NATIONAL BUREAU OF STANDARDS REPORT

2246

LOWER BOUNDS FOR THE RANK AND LOCATION OF THE EIGENVALUES OF A MATRIX

by

Ky Fan and A. J. Hoffman

NBS

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.

Electricity. Resistance Measurements. Inductance and Capacitance. Electrical Instruments. Magnetic Measurements. Applied Electricity. Electrochemistry.

Optics and Metrology. Photometry and Colorimetry. Optical Instruments. Photographic Technology. Length. Gage.

Heat and Power. Temperature Measurements. Thermodynamics. Cryogenics. Engines and Lubrication. Engine Fuels. Cryogenic Engineering.

Atomic and Radiation Physics. Spectroscopy. Radiometry. Mass Spectrometry. Solid State Physics. Electron Physics. Atomic Physics. Neutron Measurements. Infrared Spectroscopy. Nuclear Physics. Radioactivity. X-Rays. Betatron. Nucleonic Instrumentation. Radiological Equipment. Atomic Energy Commission Instruments Branch.

Chemistry. Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Gas Chemistry. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

Mechanics. Sound. Mechanical Instruments. Aerodynamics. Engineering Mechanics. Hydraulics. Mass. Capacity, Density, and Fluid Meters.

Organic and Fibrous Materials. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Organic Plastics. Dental Research.

Metallurgy. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion.

Mineral Products. Porcelain and Pottery. Glass. Refractories. Enameled Metals. Concreting Materials. Constitution and Microstructure. Chemistry of Mineral Products.

Building Technology. Structural Engineering. Fire Protection. Heating and Air Conditioning. Floor, Roof, and Wall Coverings. Codes and Specifications.

Applied Mathematics. Numerical Analysis. Computation. Statistical Engineering. Machine Development.

Electronics. Engineering Electronics. Electron Tubes. Electronic Computers. Electronic Instrumentation.

Radio Propagation. Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Frequency Utilization Research. Tropospheric Propagation Research. High Frequency Standards. Microwave Standards.

Ordnance Development. Electromechanical Ordnance. Ordnance Electronics. These three divisions are engaged in a broad program of research and development in advanced ordnance. Activities include basic and applied research, engineering, pilot production, field testing, and evaluation of a wide variety of ordnance matériel. Special skills and facilities of other NBS divisions also contribute to this program. The activity is sponsored by the Department of Defense.

Missile Development. Missile research and development: engineering, dynamics, intelligence, instrumentation, evaluation. Combustion in jet engines. These activities are sponsored by the Department of Defense.

● Office of Basic Instrumentation ● Office of Weights and Measures.

LOWER BOUNDS FOR THE RANK AND LOCATION OF THE EIGENVALUES OF A MATRIX

by

Ky Fan and A. J. Hoffman

University of Notre Dame
American University
National Bureau of Standards

<NBS>

LOWER BOUNDS FOR THE RANK AND LOCATION OF THE EIGENVALUES OF A MATRIX*

by

Ky Fan and A. J. Hoffman

This paper contains several remarks devoted to the problems mentioned in the title. The results of $\S 1$ apply to arbitrary n x n matrices with complex coefficients, those of $\S 2$ apply to normal matrices only.

1

In this section we consider the following problems concerning an arbitrary n x n matrix $A = (a_{ij})$ with complex coefficients:

> Problem 1: <u>Find lower bounds for the rank of A that can be calculated in a simple manner from the coefficients a_{ij}.</u>

> Problem 2: <u>Find n non-negative numbers $\rho_1, \ldots, \rho_n$ such that every eigenvalue of A lies in one or more of the n circular disks</u>
> $$|\lambda - a_{ii}| \leq \rho_i \quad (i = 1, \ldots, n).$$

Most of the results in the literature relevant to Problem 1 are sufficient conditions for a matrix to be non-singular,

such as Hadamard's theorem that $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$ $(1 \leq i \leq n)$ implies that A is non-singular. As mentioned in [8], the application of this theorem to $A - \lambda I$ implies the theorem of S. Geršgorin [3] that setting $\rho_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$ $(1 \leq i \leq n)$ is a solution to Problem 2 (See [8] and [10] for an extensive bibliography on solutions to Problem 2). A generalization by A. Ostrowski [5] of the theorem of Hadamard-Geršgorin is the following: If p and q are positive, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha_1, \ldots, \alpha_n$ are n positive numbers such that

$$\sum_{i=1}^{n} \frac{1}{1 + \alpha_i} \leq 1, \tag{1.1}$$

then $|a_{ii}| > \alpha_i^{1/q} \left(\sum_{j=1, j\neq i}^{n} |a_{ij}|^p \right)^{1/p} (1 \leq i \leq n)$ implies that A is non-singular; or, equivalently, setting

$\rho_i = \alpha_i^{1/q} \left(\sum_{j=1, j\neq i}^{n} |a_{ij}|^p \right)^{1/p} (1 \leq i \leq n)$ is a solution to Problem 2. (Hadamard-Geršgorin's theorem is the limiting case p = 1 of this result). An improvement of Hadamard-Gersgorin's theorem is a theorem of Taussky [9] and P. Stein [6] which asserts that if λ is an eigenvalue with s linearly independent eigenvectors corresponding to it, then λ lies in at least s of the Hadamard-Geršgorin disks. Theorem D of [6] points out that consideration of $\lambda = 0$ implies a theorem about the rank. The following theorem, formulated in terms of Problem 1, is a slight extension of this result:

THEOREM 1.1. Let $A = (a_{ij})$ be an n x n matrix. For any two integers i,m $(1 \leq i \leq n, 2 \leq m \leq n)$, let $b_i(m)$ be the

maximum sum of the moduli of m-1 distinct off-diagonal elements of the i-th row of A, i.e.

$$b_i(m) = \max_{j_1,\ldots,j_{m-1} \neq i} (|a_{ij_1}| + \ldots + |a_{ij_{m-1}}|). \quad (1.2)$$

If, for some m,

$$|a_{ii}| > b_i(m) \quad (1.3)$$

holds for at least m distinct indices i, then the rank r of A is at least m.

Proof: Consider the m x m submatrix B obtained from A by deleting the rows with indices for which (1.3) is not satisfied, and the corresponding columns. By (1.2) and (1.3), Hadamard's theorem applies to B, hence B is non-singular, which implies $r \geq m$.

If λ is an eigenvalue of A with s linearly independent eigenvectors corresponding to it, then the rank of A- λ I is n-s, and the theorem of Taussky and Stein follows at once from Theorem 1.1. It is also easy to see how Theorem 1.1 and the theorem of Taussky and Stein can be generalized by using Ostrowski's theorem quoted above. They may also be extended by using another theorem of Ostrowski [4] which asserts that if ρ_i and σ_i are the radii of the Hadamard-Geršgorin disks for A and A* respectively, and $0 \leq \alpha \leq 1$, then $\tau_i = \rho_i^\alpha \sigma_i^{1-\alpha} (1 \leq i \leq n)$ also is a solution of Problem 2.

THEOREM 1.2. For any n x n matrix $A = (a_{ij})$ of rank r,

we have

$$\sum_{i=1}^{n} \frac{|a_{ii}|^2}{\sum_{j=1}^{n} |a_{ij}|^2} \leqq r. \tag{1.4}$$

(Whenever $\frac{0}{0}$ occurs on the left-hand side, we agree to put $\frac{0}{0} = 0$.)

Proof: Let a_i denote the i-th row vector of A and e_i the i-th unit vector. Both sides of (1.4) remain unchanged, if we multiply any row of A by a number $\neq 0$. Hence we may assume that for each i, $\|a_i\|^2 = \sum_{j=1}^{n} |a_{ij}|^2 = 1$ or 0. We have to prove, under this assumption, that $\sum_{i=1}^{n} |(a_i, e_i)|^2 \leqq r$.

As A is of rank r, we can find n orthonormal vectors x_1, $x_2, \ldots, x_n$ such that

$$(a_i, x_j) = 0 \quad \text{for} \quad \begin{cases} 1 \leqq i \leqq n \\ r + 1 \leqq j \leqq n. \end{cases}$$

For each i, we have

$$(a_i, e_i) = \sum_{j=1}^{n} (a_i, x_j)(\overline{e_i, x_j}) = \sum_{j=1}^{r} (a_i, x_j)(\overline{e_i, x_j}),$$

and therefore

$$|(a_i, e_i)|^2 \leqq \left(\sum_{j=1}^{r} |(a_i, x_j)|^2\right)\left(\sum_{j=1}^{r} |(e_i, x_j)|^2\right).$$

Since $\sum_{j=1}^{r} |(a_i, x_j)|^2 = \|a_i\|^2 = 1$ or 0, we have

$$|(a_i, e_i)|^2 \leqq \sum_{j=1}^{r} |(e_i, x_j)|^2. \quad (1 \leqq i \leqq n)$$

Consequently

$$\sum_{i=1}^{n} |(a_i, e_i)|^2 \leqq \sum_{j=1}^{r} \sum_{i=1}^{n} |(e_i, x_j)|^2 = \sum_{j=1}^{r} \|x_j\|^2 = r.$$

THEOREM 1.3. <u>For any</u> n x n <u>matrix</u> $A = (a_{ij})$ <u>of rank</u> r, <u>we have</u>

$$\sum_{i=1}^{n} \frac{|a_{ii}|}{\sum_{j=1}^{n} |a_{ij}|} \leq r. \qquad (1.5)$$

(We keep our agreement $\frac{0}{0} = 0$).

Proof: For the reason explained at the beginning of the proof of Theorem 1.2, we may assume

$$a_{ii} \geq 0 \quad (1 \leq i \leq n) \qquad (1.6)$$

and

$$\sum_{j=1}^{n} |a_{ij}| = 1 \text{ or } 0. \quad (1 \leq i \leq n) \qquad (1.7)$$

It suffices to prove that $\sum_{i=1}^{n} a_{ii} \leq r$ holds for any matrix $A = (a_{ij})$ with rank r and satisfying (1.6), (1.7).

By Hadamard–Gerśgorin's theorem mentioned above, (1.7) implies that all eigenvalues of A have modulus ≤ 1. Therefore $\sum_{i=1}^{n} a_{ii}$ is not larger than the number of non-zero eigenvalues of A. But the number of non-zero eigenvalues of A is $\leq r$[(*)], hence $\sum_{i=1}^{n} a_{ii} \leq r$.

(*) Let T be a non-singular matrix such that $T^{-1}AT$ is triangular. Then the rank of A is same as the rank of $T^{-1}AT$ and the latter is obviously $\geq$ the number of non-zero eigenvalues of A.

Theorem 1.3 may be generalized as follows:

THEOREM 1.4. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. __Let__ $\alpha_1, \alpha_2, \ldots, \alpha_n$ __be__ n __positive numbers satisfying__ (1.1). __Then for any__ n x n __matrix__ $A = (a_{ij})$ __of rank__ r, __we have__

$$\sum_{i=1}^{n} \frac{|a_{ii}|}{|a_{ii}| + \alpha_i^{1/q} \left(\sum_{\substack{j=1 \\ j \neq 1}}^{n} |a_{ij}|^p \right)^{1/p}} < r. \tag{1.8}$$

(We keep our agreement $\frac{0}{0} = 0$.)

__Proof:__ The proof is similar to that of Theorem 1.3. Here it suffices to show that, if a matrix $A = (a_{ij})$ satisfies (1.6) and

$$|a_{ii}| + \alpha_i^{1/q} \left(\sum_{\substack{j=1 \\ j \neq 1}}^{n} |a_{ij}|^p \right)^{1/p} = 1 \text{ or } 0, \quad (1 \leq i \leq n) \tag{1.9}$$

then $\sum_{i=1}^{n} a_{ii} \leq r$.

According to Ostrowski's theorem [5] mentioned above, (1.9) implies that all eigenvalues of A have modulus ≤ 1. This fact together with (1.6) implies $\sum_{i=1}^{n} a_{ii} \leq r$.

Theorem 1.3 may be regarded as the limiting case $p \to 1$ of Theorem 1.4. Because of its relative simplicity, we state explicitly the other limiting case $p \to \infty$ of Theorem 1.4:

COROLLARY 1.1. __If__ n __positive numbers__ $\alpha_1, \alpha_2, \ldots, \alpha_n$ __satisfy inequality__ (1.1), __then for any__ n x n __matrix__ $A = (a_{ij})$ __of rank__ r, __we have__

$$\sum_{i=1}^{n} \frac{|a_{ii}|}{|a_{ii}| + \alpha_i \max_{\substack{1 \leq j \leq n \\ j \neq i}} |a_{ij}|} \leq r. \tag{1.10}$$

We turn now to results concerning Problem 2.

THEOREM 1.5.   <u>Let</u> $A = (a_{ij})$ <u>be an</u> $n \times n$ <u>matrix</u>.   <u>Let</u> $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.   <u>If</u> $\alpha > 0$ <u>satisfies</u>

$$\sum_{i=1}^{n} \frac{(\sum_{j \neq i} |a_{ij}|)^q}{(\sum_{j \neq i} |a_{ij}|^p)^{q/p}} \leq \alpha^q (1 + \alpha^q) \tag{1.11}$$

(where we keep our agreement $\frac{0}{0} = 0$), <u>then every eigenvalue</u> λ <u>of</u> A <u>lies in at least one of the</u> n <u>circular disks</u>:

$$|\lambda - a_{ii}| \leq \alpha (\sum_{j \neq i} |a_{ij}|^p)^{\frac{1}{p}} . \quad (1 \leq i \leq n) \tag{1.12}$$

<u>Proof</u>: Let λ be an eigenvalue of A and let $\{x_1, x_2, \ldots, x_n\}$ be a corresponding eigenvector:

$$\lambda x_i = \sum_{j=1}^{n} a_{ij} x_j. \quad (1 \leq i \leq n) \tag{1.13}$$

We may assume that

$$\sum_{i=1}^{n} |x_i|^q = 1. \tag{1.14}$$

Suppose, if possible that

$$|\lambda - a_{ii}| > \alpha (\sum_{j \neq i} |a_{ij}|^p)^{1/p} . \quad (1 \leq i \leq n) \tag{1.15}$$

Then for any index i such that $x_i \neq 0$, we have by (1.15), (1.13), (1.14):

$$\alpha^q (\sum_{j \neq i} |a_{ij}|^p)^{q/p} |x_i|^q < |\lambda - a_{ii}|^q |x_i|^q =$$

$$|\sum_{j \neq i} a_{ij} x_j|^q \leq (1 - |x_i|^q)(\sum_{j \neq i} |a_{ij}|^p)^{q/p}$$

and therefore

$$|x_i|^q < \frac{1}{1 + \alpha^q} . \tag{1.16}$$

This inequality is of course also true when $x_i = 0$. It follows that

$$\left| \sum_{j \neq i} a_{ij} x_j \right| \leq \left(\frac{1}{1 + \alpha^q} \right)^{1/q} \sum_{j \neq i} |a_{ij}|$$

or

$$|\lambda - a_{ii}|^q |x_i|^q \leq \frac{1}{1 + \alpha^q} \left(\sum_{j \neq i} |a_{ij}| \right)^q. \qquad (1 \leq i \leq n)$$

Combining (1.15) with the last inequality, we infer that

$$\alpha^q \left(\sum_{j \neq i} |a_{ij}|^p \right)^{q/p} |x_i|^q < \frac{1}{1 + \alpha^q} \left(\sum_{j \neq i} |a_{ij}| \right)^q$$

holds for every index i such that $x_i \neq 0$. Hence

$$|x_i|^q < \frac{1}{\alpha^q (1 + \alpha^q)} \cdot \frac{\left(\sum_{j \neq i} |a_{ij}| \right)^q}{\left(\sum_{j \neq i} |a_{ij}|^p \right)^{q/p}}$$

holds whenever $x_i \neq 0$. The weaker inequality

$$|x_i|^q \leq \frac{1}{\alpha^q (1 + \alpha^q)} \cdot \frac{\left(\sum_{j \neq i} |a_{ij}| \right)^q}{\left(\sum_{j \neq i} |a_{ij}|^p \right)^{q/p}}$$

is of course satisfied for all i. Thus, summing this inequality over i and using (1.14), we get an inequality contradicting (1.11).

As limiting case $p \to \infty$ of Theorem 1.5, we have

COROLLARY 1.2. <u>Let</u> $A = (a_{ij})$ <u>be an</u> n x n <u>matrix.</u> <u>If</u> $\alpha > 0$ is such that

$$\sum_{i=1}^{n} \frac{\sum_{j \neq i} |a_{ij}|}{\underset{j \neq i}{\mathrm{Max}} \, |a_{ij}|} \leq \alpha (1 + \alpha) \qquad (1.17)$$

(where we keep our agreement $\frac{0}{0} = 0$), <u>then every eigenvalue</u> <u>of</u> A <u>lies in at least one of the</u> n <u>circular disks</u>

$$|\lambda - a_{ii}| \leq \alpha \underset{j \neq i}{\mathrm{Max}} \, |a_{ij}|. \qquad (1 \leq i \leq n) \qquad (1.18)$$

In the same way that Theorems 1.3, 1.4 are derived from Hadamard-Geršgorin's theorem and Ostrowski's theorem respectively, other lower bounds for the rank can be derived from our Theorem 1.5. In particular, as a consequence of Corollary 1.2, we have

COROLLARY 1.3. Let $A = (a_{ij})$ be an n x n matrix of rank r. If α is a positive number satisfying (1.17), then

$$\sum_{i=1}^{n} \frac{|a_{ii}|}{|a_{ii}| + \alpha \underset{j \neq i}{\text{Max}} |a_{ij}|} \leq r. \tag{1.19}$$

Using theorem 1.2, we may improve the case p = 2 of Ostrowski's theorem [5].

THEOREM 1.6. Let λ be an eigenvalue of an n x n matrix $A = (a_{ij})$ and let s be the number of linearly independent eigenvectors corresponding to λ. If n positive numbers $\beta_1, \beta_2, \ldots, \beta_n$ are such that

$$\sum_{i=1}^{n} \frac{1}{1 + \beta_i} \leq s, \tag{1.20}$$

then λ lies in at least one of the n disks:

$$|\lambda - a_{ii}| \leq \beta_i^{\frac{1}{2}} (\sum_{j \neq i} |a_{ij}|^2)^{\frac{1}{2}}. \qquad (1 \leq i \leq n) \tag{1.21}$$

Proof: Since s is the number of linearly independent eigenvectors of A corresponding to the eigenvalue λ, the rank of the matrix $A - \lambda I$ is n−s. Applying Theorem 1.2 to $A - \lambda I$, we get

$$\sum_{i=1}^{n} \frac{|a_{ii} - \lambda|^2}{|a_{ii} - \lambda|^2 + \sum_{j \neq i} |a_{ij}|^2} \leq n-s.$$

As (1.20) may be written

$$n - s \leq \sum_{i=1}^{n} \frac{\beta_i}{1 + \beta_i} ,$$

we have

$$\sum_{i=1}^{n} \frac{|a_{ii} - \lambda|^2}{|a_{ii} - \lambda|^2 + \sum_{j \neq i} |a_{ij}|^2} \leq \sum_{i=1}^{n} \frac{\beta_i}{1 + \beta_i} .$$

Therefore at least one index i satisfies

$$\frac{|a_{ii} - \lambda|^2}{|a_{ii} - \lambda|^2 + \sum_{j \neq i} |a_{ij}|^2} \leq \frac{\beta_i}{1 + \beta_i} ,$$

which is precisely (1.21).

Similarly, from Theorem 1.3, we can derive the following theorem:

THEOREM 1.7. _Let_ λ _be an eigenvalue of an_ n x n _matrix_ $A = (a_{ij})$ _and let_ s _be the number of linearly independent eigenvectors corresponding to_ λ. _If_ n _positive numbers_ $\beta_1, \beta_2, \ldots, \beta_n$ _satisfy_ (1.20), _then_ λ _lies in at least one of the_ n _disks:_

$$|\lambda - a_{ii}| \leq \beta_i \sum_{j \neq i} |a_{ij}|. \quad (1 \leq i \leq n) \qquad (1.22)$$

It is interesting to compare this result with the theorem of Taussky and Stein discussed in connection with Theorem 1.1 above.

§2

Several inclusion theorems for the eigenvalues of a normal matrix are already known. Among them, recent results are due to L. Collatz [1], H. Wielandt [12], A. G. Walker and J. D. Weston [11]. All these results are concerned with a single eigenvalue. In this § we shall give inclusion theorems concerning k eigenvalues of an n x n normal matrix, k being any positive integer $\leq$ n. Our results are based on the following extremal property of the eigenvalues of normal matrices: Let the eigenvalues $\lambda_i (1 \leq i \leq n)$ of a normal matrix N be so arranged that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. Then for any positive integer k $\leq$ n, we have

$$\sum_{i=1}^{k} |\lambda_i|^2 = \text{Max} \sum_{j=1}^{k} \| N x_j \|^2, \qquad (2.1)$$

$$\sum_{i=1}^{k} |\lambda_{n-i+1}|^2 = \text{Min} \sum_{j=1}^{k} \| N x_j \|^2, \qquad (2.2)$$

where, for both maximum and minimum, $x_1, x_2, \ldots, x_k$ runs over all sets of k orthonormal vectors in the unitary n-space. As N*N is a Hermitian matrix with eigenvalues $|\lambda_i|^2$ $(1 \leq i \leq n)$, this follows from a similar property for the eigenvalues of Hermitian matrices ([2], Theorem 1).

THEOREM 2.1. Let N be an n x n normal matrix with eigenvalues λ_i $(1 \leq i \leq n)$. Let $x_1, x_2, \ldots, x_k$ be k $(\leq n)$ orthonormal vectors and γ be a complex number, δ a non-negative real number such that

$$\sum_{j=1}^{k} \| (N - \gamma I)x_j \|^2 \leqq \delta .$$ (2.3)

Then there exist k <u>distinct indices</u> $\nu_1, \nu_2, \ldots, \nu_k$ <u>among</u> 1,2,...,n <u>such that</u>

$$\sum_{i=1}^{k} |\lambda_{\nu_i} - \gamma|^2 \leqq \delta .$$ (2.4)

<u>The theorem remains true, when both signs "$\leqq$" in</u> (2.3), (2.4) <u>are reversed.</u>

Proof: Consider the normal matrix $N - \gamma I$, whose eigenvalues are $\lambda_i - \gamma$ ($1 \leqq i \leqq n$). If we rearrange the λ_i's into $\lambda_{\nu_1}, \lambda_{\nu_2}, \ldots, \lambda_{\nu_n}$ such that

$$|\lambda_{\nu_1} - \gamma| \leqq |\lambda_{\nu_2} - \gamma| \leqq \cdots \leqq |\lambda_{\nu_n} - \gamma|,$$

then applying the minimum property (2.2) to $N - \gamma I$, we obtain

$$\sum_{i=1}^{k} |\lambda_{\nu_i} - \gamma|^2 \leqq \sum_{j=1}^{k} \| (N - \gamma I)x_j \|^2 .$$ (2.5)

In view of (2.3), (2.5) implies (2.4).

Similarly one proves the second part of the theorem by applying the maximum property (2.1) to $N - \gamma I$.

Remark: Let α and β be any two complex numbers. Put

$$\gamma = \frac{\alpha + \beta}{2} , \qquad \delta = k \left| \frac{\alpha - \beta}{2} \right|^2$$

Then in the theorem just proved, condition (2.3) becomes

$$\text{Re} \sum_{j=1}^{k} ((N - \alpha I)x_j, (N - \beta I)x_j) \leqq 0.$$ (2.6)

Using this form (2.6) of condition (2.3), we see that the case k = 1 of Theorem 2.1 implies the following result:

COROLLARY 2.1. <u>Let</u> N <u>be an</u> n x n <u>normal matrix</u>. <u>Let</u> x = $\{x_1, x_2, \ldots, x_n\}$ <u>be a vector different from the zero-vector</u>. <u>Let</u> Nx = $\{y_1, y_2, \ldots, y_n\}$. <u>If</u> α, β <u>are two complex numbers such that</u>

$$\text{Re} \sum_{i=1}^{n} (y_i - \alpha x_i)(\overline{y_i - \beta x_i}) \leqq 0, \qquad (2.7)$$

<u>then</u> N <u>has at least one eigenvalue in the disk</u>

$$\left| \lambda - \frac{\alpha + \beta}{2} \right| \leqq \left| \frac{\alpha - \beta}{2} \right|$$

<u>The corollary remains true, when both signs "$\leqq$" are reversed.</u>

This result improves slightly an inclusion theorem given independently by H. Wielandt [12] and Walker-Weston [11]. These authors assume

$$\text{Re} \left[(y_i - \alpha x_i)(\overline{y_i - \beta x_i}) \right] \leqq 0, \quad (1 \leqq i \leqq n)$$

which is stronger than (2.7). On the other hand, the inclusion theorem of Wielandt-Walker-Weston generalizes an inclusion theorem of L. Collatz [1] concerning real symmetric matrices.

THEOREM 2.2. <u>Let</u> N <u>be an</u> n x n <u>normal matrix with eigenvalues</u> $\lambda_i (1 \leqq i \leqq n)$. <u>For any</u> k($\leqq$n) <u>orthonormal vectors</u> $x_1, x_2, \ldots, x_k$ <u>in the unitary</u> n-<u>space, there exist</u> k <u>distinct indices</u> $\nu_1, \nu_2, \ldots, \nu_k$ <u>among</u> 1, 2, \ldots, n <u>such that</u>

$$\sum_{i=1}^{k} \left| \lambda_{\nu_i} - \frac{1}{k} \sum_{j=1}^{k} (Nx_j, x_j) \right|^2 \leq$$

$$\sum_{j=1}^{k} \| Nx_j \|^2 - \frac{1}{k} \left| \sum_{j=1}^{k} (Nx_j, x_j) \right|^2. \tag{2.8}$$

There also exist k distinct indices ν_1, ν_2,..., ν_k satisfying the reversed inequality of (2.8).

Proof: Let

$$\check{\gamma} = \frac{1}{k} \sum_{j=1}^{k} (Nx_j, x_j),$$

$$\delta = \sum_{j=1}^{k} \| Nx_j \|^2 - \frac{1}{k} \left| \sum_{j=1}^{k} (Nx_j, x_j) \right|^2.$$

One sees that

$$\sum_{j=1}^{k} \| (N - \check{\gamma} I)x_j \|^2 = \delta,$$

which incidentally implies that $\delta \geq 0$.

Hence the first part of Theorem 2.1 implies the existence of ν_1,..., ν_k satisfying (2.8), and the second part of Theorem 2.1 implies the existence of ν_1,..., ν_k satisfying the reverse of inequality (2.8).

The case $k = 1$ of Theorem 2.2 is a known result obtained independently by Wielandt [12] and Walker-Weston [11].

For its simplicity, we state explicitly the following special case of the case $k = 1$ of Theorem 2.2. Let $N = (a_{ij})$ be an n x n normal matrix. Let D_i denote the disk $|z - a_{ii}| \leq (\sum_{j \neq i} |a_{ij}|^2)^{\frac{1}{2}}$. Then for each $i = 1,2,...,n$, D_i contains at least one eigenvalue of N and the interior of D_i never contains all n eigenvalues of N.

References

[1] L. Collatz, "Einschliessungssatz für die charakteristischen Zahlen von Matrizen," _Math. Zeitschr._, 48, (1942), 221-226.

[2] K. Fan, "On a theorem of Weyl concerning Eigenvalues of linear transformations, I," _Proc. Nat. Acad. Sci. USA_, 35(1949) 652-655.

[3] S. Geršgorin, "Über die Abgrenzung der Eigenwerte einer Matrix," _Izv. Akad. Nauk S.S.S.R._, 7(1931), 749-754.

[4] A. Ostrowski, "Über das Nichtverschwinden einer Klasse von Determinanten und die Lokalisierung der charakteristischen Wurzeln von Matrizen," _Compositio Math._ 9(1951), 209-226.

[5] A. Ostrowski, "Sur les conditions générales pour la régularité des matrices," _Rend. di Matem.e delle sue appl._, 10(1951), 156-168.

[6] P. Stein, "A Note on Bounds of Multiple Characteristic Roots of a Matrix," _Journal of Research of the National Bureau of Standards_, 48(1952), 59-60.

[7] O. Taussky-Todd, "Bounds for characteristic roots of matrices," _Duke. Math. J._, 15(1948), 1043-1044.

[8] O. Taussky-Todd, "A recurring theorem on determinants," _Amer. Math. Monthly_, 56(1949), 672-676.

[9] O. Taussky-Todd, "Bounds for characteristic roots of matrices II," _Journal of Research of the National Bureau of Standards_, 46(1951), 124-125.

[10] O. Taussky-Todd, "Bibliography on Bounds for Characteristic Roots of Finite Matrices," _National Bureau of Standards Report_ 1162, September 1951.

[11] A. G. Walker and J. D. Weston, "Inclusion theorems for the eigenvalues of a normal matrix," _Jour. London Math. Soc._, 24(1949), 28-31.

[12] H. Wielandt, "Ein Einschliessungssatz für charakteristische Wurzeln normaler Matrizen," _Archiv der Math._, 1(1948), 348-352.

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.