NATIONAL BUREAU OF STANDARDS REPORT

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THE PROPER VALUES OF THE

SUM AND PRODUCT OF SYMMETRIC MATRICES

by

V. B. Lidskiĭ

Translated from the Russian by Curtis D. Benster

Edited by

Wallace Givens and George E. Forsythe

NBS

U. S. DEPARTMENT OF COMMERCE
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THE PROPER VALUES OF THE SUM AND PRODUCT OF SYMMETRIC MATRICES[1]

by

V. B. Lidskiĭ

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1. Let A and B be two arbitrary symmetric matrices with given proper values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

Let us raise the question: what are the proper values of their sum? The stated question can be formulated more exactly as follows: what are the systems of proper values of the matrices

$$S = A + B = U^{-1}\Lambda U + V^{-1}MV \quad , \tag{1}$$

where Λ and M are given diagonal matrices with real elements $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and U and V, independently of one another, run through the group of orthogonal matrices. In case A and B are positive definite matrices, the analogous question about the proper values of the product AB is interesting. The answer to the

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[1]Doklady Akademii Nauk SSSR, 1950, Vol. 75, No. 6, pp. 769-772.

second question has been obtained by I. M. Gel'fand and M. A. Naĭmark [1]. An elementary investigation of the stated questions is presented below. We shall first consider the problem of the proper values of a sum.

2. We shall put in correspondence with the system of proper values of each of the matrices S a point in an auxiliary n-dimensional space $R^{(n)}$ with coordinates $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ [arranged comparatively: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$]. The set so obtained will be designated as E.

Consider the 2(n!) points in $R^{(n)}$

$$\bar{a}_j(\lambda_1 + \mu_1', \lambda_2 + \mu_2', \cdots, \lambda_n + \mu_n'), \quad j = 1, 2, \cdots, n!,$$

$$\bar{b}_i(\mu_1 + \lambda_1', \mu_2 + \lambda_2', \cdots, \mu_n + \lambda_n'), \quad i = 1, 2, \cdots, n!,$$

where $\mu_1', \mu_2', \cdots, \mu_n'$ and $\lambda_1', \lambda_2', \cdots, \lambda_n'$ are certain permutations of the proper values of B and A, and all the permutations are taken.

Let us denote by K_a the closed convex hull stretched upon $\bar{a}_j$, and by K_b that upon $\bar{b}_i$. Furthermore let L be the intersection of K_a and K_b. The following theorem holds with regard to the proper values of the matrix S.

THEOREM 1. _The set E consisting of the points with coordinates_ $(\sigma_1, \sigma_2, \cdots, \sigma_n)$, _where_ $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ _are the proper values of the matrix_ S, _is contained in_ L.

We shall outline the proof, which is carried through by induction on the hypothesis that the theorem is valid for matrices of order $k < n$. For $n = 2$ the validity of the theorem is directly verifiable.

It is obvious that, without altering the problem, in (1) U, for example, can be considered to be the unit matrix. E can then be regarded as a bounded and continuous image of the group of orthogonal matrices over which V runs. On the strength of the compactness of the latter, E contains all its boundary points. We note that if V is an orthogonal matrix that is close, element-for-element, to the orthogonal matrix V_o, we have

$$\tilde{V} = V_o^{-1} \cdot V = \sum_{o}^{\infty} \frac{\alpha^k P^k}{k!} \quad , \tag{2}$$

where αP is a skew-symmetric matrix, the $\frac{1}{2}n(n - 1)$ independent elements of which, $\alpha\delta_{ij} = -\alpha\delta_{ji}$, may be regarded as local coordinates of a neighborhood of V_o. It is easily shown that, in the neighborhood of those V to which correspond the matrices S which do not have multiple proper values, our mapping is differentiable in the sense that there exist total differentials of the proper values σ_k as functions of the local coordinates. The computations lead to the following expressions for the differentials of the proper values of the matrix S:

$$d\sigma_k = 2 \sum_{i>j} \gamma_{ij} w_{ki} w_{kj} \quad , \tag{3}$$

where* $\gamma_{ij} = \alpha\delta_{ij}(\lambda_j - \lambda_i)$ and w_{ki} are the components of the k-th proper vector of S.

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*We assume $\lambda_i \neq \lambda_j$, $i \neq j$. Subsequently we are easily freed of this condition.

It is obvious that, without altering the problem, in (1) U, for example, can be considered to be the unit matrix. E can then be regarded as a bounded and continuous image of the group of orthogonal matrices over which V runs. On the strength of the compactness of the latter, E contains all its boundary points. We note that if V is an orthogonal matrix that is close, element-for-element, to the orthogonal matrix V_o, we have

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Let us for convenience consider that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$ and $\mu_1 + \mu_2 + \cdots + \mu_n = 0$. This is always possible to contrive by an addition of scalar matrices, which have no influence on the essence of the problem. In such a case for all V we shall have:

$$\sigma_1 + \sigma_2 + \cdots + \sigma_n = \text{trace } S = \text{trace } A + \text{trace } B = 0 \quad .$$

Accordingly E is situated in an $(n - 1)$-dimensional hyperplane of $R^{(n)}$: $x_1 + x_2 + \cdots + x_n = 0$. We shall seek the boundary points of the set E. Let us agree to call points of E not containing equal coordinates (in their neighborhood the mapping is differentiable) non-multiple, and in the contrary case, multiple. Obviously a non-multiple point can be a boundary point only if the rank of the corresponding matrix of the transformation (3) is strictly less than $(n - 1)$. It can be proved that if r is the rank of the matrix of the transformation (3), and $r < n - 1$, the corresponding matrix S is resolvable into the direct sum of matrices of orders r and $(n - r)$. On the strength of the induction hypothesis the corresponding point lies in L.

We shall show that L generally contains all non-multiple points of E. For this we note that the following easily proved fact obtains: if the point $\bar{o}(0, 0, \cdots, 0)$ (the origin of coordinates of $R^{(n)}$), corresponding to $S = 0$, is contained in E, then it belongs to L. We note further that, in view of the convexity of L, every straight line of the plane $x_1 + x_2 + \cdots + x_n = 0$ either does not intersect L or has a segment $[\bar{a}, \bar{b}]$ in common with L. Let there now be found a non-multiple point $\bar{\sigma}$ such that $\bar{\sigma} \in E$ and $\bar{\sigma} \bar{\in} L$. Let us draw through it a straight line from the origin of coordinates. Since $\bar{\sigma}$ lies in E, together with

a whole neighborhood, on the line an interval can be found belonging to E and containing $\bar{\sigma}$. It is easily seen that, whatever may be the arrangement of the points $\bar{o}$, $\bar{a}$, $\bar{b}$ on the line, the interval has neither a right nor a left boundary point, which contradicts the closedness of E. Since in any neighborhood of a multiple point there is always located a non-multiple point, and since L is closed, $E \subset L$.

Below is adduced one important particular case, when E = L. We formulate one proposition necessary for what is to follow, which is also of independent value.

THEOREM 2. In order that S have no multiple proper values for any U and V, the fulfillment of one of the following inequalities is sufficient: either

(4')
$$|\mu_k - \mu_l| < |\lambda_i - \lambda_j| \quad , \quad i \neq j \quad ,$$
or
(4")
$$|\lambda_i - \lambda_j| < |\mu_k - \mu_l| \quad , \quad k \neq l \quad .$$

Proof. The sufficiency of condition (4) is a corollary to Theorem 1. Indeed, in this case, for example, given (4'), K_a does not even intersect one of the hyperplanes of $R^{(n)}$, $x_i = x_j$, $i \neq j$. We note that condition (4) is in a certain sense necessary for Theorem 2. Indeed, if multiple proper values do not occur for any U and V, the order of quantitative succession of the proper values of the matrix S is preserved, namely, a large proper value continuously passes into a larger. On the assumption that neither of conditions (4) is satisfied, it is not difficult to choose V_1 and V_2 such that on passing from V_1 to V_2 the order of succession of the proper values is violated.

THEOREM 3. *If one of conditions (4) is satisfied, then*

$$E = L \quad . \tag{5}$$

The proof is carried through by induction. For $n = 2$ formula (5) is verified directly. We note that on the strength of Theorem 2 the mapping is everywhere differentiable. For definiteness we shall consider (4') satisfied; then it is easily seen that $L = K_a$. On the strength of the remark made earlier, only those boundary points of E can serve which correspond to cellularized S's. Using the induction hypothesis, it is easy to satisfy oneself that the points to which cellularized S's correspond are disposed on $(n - 2)$ dimensional facets and diagonal hyperplanes of K_a; the latter on the strength of this same hypothesis consist wholly of the points of E.

For the proof of the theorem it turns out to be sufficient to vary the trace of one of the cells of S, keeping its value constant at all points of the corresponding facet or diagonal hyperplane of K_a. The first differential of the trace of a cell, as follows from (3), is equal to 0. The computation of the second differential of the trace of a cell leads to the following quadratic form with respect to the local coordinates introduced by us:

$$d^2\left(\sum_{p=1}^{r} \sigma_p\right) = \tag{6}$$

$$= \left(\sum_{p=1}^{r} \sum_{k=r+1}^{n} \frac{\gamma_{pk}^2}{\lambda_k - \lambda_p} + \sum_{\chi, s=r+1}^{n} \sum_{i,j=1}^{r} \gamma_{i\chi}\gamma_{js} \sum_{p=1}^{r} \sum_{k=r+1}^{n} \frac{w_{pi} w_{pj} w_{k\chi} w_{ks}}{\sigma_p - \sigma_k}\right) ,$$

where we have put $\gamma_{ij} = \alpha\delta_{ij}(\lambda_j - \lambda_i)$.

An investigation of the $r(n - r)$-th order matrix of this quadratic form establishes that the form (6) is sign-determinate only at the facets of K_a. And this proves the theorem.

3. On the assumption that A and B are positive definite matrices, the question concerning the proper values of $AB = P$ is solved analogously. The computations lead to the following expressions for the differentials of the logarithms of the proper values of P:

$$d \ln \pi_k = \sum_{i>j} \alpha\delta_{ij} \left(\frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i}\right) w_{ki} w_{kj} \quad ,$$

in complete accord with (3). In this case the validity of Theorems 1, 2 and 3 can be proved; the point with coordinates $(\ln \pi_1, \ln \pi_2, \cdots, \ln \pi_n)$ need only be put in correspondence with the proper values of the matrix P. The corresponding changes should also be made in the construction of K_a and K_b, stretching the convex hulls upon the points

$$\bar{a}_j(\ln \lambda_1 + \ln \mu_1', \ln \lambda_2 + \ln \mu_2', \cdots, \ln \lambda_n + \ln \mu_n') \quad ,$$

$$\bar{b}_i(\ln \mu_1 + \ln \lambda_1', \ln \mu_2 + \ln \lambda_2', \cdots, \ln \mu_n + \ln \lambda_n') \quad .$$

We also observe that all the results expounded are transferable to the case of Hermitian matrices.

In conclusion I express my gratitude to Professor I. M. Gel'fand for assistance and direction in the work.

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1. I. M. Gel'fand and M. A. Naĭmark. ["The relation between the unitary representations of the complex unimodular group and its unitary subgroup" (Russian)], Izvestiia Akad. Nauk SSSR, ser. matem., 14, (1950), pp. 239 [-260].

[February 5, 1953]

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

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