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ON STRONGLY CONTINUOUS STOCHASTIC PROCESSES

by

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U. S. DEPARTMENT OF COMMERCE
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On strongly continuous stochastic processes*

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On strongly continuous stochastic processes*

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1. Introduction

In this paper we study strongly continuous stochastic processes and give first a condition which assures that the increments of a stochastic process $y(t)$ are normally distributed. This condition is of interest in connection with the study of the Wiener process. This process affords a particularly simple model for certain phenomena because it has normally and independently distributed increments. It is shown in the present paper that the normality of the increments follows from a certain continuity property. Later we give a characterization of the Wiener process and discuss also another process with independent increments. We give next the definitions which are necessary for the formulation of our theorems.

The increment of a stochastic process $y(t)$ over the time interval $(t, t + \tau)$ is the random variable $y(t + \tau) - y(t)$.

A stochastic process is said to be a process with independent increments if the increments over non-overlapping time invertals are completely independent of each other.

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A process $y(t)$ is said to be <u>strongly continuous</u> [7] in the closed interval $[a,b]$ if to every $\varepsilon > 0$ and $\eta > 0$ there exists a $\delta = \delta(\varepsilon, \eta)$ such that for every finite set S of points contained in $[a,b]$

$$(1) \qquad P[\mathscr{E}(\delta, \varepsilon, S)] \geq 1 - \eta.$$

Here $\mathscr{E}(\delta,\varepsilon,S)$ in the event that the inequalities $|y(t_i)-y(t_k)| \leq \varepsilon$ are simultaneously satisfied for all pairs (t_i, t_k) with $|t_i - t_k| < \delta$ belonging to a finite set S of points contained in $[a,b]$. The symbol $P[\ldots]$ stands here and in the following for the probability of the event $[\ldots]$ in the brackets.

We shall write occasionally $\mathscr{E}_y(\delta,\varepsilon,S)$ for the event $\mathscr{E}(\delta,\varepsilon,S)$ if it is necessary to point out that the event refers to the process $y(t)$. We are now in a position to formulate the condition mentioned in the introductory section.

<u>Theorem 1</u>: Let $y(t)$ be a stochastic process and assume that

(i) $y(t)$ is a process with independent increments;

(ii) $y(t)$ is strongly continuous in the interval $[a,b]$.

Then $y(b) - y(a)$ is normally distributed.

A theorem of this type is due to P. Lévy [5] (theorem 18, 3). However it might be of interest to give a different proof using an ε,δ condition for the definition of strong continuity. This definition is due to H. B. Mann [7].

2. <u>Khintchine's theorem</u>.

For the proof we need a theorem of A. Khintchine which gives conditions for the convergence to the normal distribution.

This theorem is given in a book by Khintchine [4], published in the Russian language. The following formulation was taken from a paper by B. V. Gnedenko [3] which is available in an English translation.

We consider a sequence of sequences $X_{n1}, X_{n2}, \ldots, X_{nk_n}$ ($n = 1, 2, \ldots$ ad inf) of random variables which are independent within each sequence. The random variables X_{ns} are said to be __infinitesimal__ if for any $\varepsilon > 0$ the relation $\lim_{n \to \infty} P[|X_{ns}| > \varepsilon] = 0$ holds uniformly in s $(1 \leq s \leq k_n)$.

We denote by $F_{ns}(x)$ the distribution function of the random variable X_{ns} and state

__Khintchine's theorem__: If the distributions of the sums

(2) $$T_n = X_{n1} + X_{n2} \ldots + X_{nk_n}$$

of independent (within each sequence) infinitesimal random variables X_{ns} $(1 \leq s \leq k_n)$ converge to a limiting distribution, then the necessary and sufficient condition for the limiting distribution to be normal is that for any $\varepsilon > 0$

(3) $$\lim_{n \to \infty} \sum_{s=1}^{k_n} \int_{|x| \geq \varepsilon} dF_{ns}(x) = 0 .$$

For the proof the reader is referred to Gnedenko's paper [3].

3. __Proof of theorem 1__

We consider a sequence $\{S_n\}$ of subdivisions of the interval $[a,b]$. For the sake of simplicity we let S_n be the sub-

division $S_n = (t_0^n, t_1^n, t_2^n, \ldots, t_n^n)$ of $[a,b]$ into n equal parts, that is we put $t_j^n = a + (b-a)j/n$ $(j=0,1,2,\ldots,n)$. We write $x_{n,j} = y(t_j^n) - y(t_{j-1}^n)$ for $j = 1,2,\ldots,n$ and show first that the random variables $x_{n,j}$ are infinitesimal.

By assumption (ii) the process $y(t)$ is strongly continuous in $[a,b]$. It is therefore possible to determine for every $\varepsilon > 0$ and $\eta > 0$ a $\delta = \delta(\varepsilon,\eta)$ such that for every subdivision S_n

(4) $$P[\mathscr{E}(\delta,\varepsilon,S_n)] \geq 1 - \eta$$

We next choose a number $N=N(\varepsilon,\eta)$ such that $N > \frac{(b-a)}{\delta(\varepsilon,\eta)}$. For any $n \geq N$ the event $\mathscr{E}(\delta,\varepsilon,S_n)$ implies that the n inequalities $|x_{n,j}| \leq \varepsilon$ $(j=1,2,\ldots n)$ are simultaneously satisfied. We conclude then from (4) that

(5) $1 - \eta \leq P[|x_{n,j}| \leq \varepsilon\ ;\ j = 1,2,\ldots n]$ and also

(6) $1 - \eta \leq P[|x_{n,j}| \leq \varepsilon\ ]$ for $j = 1,2,\ldots n$, if only $n \geq N$.

We denote here by $P[|x_{n,j}| \leq \varepsilon\ ;\ j = 1,2,\ldots,n]$ the probability that the n inequalities $|x_{n,j}| \leq \varepsilon$ $(j = 1,2,\ldots,n)$ hold simultaneously (*)

(*) If $R_i (i=1,2,\ldots,n)$ are n events then $P[R_i;\ i=1,2,\ldots,n]$ means the probability that all n events occur simultaneously. Thus $P[R_i;\ i=1,2,\ldots,n] \geq 1-\eta$ means that the probability of the simultaneous occurrence of all n events is at least equal to $1 - \eta$. This should be carefully distinguished from the statement $P[R_i] \geq 1-\eta\ (i=1,2,\ldots,n)$ which means that the probability of the occurrence of each single event is at least equal to $1-\eta$, this statement does not imply anything about the probability of the joint occurrence of the n events.

For every $\varepsilon > 0$ and $\eta > 0$ it is therefore possible to find an $N = N(\varepsilon, \eta)$ such that for $n \geq N$

(7) $$P[|x_{n,j}| > \varepsilon] \leq \eta \text{ for } j = 1,2,\ldots,n.$$

This shows that $P[|x_{n,j}| > \varepsilon]$ converges (uniformly in j) to zero as $n \to \infty$, or in other words the $x_{n,j}$ are infinitesimal random variables. We consider the sequence of random variables

(8) $$T_n = x_{n,1} + x_{n,2} + \ldots + x_{n,n}$$

We have shown that T_n is the sum of infinitesimal random variables which are by assumption independent (within each sequence). Since

$$T_n = \sum_{j=1}^{n} x_{n,j} = \sum_{j=1}^{n} [y(t_j^n) - y(t_{j-1}^n)] = y(t_n^n) - y(t_o^n) = y(b) - y(a)$$

we see that the limiting distribution of the T_n is the distribution of the random variable $y(b) - y(a)$.

We have already shown that it is possible to find for every $\varepsilon > 0$ and $\eta > 0$ a $N = N(\varepsilon, \eta)$ such that

$$P[|x_{n,j}| \leq \varepsilon \ ; \ j = 1,\ldots,n] \geq 1 - \eta$$

for $n \geq N$.

Since the increments $x_{n,j}$ are independently distributed [by assumption (i)] it follows that also

(9) $$P[|x_{n,j}| \leq \varepsilon \ ; \ j = 1,2,\ldots,k] \geq 1 - \eta.$$

where ε and η are arbitrary positive numbers and $n \geq N$ while k is an integer less than n.

For any subdivision S_n of [ab] into n equal parts we introduce the random variable

(10) $M_{abS_n} = \max [\,|x_{n,1}|\,,\ |x_{n,2}|,\ldots,\ |x_{n,n}|\,].$

The statement that the n inequalities $|x_{n,j}| \leq \varepsilon$ for $j = 1,2,\ldots n$ hold simultaneously is equivalent to the statement $M_{abS_n} \leq \varepsilon$. Therefore

$$P[M_{abS_n} \leq \varepsilon\,] = P[\,|x_{n,j}| \leq \varepsilon\,;\ j = 1,2,\ldots,n] \geq 1 - \eta$$

and also

(11) $$P[M_{abS_n} > \varepsilon\,] \leq \eta,$$

for all $\varepsilon > 0$ and $\eta > 0$, provided that $n \geq N$.

We introduce the following n events:

$B_1^{(n)}$ is the event that the inequality $|\,x_{n,1}| > \varepsilon$ holds.

$B_j^{(n)}$ (for $j = 2,3,\ldots,n$) is the event that the j inequalities

$|x_{n,1}| \leq \varepsilon$, $|x_{n,2}| \leq \varepsilon$,... $|x_{n,j-1}| \leq \varepsilon$, $|x_{n,j}| > \varepsilon$ hold simultaneously. The events $B_1^{(n)}$, $B_2^{(n)}$,...,$B_n^{(n)}$ are mutually exclusive and exhaust all the cases for which $M_{abS_n} > \varepsilon$. Therefore we see from (11)

(12) $$\eta \geq P[M_{abS_n} > \varepsilon\,]\, j = \sum_{j=1}^{n} P[B_j^{(n)}]$$

for all $\varepsilon,\eta > 0$ provided that $n \geq N$.

Since the random variables $x_{n,1},\ldots,x_{n,n}$ are completely independent we have for $j = 2,\ldots,n$

$$P[B_j^n] = P[\,|x_{n,k}| \leq \varepsilon\,;\ k=1,2,\ldots(j-1)]\ P[\ x_{n,j}\ > \varepsilon].$$

It follows therefore from (9) that

$$(13) \qquad P[B_j^n] \geq (1-\eta)\ P[\,|x_{n,j}| > \varepsilon] \quad (j=2,\ldots,n)$$

for any $\varepsilon > 0$ and $\eta > 0$ if only $n \geq N$ so that

$$(14) \qquad \sum_{j=1}^{n} P[B_j^n] \geq (1-\eta) \sum_{j=1}^{n} P[\,|x_{n,j}| > \varepsilon\,]$$

for any $\varepsilon > 0$ and $\eta > 0$ provided that $n \geq N$.

From (12) and (14) we see that

$$(15) \qquad \eta\,/(1-\eta) \geq \sum_{j=1}^{n} P[\,|x_{n,j}| > \varepsilon\,]$$

for any $\varepsilon > 0$, $\eta > 0$ if $n \geq N$. Hence we conclude that

$$\lim_{n \to \infty} \sum_{j=1}^{n} P[\,|x_{n,j}| > \varepsilon\,] = 0 \quad \text{or}$$

$$(16) \qquad \lim_{n \to \infty} \sum_{j=1}^{n} \int_{|x|>\varepsilon} dF_{n,j}(x) = 0$$

But (16) is exactly Khinchine's condition (3) so that we have shown that the limiting distribution of the T_n, that is the distribution of $y(b) - y(a)$ is normal.

4. Some properties of strongly continuous processes.

In this section we derive several lemmas.

Lemma 1. If the process $y(t)$ is strongly continuous in the interval $[a,b]$ then it is continuous at every point of the interval, that means $\operatorname{plim}_{\tau \to 0} y(t+\tau) = y(t)$ if $a \leq t \leq b$ and $a < t+\tau < b$.

Proof: We take for the set S the set consisting of the two points t and $t+\tau$ where $t+\tau$ is an interior point of $[a,b]$.

The event $\mathcal{E}(\delta,\varepsilon,S)$ is then the event that $|y(t+\tau)-y(t)| \leq \varepsilon$ for $|\tau| < \delta$. The strong continuity of the y(t) process implies that for every $\varepsilon > 0$, and $\eta > 0$ there exists a $\delta = \delta(\varepsilon,\eta)$ such that $P(|y(t+\tau) - y(t)| \geq \varepsilon) \geq 1-\eta$ for $|\tau| < \delta$. By the definition of convergence in probability this means that

$$\operatorname*{plim}_{\tau\to 0} [y(t+\tau) - y(t)] = 0$$

<u>Lemma 2</u>. Let y(t) be a stochastic process and denote by $m(t) = E\, y(t)$ its mean value function. Assume that

(i) y(t) is a process with independent increments

(ii) y(t) is strongly continuous in the interval [a,b].

Then the mean value function m(t) is continuous in every point of the closed interval [a,b].

Proof: Let t be a point of [a,b] and assume that $t + \tau$ is an interior point of [a,b]. The process y(t) is then strongly continuous in the interval $[t, t+\tau]$ and we see from theorem 1 that $y(t+\tau) - y(t)$ is normally distributed. Therefore the random variable $y(t+\tau) - y(t)$ has only one median which is identical with its mean $m(t+\tau) - m(t)$. According to lemma 1 the random variable $y(t+\tau) - y(t)$ converges in probability to the sure number zero, we can therefore conclude from a result of Slutsky [see for instance Fréchet [2] pp 186–187] that $\lim_{\tau\to 0} [m(t+\tau) - m(t)] = 0$ which proves the lemma.

<u>Lemma 3</u>. Let y(t) be a stochastic process, defined for $t \geq 0$, and denote by $f(t,\tau)$ the variance of the increments $y(t+\tau)-y(t)$.

Assume that

(i) $y(t)$ is a process with independent increments

(ii) $y(t)$ is strongly continuous in every interval $[a,b]$ $(0 \leq a < b)$

then the function $f(t,\tau)$ is continuous in τ.

Proof: We have clearly

$$y(t+\tau+\lambda) - y(t) = [y(t+\tau+\lambda) - y(t+\tau)] + [y(t+\tau)-y(t)].$$

On account of the assumed independence of the increments on the right-hand side of this equation we have

$$f(t,\tau+\lambda) - f(t,\tau) = f(t+\tau,\lambda). \tag{17}$$

We know from theorem 1 that the increment $y(t+\tau+\lambda)-y(t+\tau)$ is normally distributed. Lemma 1 shows that it converges in probability to zero. Its variance $f(t+\tau,\lambda)$ must therefore converge to zero as λ goes to zero. We see then from (17) that $\lim\limits_{\lambda\to 0} [f(t,\tau+\lambda) - f(t,\tau)] = 0$ so that lemma 3 is proven.

5. Characterization of the Wiener process (*)

A process $x(t)$ is said to be a Wiener process if

(i) it is a process with independent increments and initial value $x(0) = 0$.

(ii) the increment $x(t+\tau) - x(t)$ is normally distributed with mean zero and variance $c\tau$, (where $c > 0$).

We are now ready to characterize the Wiener process.

Theorem 2. A stochastic process $y(t)$ is a Wiener process

(*) The Wiener process is also called fundamental random process.

if and only if the following three conditions are satisfied

(a) $y(t)$ is a strongly continuous process with independent increments and initial value $y(0) = 0$.

(b) The mean value function of the process is identically zero.

(c) The variance of the increments $y(t + \tau) - y(t)$ depends only on the length τ of the interval over which the increment is taken but is independent of the location of the interval on the time axis.

Proof: The necessity of these conditions follows from well known properties of the Wiener process (see [6] or [7]) so that we have only to show that the conditions of theorem 2 are sufficient. From (a) it is immediately seen that property (i) of the Wiener process is satisfied. It follows from theorem 1 that the increments $y(t + \tau) - y(t)$ are normally distributed while (b) shows that the mean of the distribution of the increments is zero. We have therefore only to show that the variance of the increment $y(t + \tau) - y(t)$ is $c\tau$. According to condition (c) the function $f(t,\tau)$ depends only on τ so that we may write $f(t,\tau) = f(\tau)$. From (17) we see that $f(\tau)$ satisfies the functional equation

(18) $$f(\tau + h) = f(\tau) + f(h).$$

Lemma 3 states that $f(\tau)$ is a continuous function. It is well known that the only continuous solution of (18) is $f(\tau)=c\tau$. Since $f(\tau)$ is by definition a variance we see finally that $c > 0$.

6. Processes with factorable variance function of the increments.

In this section we give an example which shows that in a strongly continuous process with independent increments the variance $f(t,\tau)$ need not be independent of t.

Theorem 3. Let $y(t)$ be a stochastic process, defined for $t \geq 0$ and denote by $f(t,\tau)$ the variance of the increment $[y(t+\tau) - y(t)]$. Assume that the mean value function of $y(t)$ is identically zero and that the initial value is zero, i.e. $Ey(t) \equiv 0$ and $y(0) = 0$.

The process $y(t)$ is a Gaussian process with mean value function zero and covariance function

(19) $$\sigma_{y(t_1)y(t_2)} = \sigma^2_{y(t)} = c\,\frac{e^{Kt}-1}{e^{K}-1}$$

where $t = \min(t_1,t_2)$ and $c > 0$ if and only if

(i) $y(t)$ is a process with independent increments and is strongly continuous in every finite interval $[a,b]$ where $0 \leq a < b$,

(ii) $f(t,\tau) > 0$ for $t \geq 0$ and $\tau > 0$

(iii) $f(t,\tau) = h(t)\, g(\tau)$

We first prove the sufficiency of the conditions.

By means of theorem 1 we conclude from assumptions (i) and from $y(0) = 0$ that the process is Gaussian, by assumption its mean value function is identically zero. The purpose of assumption (ii) is to exclude some "improper" processes. To see this we assume that there exist values t_0 and $\tau_0 > 0$ such

that $f(t_0, \tau_0) = 0$. From the independence of the increments and the additivity of the variance it follows then that $f(t_0, \tau_1) = 0$ if $\tau_1 \leq \tau_0$ and also $f(t_0 + \tau_1, \tau_2) = 0$ if $\tau_1 + \tau_2 \leq \tau_0$. The increments $y(t_0 + \tau_1 + \tau_2) - y(t_0 + \tau_1)$ have therefore a degenerate distribution if $\tau_1 + \tau_2 \leq \tau_0$. We conclude therefore from $Ey(t) = 0$ that $y(t) = y(t_0)$ if $t_0 \leq t \leq t_0 + \tau_0$. Assumption (ii) excludes therefore processes which do not change over a fixed interval. We proceed now to derive formula (19). From assumption (iii) and from (17) we see that

$$h(t)\,[g(\tau + \lambda) - g(\tau)] = h(t + \tau)\, g(\lambda) \tag{20}$$

Further we conclude from (ii) that $h(t) \neq 0$ for all $t \geq 0$ while $g(\lambda) \neq 0$ for all $\lambda > 0$. We may therefore rewrite (20) as

$$\frac{g(\tau + \lambda) - g(\tau)}{g(\lambda)} = \frac{h(t + \tau)}{h(t)} \tag{21}$$

The left-hand side of this equation is independent of t but depends on τ and λ, while the right-hand side of (21) is a function of τ and t but is independent of λ. This is only possible if the quotient on either side is a function only of τ. We denote this function by $\rho(\tau)$ and obtain

$$h(t + \tau) = \rho(\tau)\, h(t). \tag{22}$$

If we set $t = 0$ we have $h(\tau) = h(0)\, \rho(\tau)$ hence $h(t + \tau) = h(0)\, \rho(t + \tau)$. Substituting these expressions into (22) we obtain

$$h(t+\tau) = h(0)\ \rho(t+\tau) = \rho(\tau)\ h(t) = \rho(\tau)\ h(0)\ \rho(t)$$

or

(23) $$\rho(t+\tau) = \rho(t)\ \rho(\tau)\ .$$

Since by lemma 3 $g(\tau)$ is a continuous function of τ the same is true for $\rho(\tau)$ so that the only solution of (23) is

(24) $$\rho(t) = e^{Kt}$$

and therefore

(25) $$h(t) = h(0)\ e^{Kt}$$

From (21) and (24) we obtain the functional equation for $g(\tau)$

(26) $$g(\tau+\lambda) = g(\tau) + e^{K\tau} g(\lambda)$$

In case $K = 0$, this reduces to

(26a) $$g(\tau+\lambda) = g(\tau) + g(\lambda)$$

and the process $y(t)$ is then a Wiener random process. We may therefore assume in the following that $K \neq 0$. We obtain easily from (26)

$$g(t_1+t_2 \dots +t_n) = g(t_1) + e^{Kt_1}\ g(t_2) + e^{K(t_1+t_2)} g(t_3) + \dots$$
$$+ e^{K(t_1+t_2+\dots+t_{n-1})}\ g(t_n).$$

If in particular $t_1 = t_2 = \dots = t_n = t$ we have

(27) $$g(nt) = g(t) \sum_{s=0}^{n-1} e^{Kst} = g(t)\ \frac{e^{Knt}-1}{e^{Kt}-1}$$

Setting here $t = \frac{1}{n}$ we obtain for integer n

$g(\frac{1}{n}) = g(1)\,\frac{e^{K/n}-1}{e^{K}-1}$. In the same manner we obtain for integers m and n

(28) $$g(\tfrac{m}{n}) = g(1)\,\frac{e^{mK/n}-1}{e^{K}-1}$$

From the continuity of the function $g(\lambda)$ we conclude finally that for any real λ

(29) $$g(\lambda) = g(1)\,\frac{e^{\lambda K}-1}{e^{K}-1} \qquad (\text{if } K \neq 0).$$

The variance $f(t,\tau)$ of the increment $y(t+\tau) - y(t)$ is now immediately obtained from assumption (iii) and from (25) and (29) and is

(30) $$f(t,\tau) = c\ e^{Kt}\,\frac{e^{K\tau}-1}{e^{K}-1}$$

where we wrote for brevity $c = h(0)\ g(1) = f(0,1)$. By assumption $y(0) = 0$ hence $y(t) = y(t) - y(0)$ so that the variance of $y(t)$ is obtained by substituting $t = 0$ and $\tau = t$ into (30) hence $\sigma^2_{y(t)} = c\frac{e^{Kt}-1}{e^{K}-1}$. The formula (19) for the covariance function follows then easily from the independence of the increments.

We proceed to prove that the conditions are necessary. We assume therefore that $y(t)$ is a Gaussian process with mean value function identically zero and initial value zero and variance function given by (19) and show that (i), (ii) and (iii) hold.

By an elementary computation we obtain from (19)
$f(t,\tau) = E[y(t+\tau)-y(t)]^2 = c\ e^{Kt}\,\frac{e^{K\tau}-1}{e^{K}-1}$ so that (ii) and (iii) are satisfied.

Let now t_1, t_2, t_3, t_4 be four real numbers such that $0 \leq t_1 < t_2 \leq t_3 < t_4$, then the intervals $[t_1, t_2]$ and $[t_3, t_4]$ are nonoverlapping. Using (19) we see easily that

$$E[y(t_2)-y(t_1)]\,[y(t_4)-y(t_3)] = 0.$$

The increments over nonoverlapping intervals are therefore uncorrelated; since the process is Gaussian they are also independent.

We complete the proof by showing that the process is strongly continuous. For this we use an idea of J. L. Doob [1]. We write

$$V(t) = c\,\frac{e^{Kt}-1}{e^{K}-1} \tag{31a}$$

and

$$W(t) = \frac{1}{K}\ln\,[1 + (e^{K}-1)t] \tag{31b}$$

then

$$V[W(t)] = ct\ .$$

We define a new process $\xi(t')$ by setting

$$\xi(t') = y(t) \text{ where } t = W(t') \text{ or } t' = \frac{V(t)}{c} \tag{32}$$

The function $t = W(t')$ maps nonoverlapping intervals onto nonoverlapping intervals, it follows therefore from the assumptions about the $y(t)$ process that $\xi(t')$ is a process with independently and normally distributed increments and mean value function zero. Moreover it is easily seen that $\xi(0) = 0$ and that $E\,\xi(t')\,\xi(t'+\tau') = c\,t'$ for $\tau' > 0$ so that $\mathrm{Var}\,[\,\xi(t'+\tau') - \xi(t')] = c\,\tau'$. It follows then

from the definition given at the beginning of section 5 that (32) transforms the process $y(t)$ into a Wiener process. (In case $K < 0$, $y(t)$ is transformed into a segment of a Wiener process). A t-interval $[a,b]$ and a set S of points t_ν of $[a,b]$ are mapped by (32) on a t'-interval and a set S' of points t'_ν. Since $V(t)$ is uniformly continuous in any closed interval $[a,b]$ it is possible to determine for a given $\delta' > 0$ a $\delta = \delta(\delta')$ such that the event $\mathcal{E}_\xi(\delta', \varepsilon, S')$ implies the event $\mathcal{E}_y(\delta, \varepsilon, S)$. Since (theorem 2) the process $\xi(t')$ is strongly continuous it follows that $y(t)$ is also strongly continuous.

7. REFERENCES

[1] J. L. Doob, The elementary Gaussian processes, Annals Math. Stat. 15, 229-282 (1944).

[2] M. Fréchet, Généralités sur les Probabilités. Éléments aléatoires. 2nd ed. Gauthiers-Villars, Paris 1950.

[3] B. V. Gnedenko, Limit Theorems for Sums of Independent Random Variables. American Mathematical Society Translation No. 45, New York (1951). Translated from Uspehi Matem. Nauk 10, 115-165 (1944).

[4] A. Khintchine, Limiting distributions for sums of independent random variables. (in Russian) Moscow-Leningrad (1938).

[5] Paul Lévy, Processus stochastiques et mouvement Brownien. Gauthiers-Villars, Paris 1948.

[6] H. B. Mann, Estimation of parameters in certain stochastic processes. Sankhya vol. 11, 97-106 (1951).

[7] H. B. Mann, Introduction to the theory of stochastic processes depending on a continuous parameter. In press as No. 24 in the Applied Mathematics Series, National Bureau of Standards. Published by the United States Government Printing Office, Washington, D.C. (1953).

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