NATIONAL BUREAU OF STANDARDS REPORT

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TABLES USEFUL IN ESTIMATING THE MEAN VALUE FUNCTION OF A FUNDAMENTAL RANDOM PROCESS

by

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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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1. Introduction. Let y(t) be a stochastic process and assume that

(1.1) $$y(t) = x(t) + f(t)$$

(1.2) $$f(t) = K_1\phi_1(t) + K_2\phi_2(t) + \dots + K_s\phi_s(t)$$

where x(t) is a fundamental random process.

In a forthcoming paper [4] H. B. Mann assumes that the y(t) process is observed over an interval [0,T] and developes a method for finding the maximum likelihood estimates for the parameters $K_1, K_2, \dots, K_s$ of the mean value function f(t)**. In a technical note [5] he gives a brief exposition of his results and also explicit formulae for the practically important case where f(t) is a polynomial in t. As in least squares estimates it seems also here advantageous to use orthogonal polynomials. However, it is here necessary to assume that the derivatives $\phi'_j(t)$ are the first s polynomials of a system orthogonal with respect to the weight function 1 over an interval [0,T]. The functions $\phi_j(t)$ become then the integrals of Legendre polynomials adapted to the interval [0,1]. As a consequence the mean value function must have initial and terminal value zero, i.e.,

(1.3) $$f(0) = f(T) = 0 .$$

While (1.3) will apply in certain physical situations its failure will in general prevent the use of orthogonal polynomials in estimating a mean value function***. It will then be necessary

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**The $\phi_j(t)$ are known functions, and are subject to certain restrictions, stated in [4] and [5].

***Ordinarily the condition (1.3) could be obtained by a rotation of axis, but in the situation under consideration such a rotation would disturb the nature of the random process, and hence can not be used.

to assume that $\phi(t) = t^j$ and to use the methods developed for this case.

Let us therefore suppose that

$$(1.4) \qquad f(t) = K_1 t + K_2 t^2 + \dots + K_s t^s$$

According to the formulae given in [4] and [5] the estimate $\hat{K}_j$ of K_j is given by

$$(1.5) \qquad \hat{K}_j = \sum_{r=1}^{s} \Phi^{jr}\, r \int_0^T t^{r-1} dy(t) \qquad (j = 1,2,\dots,s)$$

Here $\Phi^{jr} = \frac{S^{jr}}{jrT^{j+r-1}}$ where the matrix $\| S^{jr} \|$ is the inverse of the matrix $\left\| \frac{1}{j+r-1} \right\|_{j,r = 1,\dots,s}$. The purpose of the present note is to simplify the computations necessary for obtaining the estimates (1.5). This is done by studying first the matrix $\left\| \frac{1}{i+j-1} \right\|$ and giving formulae for finding its inverse. Tables of the coefficients S_n^{jr} are then computed for n = 2(1)10. These tables will be found at the end of this report. The same matrix occurs in least square theory when an integral is minimized instead of a sum. The inversion of this and related matrices was studied by A. R. Collar [2], [3] it is however believed that the proofs given here are somewhat more elementary.

2. <u>A theorem due to A. Cauchy</u>. The following theorem is due to A. Cauchy [1]. Let $a_1,\dots,a_n$, $b_1,\dots,b_n$ be 2n numbers and consider the determinant whose elements are of the form $1/(a_i+b_k)$ $(i,k = 1,2,\dots,n)$. Then

$$(2.1) \qquad \left\| \frac{1}{a_i+b_k} \right\|_{i,k=1,\dots,n} = \frac{\prod_{j>k}^{1..n} (a_j-a_k)(b_j-b_k)}{\prod_{j,k}^{1..n} (a_j+b_k)}$$

This theorem as well as an indication of its proof may also be found in [7], page 98, problem 3.

3. <u>Inversion of a certain matrix</u>. In the introduction we proposed to consider the matrix

$$(3.1) \qquad S_n = \left\| \frac{1}{i+j-1} \right\|_{i,j=1,\dots,n}$$

and to find its inverse. Clearly the determinant of S_n as well as all its minors are of the form discussed in Section 2. We can therefore compute the inverse S_n^{-1} by applying Cauchy's theorem. We denote by Δ_n^{ij} the minor of the element in the i-th row and and j-th column of the determinant Δ_n of the matrix S_n. If we write S_n^{ij} for the element in the i-th row and j-th column of the inverse S_n^{-1} then

$$S_n^{ij} = (-1)^{i+j} \frac{\Delta_n^{ij}}{\Delta_n} \quad . \tag{3.2}$$

If we use (2.1) to find Δ_n^{ij} and Δ_n we obtain by an elementary computation

$$\Delta_n = \frac{\left(\prod_{k=1}^{n-1} k!\right)^2}{n^n \prod_{k=1}^{n-1} (n^2-k^2)^{n-k}} \tag{3.3}$$

and

$$\Delta_n^{ij} = \frac{(n+i-1)!(n+j-1)!}{[(i-1)!(j-1)!]^2 n!(n-i)!(n-j)!} \frac{\left[\prod_{k=1}^{n-1} k!\right]^3}{\prod_{k=1}^{n-1} (n+k)!} \frac{1}{j+i+1} \tag{3.4}$$

It is easy to show inductively that

$$\frac{n^n}{n!} \prod_{k=1}^{n-1} \frac{(n^2-k^2)^{n-k} k!}{(n+k)!} = 1 \tag{3.5}$$

so that we obtain from (3.2), (3.3) and (3.4)

$$S_n^{ij} = \frac{(-1)^{i+j}}{i+j-1} \frac{(n+i-1)!(n+j-1)!}{[(i-1)!(j-1)!]^2 (n-i)!(n-j)!} \quad . \tag{3.6}$$

4. <u>Formulae for the numerical computation of the S_n^{ij}</u>. It is quite convenient to compute the S_n^{ij} recursively. We obtain immediately from (3.6)

$$S_{n+1}^{ij} = \frac{(n+i)(n+j)}{(n+1-i)(n+1-j)} S_n^{ij} \quad \text{for } i,j=1,2,\ldots,n \quad . \tag{4.1}$$

By means of (4.1) it is possible to compute from the elements of S_n^{-1} the elements of S_{n+1}^{-1} in the first n rows and columns. We still have to determine the elements of the last column. We can do this by deriving a relation between the elements of S_n^{-1} and the coefficients of certain orthogonal polynomials. Consider a polynomial of degree m

$$P_m(x) = 1 + a_{m,1}x + \dots + a_{m,m}x^m \quad (a_{m,m} \neq 0) \tag{4.2}$$

such that

$$\int_0^1 x^k P_m(x)dx = 0 \quad \text{for } k = 0,1,\dots,(m-1) \quad . \tag{4.3}$$

Equation (4.3) determines the polynomials $P_m(x)$, they are the Legendre polynomials adapted to the interval $0 \leq x \leq 1$. They are obtained from the Legendre polynomials formed for the interval $-1 \leq x \leq +1$ by a linear transformation of the variables. They are discussed and to a certain extent tabulated in [6]. This system of polynomials is orthogonal and we have

$$\int_0^1 P_m(x)P_n(x)dx = \begin{cases} 0 \text{ if } m \neq n \\ \frac{1}{2m+1} \text{ if } m = n \end{cases} \quad . \tag{4.4}$$

Moreover,

$$a_{m,k} = (-1)^k \binom{m}{k}\binom{m+k}{k} \quad . \tag{4.5}$$

For the proof of (4.4) and (4.5) the reader is referred to W. E. Milne's book [6] chapter IX, Section 69.

We write $a_{m,0} = 1$ and substitute (4.2) and (4.3) and obtain, considering also (4.4),

$$\sum_{j=0}^{m} \frac{a_{m,j}}{k+j+1} = \frac{1}{(2m+1)a_{m,m}} \delta_{k,m} \quad (k=0,1,\dots,m) \quad ,$$

where $\delta_{k,m}$ is the Kronecker delta. We rewrite these equations as

$$\sum_{j=1}^{m+1} \frac{a_{m,j-1}}{k+j-1} = \frac{1}{(2m+1)a_{m,m}} \delta_{k,m+1} \quad (k=1,\dots,m+1) \quad . \tag{4.6}$$

Considering the obvious relation

$$\sum_{k=1}^{m+1} \frac{1}{j+k-1} S_{m+1}^{hk} = \delta_{hk}$$

we solve the system (4.6) and obtain

$$a_{m,k-1} = \frac{1}{(2m+1)a_{m,m}} S_{m+1}^{m+1,k}$$

or

(4.7) $$S_{m+1}^{m+1,k} = (2m+1)\ a_{m,k-1} = S_{m+1}^{k,m+1} \quad .$$

From (4.7) and (4.5) we see then that for $k = 1, \ldots, (m+1)$

(4.8) $$S_{m+1}^{m+1,k} = S_{m+1}^{k,m+1} = (-1)^{m+k-1}\ (2m+1)\binom{2m}{m}\binom{m}{k-1}\binom{m+k-1}{k-1} .$$

Formulae (4.1) and (4.8) can be used to compute systematically tables of the elements of S_m^{-1} .

5. <u>An alternate method for computing S_m^{-1}</u>. We normalize the polynomials (4.2) and obtain

(5.1) $$Q_m(x) = \sqrt{2m+1}\ P_m(x) = \sum_{k=0}^{m} b_{m,k}x^k \quad (m=0,1,2,\ldots)$$

It is seen from (4.5) and (4.4) that

(5.2) $$b_{m,k} = (-1)^k\sqrt{2m+1}\binom{m}{k}\binom{m+k}{k}$$

and

(5.3) $$\int_0^1 Q_m(x)Q_n(x)dx = \delta_{m,n} \quad .$$

We consider the matrix

$$B_N = \left\| b_{m,k} \right\|_{m,k=0,\ldots,(N-1)} \quad .$$

It follows then from (5.3) that

(5.4) $$B_N S_N B_N' = I \quad .$$

Here I denotes the identity matrix and B_N' the transposed of B_N. From (5.4) we see that

(5.5) $$S_N^{-1} = B_N^{-1} B_N \quad .$$

If tables of the coefficients of the polynomials $P_m(x)$ are known it is possible to find S_N^{-1} from (5.5). On the other hand one could also derive (3.6) from (5.5) and (5.2).

6. Description of Tables and their preparation. Tables are given for the inverse of the matrix S_n [see equation (3.1)] for n=2,..,10. That is the quantities S_n^{ij} [see equation (3.2)] are tabulated for $1 \leq i \leq j \leq n \leq 10$.

Since the matrix S_n is symmetric its inverse is also, hence in the following tables S_n^{ij} is given only for $j \geq i$.

These tables were obtained by the use of equations (4.1), (4.7) and (4.8).

The tables were checked by using the relationships:

$$\sum_{i=1}^{n} S_n^{ij}/(i+k-1) = \delta_{jk} \quad (j,k = 1,\ldots,n). \tag{6.1}$$

All of the above relations were checked for j=k, and supplementarily a few others were examined and found correct.

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TABLES $T_n = S_n^{-1}$

T_2

4	-6
-6	12

T_3

9	-36	30
-36	192	-180
30	-180	180

T_4

16	-120	240	-140
-120	1200	-2700	1680
240	-2700	6480	-4200
-140	1680	-4200	2800

T_5

25	-300	1050	-1400	630
-300	4800	-18900	26880	-12600
1050	-18900	79380	-1 17600	56700
1400	26880	-1 17600	1 79200	-88200
630	-12600	56700	-88200	44100

T_6

36	-630	3360	-7560	7560	-2772
-630	14700	-88200	2 11680	-2 20500	83160
3360	-88200	5 64480	-14 11200	15 12000	-5 82120
-7560	2 11680	-14 11200	36 28800	-39 69000	15 52320
7560	-2 20500	15 12000	-39 69000	44 10000	-17 46360
-2772	83160	-5 82120	15 52320	-17 46360	6 98544

T_7

49	-1176	8820	-29400	48510	-38808	12012
-1176	37632	-3 17520	11 28960	-19 40400	15 96672	-5 04504
8820	-3 17520	28 57680	-105 84000	187 11000	-157 17240	50 45040
-29400	11 28960	-105 84000	403 20000	-727 65000	620 92800	-201 80160
48510	-19 40400	187 11000	-727 65000	1334 02500	-1152 59760	378 37800
-38808	15 96672	-157 17240	620 92800	-1152 59760	1005 90336	-332 97264
12012	-5 04504	50 45040	-201 80160	378 37800	-332 97264	110 99088

T_8

64	-2016	20160	-92400	2 21760	-2 88288	1 92192	-51480
-2016	84672	-9 52560	46 56960	-116 42400	155 67552	-105 94584	28 82880
20160	-9 52560	114 30720	-582 12000	1496 88000	-2043 24120	1412 61120	-389 18880
-92400	46 56960	-582 12000	3049 20000	-8004 15000	11099 08800	-7769 36160	2162 16000
2 21760	-116 42400	1496 88000	-8004 15000	21344 40000	-29967 53760	21189 16800	-5945 94000
-2 88288	155 67552	-2043 24120	11099 08800	-29967 53760	42499 41696	-30300 51024	8562 15360
1 92192	-105 94584	1412 61120	-7769 36160	21189 16800	-30300 51024	21754 21248	-6183 77760
-51480	28 82880	-389 18880	2162 16000	-5945 94000	8562 15360	-6183 77760	1766 79360

T_9

81	-3240	41580	-2 49480	8 10810	-15 13512	16 21620	-9 26640	2 18790
-3240	1 72800	-24 94800	159 66720	-540 54000	1037 83680	-1135 13400	658 94400	-157 52880
41580	-24 94800	384 19920	-2561 32800	8918 91000	-17481 06360	19423 40400	-11416 20480	2756 75400
-2 49480	159 66720	-2561 32800	17563 39200	-62432 37000	1 24309 78560	-1 39848 50880	83026 94400	-20216 19600
8 10810	-540 54000	8918 91000	-62432 37000	2 25450 22500	-4 54507 65360	5 16485 97000	-3 09188 88000	75810 73500
-15 13512	1037 83680	-17481 06360	1 24309 78560	-4 54507 65360	9 25542 85824	-10 60517 85840	6 39307 46880	-1 57686 32880
16 21620	-1135 13400	19423 40400	-1 39848 50880	5 16485 97000	-10 60517 85840	12 23674 45200	-7 42053 31200	1 83967 38360
-9 26640	658 94400	-11416 20480	83026 94400	-3 09188 88000	6 39307 46880	-7 42053 31200	4 52299 16160	-1 12633 09200
2 18790	-157 52880	2756 75400	-20216 19600	75810 73500	-1 57686 32880	1 83967 38360	-1 12633 09200	28158 27300

T_{10}

100	-4950	79200	-6 00600	25 22520	-63 06300	96 09600	-87 51600	43 75800	-9 23780
-4950	3 26700	-58 80600	475 67520	-2081 07900	5351 34600	-8324 31600	7701 40800	-3898 83780	831 40200
79200	-58 80600	1129 07520	-9513 50400	42810 76800	-1 12378 26600	1 77585 40800	-1 66350 41280	85065 55200	-18290 84400
-6 00600	475 67520	-9513 50400	82450 36800	-3 78756 37800	10 10017 00800	-16 16027 21280	15 29079 55200	-7 88431 64400	1 70714 54400
25 22520	-2081 07900	42810 76800	-3 78756 37800	17 67529 76400	-47 72330 36280	77 12857 15200	-73 58695 34400	38 20861 04400	-8 32233 40200
-63 06300	5351 34600	-1 12378 26600	10 10017 00800	-47 72330 36280	130 15446 44400	-212 10357 16800	203 77925 56800	-106 43827 19400	23 30253 52560
96 09600	-8324 31600	1 77585 40800	-16 16027 21280	77 12857 15200	-212 10357 16800	348 06739 96600	-336 39750 14400	176 60868 82560	-38 83755 87600
-87 51600	7701 40800	-1 66350 41280	15 29079 55200	-73 58695 34400	203 77925 56800	-336 39750 14400	326 78614 42560	-172 32863 07600	38 04495 55200
43 75800	-3898 83780	85065 55200	-7 88431 64400	38 20861 04400	-106 43827 19400	176 60868 82560	-172 32863 07600	91 23280 45200	-20 21138 26200
-9 23780	831 40200	-18290 84400	1 70714 54400	-8 32233 40200	23 30253 52560	-38 83755 87600	38 04495 55200	-20 21138 26200	4 49141 83600

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