denote respectively the solutions of the Dirichlet problem and its difference analogue vanishing on S except for $y = 0$. If $y = 0$ the values are determined by $f_5(x)$. We now prove

Theorem 6.1. If $(x,y)\in\Omega_L$ and $0 \leq x \leq a$, $\epsilon \leq y \leq b$, then

(6.2) $$|U_5(x,y)-u_5(x,y)| \leq 8M^*C\lambda(h/a)^2 \quad (\epsilon/a)^{-3}$$

where

(6.3) $$\lambda = \frac{6e}{\alpha^3\pi^3} + \frac{27\,\epsilon/a}{\alpha^2\pi^2e^2}$$

$$M^* = \operatorname*{Max}_{0\leq x\leq a} |g(x)|$$

<u>and where</u> C <u>and</u> α <u>are determined by</u> (3.10) <u>and</u> (3.12) <u>respectively.</u>

<u>Proof.</u> It can be verified that

(6.4) $$u_5(x,y) = \sum_{n=1}^{A} d_{A,n}\, A_n \sin(n\pi x/a)\Gamma_n(y)$$

(6.5) $$U_5(x,y) = \sum_{n=1}^{A} d_{A,n}\, A_n \sin(n\pi x/a)\Gamma_m(y)$$

where m and n are related by (3.7) and where $\Gamma_m(y)$ is defined by (3.16). Evidently we have

(6.6) $$|U_5(x,y)-u_5(x,y)| \leq \sum_{n=1}^{A} |d_{A,n}\, A_n|\, |\Gamma_m(y)-\Gamma_n(y)|$$

By Lemma 3.3 we have

(6.7) $$\left|\frac{d}{dm}\Gamma_m(y)\right| \leq (\pi y/a)\coth(m\pi b/a)\exp(-m\pi y/a)$$

$$(0 \leq y \leq b), \quad (m \geq 0).$$

By Lemma 3.2 we have $m \geq \alpha n$ for $1 \leq n \leq A$, and

(6.8) $$\left|\frac{d}{dm}\Gamma_m(y)\right| \leq (\pi y/a)\coth(\alpha\pi b/a)\exp(-\alpha n\pi y/a).$$

Using Lemma 3.1 we get, as in the proof of Lemma 3.5

$$|\Gamma_m(y)-\Gamma_n(y)| \leq C\,\alpha e(h/a)^2(\pi y/a)n^3\exp(-\alpha n\pi y/a).$$

Therefore we have

$$\sum_{n=1}^{A} |\Gamma_m(y)-\Gamma_n(y)| \leq \sum_{n=1}^{\infty} |\Gamma_m(y)-\Gamma_n(y)|$$

$$\leq C\,\alpha e(h/a)^2(\pi y/a) \sum_{n=1}^{\infty} n^3\exp(-\alpha n\pi y/a)$$

which is convergent for $y > 0$.

Now the following statement can be verified easily: if $G(x)$ is a continuous non-negative function for $x \geq 0$, non-decreasing for $0 \leq x \leq X$, and non-increasing for $X \leq x$, then

$$\sum_{n=1}^{\infty} G(n) \leq \int_0^\infty G(t)dt + G(X).$$

Therefore since for $\gamma > 0$, we have

$$\operatorname*{Max}_{x\geq 0} [x^3\exp(-\gamma x)] = (3/\gamma e)^3$$

it follows that

$$\sum_{n=1}^{\infty} n^3 \exp(-\alpha n\pi y/a) \leq \int_0^\infty t^3\exp(-\alpha\pi yt/a)dt + (3a/\alpha\pi ye)^3$$

$$\leq 6(a/\alpha\pi y)^4 + (3a/\alpha\pi ye)^3.$$

Therefore we get

$$(6.9)\qquad \sum_{n=1}^{A} |\Gamma_m(y)-\Gamma_n(y)| \leq \lambda C(h/a)^2(y/a)^{-3}$$

$$\leq \lambda C(h/a)^2(\epsilon/a)^{-3}$$

since $y \geq \epsilon$.

For the Fourier coefficients, since $|g(t)| \leq M^*$, we have

$$(6.10) \qquad \left| A_n \right| = \left| (2/a) \int_0^a g(t)\sin(n\pi t/a)dt \right|$$
$$\leq (2/a) \int_0^a |g(t)|\, dt < 2M^*.$$

Jackson [6] proved that

$$\left| 1 - d_{A,n} \right| \leq 3n/A \leq 3$$

hence

$$(6.11) \qquad |d_{A,n}| \leq 4.$$

From (6.6), (6.9), (6.10) and (6.11) the theorem now follows.

Corollary. For $(x,y)\in \Omega_L$ and $0 \leq x \leq a$, $\epsilon \leq y \leq b$ we have

$$(6.12) \qquad \left| U_3(x,y)-u_3(x,y) \right| \leq 8M^*C\,\lambda\,(h/a)^2(\epsilon/a)^{-3}$$
$$+ 2K\,\omega_3(2h)$$

where K is a positive absolute constant less than[10] 3.

Proof. $|U_3(x,y)-u_3(x,y)| \leq |U_3(x,y)-U_5(x,y)|$
$+|U_5(x,y)-u_5(x,y)| + |u_5(x,y)-u_3(x,y)|$.

By the maximum and minimum principles for harmonic and d.h. functions we have for $(x,y)\in \Omega_L$

$$|U_3(x,y)-U_5(x,y)|, |u_3(x,y)-u_5(x,y)| \leq \max_{0\leq x\leq a} |f_5(x)-g(x)|.$$

10. See Jackson [7] page 8.

Using a theorem of Jackson [7] page 7 we obtain

$$\max_{0\le x\le a} |f_5(x)-g(x)| \le K\omega_3(2a/A)$$

where K is a positive absolute constant less than 3. The corollary now follows from the theorem.

We shall now derive an error bound for the original problem where we no longer assume that the boundary values vanish at the corners or along the sides of the given rectangle.

<u>Theorem 6.2.</u> <u>For all</u> $(x,y)\in \Omega_L$ <u>we have</u>

$$(6.13)\qquad |U(x,y)-u(x,y)| \le \operatorname{Max}\left\{I;\ 2\omega_L^*\left[r\left(\frac{hk}{r^2}\right)^{2/7}\right]\right\}$$

<u>where</u>

$$I = 16M[\lambda_a C_a \sigma^{-6/7}(a/b)^{9/14}(h/a)^{2/7}$$

$$+\lambda_b C_b \sigma^{6/7}(b/a)^{9/14}(k/b)^{2/7}]$$

$$+4K\left\{\omega(2h)+\omega(2k)+2M[h/a+k/b]\right\}$$

$$(6.14)\qquad r=(ab)^{1/2}$$

$$(6.15)\qquad C_a = \frac{\pi^2(1+\sigma^2)}{24\alpha e}\coth(\alpha\pi b/a),\quad C_b = \frac{\pi^2(1+\sigma^{-2})}{24\beta e}\coth(\beta\pi a/b)$$

$$(6.16)\qquad \alpha = (2/\pi\sigma)\sinh^{-1}(\sigma),\quad \beta=(2\sigma/\pi)\sinh^{-1}(\sigma^{-1})$$

$$(6.17)\qquad \lambda_a = \frac{6e}{\alpha^3\pi^3}+\frac{27/2}{\alpha^2\pi^2e^2}$$

$$\lambda_b = \frac{6e}{\beta^3\pi^3}+\frac{27/2}{\beta^2\pi^2e^2}$$

$$(6.18)\qquad M = \max_{(x,y)\in S} u(x,y)-\min_{(x,y)\in S} u(x,y)$$

(6.19) $\omega_L{}^*(\delta) = \omega\,[2^{1/2}\{(r\delta)^{1/2} + h+k\}] + M\,\nu(1/2)(\delta/r)^{1/2}$

and $\omega(\delta)$ is the modulus of continuity of $u(x,y)$ on S. The function $\nu(b/a)$ is defined by (4.6) and $\nu(1/2)$ equals 3.97 approximately.

Proof.

Lemma 6.1. If $u_1(x,y)$ is defined by (3.1), then for all $(x,y)\in\Omega$ we have

$$|u(x,y) - u_1(x,y)| \le M.$$

Proof. Since $u(x,y)$ and $u_1(x,y)$ are harmonic, the maximum of $|u(x,y)-u_1(x,y)|$ is assumed on S. We can assume, without loss of generality that the maximum occurs at a point whose ordinate is zero. By (3.1) we have

$$u_1(x,0) = (1-x/a)u(0,0) + (x/a)u(a,0)$$

and

$$|u(x,0)-u_1(x,0)| \le |1-x/a|\ |u(x,0)-u(0,0)| + |x/a|\ |u(x,0)-u(a,0)|.$$

But

$$|u(x,0)-u(0,0)|\ ,\ |u(x,0)-u(a,0)| \le \max_{0\le x\le a} u(x,0) - \min_{0\le x\le a} u(x,0) \le M,$$

and the lemma follows.

The above result cannot be improved as one can show by considering the case where $u_1(x,y)$ is identically zero and $u(x,y) \ge 0$.

From Lemma 6.1 we conclude that

$$|u_3(x,y)| \le M.$$

In particular, we have

$$|g(x)| \leq M;$$

hence

(6.20) $$M^* \leq M.$$

Lemma 6.2.

(6.21) $$\omega_3(\delta) \leq \omega(\delta) + \delta M/a.$$

Proof. Evidently we have

$$\omega_3(\delta) \leq \omega(\delta) + \omega_1(\delta)$$

where $\omega_1(\delta)$ denotes the modulus of continuity of $u_1(x,0)$ considered as a function of x. But since

$$u_1(x + \delta, 0) - u_1(x,0) = (\delta/a)[u(a,0) - u(0,0)]$$

we have

$$\omega_1(\delta) \leq \delta M/a$$

and the lemma follows.

Lemma 6.3. If $\epsilon \leq x \leq a-\epsilon$, $\epsilon \leq y \leq b-\epsilon$, $(x,y) \in \Omega_L$, then

(6.22) $$|U(x,y)-u(x,y)| \leq 16M\{\lambda_a C_a (h/a)^2(\epsilon/a)^{-3}$$

$$\lambda_b C_b (k/b)^2 (\epsilon/b)^{-3}\} + 4K[\omega(2h) + \omega(2k)$$

$$+ 2M(h/a + k/b)].$$

Proof. We observe that

$$|U(x,y)-u(x,y)| = |U_2(x,y)-u_2(x,y)|.$$

The function $[U_2(x,y)-u_2(x,y)]$ is the sum of $[U_3(x,y)-u_3(x,y)]$ and three other terms of similar type. The lemma follows from (6.12), from (6.20) and from Lemma 6.2. (We note that from the hypothesis we have ϵ/a, $\epsilon/b \leq 1/2$).

Lemma 6.3 affords us an error bound for those points of Ω_L which are at a distance of not less than ϵ from the boundary. In order to obtain a uniform error bound we now consider those points which are within ϵ of the boundary.

Since $4/\pi < \nu(1/2)/2$ it follows from Theorem 5.1, (with $D=r$), and Theorem 5.4 that the moduli of continuity of $U(x,y)$ and $u(x,y)$ do not exceed the upper bound for $\omega_L{}^*(\delta)$ given by Theorem 5.4, provided $\delta \leq (ab)^{1/2}$. If $(x,y) \in \Omega_L$ and if (x,y) is within ϵ of a point of S, then there exists a point of S_L at a distance not greater than ϵ from (x,y). We remark that every point of Ω_L is within $(ab)^{1/2}$ of some point of S_L. Since $U(x,y) = u(x,y)$ on S_L we have $|U(x,y)-u(x,y)| \leq 2\omega^*{}_L(\epsilon)$. If I denotes the right member of (6.22), then we have

$$(6.23) \qquad |U(x,y) - u(x,y)| \leq \operatorname{Max}\left\{I;\ 2\omega_L^*(\epsilon)\right. .$$

In order that the error in Ω_L should approach zero with the highest power of $\gamma = (hk)^{1/2}$ whenever $\omega(\delta)/\delta$ is bounded as a function of δ, we choose

$$\epsilon = r(hk/r^2)^{2/7}.$$

Evidently we have

$$\epsilon/a = \sigma^{2/7}(h/a)^{4/7}(b/a)^{3/14}$$
$$\epsilon/b = \sigma^{-2/7}(k/b)^{4/7}(a/b)^{3/14}.$$

The theorem now follows from (6.23), Lemma 6.3, and Theorem 5.4.

For the special case of the unit square Theorem 6.2 and Theorem 5.4, Corollary, give

__Corollary.__ __If__ $a = b = \sigma = 1$ __then we have__

$$|U(x,y)-u(x,y)| \leq \text{Max}\{J_1; J_2\}$$

__where__

$$J_1 = 32\, M\, \lambda\, C\, h^{2/7} + 8K\, [\, \omega\,(2h) + 4Mh]$$
$$J_2 = 2\, \omega\, [2^{1/2}(h^{2/7} + h)] + M\, \nu\, (1/2)h^{2/7}$$
$$C = (\pi^2/12\, \alpha\, e)\, \coth(\alpha\pi) \backsim .536$$
$$\alpha = (2/\pi)\sinh^{-1}(1) \quad \backsim \quad .564$$
$$\lambda = \frac{6e}{\alpha^3\pi^3} + \frac{27/2}{\alpha^2\pi^2 e^2} \quad \backsim \quad 3.512\ .$$

Substituting numerical values we obtain

$$J_1 \leq 61\, M\, h^{2/7} + 24\, \omega(2h) + 96M\, h$$
$$J_2 \leq 2\omega[2^{1/2}(h^{2/7} + h)] + 4M\, h^{2/7}$$

The above expressions for J_1 and J_2 represent a slight improvement in the formulas previously given by the authors [14].

§ 7. Other Regions

In this section we consider regions of the type described in § 2 other than rectangles. The case of two overlapping rectangles is studied in detail as an illustration of the method which can be extended to more complicated regions. We first prove a theorem valid for two overlapping regions whether rectangles or not.

Theorem 7.1. Let $\Omega = \Omega' \cup \Omega''$ where $R' \cap R''$ is not empty and where neither region includes the other. Let $u(x,y)$ and $U(x,y)$ denote respectively the solution of the Dirichlet problem and its finite difference analogue for Ω. Let Ω_L, Ω'_L and Ω''_L denote the points of $L[h,k]$ belonging to Ω, Ω' and Ω'' respectively. Let

$$T'_L = S'_L \cap R''_L$$
$$T''_L = S''_L \cap R'_L$$

and let

$$\mu_1 = \max_{(x,y) \in T'_L} H_L[(x,y), T''_L, \Omega''_L]$$

$$\mu_2 = \max_{(x,y) \in T''_L} H_L[(x,y), T'_L, \Omega'_L].$$

Next, let U' and U'' denote functions d.h. in Ω'_L and Ω''_L, respectively, and equal to u on S'_L and S''_L respectively. Then for $(x,y) \in \Omega_L$ we have

$$|U(x,y) - u(x,y)| \leq \mathrm{Max}(A,B)$$

where

$$A = \frac{E'(1 + \mu_1 - \mu_1 \mu_2) + E''}{1 - \mu_1 \mu_2}$$

$$B = \frac{E''(1 + \mu_2 - \mu_1 \mu_2) + E'}{1 - \mu_1 \mu_2}$$

$$E' = \max_{(x,y) \in R'_L} |U'(x,y) - u(x,y)|$$

$$E'' = \max_{(x,y) \in R''_L} |U''(x,y) - u(x,y)|.$$

Proof. Let

$$F' = \max_{(x,y) \in T'_L} |U(x,y) - u(x,y)|$$

$$F'' = \max_{(x,y) \in T''_L} |U(x,y) - u(x,y)| .$$

Clearly, for $(x,y) \in T'_L$

$$|U(x,y) - u(x,y)| \leq |U(x,y) - U''(x,y)| + |U''(x,y) - u(x,y)|$$

$$\leq \mu_1 \max_{(x,y) \in T''_L} |U(x,y) - U''(x,y)| + |U''(x,y) - u(x,y)| .$$

Hence by the maximum and minimum principles for d.h. functions

$$F' \leq \mu_1 F'' + E'' .$$

Similarly

$$F'' \leq \mu_2 F' + E' .$$

Therefore

$$F' \leq \frac{\mu_1 E' + E''}{1 - \mu_1 \mu_2} ,$$

$$F'' \leq \frac{\mu_2 E'' + E'}{1 - \mu_1 \mu_2} .$$

By the maximum and minimum principles

$$\max_{(x,y) \in \Omega'_L} |U(x,y) - u(x,y)| \leq E' + F'$$

$$\max_{(x,y) \in \Omega''_L} |U(x,y) - u(x,y)| \leq E'' + F''$$

and the theorem follows.

The proof of this theorem was, of course, motivated by consideration of the difference analogue of the Schwarz Alternating Process. Indeed, in order to solve the difference equation for Ω_L one might guess values for $U(x,y)$ on $S'_L \cap R''_L$ and then solve the difference equation in R'_L obtaining in particular values on $S''_L \cap R'_L$. Next one solves the difference equation for R''_L, using the computed values on $S''_L \cap R'_L$, and obtains new values for $S'_L \cap R''_L$. This process can be repeated and the successive values thus obtained converge to the exact solution of the difference equation. Moreover, the rapidity of convergence can be estimated if μ_1 and μ_2 are known. In fact one can show that after a complete iteration the maximum error is reduced by a factor of $(1-\mu_1\mu_2)$.

If for Ω' and Ω'' the quantities E' and E'' are known whenever the modulus of continuity and oscillation of $u(x,y)$ are known on S' and S'', then by using Theorem 5.1 or Theorem 5.3, E' and E'' can be computed provided the modulus of continuity of $u(x,y)$ on S is known. If, moreover, μ_1 and/or μ_2 are known, an error bound for the composite region can be obtained.

We consider now the case of two overlapping rectangles Ω_1 and Ω_2 with sides parallel to the coordinate axes. If the intersection of the interiors is not empty and if neither region is a subset of the other, then the following cases can occur.

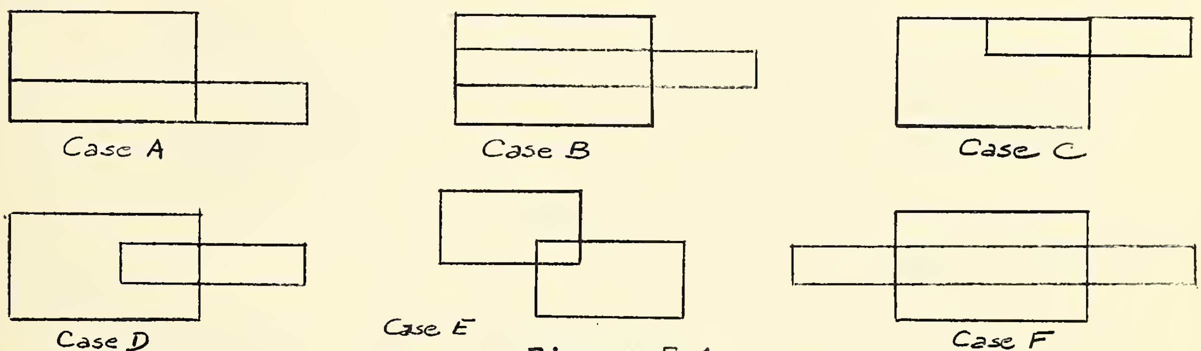

Figure 7.1

The composite regions of Cases C and D are of the same type as those of Cases A and B respectively. The problem of determining μ_1 and μ_2 became problems in d.h.m. for the rectangle of the following types:

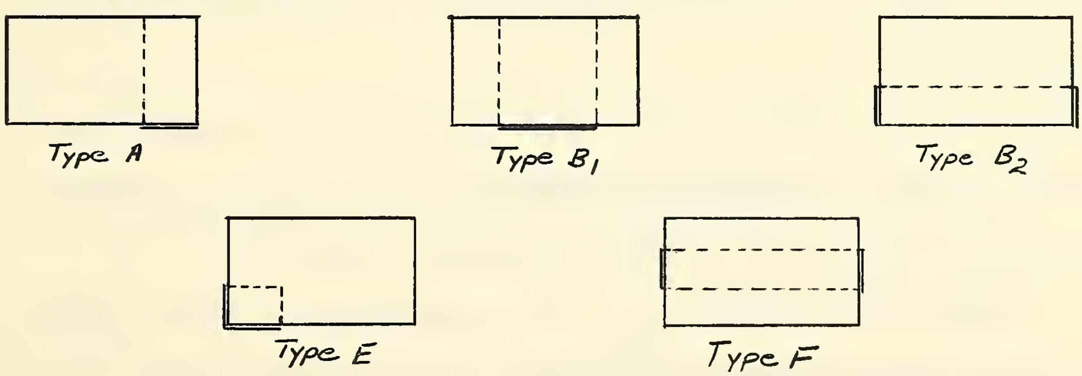

Figure 7.2

Upper bounds for d.h.m. are required in points on the closed dotted lines for the open arcs on the boundary indicated by heavy lines.

By (4.3), these upper bounds are not greater than the upper bounds for the following problems:

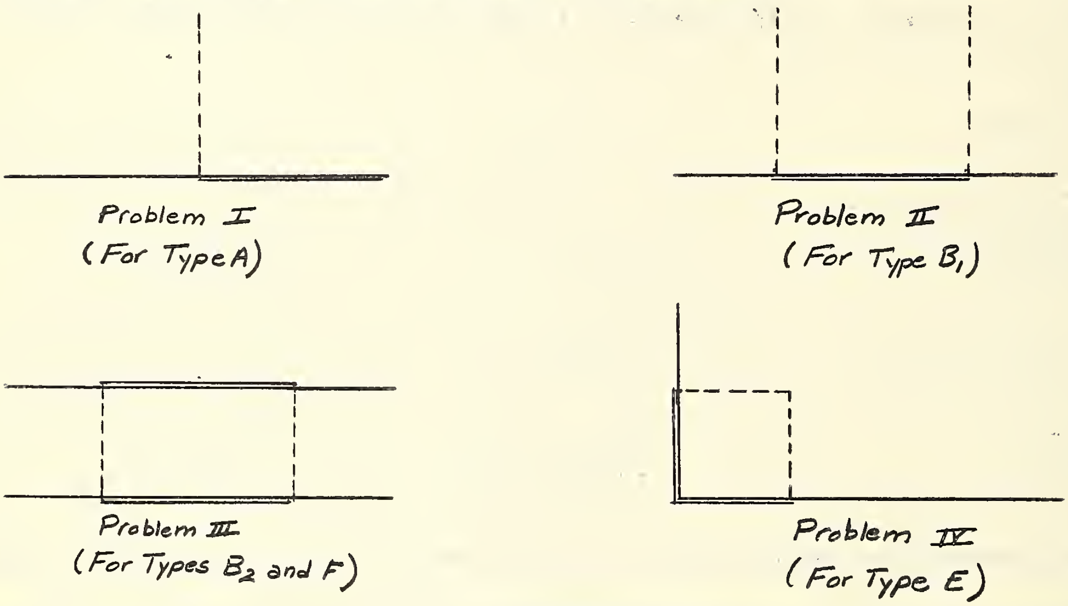

Figure 7.3

For problem I, since the harmonic measure of the open arc is required, 1/2 is an upper bound on the dotted line, by symmetry.

Theorem 7.2. Let Ω_L be bounded in part by a segment of a line ℓ. Let R_L contain no point of ℓ. Let ℓ' denote a line perpendicular to ℓ and containing a point of $\ell \cap S_L$. Let S' denote any subset of $\ell \cap S_L$ contained in one of the open half planes bounded by ℓ'. If (x,y) is any point of $R_L \cap \ell'$ then

$$H_L[(x,y),\ S',\Omega_L] \le 1/2,$$

Proof. The theorem follows easily by (4.2), (4.3), and Problem I.

As a corollary we have an upper bound of 1/2 for Problem II. Also, for Problem III if we extend the lines whose harmonic measure is required to the right, we get by symmetry as in Problem I, 1/2 as an upper bound for the d.h.m. on the left dotted line. Similarly 1/2 is an upper bound for the d.h.m. on the right dotted line.

For Problem IV the situation is somewhat more complicated. We prove

Theorem 7.3. Let Ω denote the region: $x \ge 0$, $y \ge 0$, and let Ω_L denote the subset of $L[h,k]$ contained in Ω. Let I_a, I_b denote respectively, the regions $0 < x < a$, $y = 0$ and $0 < y < b$, $x = 0$, where the point (a,b) belongs to $L[h,k]$. We have

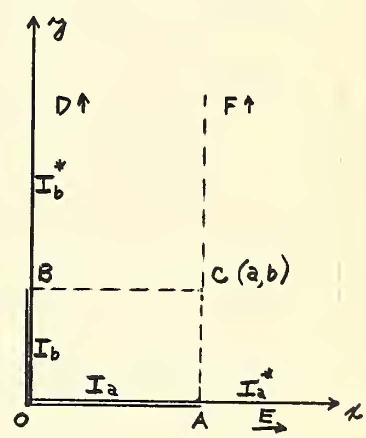

Figure 7.4

$$(7.1)\qquad H_L[(x,b),I_a \cup I_b,\Omega_L] \le \text{Max}\ \left[\frac{1+w}{2};\ 1 - \frac{(1-w)b}{2y_0}\right]$$

where

$$(7.2)\qquad w = (2/\pi)\log \frac{1 + \exp(-\pi/2)}{1-\exp(-\pi/2)}$$

Here y_0 is the smallest number not less than $a/2\alpha$ such that y_0/k is an integer. The quantities α and σ are defined by (3.12) and (2.3) respectively.

Proof. Let I^*_a, I^*_b denote respectively the regions $x \geq a$, $y = 0$ and $y \geq b$, $x = 0$. We first prove

Lemma 7.1. If $0 \leq x$, $0 \leq y$, *then*

$$(7.3) \qquad H_L[(x,b),\ I^*_b,\Omega_L] \geq H_L[(x,b),\ I_b,\Omega_L]$$

and

$$(7.4) \qquad H_L[(a,y),\ I^*_a,\Omega_L] \geq H_L[(a,y),\ I_a,\Omega_L].$$

Proof. By Theorem 7.2 we have

$$H_L[(x,b),\ y \geq b \cap x = 0,\ x \geq 0] \geq H_L[(x,b), y < b \cap x = 0, x \geq 0].$$

Hence we certainly have

$$H_L[(x,b),\ \{y \geq b \cap x = 0\} \cup \{-b < y < 0 \cap x = 0\},\ x \geq 0]$$
$$\geq H_L[(x,b),\ \{0 < y < b \cap x = 0\} \cup \{ y \leq -b \cap x = 0\},\ x \geq 0].$$

But by a sign-changing reflection about the line $y = 0$ we can show that the right member of the above expression minus the left member equals

$$H_L[(x,b), I^*_b,\Omega_L] - H_L[(x,b),\ I_b,\Omega_L] \geq 0.$$

This proves (7.3). Evidently (7.4) can be proved by the same method.

From Lemma 7.1 we conclude that

$$(7.5) \qquad H_L[(x,b), I_a \cup I_b, \Omega_L] \leq H_L[(x,b), I_a, \Omega_L]$$
$$+ (1/2)H_L[(x,b),\ 0 < y \cap x=0, \Omega_L].$$

But by (7.4) we have

$$(1/2)H_L[(a,y), 0 < y \wedge x = 0, \Omega_L] + H[(a,y), I_a, \Omega_L] \leq 1/2.$$

Therefore the d.h. function

$$(7.6)\quad W(x,y) = H_L[(x,y), I_a, \Omega_L] + (1/2)H_L[(x,y), 0 < y \wedge x = 0, \Omega_L]$$

satisfies the conditions

$$(7.7)\quad \begin{cases} W(x,0) = 1, & (0 < x < b) \\ W(0,y) = 1/2, & (0 < y) \\ W(a,y) \leq 1/2, & (0 \leq y). \end{cases}$$

Let Ω^* denote the region $0 \leq x \leq a \wedge y \geq 0$, and let Ω^*_L denote the corresponding subset of $L[h,k]$. Evidently

$$(7.8)\quad W(x,y) \leq U_1(x,y)$$

where

$$(7.9)\quad \begin{cases} \text{(a)} & U_1(x,y) \text{ is d.h. and bounded in } R^*_L \\ \text{(b)} & U_1(x,0) = 1, \ (0 < x < a) \\ \text{(c)} & U_1(0,y) = U_1(a,y) = 1/2, \ (y \geq 0). \end{cases}$$

We now prove

<u>Lemma 7.2</u>. <u>Let</u> $U_2(x,y)$ <u>be</u> d.h. <u>and bounded for</u> $(x,y) \in R^*_L$ and such that

$$U_2(x,0) = 1, \quad (0 < x < 1)$$

$$U_2(0,y) = U_2(a,y) = 0, \quad (y \geq 0).$$

We have

$$(7.10)\quad U_2(x,c) \leq \text{Max}[w;\ 1-(1-w)c/y_0].$$

<u>Proof</u>. Substituting in (3.5) and (3.6) and taking the limit as b becomes infinite we get

$$U_2(x,y) = \sum_{\substack{n=1 \\ n\ \text{odd}}}^{A} (2/A)\cot(n\pi/2N)\sin(n\pi x/a)\exp(-m\pi y/a)$$

where $Ah = a$.

By Lemma 3.2 we have $m \geq \alpha n$. Also, since $\cot[n\pi/2A] \leq 2A/n\pi$ for $n < A$ we have

$$U_2(x,y) \sum_{\substack{n=1 \\ n\ \text{odd}}}^{A} (4/\pi\ n)\exp(-\pi\alpha yn/a)$$

$$\leq \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} (4/\pi\ n)\exp(-\pi\alpha\ yn/a)$$

$$= (2/\pi) \log \frac{1 + \exp(-\pi\alpha\ y/a)}{1 - \exp(-\pi\alpha\ y/a)} .$$

Evidently if $y \geq a/2\alpha$, we have

(7.11) $$U_2(x,y) \leq w.$$

Now since $0 < w < 1$, the function

(7.12) $$U_3(x,y) = 1-(1-w)y/y_o$$

is d.h. in R^*_L and is not less than $U_2(x,y)$ for $S^*_L \cap y \leq y_o$ and for $y = y_o$. Therefore, for $y \leq y_o$ we have

(7.13) $$U_2(x,y) \leq 1-(1-w)y/y_o$$

and the lemma follows.

Now $U_2(x,y) = 2U_1(x,y) - 1$. We also have by (7.5), (7.6) and (7.8)

$$H_L[(x,b), I_a \cup I_b, \Omega_L] \leq W(x,b) \leq U_1(x,b)$$

and the theorem now follows.

In applying the theorem to Problem IV we note that we can assume the value zero at the point (0,0) without changing the d.h.m.

Thus an error bound can be obtained for the case of two overlapping rectangles. The above methods can also be used to find error bounds for other regions of the type considered in § 2.

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