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NATIONAL BUREAU OF STANDARDS REPORT

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A NEW METHOD OF ANALYZING EXTREME-VALUE DATA

by

Julius Lieblein

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A NEW METHOD OF ANALYZING EXTREME-VALUE DATA

by

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Statistical Engineering Laboratory

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FOREWORD

This report develops various aspects of a new method of treating extreme-value data. This method, based on order statistics, has a number of advantages over existing procedures, and will be useful in the efficient handling of small or large sets of extreme gust-load data and extreme data in many other fields as well.

The present report completes work under a research project aimed at the improved application of the theory of extreme values to the analysis of gust loads of airplanes. This project, supported by the National Advisory Committee for Aeronautics, was carried out by Julius Lieblein under the general supervision of Dr. Churchill Eisenhart, Chief of the Statistical Engineering Laboratory. The Statistical Engineering Laboratory is Section 11.3 of the National Applied Mathematics Laboratories (Division 11, National Bureau of Standards), and is concerned with the development and application of modern statistical methods in the physical sciences and engineering.

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Chief, National Applied
Mathematics Laboratories

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A NEW METHOD OF ANALYZING EXTREME-VALUE DATA*

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1. SUMMARY

A new method is presented and proposed for analyzing extreme-value data which may arise in a wide variety of applications.

Classical applications of statistical methods, which usually concern average values, are inadequate when the quantity of interest is the <u>largest</u> (or smallest) in a set of magnitudes. This is the situation in a number of fields, e.g., gust loads of an airplane in flight, highest temperatures or lowest pressures in meteorology, floods and droughts in hydrology, breaking strengths in materials testing, breakdown voltage of capacitors, and human life spans, in all of which applications of methods for dealing with extremes have already been made.

Discussion of the proposed method is preceded by the necessary statistical theory (Section 4) which also furnishes a basis for evaluating the new method in relation to existing ones. The techniques described provide a simple means for estimating the

* Preparation of this report was sponsored by the National Advisory Committee for Aeronautics.

necessary parameters, making predictions from the fitted curve, estimating the reliability, and evaluating the efficiency of the method in relation to other methods. Moreover, these quantities are all produced by a single set of computations involving just two worksheets. This background material is not essential to an application of the method, and may be omitted if desired. The method itself is summarized for practical convenience and illustrated step-by-step in Section 5, and compared with present procedures in Section 6. The latter section also discusses the advantages of the proposed method, chief among which are:*

(1) For the first time there is available an unbiased estimator of known efficiency;

(2) The proposed estimator appears to be more efficient than a simplified form of the Gumbel estimator in many practical cases, namely, for samples

* The technical terms used here are defined and discussed in the main text, and they can be located with the aid of the list of symbols in Section 3.

of about 20 or more, and P = .95 or more. The improvement in efficiency increases with increasing P or increasing sample size. When compared with the original Gumbel estimator, the proposed one is up to twice as efficient.

(3) The confidence intervals have a more valid theoretical basis and are in many cases narrower than the ones in the Gumbel method.

Included in the report are several appendices giving mathematical developments not given in the text.

2. INTRODUCTION

The statistical theory of extreme values has been found to have wide applicability in many diverse fields, for example, meteorological extremes, floods, droughts, breaking strength of textiles and other types of materials, span of human life, gust loads experienced by an airplane in flight, breakdown voltage of capacitors.

The two existing methods of analyzing extreme-value data have several limitations, discussed in the body of this report. One of these methods is known as the method of maximum likelihood and has been described by B. F. Kimball (references 9, 10). The other, the method of moments, has been developed by E. J. Gumbel (references 2, 3, 6), and its application to gust-load problems has been discussed in detail in a previous NACA Technical Note by Harry Press (reference 18).

The present report gives a new method for dealing with the problem of analyzing extreme measurements, treated in the above Technical Note, which has certain advantages over the existing methods. The method of application is presented in detail, together with the necessary worksheets and other data, and the new method is compared with the method of moments previously in use. For definiteness, the discussion is at time presented in terms of application to gust loads, but the methods are also applicable to other fields where extreme values occur.

3. PRINCIPAL SYMBOLS (Listed Alphabetically)

Note: By "samples" are meant independent random samples from the extreme-value distribution

		Equation no., etc.
a_i, b_i, $i = 1,2,\ldots,n$	Numerical quantities entering into weights of order-statistics estimator for sample of n	(4.17); Table I
$\text{cov}(\bar{y},s)$ or $\sigma(\bar{y},s)$	Covariance of mean and standard deviation in samples of n from reduced distribution	Page 110 (Appendix E)
$E(\ldots)$	Mathematical expectation (or mean value) of (...)	Various, e.g., (4.6)
E_m, E_n	Efficiency of order-statistics estimator for subgroups of m observations, or for samples of n	(4.24), (4.19); Table III, Part B
$E(s)$	Mean value of standard deviation in samples of n from reduced distribution	Page 109 (Appendix E)
$F(x) = F(x;u,\beta) = \exp[-e^{-(x-u)/\beta}]$	Probability (cumulative) distribution function (c.d.f.) of extreme-value distribution with two parameters	(4.1)
$f(x) = \frac{dF(x)}{dx}$	Density (or frequency) function of extreme-value distribution F(x)	Page 11; Figure 1

k	Number of equal subgroups of size m contained in sample of n.	Pages 33, 36
m	Size of one of the k equal subgroups contained in sample of n	Pages 33, 36
m'	Size of remainder subgroup in sample of n that is left after k equal subgroups of m are taken, i.e.: $n = km + m'$	Page 36
MSE(...)	Mean square error of (...); equals variance plus square of bias	(4.13)
n	Sample size (N denotes sample size in Gumbel method)	Page 23
P	Probability level associated with a predicted value	Page 13
Q_o	Numerator in Cramér-Rao lower bound; $Q_{LB} = Q_o/n$	(4.24); Table III, Part A
Q_{LB}	Cramér-Rao lower bound to variance of unbiased estimator of parameter ζ_P (see Q_o)	Page 26
Q_m, Q_n	Variance of order-statistics "sub-estimator" for subgroups of m, or of estimator for sample of n.	(4.23) (4.18)

r	Rank of r-th observation (counted from smallest) in samples of n when arranged in ascending order from smallest to largest observation	Page 54
$R(T_1, T_2) = \frac{MSE(T_2)}{MSE(T_1)}$	Relative efficiency of estimator T_1 to T_2 (greater than unity when T_1 is more efficient)	(E.7)
$s = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2}$	Standard deviation (s.d.) of sample of n from reduced distribution	Page 109 (Appendix E)
$s_x = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}$	S.d. of sample of n from original distribution	(4.11)
$T_i, i=1,2,\ldots,k$	Order statistics "sub-estimator" for i-th of the k equal-size subgroups in samples of n	(4.21)
$\bar{T} = \frac{1}{k} \sum_{i=1}^{k} T_i$	Average of the sub-estimators for the k equal-size subgroups	(4.22)
T'	Sub-estimator for remainder subgroup (see m')	(4.25)
t, t'	Weights for $\bar{T}$ and T' in grand estimator for sample: $\xi_P = t\bar{T} + t'T'$	(4.26), (4.27)
$\hat{u} = x - \frac{\bar{y}_n}{\sigma_n} s_x$	Gumbel's original estimator of mode u for sample of n	(E.1)

$\hat{u}' = \bar{x} - \frac{\sqrt{6}}{\pi}\gamma s_x$	Simplified expression used to represent Gumbel's estimator of mode u	(E.2)
u	Mode or location parameter of extreme-value distribution	Page 12
x	Random variable ("unreduced") having extreme-value distribution F(x)	(4.1), (4.5)
$x_1, x_2, \ldots, x_n$	The n order-statistics in sample of n, i.e., the observations ranked in ascending order	Page 23
$x_{\lambda n}, x_{\mu n}, x_{\nu n}, 0 < \lambda < \mu < \nu < 1$	Three selected order-statistics in Mosteller method for very large samples of n	Page 86 (Appendix C)
$\bar{x}$	Sample mean in sample from original ("unreduced") distribution	Page 20
y, y_p	Reduced variate	Page 14
β	Scale parameter of extreme-value distribution F(x)	Page 12
$\hat{\beta} = \frac{s_x}{\sigma_n}$	Gumbel's original estimator of β for sample of n	(E.1)
$\hat{\beta}' = \frac{\sqrt{6}}{\pi} s_x$	Simplified expression used to represent Gumbel's estimator of β	(E.2)
$\gamma = 0.57721, 56649$	Euler's constant	(4.8)

$\Delta_{x,n} = 1.141\beta$	Half-width of 68% confidence interval in Gumbel method	(D.1)
$\Delta' = 1.141 B_p \beta$	Half-width of 68% confidence interval when modified by probability factor	(D.11)
Δ_o	Half-width of 68% confidence interval in method of order statistics	Table VIII
$\mu_x = E(x)$	First moment or mathematical expectation of random variable x	(4.6)
$\mu_2 = \sigma_y^2$	Variance of reduced distribution	(4.9)
$\xi_P = u + \beta y_P$	The 100P-percent point of the extreme-value distribution F(x)	Page 14
$\hat{\xi}_G$	Simplified expression used to represent Gumbel estimator of ξ_P	(E.3)
$\sigma^2(s)$	Variance of standard deviation in samples of n from reduced distribution	Table IX
σ_x^2, σ_y^2	Population variance of x, y	(4.7)
$\Phi(x) = \frac{r}{n+1}$	Plotting position of r-th observation ranked from smallest	Page 54
$\Phi(y) = \exp(-e^{-y})$	C.d.f. of reduced extreme-value distribution	(D.4); Page 112 (Appendix E)

4. STATISTICAL THEORY

4.1 Extreme-value distribution and meaning of parameters.

The method of analysis presented herein is based upon the assumption that the observed maxima to be analyzed are independent observations from a statistical distribution of the form

$$F(x) = F(x;u,\beta) = \exp(-e^{-(x-u)/\beta}) \quad . \qquad (4.1)$$

This is the <u>cumulative</u> (or ogive) form of the distribution, which expresses the chance that an observed extreme value (gust load, for example) will not exceed x in value. The more familiar concept of frequency or density function, $f(x) = F'(x)$, for this distribution may be obtained by differentiation but is rather cumbersome (see Appendix A) and is not needed for present purposes. The general shape of the density function $f(x)$ is shown in Figure 1. The meaning of the various quantities indicated is explained below. A more detailed graph for the case where the parameters are $u = 0$, $\beta = 1$ (the "reduced" extreme-value distribution) is plotted in Figure 2.

The distribution (4.1) has been studied extensively by Gumbel (among others) (references 2,3,6) and

is known as the asymptotic distribution of largest values. We shall refer to it briefly as the extreme-value distribution. The significance of the term "asymptotic" is as follows. If the underlying distribution of all (not merely the largest) gust loads (e.g., effective gust velocity, normal acceleration) is considered, then the largest values in repeated large samples from this distribution have a distribution of their own which, as the sample size becomes larger and larger, approaches closer and closer (in a certain sense) to a limiting distribution. This limiting distribution is, according to evidence presented in reference 18, of the form (4.1), with $1/\beta$ replacing the parameter α used in the reference.

The parameters of the extreme-value distribution are depicted in Figure 1. The quantity u is the mode or highest point of the (frequency) distribution. The quantity β is a scale parameter, analogous to the standard deviation σ in the case of the normal distribution. In fact, β equals $\sqrt{6}/\pi$ (about 3/4) times the standard deviation of the extreme-value distribution.

Although the two parameters u, β completely specify the distribution, it is desirable

to introduce another quantity $\xi = u + \beta y$ which is a linear combination of the parameters u and β (and therefore, since we shall assign known values to y, itself a parameter),* and makes it possible to estimate u and β simultaneously, rather than in terms of two separate problems. Thus if we can estimate as a + by with a, b known, then we can read off at once the values u = a, β = b.

The parameter ξ has another highly important meaning. In Figure 1 the area P under the distribution to the left of the ordinate erected at ξ represents the probability that a value larger than ξ will not occur. If ξ is very large, then P very nearly equals the whole area, unity, which means an observation is almost certain not to exceed ξ ; in other words, a larger value of ξ will occur only very rarely. Thus if P = .99, then the corresponding value of has a chance of only .01 of being exceeded. To denote this dependence of ξ upon the probability

* That is, we are concerned with the transformed parameters (ξ, β'), obtained from the original parameters (u,β) by the linear transformation $\xi = u + \beta y, \beta' = \beta$. We shall henceforth give attention only to the first parameter ξ , disregarding the second parameter β' of the transformed pair (ξ, β'). Whenever it should become necessary to refer to β', however, the prime will be dropped for simplicity. (See second footnote to page 26).

P we use a subscript: ξ_P. This parameter is called a percentage point or the 100P-percent point of the (extreme-value distribution). If we can estimate ξ_P for different probability levels such P = .90, .95, .99, etc., then these values are precisely the predictions we desire for, say, gust-load accelerations that will be exceeded (on the average) only 10, 5, 1, etc., respectively, times in 100.

The explicit relationship between ξ_P and P can be determined by means of formula (4.1) for the extreme-value distribution. If x is put equal to ξ_P, then P, the probability of not exceeding this value, is simply $F(\xi_P)$. Thus

$$P = F(\xi_P) = \exp(-e^{-(\xi_P - u)/\beta}) = \exp(-e^{-y}), \quad (4.2)$$

since $\xi_P = u + \beta y$. Hence, for a given (usually large) probability P, the corresponding ξ_P is obtained by finding y from the relation (4.2), and then writing

$$\xi_P = u + \beta y_P \quad , \qquad (4.3)$$

where the subscript P has been added to y to denote dependence on P. Comparison of the right members of (4.1) and (4.2) shows that the quantity y bears the following simple relation to the corresponding variable x in (4.1):

$$y = \frac{x - u}{\beta} \quad , \tag{4.4}$$

or

$$x = u + \beta y \quad . \tag{4.5}$$

Also, if in (4.1) we set $u = 0$, $\beta = 1$, then x has the same distribution as given by the right-hand side of (4.2). In other words, y as defined by (4.4) or (4.5) has an extreme-value distribution whose parameters have the extremely simple values $u = 0$, $\beta = 1$. Thus y is called the reduced variate* and is perfectly analogous to the standardized variate $t = (x - \mu)/\sigma$ of normal distribution theory. The distribution of y in (4.2), called the reduced distribution, has been tabulated in Table 2 of reference 15, which also contains a table of the inverse function as well as a number of other tables

* The variate x is sometimes referred to as the original or "unreduced" variate.

related to application of extreme-value theory.

From the above discussion it is evident that the solutions of both the problems of estimation and prediction are embodied in the one quantity $\xi_P = u + \beta y_P$. Estimation of this quantity will be the main objective of the remainder of this report.

4.2 Determination of method of estimation

To avoid confusion, we distinguish between a function of sample variables $x_1, x_2, \ldots, x_n$, such as the sample mean $g(x_1, x_2, \ldots, x_n) = \bar{x} = (x_1 + x_2 + \ldots + x_n)/n$, and the numerical values $g_o = g(\overset{o}{x}_1, \overset{o}{x}_2, \ldots, \overset{o}{x}_n)$ assumed by the function when the actual values of the observations $x_i = \overset{o}{x}_i$ are substituted into the function. If the <u>function</u> is used to estimate a parameter, we shall call it an <u>estimator</u> of the parameter; the particular <u>numerical</u> value assumed in a given case shall be called an <u>estimate</u>.

In searching for estimators the first step is to seek what are known as <u>sufficient statistics</u>. A definition of this concept may be found in any

advanced text on statistical theory, for example reference 8, Vol. II, p. 81; but the feature of importance here is that given a set of joint sufficient statistics, that is, certain functions of the sample observations, it is often possible to deduce from them an estimator with certain desirable properties, provided that the number of such functions does not depend upon sample size. If it turns out that the only set of sufficient statistics is the trivial set consisting of the n functions $t_i(x_1, \ldots, x_n) \equiv x_i$, $i=1, \ldots, n$, i.e. the n sample observations themselves, then obviously this furnishes no guide whatever for constructing functions of the x's which are optimum estimators.

Investigation reveals that, unfortunately, joint sufficient statistics do not exist for the two parameters of the extreme-value distribution. A proof of this fact (which was conjectured by B. F. Kimball, reference 9, p. 299) has been discovered by I. Richard Savage of the Statistical Engineering Laboratory of the National Bureau of Standards, and is presented in Appendix A.

It may be noted that Kimball (*ibid.*) has studied a broader concept called "set of statistical estimation functions" whereby the estimators of the parameters are given, not by explicit formulas involving only the sample values, but implicitly as the solutions of a set of simultaneous equations, for example, the classical maximum-likelihood equations. Unfortunately, such estimators do not seem to lend themselves to the procedure referred to above for constructing optimum estimators, and there seems to be no analytical means of accurately evaluating the important characteristics of bias and efficiency, defined below, for such estimators in the case of *finite* samples. (Although these estimators may be asymptotically optimum, i.e., for infinitely large samples, this need not be the case for samples of finite size.)

A second method of approach to the problem of estimation is the classical one known as the *method of moments*. In the case of the extreme-value population this method is as follows.

The first two moments of the extreme-value population (4.1) are

$$\mu_x = E(x) = u + \beta E(y) \tag{4.6}$$

$$\sigma_x^2 = E[x-E(x)]^2 = \beta^2 E[y-E(y)]^2 = \beta^2\sigma_y^2, \tag{4.7}$$

where y has the reduced extreme-value distribution (4.2), E denotes mathematical expectation, and σ^2 is the variance, the second moment about the mean. Using the moments of the reduced distribution (see, e.g., reference 8, Vol. I, page 221),

$$E(y) = \mu_1' = \gamma = .577216 \text{ (Euler's constant)} \tag{4.8}$$

$$\sigma_y^2 = \mu_2 = \frac{\pi^2}{6} = 1.644934 , \tag{4.9}$$

we obtain

$$\mu_x = u + \gamma\beta, \quad \sigma_x = \frac{\pi}{\sqrt{6}}\beta , \tag{4.10}$$

relations which express the population moments in terms of the population parameters. Therefore, if we had good estimators of the population moments, we could readily find the parameters. This fact constitutes the essence of the method of moments. It consists in treating the sample as an adequate representation of the population, replacing the population moments in the expressions which relate

them to the parameters by the corresponding sample moments, e.g., μ_x by the sample mean $\bar{x}$, and σ_x by the sample standard deviation

$$s_x = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \quad . \tag{4.11}$$

This gives $\bar{x} = u + \gamma\beta$, $s_x = (\pi/\sqrt{6})\beta$, which yield the moment estimators of the parameters:

$$\text{for } \beta,\ \hat{\beta} = (\sqrt{6}/\pi)s_x \ ;$$

$$\text{for } u,\ \hat{u} = \bar{x} - \gamma(\sqrt{6}/\pi)s_x \quad . \tag{4.12}$$

These are essentially the estimators which form the basis of Gumbel's method (reference 6, Lecture 3, eq. (3.29), with $u_n = \hat{u}$, $1/\alpha_n = \hat{\beta}$)*. This method is justified by the fact that under general conditions the estimator functions $\hat{u} = \hat{u}(x_1, x_2, \ldots, x_n)$, $\hat{\beta} = \hat{\beta}(x_1, x_2, \ldots, x_n)$ in (4.12) approach (in a certain sense) the values of the corresponding parameters u, β as the sample size becomes infinite.

* The actual estimators used in the Gumbel method are slightly more complicated (reference 6, Lecture 3, eq. (3.39), but the difference is not important at this point. (See Appendix E).

This method has apparently given satisfactory results in practice. It is, however, subject to an important limitation. In studying estimators it is highly desirable to know something about their probability distributions--if not the exact density functions, then at least their means and variances. The mean value (mathematical expectation) of an estimator indicates whether <u>on the average</u> the estimates given by it are too high or too low relative to the actual values of the parameter estimated --in other words, whether there is any <u>bias</u> in using the estimator. Similarly, the variance indicates how much the estimates scatter among themselves and is the basis for constructing a measure of <u>efficiency</u> which makes it possible to compare the performances of different estimators. A more useful concept for some purposes than variance is mean square error (abbreviated MSE) which measures how far the estimates deviate, on the average, not from their own mean, but from the quantity--the parameter--which they are supposed to measure. There is a simple relationship between variance and MSE, namely,

$$\text{Mean square error} = \text{variance} + (\text{bias})^2. \qquad (4.13)$$

Thus, for unbiased estimators, variance and MSE are identical, and for brevity we shall use the term "variance" in such cases. But it should be remembered that the concept in view is actually the MSE. This becomes especially important later when biased estimators are discussed (Appendix E), and variance and MSE are no longer identical.

If we try to determine the mean (or expected) values of the estimators, u, β, in (4.12) we find that statistically these functions are quite complicated, leading to very difficult multiple integrals which apparently can be evaluated accurately only by large-scale numerical integration.* This difficulty evidently persists if we are interested in the parameter $\xi_p = u + \beta y_p$ instead of u or β separately.

4.3 Order-statistics approach for small samples

Apparently the only method of estimation which avoids this difficulty is the <u>method of order statistics</u>.

* Shorter methods of limited accuracy are possible and have been used in this report for comparison purposes. (See Section 6.1 below and Appendix E.)

If the values in a sample of n observations are arranged in, say, increasing order of size, and denoted by $x_1, x_2, \ldots, x_n$, $x_1 \leq x_2 \leq \ldots, \leq x_n$, then these values x_i are called order statistics. The smallest is called the first order statistic; the middle one (if n is odd), the median; the one which is one-fourth the way up from the bottom, the first quartile, etc. (If there are several equal ones then suitable modifications are made in the definitions). There is an extensive literature on this subject, chief among which is the comprehensive survey in reference 20.

Order statistics provide rapid and practical methods of analyzing data. The range, $x_n - x_1$, is a very common illustration from quality control. It is simply the difference of two order statistics, the largest and smallest, and its properties have been extensively studied for samples from the normal distribution. The range has been found to yield estimates of the standard deviation of the population that often compare very favorably with the theoretically best obtainable. More general linear functions, $C_1x_1 + C_2x_2 + \ldots + C_nx_n$, which give weight to every

sample value, have also been studied (reference 19), and values of the coefficients have been found which make it possible to estimate very simply and remarkably well certain quantities which previously were obtained only by more complicated calculations.

We shall carry over and extend this procedure to the case of samples from the extreme-value distribution (4.1). The method will in many respects follow the general approach used in reference 19 for several other distributions. The aim is to determine the weights w_i, $i=1, \ldots, n$, for all the n order statistics in a sample of size n so that the linear estimator

$$L = \sum_{i=1}^{n} w_i x_i \tag{4.14}$$

has the properties we desire, namely:

(1) The mathematical expectation equals the parameter to be estimated, i.e. the estimator is unbiased:

$$E(L) = \xi_p; \tag{4.15}$$

(2) The MSE, which in this case is the same as the variance, is as small as possible, consistent with (1):

$$MSE(L) = \sigma^2(L) = E[L-E(L)]^2$$
$$= \text{a minimum.} \qquad (4.16)$$

An estimator L which satisfies these two conditions will be denoted by ξ_p, a notation suggested by condition (1).

Condition (2) is equivalent to saying that the estimator ξ_p is as efficient as possible under the given conditions. This concept will be discussed below.

The mathematical formulation of this minimum-variance problem is developed in Appendix B, and the solutions (the weights) are shown in Table I for n = 2 to n = 6. The case n greater than 6 is discussed in the next Section. For each given value of n, n weights $w_1, w_2, \ldots, w_n$ are determined that depend on the quantity y_p that occurs in the parameter $\xi_p = u + \beta y_p$ to be estimated. The w_i are each of the following form:

$$w_i = a_i + b_i y_p \quad , \qquad i = 1, 2, \ldots, n \quad . \qquad (4.17)$$

Substituting these weights for given n into (4.16) actually gives the minimum value, Q_n, that the variance can attain under the above conditions, and this value depends upon y_p quadratically:

$$V_{min} = Q_n = (A_n y_p^2 + B_n y_p + C_n)\beta^2 \quad . \qquad (4.18)$$

Table I gives the values a_i, b_i, A_n, B_n, C_n, which have been found by exact computation methods as indicated in Appendix B and Table II there described. The quantities $V_{min} = Q_n$ are shown in Table III, Part A.

As sample size increases, the estimation is expected to improve and the variance to diminish. In order to have a convenient standard of comparison, in the case of unbiased estimators,* we scale all variances by dividing into a theoretically specified variance (Q_{LB}), known as the "Cramér-Rao lower bound" (reference 1, pp. 480, equation (32.3.3a),** which

* For biased estimators, see Appendix E.

** This Cramér-Rao bound is given for the case where the distribution has only one parameter to be estimated. For the extreme-value distribution with the two parameters (ξ,β) we can regard β as a "nuisance parameter" and thus obtain a "Cramér-Rao bound" for ξ, the expression for which will involve β (see first footnote to Table III). This procedure is based on the "method of nuisance parameters" discussed in reference 11. (See also footnote to page 13.)

is less than or at most equal to the variance of any (unbiased) estimator of the parameter in question* . The result is then an absolute number between 0 and 1 which, when expressed as a percentage, is called the efficiency of the estimator for samples of n:

$$\text{Efficiency (L)} = E_n(L) = \frac{Q_{LB}}{Q_n} . \qquad (4.19)$$

The quantities E_n, which evidently depend upon y_P, and therefore upon P, are given for n = 2 to 6 for selected values of the probability P in Table III, Part B. Part A contains the numerical values of the variances Q_n and the lower bound Q_{LB} in terms of the parameter β^2. The expression for Q_{LB} has been implicitly given in reference 10, p. 113, and is indicated in the first footnote to Table III, Part A of this report.

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* There may or may not exist estimators whose variances reach the lower limit Q_{LB}. If (as may happen) there exists a $Q' > Q_{LB}$ such that the variance of every estimator is $\geq Q'$ (and of course $> Q_{LB}$), then Q' may be substituted for Q_{LB} in the numerator of the above expression for efficiency (4.19) without the fraction exceeding 1. The investigation of the existence of Q' is too complex a matter for purposes of this report. However, the only effect of using a lower bound Q_{LB} which is too low is to understate the efficiency, so that the results are on the safe, conservative side.

Two points should be noted about the choice of probability levels shown in Table III. The value $P = .36788 = 1/e$, which corresponds to $y_P = 0$, is important because it gives the mode, one of the desired parameters of the distribution. This is evident from the fact that the parameter we are estimating is $\xi_P = u + \beta y_P = u$, the mode, for $y_P = 0$. Similarly, the limiting value $P = 1$ corresponds to the scale parameter β. This may be seen as follows. If P approaches 1, the ξ_P and y_P both become indefinitely large, but their ratio $\xi_P' = \xi_P/y_P = (u/y_P)+\beta$ may be considered to be a new parameter which approaches β, since the mode u remains fixed and finite (as does also β). Hence we may estimate β by first estimating ξ_P' for arbitrary P and then letting P approach 1.

Now from (4.14) and (4.17), the linear estimator $L = \hat{\xi}_P$ is of the form

$$\hat{\xi}_P = f_1 + y_P f_2 \qquad (4.20)$$

where f_1 and f_2 are functions of the sample values which do not involve y_P. By the preceding remark, we can then estimate the parameter β by writing down

the corresponding estimator of ξ_P',

$$\hat{\xi}_P' = \frac{1}{y_P}\hat{\xi}_P = \frac{f_1}{y_P} + f_2 ,$$

and letting P approach 1, obtaining

$$\hat{\xi}_P = f_2$$

as the corresponding estimator of β.

In other words, an estimator $\hat{\xi}_\beta$ of β may be obtained by simply taking the <u>coefficient of y_P</u> in $\hat{\xi}_P$ when written in the form (4.20). Similarly, the variance of $\hat{\xi}_\beta$ is the <u>coefficient of y_P^2</u> in the variance of $\hat{\xi}_P$. This may readily be seen as follows. From (4.20), we have

$$\sigma^2(\hat{\xi}_P) = \sigma^2(f_1) + 2y_P \operatorname{cov}(f_1, f_2) + y_P^2 \sigma^2(f_2) = A + By_P + Cy_P^2 ,$$

where A, B, C are quantities which do not involve y_P (though they may involve β, in general); thus, as P approaches 1 and y_P increases without limit,

$$\sigma^2(\hat{\xi}_P') = \frac{1}{y_P^2}\sigma^2(\hat{\xi}_P) = \frac{A}{y_P^2} + \frac{B}{y_P} + C \rightarrow C,$$

the coefficient of y_P^2 in $\sigma^2(\hat{\xi}_P)$. From this it follows also that the efficiency of the estimator $\hat{\xi}_\beta$, being a ratio of variances, is simply the ratio of the coefficients of y_P^2, the other terms being disregarded.

These facts applied to the estimator $\hat{\xi}_P$ make it possible to avoid a separate treatment for the two parameters u and β. Their estimators are each represented by a single line in a table (such as Table III) showing values for various probability levels:

$P = .36788$ (or $y_P = 0$) gives u; $P = 1$ (or $y_P = \infty$) gives β.

The concepts of variance and efficiency have also a more concrete, practical significance. The lower bound to the variance Q_{LB} has the form $Q_{LB} = Q_o/n$, where Q_o is a quadratic function of y_P, but is independent of sample size n. For two samples of sizes n' and n", the variances Q'_{LB}, Q''_{LB} are in the ratio

$$\frac{Q'_{LB}}{Q''_{LB}} = \frac{n''}{n'} ,$$

i.e., inversely proportional to sample size. Similarly, if we had two estimators for the same sample size, we could form the ratio of their variances and think of it as representing an (inverse) ratio of (hypothetical) sample sizes. Thus, if for a sample of 20, the variance Q' of one estimator were one-half the variance Q'' of an alternative estimator, then the first estimator would require a sample of only 10 to give as much information as could be obtained with the second from a sample of 20. This saving of half the number of observations is expressed by saying that the first estimator is twice as efficient as the second. In general, a saving of the fraction p of the observations makes one estimator $1/(1-p)$ times as efficient as a second.

The efficiencies of the estimators $\hat{\xi}_p$ in Table III are more conveniently compared in graphical form, as in Figure 3. The heavy horizontal line at the top indicates perfect or 100 percent efficiency, and the rising curves as n increases show how closely the estimator is approaching the standard of perfection. The most outstanding fact is that, in marked contrast to a theoretical, perfect estimator, the efficiency of

the actual estimator $\hat{\xi}_P$ depends upon the probability P, being largest for the middle ranges .40 to .60 and dropping considerably at the ends near 0 and 1. Since analysis of extreme (largest) data is concerned chiefly with the larger magnitudes associated with very small probabilities of occurring or being exceeded, we shall limit our interest to the range above $P = .90$. For $n = 6$ the efficiency exceeds the 80 percent level for all values of P in this range that are apt to occur in paractice (i.e., $P < .999$). In view of the satisfactory values of efficiency, further calculation for $n > 6$ did not appear warranted at this time, particularly since it became apparent that the labor of computation would increase out of all proportion to the rapidly diminishing improvement in efficiency.

Of course, most samples of observations are larger than the trivial size of 6, and the question arises of how to handle the larger samples. This is treated in the next sub-section.

4.4 Extension to larger samples

The key to handling samples with more than six observations is to treat them as sets or sub-groups of samples of 6 (or if necessary, 5). If a sample size is not an exact multiple of 6 or of 5, then the sample may be treated as consisting either of subgroups of 6 with an odd group remaining having less than 6 items, or of subgroups of 5 with a remaining group of 6. We first deal with the simpler case where n is an exact multiple of 5 or 6.

Case I. Sample size an exact multiple of 5 or 6. Suppose, in general, n = km, where m is the size of subgroup, which need not be 6, and k is the number of subgroups in the sample. If the sample is so divided into subgroups that the observations in one subgroup may be considered to be statistically independent of those in any other subgroup, then it is legitimate to treat the sample as consisting of k independent subsamples, each of size m.

One way of obtaining independent groups is by use of random numbers. This, however, will lose valuable information embodied in the order in which the data were actually observed. If the data are

truly random, so that, for example, there are no seasonal effects, then this implies that subgroups formed in the order in which the data are observed— the first m values observed put into the first group, the next m into the second, etc.—should be independent. This assumption, of course, underlies the entire method of estimation described in this report, and we shall adopt it in our procedures.

From each subgroup we form the "sub-estimator"

$$T_i = \sum_{j=1}^{m} w_j x_j \quad , \quad i = 1, 2, \ldots, k \quad , \tag{4.21}$$

where the weights $w_1, w_2, \ldots, w_m$ are those taken from Table I for sample size m and are the <u>same for each subgroup of m values</u> (but, of course, are different for different sizes m). These k sub-estimators T_i are then combined by simple averaging to form the grand sample estimator:

$$\bar{T} = \frac{1}{k} \sum_{i=1}^{k} T_i \quad . \tag{4.22}$$

The variance of this estimator is simply

$$\text{Var}\ (\bar{T}) = \frac{1}{k} Q_m \quad , \tag{4.23}$$

since we are taking the variance of a mean of k independent quantities T_i, each of which has the same variance* $Var(T_i) = Q_m$; Q_m denotes the variance tabulated in Table III, Part A, for m = 2, 3, 4, 5, 6.

The efficiency of $\overline{T}$ is, since n = km,

$$E = \frac{Q_{LB}}{Var(\overline{T})} = \frac{\frac{1}{km} Q_o}{\frac{1}{k} Q_m} = \frac{\frac{1}{m} Q_o}{Q_m} = E_m \quad , \qquad (4.24)$$

Where $Q_{LB} = Q_o/n = Q_o/km$, and Q_o is independent of n. Thus we have the important fact that <u>if a sample is broken into equal-size subgroups, the efficiency of the order-statistics estimator depends only upon the size (m) of the subgroup (and, of course, P).</u>

Since, according to Table III, Part B, efficiency increases with sample (or subgroup) size, it follows that when there is a choice, a sample should be broken into subgroups as large as possible for best efficiency, i.e. into subgroups of 6. If this is not possible, but if the sample size n is an exact multiple of 5, then subgroups of 5 may be used with not much loss in

* These variances are equal because they depend only upon m, P, and β, which are constant for all the subgroups of the same sample.

efficiency. The last two columns in Table III, Part B, show that the loss is 2.4 percent (= .8647 - .8404) at P = .95, and rises to a maximum of 3.8 percent for the limiting value P = 1.

Case II. Sample size not an exact multiple of 5 or 6. In most cases, of course, the sample size will have a remainder when divided by both 5 and 6. There are then a great variety of choices as to how to partition n into subgroups of 6 and 5 and perhaps other sizes. Many of these possibilities have been examined, the aim being to establish as simple rules as possible without too great a loss in efficiency. Fortunately, most of the methods of partitioning a sample of given size n do not lead to greatly different efficiencies. Thus the following rules can be laid down for $n \geq 7$ ($n \leq 6$ does not involve breaking into subgroups) based on writing n in the form either $6k + m'$ or $5p + 6$, where $m' < 6$:

(a) n = 7 up to large values.

(i) Use $n = 6k + m'$: split up into k subgroups of 6 and a remainder subgroup of $m' < 6$ items, unless n = 31, 61, etc., i.e., a multiple of 30 plus 1.

(ii) If $n = 30k + 1$, write it as $n = (30k-5) + 6 = (6k-1) \times 5 + 6$, i.e. split sample into $(6k-1)$ subgroups of 5 and a "remainder" subgroup of 6.

(b) n extremely large. If sample size is of the order of several hundred or more, so that the number of subgroups is of the order of 50 or 100, then the amount of computation becomes increasingly laborious. For such very large samples of extremes, which are rather rare, a short-cut method is available which is explained in Appendix C. While its efficiency is substantially less than the longer method presented here, it is nevertheless of practical value inasmuch as the loss in efficiency, which in practical terms means an effective loss in number of observations, is not very important when a very extensive amount of data happens to be available.

The variance and efficiency of an estimator for most sample sizes (Rule (a)) can be discussed readily in general terms. Assume that $n = km + m'$ represents the separation of the sample into two parts: one consisting of k equal subgroups of size $m = 5$ or 6; the other consisting of the remainder subgroup of size $m' < m$ except for the exceptional case where $m = 5$, $m' = 6$ (Case II, rule (a)(ii)). The average, $\overline{T}$, is formed from the first part as described under Case I. Then a sub-estimator T' is formed from the remainder subgroup of m' values using the weights w_i' for samples of size m':

$$T' = \sum_{i=1}^{m'} w_i' x_i' \quad , \qquad (4.25)$$

where x_i', $i = 1, 2, \ldots, m'$, denotes the m' values in the subgroup. Finally a weighted average of $\overline{T}$ and T' is formed, and this is the grand sample estimator $\hat{\xi}_P$:

$$\hat{\xi}_P = t\overline{T} + t'T' \quad , \qquad (4.26)$$

where the multipliers are*

$$t = \frac{km}{n} \quad , \; t' = \frac{m'}{n} = 1 - t \quad . \qquad (4.27)$$

Since all the subgroups are independent, so are $\bar{T}$ and T'; whence

$$Var(\hat{\xi}_P) = \frac{t^2}{k} Q_m + t'^2 Q'_m$$

since the variance of the mean T is $\frac{1}{k} Q_m$.

From the above it is evident that once the partitioning of sample size n into $n = km + m'$ is determined, the variance and efficiency may be obtained except (in the case of the variance) for a factor β^2 which must be estimated from the data. Table IV lists for convenience the efficiencies at two probability levels, $P = .99$ and the limiting value $P = 1$, for most of the sample sizes that

* Other multipliers are possible. In particular, there is an optimum set of multipliers which produces an unbiased estimator ξ_P with slightly smaller variance, and hence slightly greater efficiency. The optimum multipliers are, however, less simple than the proportional ones — for example, they are not constants but depend on P — and the gain in efficiency is not great. This was shown by a number of trials and by the fact that in any event, the efficiency cannot exceed that for the larger subgroup size, E_m (or $E_{m'}$, if $m' > m$), and does not differ much from it if the total sample size n is at all sizable, say > 20.

may occur in practice with gust-load data, provided the sample is split up according to the above rules. The levels P = .99 and P = 1 furnish a convenient basis for comparing the efficiencies of two different partitions of the sample size. At this end of the probability scale the difference between the two efficiencies decreases monotonically as P decreases. Thus, if the difference in efficiencies is 3 percent at P = .99 and 4 percent at P = 1, then the difference is between 3 and 4 percent at P = .995, say, and at P = .95 and under is apt to be substantially below 3 percent, a difference negligible for practical purposes. The partitions shown in Table IV are those recommended by rule (a) above. In certain cases, the efficiencies of alternative partitions are shown in the footnotes to Table IV for use in case the extra few percentage points in efficiency are considered to be worth a little loss of simplicity in computation.

There are some useful a priori guides for judging the efficiency in any given case even beyond the limit n = 40 of Table IV. Thus, if $n = km + m'$, it is clear that the efficiency cannot exceed that for the subgroup sizes m and m', but must lie

somewhere between the efficiencies corresponding to these two sample sizes. If m and m' are not far apart, then, regardless of the number of subgroups k, the efficiency is determined between narrow limits. Again, if k is substantial, say near 10 or more, then the efficiency is practically that for the larger sample size m. Of course the maximum efficiency obtainable by the procedure outlined here is for Case I when the sample size is an exact multiple of 6. For P = .99 the efficiency in such case is 83.2, and for P = 1, it is 76.8. If any given partition results in efficiencies within, say, 2 or 3 percent of these values, then there is nothing significant to be gained by using any other partition, unless it is such as to simplify the computation.

5. SUMMARY OF PROCEDURES

The above method of analysis will now be summarized for ease of reference. The use of the method has been considerably simplified by the construction of specially designed worksheets, represented by the pair of blank specimen forms following this page. A completely filled-out pair (Worksheets 1 and 2) will be found immediately preceding the tables at the end of this report. With the aid of such worksheets about two hours should be sufficient for all the calculations for a moderate-size sample, such as the sample of 23 observations analyzed below, and it has been found that this period is even sufficient to include the graphical analysis also presented.

The materials needed for application of the method, besides Worksheets 1 and 2 and a sheet of extreme probability paper, are, in the order in which needed:

(i) Table IV, showing efficiencies for various methods of splitting sample into subgroups.

(ii) Table I, giving the weights a_i, b_i.

(iii) Table III, furnishing the quantities Q_0, Q_2, Q_3, Q_4, Q_5, Q_6.

ESTIMATION OF EXTREMES

Worksheet 1 - Determination of Estimators

Source:________________________

Computer:________________

Date:________________

Record No.	Observed extremes

I. SUBGROUP SIZES AND PROPORTIONALITY FACTORS:

$n =$ ___ $= k\,m + m'$ $= $ __ x __ + _ $t = km/n = 0.$____ $t^2/k = 0.$____ $t' = m'/n = 0.$____ $t'^2 = 0.$____

$k =$ ___ $m =$ ___ $m' =$ ___

IIA. MAIN SUBGROUPS:

Weights a_i, b_i (from Table I)

$i =$	1	2	3	4	5	6	Check sum
$a_i =$							
$b_i =$							

Observations x_i in increasing order from $i = 1$ to $i = m$

Subgroup No.	x_1	x_2	x_3	x_4	x_5	x_6	Check sum	$\Sigma a_i x_i$	$\Sigma b_i x_i$
1									
2									
3									
4									
5									
.									
.									
k									
Sum									

$$\bar{T} = \Sigma a_i x_i/k + (\Sigma b_i x_i/k) y_P = \_\_\_ + \_\_\_\ y_P$$

IIB. REMAINDER SUBGROUP:

Weights a_i', b_i' (from Table I)

$i =$	1	2	3	4	5	6	Check sum
$a_i =$							
$b_i =$							

Observations x_i' in increasing order from $i = 1$ to $i = m'$

x_1'	x_2'	x_3'	x_4'	x_5'	x_6'	Check sum	$\Sigma a_i' x_i'$	$\Sigma b_i' x_i'$

$$T' = \Sigma a_i' x_i' + (\Sigma b_i' x_i') y_P = \_\_\_ + \_\_\_\ y_P$$

III. ESTIMATORS:

$$\hat{\xi}_P = t\,\bar{T} + t'T' = \_\_\_ + \_\_\_\ y_P$$

$u =$ ____, $\beta =$ ____

ESTIMATION OF EXTREMES

Worksheet 2 - Predicted values, confidence band, efficiency, plotting positions

P	y_P	PREDICTED VALUES $\hat{\xi}_P =$ ___ + ___ y_P	$Q_m = Q$ _ (from Table III)	$Q_{m'} = Q$ _	$\mathrm{Var}(\hat{\xi}_P) = \frac{t^2}{k} Q_m + t'^2 Q_{m'}$	68%-CONFIDENCE BAND HALF-WIDTH $\sigma(\hat{\xi}_P) = \sqrt{\mathrm{Var}(\hat{\xi}_P)}$	$Q_{LB} = Q_o/n$ (Q_o from Table III)	EFFICIENCY $E = \frac{Q_{LB}}{\mathrm{Var}(\hat{\xi}_P)}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
			$t^2/k =$ ___	$t'^2 =$ ___		$\beta =$ ___		
Estimate of u:								
.36788	0		β^2	β^2	β^2		β^2	
.50	0.36651		β^2	β^2	β^2		β^2	
.90	2.25037		β^2	β^2	β^2		β^2	
.95	2.97020		β^2	β^2	β^2		β^2	
.99	4.60016		β^2	β^2	β^2		β^2	
.999	6.90726		β^2	β^2	β^2		β^2	
Estimate of β: 1	- - -	- - -	$y_P^2\beta^2$	$y_P^2\beta^2$	$y_P^2\beta^2$	- - -	$y_P^2\beta^2$	

PLOTTING POSITIONS

Observed extremes in increasing rank from 1 to n = __

Rank r	Observed Extreme	Plotting Position r/(n+1)	Rank r	Observed Extreme	Plotting Position r/(n+1)	Rank r	Observed Extreme	Plotting Position r/(n+1)
1			11			21		
2			12			22		
3			13			23		
4			14			24		
5			15			25		
6			16			.		
7			17			.		
8			18			.		
9			19			n		
10			20			Sum	___	

The <u>assumptions</u> upon which the method is based are that the data in the given sample (arranged in the order in which observed*) may be treated as independent random observations all from the same population

$$F(x) = \exp(-e^{-(x-u)/\beta})$$

(in cummulative form), with constant unknown parameters u, β to be estimated.

For concreteness, the rules below refer to an actual example, worked out in Worksheets 1, 2, and Figure 4, consisting of the 23 maximum positive acceleration increments observed in 23 flights of an airplane and identified as "NACA-Langley-Sample III," which are listed in the column headed "Observed extremes (+Δn)" in Worksheet 1. These data are assumed to be given in the order of observation, so that under the above assumptions this arrangement may be considered to be a random one.

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* If the observations are not available in their original order, it will first be necessary to randomize them by use of a table of random numbers.

Each rule (except Nos. 2 and 7, which are subdivided) consists of a single paragraph and this is followed by a detailed explanation of its use, inserted for convenience of the user. This makes the list unavoidably lengthy, but the rules themselves are brief and simple to apply.

Before starting the calculations, it is desirable to plot the data on special probability paper according to the directions in rule 6(a), "Graphical analysis," below, in order to obtain a crude judgement of how well the data fit the assumed distribution. In rearranging the data in order of size, however, care should be taken <u>not to lose the record of the original order</u> in which the data were taken; because randomness will then have to be reintroduced.

<u>Determination of estimators — Worksheet 1.</u>

1. <u>Enter the observations</u> in the second column in the order in which given. The first column is for identification purposes.

2. Determine partition of sample size (if 7 or more, but not extremely large) and split sample into subgroups according to the following rules (a), (b) or (c). If n is extremely large, say several hundred or more, see Appendix C.

   (a) If n is an exact multiple of 5 or 6, write $n = k \cdot 5$ or $n = k \cdot 6$; if both, use $n = k \cdot 6$.

   (b) If n is not an exact multiple of 5 or 6, write $n = k \cdot 6 + m'$, $m' < 6$, unless n = 31, 61, etc., i.e., one plus a multiple of 30.

   (c) If n is of the form $30k + 1$, write it as $n = (30k-5) + 6 = (6k-1)5 + 6$, i.e. split n up into (6k-1) subgroups of 5 and a "remainder" subgroup of 6.

   Once k, m, m' are determined the blanks in Section I can be filled in. At the same time, in Worksheet 2, the numerical values of m, m' should be entered as subscripts in the headings "$Q_{\_}$" and "$Q_{\_}$" for columns 4 and 5, respectively. In the worked example,

$n = 23 = 3 \cdot 6 + 5$ (rule (b)), so the data are split into 3 main subgroups of 6 and a remainder subgroup of 5.

Number of decimal places. - As a result of considerable experimentation it is recommended that all computations be carried to exactly the number of places shown for each item on the two worksheets.

3. Find estimators for the parameters ξ_p, u, by filling in the blanks and following the directions indicated in Worksheet 1, Sections IIA, IIB, III.

In Section IIA, obtain the weights a_i, b_i from Table I for n = m, the size of the main subgroups. Mark off the subgroups by any convenient means,* arrange the observations in increasing order within each subgroup and enter horizontaly opposite

* It was found convenient here to determine the subgroup size m before entering the data in the extreme left columns, so that the subgroups could be plainly indicated by means of a space after every m-th observation.

the proper subgroup number in Section IIA. Obtain the two product sums $\sum_{i=1}^{m} a_i x_i$, $\sum_{i=1}^{m} b_i x_i$ as indicated in two right-hand columns and sum all columns as shown. The two product-sums evaluated for the line labelled "Sum" will serve as a check. Form the average, $\overline{T}$, by dividing by the number, k, of main subgroups.

The work in Section IIB is analogous, except that the weights a_i', b_i' are the a_i and b_i shown in Table I for $n = m'$, the size of the remainder subgroup; also, since there is only one subgroup, averaging is unnecessary.

Section III combines the (sub)estimators $\overline{T}$ and T' with the proportionality coefficients t, t', determined in Section I, to produce the final over-all sample estimator

$$\hat{\xi}_p = t\overline{T} + t'T' = .92946 + .16774\, y_p,$$

on collecting the coefficients of y_p

and the constant terms. The estimates of the parameters u, β are read off at once from the coefficients of $\hat{\xi}_P$ and entered. This constitutes the fitting of an extreme-value distribution to the given data.

Predicted values (etc.) - Worksheet 2.

4. Compute the values of $\hat{\xi}_P$ in column 3 for the values of P and y_P shown in columns 1, 2. These values constitute the set of predictions for the respective probability levels.

Additional probability levels may be inserted between those shown, if desired. The value of $y_P = -\log_e(-\log_e P)$ is found most conveniently from Table 2 of reference 15.

5. Confidence-band half-width (68-percent control curves) are computed from the standard deviations as indicated.

The numerical values of the variances Q_m, Q_m' in columns 4 and 5 are found under these same headings in Table III, Part A, and entered as shown. The values of t^2/k, t'^2 are entered above these values, as indicated, in order to facilitate computation of the variances of the overall estimator,

$$\mathrm{Var}(\hat{\xi}_p) = \frac{t^2}{k} Q_m + t'^2 Q_m' \quad ,$$

in column 6. Column 7 gives the standard deviation of the estimator $\hat{\xi}_p$. It is most easily computed by taking the square root of the coefficient of β^2 in column 6 and multiplying by the value β found in Section III of Worksheet 1. Thus $\sigma(\hat{\xi}_p)$ for $P = .50$ is $\sqrt{.06051}$ times the value $\beta = .16774$ (written at the top of column 7 for convenience), giving the value .0413 shown.

The standard deviation of the estimator measures the reliability, that is the extent to which repeated application of the procedure

to repeated samples taken under the same conditions would give values clustering more or less closely about the unknown parameter value. For example, for a fixed probability P, in about 68 percent of the time (when the assumptions are satisfied) the computed interval ξ_P plus or minus one standard deviation will contain the true unknown parameter $\xi_P = u + \beta y_P$. For two standard deviations the percentage rises to 95*. Two curved lines, one joining the left-hand endpoints of these intervals and one joining the right-hand endpoints, are called control curves (see rule 7 for graphical analysis, below) and these two curves define a confidence band consisting of the area between them. The interval of values of the abscissa $x = \xi_P$ included between the

* These percentages are only approximate since they assume ξ_P to be normally distributed. As indicated in Appendix D, this assumption is sufficiently correct for practical purposes for samples of the order of 100 or more. This may, of course, not be the case for much smaller samples. However, normality assumptions of this kind must often be made in practice in the absence of large-scale investigations to establish more precise distributions. Results obtained in this manner have often been found to be satisfactory.

control curves, when P is given a specific value, is called a confidence interval. The standard deviation in Column 7 is thus the half-width of a 68-percent confidence band (or interval). If levels of 95 percent, etc., are desired, the values can be readily obtained by adding another column consisting of twice (etc.) the entries in column 7.

6. Efficiency is computed as follows. The values of Q_o for the indicated values of P are taken from the column headed Q_o in Table III, Part A, divided by the given sample size n, and entered in the Q_{LB} column, 8, of Worksheet 2. The efficiency is obtained by dividing this by the corresponding entry in column 7, canceling the β^2 (which was one reason for carrying it along separately), and finally entering the result in column 9.

7. Graphical analysis consists of plotting the data on suitably ruled paper, drawing the estimated straight line, drawing

in the control curves, and seeing how well the data fall within them. The method is essentially due to Gumbel (cf. reference 5).

(a) In the section of Worksheet 2 called <u>Plotting Positions</u>, arrange <u>all n</u> observations in the sample in a <u>single</u> ascending series from smallest to largest and enter them opposite the rank numbers r = 1 to n. Compute and enter the plotting positions $\phi(x) = \frac{r}{n+1}$. Then, on a sheet of <u>Extreme Probability Paper</u>* such as used in Figures 4 and 5, plot the points $(x_r, \frac{r}{n+1})$. The observation x_r is plotted on the uniform scale along the horizontal axis; the fraction $\frac{r}{n+1}$ is plotted along the nonuniform vertical scale $\phi(x)$. These points are plotted as shown in Figure 4.

* I.e., coordinate paper with one scale (x) uniformly spaced and the other (y) distorted in such manner that the extreme-value distribution $\exp(-e^{-y})$ will plot as a straight line.

(b) After the points are plotted the <u>estimated line $x = u + \beta y$</u>, i.e., $x = .9295 + .1677y$ (see Rule 3, above), is drawn through them. This is easily done from columns 2 and 3 (Worksheet 2), since column 3 gives the predicted values of $x(=\hat{\xi}_P)$ corresponding to the values of $y(=y_P)$ in column 2. An even simpler method is to take two or three widely separated values P in column 1 together with the corresponding values $\hat{\xi}_P$, plot them on the $\phi(x)$ and x scales, respectively, and draw the line through them.

(c) <u>The 68-percent control curves</u> are obtained by measuring off horizontally, at each value of P in column 1, the distance $\sigma(\hat{\xi}_P)$, taken from column 7, to the right and left of the fitted line, and then joining all the right and all the left endpoints of the intervals so formed, as in Figure 4. The area included between the two control curves

is the 68-percent confidence band. If most or all of the plotted points fall within the band, as in Figure 4, then we conclude that the fit is satisfactory and furnishes no evidence that any of the basic assumptions are violated.

(d) The fitted straight line provides the predictions for any desired probability level P.* For example, the prediction for P = .995, which means a value of acceleration increment which has only one chance in 200 of being exceeded, is obtained (in Figure 4) by reading across to the solid (fitted) line at P = .995, and down to find the value x = 1.82g. This is sufficiently close to the value 1.8176 obtained by calculation, using the value $y_{.995} = 5.2958$. The 68-percent curves give a confidence interval for this value of approximately 1.66 to 1.98. This means that there

* On the probability paper (Figures 4,5), P is denoted by $\Phi(x)$.

is a probability of about two-thirds that such an interval includes the true predicted value ξ_P that we are trying to estimate. The efficiency associated with this estimated is between 80.3 percent and 82.6 percent (column 9), sufficiently narrow limits for practical purposes. If a more accurate value for the prediction or measure of efficiency is desired, it can be readily obtained by inserting a "P = .995 line" in the first table on Worksheet 2 and performing the computations indicated in columns 2 through 9.

6. COMPARISON WITH METHOD IN PRESENT USE

It is of interest to compare the proposed order-statistics method with the method of moments of E. J. Gumbel which has been used up to now in extreme gust-load computations (reference 18). The comparison is presented in two aspects — theoretical, involving an empirical attempt to evaluate the bias and efficiency* of the Gumbel estimator; and practical, showing how the two methods work out in an actual example.

6.1 Theoretical Comparison

Only the general results will be indicated here, the details being furnished in Appendix E. The comparison consists in writing down the Gumbel estimator, a function of the observations involving the sample mean, standard deviation, and the probability factor y_p, and then obtaining the bias and the relative efficiency of the proposed order-statistics estimator to the Gumbel estimator.

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* For theoretical comparison of confidence bands, see Appendix D.

Of the two characteristics bias and efficiency, the main interest at this point is in determining the efficiency of the proposed method, since that is the important feature whereby possibilities of cost savings, through taking fewer observations, can arise. Bias is less important for this purpose, and its consideration is therefore limited to the appendix (E).

As shown in Appendix E, (Section 1 *)) relative efficiency involves the first two moments of the sample mean and sample standard deviation, and the covariance of the mean and standard deviation. Of these, only the first two moments of the sample mean can be obtained readily by standard procedures, while the remaining three quantities would require a prohibitive amount of numerical integration to evaluate accurately.

* The present discussion compares the order-statistics estimator with the Gumbel estimator $\hat{\xi}_G = x + \frac{\sqrt{6}}{\pi}(y_p - \gamma) s_x$. As explained in Appendix E, this estimator is a simplified form of Gumbel's original estimator, and is used when the sample of extremes is large. Appendix E also considers the original Gumbel estimator, which is a more complicated expression used for small samples, and shows that this estimator is both more biased and much less efficient than the simplified estimator.

Resort was therefore had to a method whereby the theoretical extreme-value distribution was represented by a large set of suitably constructed random numbers. By means of these numbers a large number of actual random samples were drawn and the results tabulated. This was carried out mechanically with high-speed IBM equipment. By using 12,000 random numbers, 1200 random samples of 10 were drawn and a single average figure for relative efficiency was computed for each set of 100 samples. All these computations were made for the single probability level P = .95. Other values of P are considered below.

The results are shown in Table V and portrayed in Figure 6. For samples of 10, the efficiency was greater for the proposed order-statistics estimator in 5 cases out of 12 (relative efficiency R' (column 8) greater than 1) and greater for the present moment estimator in 7 cases out of 12. The average of all 12 relative efficiencies was very nearly unity. These results suggest that, for samples of 10, the two methods are equally efficient.

The entire procedure was repeated for samples of 20, obtaining 6 (instead of the previous 12) values for the 6 sets of 100 samples each. As Table V (column 9) and Figure 6 show, the balance now was 5 to 1 in favor of the proposed method, with the average being 1.11, representing 11 percent greater average efficiency for the proposed method.

For samples of 30, there were 4 sets of 100 samples each, and the results (column 10) were 3 to 1 in favor of the proposed method. The average relative efficiency was 1.13, representing a 13 percent gain in average efficiency.

To see the effect of different probability levels on these results, computations were undertaken for several values of P beyond .95. However, in order to avoid needless calculation, in view of the fact that only qualitative conclusions are warranted, the above procedure was modified as follows. The sets of 100 samples were combined for each sample size, and a single overall average for relative efficiency was obtained for the 1200 samples of 30, the computations being carried out for the selected probabilities P = .95, .99, .999, and the

limiting value, unity. The results are shown in Table VI. In addition, theoretical calculations* were made to obtain the asymptotic relative efficiencies as sample size increases without limit. These values will be found at the bottom of column 9 of Table VI.

The above additional results indicate that increasing the probability P tends to increase the efficiency of the proposed method relative to Gumbel's.

It should be pointed out that these values obtained from the empirical sampling method are indicative, rather than conclusive, on account of the random variation inherent in the method, as manifest in the wide fluctuation in efficiencies shown in Table V for the individual sets of 100 samples. Nevertheless, the above results do give strong indication for the following statement:

> For samples of 10, the proposed order-statistics method is about as efficient as the method of Gumbel, while for samples of 20 or 30 or more, the proposed method is more efficient.

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* Since these calculations are mainly of theoretical interest, they have been omitted in order to keep this report from becoming unduly long.

> For P = .95 or greater, this increase in efficiency is about 12 to 15 percent for samples of 20 to 30, and ultimately rises to 25 to 30 percent for indefinitely large samples.

If, in the comparison presented above, the simplified Gumbel estimator is replaced by the original form of the estimator (Appendix E, Section 2), then the comparison becomes much more favorable to the proposed order-statistics method and it can be stated that:

> For samples of 10, 20, and 30, and P = .95 or more, the order-statistics method is up to twice as efficient as the Gumbel method using the original estimator. Moreover, this 100 percent difference in efficiency between the two methods is of sufficient magnitude not to be significantly affected by the sampling errors inherent in the method of evaluation.

6.2 Comparison based on a sample of actual observations

We shall use the same data already analyzed by the order-statistics method, consisting of the 23 maximum acceleration increments listed in Worksheet 1. For convenience we shall use a standard form of worksheet, employed by the Environmental

Protection Section of the Office of the Quartermaster Genral, Department of the Army, (reference 17) for applying the method of moments of E. J. Gumbel. To avoid confusion with the Worksheets 1 and 2 discussed previously, these new worksheets shall be referred to as Table VII—Part A and Part B. The items are filled in on both parts as directed, except that the factor N/(N-1) is ignored in Sections I and IV, since subsequent theoretical investigation has shown its use to be incorrect; also, the values x_{10} and x_{100} in Section III, and the entire Section V are not needed for our purposes. The values of σ_n and $\bar{y}_n$ in Section II are taken from a table supplied with the worksheets but omitted here.

Comparison is best shown graphically, as in Figure 5. It will be seen that in this particular case the fitted lines given by the two methods are not greatly different, the predicted values difering by amounts varying from .03 g at the P = .95 level (1 chance in 20 of being exceeded) to nearly .10 g for P = .999 (1 chance in 1,000 of being exceeded).

The most striking and significant feature about the comparison in Figure 5 is the narrowness

of the confidence band for the order-statistics method compared with that of the Gumbel mthod. This is attributable mainly to that fact that in the case of the order-statistics estimator the confidence-band width is based on the standard deviation of the estimator, computed by the methods indicated in this report, whereas in the case of the moment (Gumbel) estimator, the standard deviation, whose value is not known, is replaced by a standard deviation that can be readily calculated, but which results in an unnecessarily wide confidence band (for details see Appendix D).

6.3 Advantages and Limitations of Proposed Method

From the discussion given herein it appears that the proposed order-statistics method offers the following advantages over the method of moments now in use:

a. The proposed method provides for the first time an estimator known to be unbiased, whose efficiency can be simply and accurately evaluated.

b. The new estimator is more efficient than a simplified form of the Gumbel estimator, for samples of about 20 or more and P = .95 and more. Compared with the original form of the Gumbel estimator, the new estimator is up to twice as efficient for the same range of values of P and for samples of 10 or more.

c. The calculations necessary for the proposed method are simple and unified, giving simultaneously (i) estimates of both parameters, (ii) the predicted values corresponding to assigned probabilities and the reliability of these values, and (iii) estimates of the efficiency of the method.

d. The proposed method uses a more valid procedure for obtaining the reliability of predicted values, and this procedure yields smaller confidence intervals in many cases. (See Appendix D.)

The following two limitations of the proposed method should be kept in mind:

a. As is true of any other method of analyzing data, its use is appropriate only when the assumptions upon which it is based may be considered to be approximately satisfied, namely: all the observations constitute an independent random sample from the same population $F(x) = \exp(e^{-(x-u)/\beta})$ (in cumulative form).

b. The assumption that the data are to be available in the order in which observed is of some importance. For if the data are first rearranged, grouped or processed in any manner,

> their randomness must be considered lost. In order to use the proposed method it will then be necessary to restore randomness by use of a table of random numbers to rearrange the data. This is less desirable and the original order should therefore be preserved if possible.

This necessity of avoiding preliminary processing imposes a disadvantage on the proposed method, as compared with the Gumbel method of moments, when the sample is very large (several hundred or more, say). In the latter method the data may be grouped, simplifying the computations. The method of order statistics, on the other hand, is not applicable with grouped data - each observation must be treated on an individual basis - and hence is not suitable for occasional enormous samples, as is the Gumbel method. However, for such masses of data an even simpler method, described in Appendix C, is available.

7. CONCLUDING REMARKS

This report has developed and illustrated a new method of analyzing extreme-value data based on order statistics that is simple to use and offers certain important advantages over the method of moments of Gumbel now in use, as well as being subject to certain limitations (see Section 6.3).

In view of these considerations, this new method is recommended for practical use in place of the present method of estimation.

In developing an estimator intended to be useful and efficient a number of subsidiary questions were encountered and treated. The most important of these were (i) obtaining minimum-variance unbiased linear functions of order statistics for small samples, and (ii) finding the most feasible way of breaking up a large sample into subgroups small enough to take advantage of the results in (i). In addition, considerable attention was given to a number of theoretical points of difference between the present and proposed methods.

Such theoretical study showed that one feature of the present Gumbel method, namely determination of the confidence intervals or control curves

for large values of P, does not have a suitable theoretical basis, and that certain adjustments should be made in the formulas. These adjustments would have the effect of replacing the parallel control lines by diverging curves in the regions of high values of P, resulting in smaller confidence intervals for the more common values of P and larger intervals for the higher values of P that occur less often in practice.

The solutions to the above two main auxiliary problems have been incorporated into a simple set of tables and a pair of unified worksheets designed so that the computations show at a glance the essential quantities of interest - the actual predictions, their reliability, and the efficiency of the method. The method includes provision for showing these results graphically.

The present study has also devoted some attention to a method involving empirical random sampling and IBM tabulating equipment in cases where direct numerical evaluation is prohibitive. The use of 12,000 random numbers and from 400 to 1200 random samples was found insufficient to yield

accurate quantitative results for one form of the Gumbel estimator (the simplified form) on account of sampling variation. However, definite qualitative results in favor of the proposed method were indicated in the case of samples of 20 and 30 and theoretical calculation showed that this advantage was considerably greater for indefinitely large samples.

As a result of the experience gained in these studies it seems likely that for accurate results perhaps ten times the number of samples used (or more) should be taken and the computations performed through specialized procedures on high-speed electronic computing equipment.

Further calculation showed that in the case of the original form of the Gumbel estimator, much more definite statements were possible concerning efficiency. In this comprison the proposed estimator turned out to be up to twice as efficient as that of Gumbel, not only for the sample sizes 20 and 30, but down to samples of 10 as well. For very large samples this advantage dropped somewhat, but the proposed estimator remained at least 20 to 30 percent more efficient.

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APPENDIX A

PROOF THAT SUFFICIENT STATISTICS DO NOT EXIST FOR THE PARAMETERS OF THE EXTREME-VALUE DISTRIBUTION*

(Relates to Section 4.2 of text)

<u>Problem</u>: We have a sample of n from the extreme-value population whose density function** is

$$f(x) = ae^{-a(x-u)-e^{-a(x-u)}} ;$$

$\beta = 1/\alpha > 0$ and u are unknown and we wish to find sufficient statistics for them.

<u>Theory</u>: (1) If $t = (t_1, \ldots, t_k)$ is sufficient (i.e., is a set of jointly sufficient statistics) for $\theta = (\theta_1, \ldots, \theta_m)$ then the density function of $x = (x_1, \ldots, x_n)$ may written in the form

$$P(x, \theta) = f(t, \theta)\ g(x) \quad .$$

(2) If $t(x) = t(x')$ for sample points x and x', then

$$A \equiv \frac{P(x, \theta)}{P(x', \theta)} = \frac{g(x)}{g(x')} = h(x, x') \quad .$$

* This Appendix has been prepared by I. Richard Savage of the Statistical Engineering Laboratory, National Bureau of Standards.

** For convenience the symbol a is used in place of the parameter $(1/\beta)$ of the text.

(3) Hence for all those points where $t(x)$ has a constant value the ratio A is free of θ, and thus we can find sufficient statistics by seeing for which point sets A is constant.

(4) Evidently, if $\theta = g(\theta')$ (i.e., $\theta_i = g_i(\theta'_1,\ldots,\theta'_k)$, $i = 1,\ldots,k$) is a non-singular transformation of the parameters, then also

$$\frac{P(x,\theta')}{P(x',\theta')} = \frac{f(t,\theta')\ g(x)}{f(t,\theta')\ g(x)} = h(x,x') \quad ,$$

using the same (set of estimators) t as for θ. In other words, if a set of statistics t is sufficient for a set of parameters θ, the same set t is sufficient for any other set θ' obtained from θ by a non-singular transformation.

<u>Results</u>: We will now apply the above theory to the problem at hand and show that the largest point set on which A is constant contains n! points, that is, it takes n functions to describe t, so that the resulting sufficient statistic is the trivial set, $t = (x_1,\ldots,x_n)$ or $(\frac{\Sigma x}{n}, \frac{\Sigma x^2}{n}, \ldots, \frac{\Sigma x^n}{n})$. In other words, the only sufficient statistics are the n observations themselves, so that we do not have a basis upon which to construct optimum estimators.

<u>Analysis</u>: For the distribution f(x) we have

$$A = \exp\left\{ a\left(\sum_{i=1}^{n} x_i' - \sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} \left[e^{-a(x_i-u)} - e^{-a(x_i'-u)} \right] \right\}.$$

If A is free of the parameters β and u then it is also free of a = 1/β and u, and so are log A and $\frac{\delta^k \log A}{\delta a^k}$. Hence

$$\log A = na(\bar{x}-\bar{x}') - \sum_{i=1}^{n} [e^{-a(x_i-u)} - e^{-a(x_i'-u)}] .$$

Let u approach $-\infty$. We first find that $\bar{x} = \bar{x}'$ in order to have log A free of u and a. Next,

$$\frac{\delta^k \log A}{\delta a^k} = (-1)^{k+1} \sum_{i=1}^{n} [(x_i-u)^k e^{-a(x_i-u)} - (x_i'-u)^k e^{-a(x_i'-u)}]$$
$$= 0, \ k = 2,3,\ldots,$$

and this is true for k = 1 as well, since $\bar{x} = \bar{x}'$.

Since this is an identity in u let us set u = 0. We then get

$$\sum_{i=1}^{n} x_i^k e^{-ax_i} = \sum_{i=1}^{n} x_i'^k e^{-ax_i'} .$$

These are finite sums; and therefore, since they are identitites in <u>a</u>, it is clear, since <u>a</u> may converge to zero, that

$$\frac{\Sigma x_i^k}{n} = \frac{\Sigma x_i'^k}{n} .$$

Thus the largest set of points of constancy of A are those points which give the same sample moments, and this fact implies the desired result.

<u>Note</u>: Statement (4) above implies that the result also holds if the parameter u is replaced by $\xi_p = u + \beta y_p = u + y_p/a$.

<u>Example</u>: To show how this method works for a familiar problem consider a sample of n from a normal distribution; here

$$A = e^{-1/2\sigma^2[\Sigma(x_i-\theta)^2 - \Sigma(x_i'-\theta)^2]} ,$$

$$-2\log A = [\Sigma(x_i^2-x_i'^2) - 2\theta\Sigma(x_i-x_i')]/\sigma^2 ,$$

and clearly the necessary and sufficient condition for A to be constant for all σ^2 and θ is that $\Sigma x_i^2 = \Sigma x_i'^2$ & $\Sigma x_i = \Sigma x_i'$, which is the classical result that the first two moments are sufficient statistics.

APPENDIX B

MATHEMATICAL FORMULATION AND SOLUTION OF MINIMUM-VARIANCE PROBLEM

(Relates to Section 4.3 of text)

We consider an estimator of $\hat{\xi}_P = u + \beta y_P$ of the form

$$L = \sum_{i=1}^{n} w_i x_i , \qquad (B.1)$$

where $x_1 \leq x_2 \leq \ldots \leq x_n$ are the n order statistics of a sample of n from the extreme-value distribution (4.1), and seek to find the w_i which minimize Var(L) subject to

$$E(L) = \xi_P . \qquad (B.2)$$

The estimator L in (B.8) below with weights so determined is called the minimum-variance, unbiased, (linear) order-statistics estimator for sample size n.

Writing

$$x = u + \beta y , \qquad (B.3)$$

where y is the reduced variable corresponding to x, we also have

$$x_i = u + \beta y_i \qquad (B.4)$$

where $y_1 \leq y_2 \leq \ldots \leq y_n$ are the n order statistics of a sample of size n from the reduced distribution

$\exp(-e^{-y})$, free of parameters. From the above it follows that

$$E(x_i) = u + \beta E(y_i) \quad , \tag{B.5}$$

since u and β, though unknown, are constants not subject to sampling variation when the operation of expectation is performed. The values $E(y_i)$ have been tabulated in reference 14 for $i = n(1)\min(1,n-25)$, $n = 1(1)10(5)60(10)100$*.

These results give readily

$$E(L) = \Sigma w_i (u+\beta E y_i) = \xi_p = u + \beta y_p \quad . \tag{B.6}$$

This is required to be an identity for all values of the parameters u, β. Equating their coefficients gives the two conditions on the weights w_i :

$$\sum_{i=1}^{n} w_i = 1$$
$$\sum_{i=1}^{n} (Ey_i) w_i = y_p \quad , \tag{B.7}$$

where Ey_i are the numerical values tabulated in reference 14.

* The notation in the table cited differs from that used here: $E(y_i)$ in this report corresponds to $E(y_{n-i})$ in the table.

Turning to the variance, we have

$$\mathrm{Var}(L) = \sum_{i=1}^{n} w_i^2 \sigma_{x_i}^2 + \sum_{\substack{j=1 \\ i \neq j}}^{n} \sum_{i=1}^{n} w_i w_j \sigma_{x_i x_j} \quad .$$

From (B.4) and the properties of the variances and covariances of linear estimators we have

$$\sigma_{x_i}^2 = \beta^2 \sigma_{y_i}^2 = \beta^2 \sigma_i^2 , \qquad \sigma_{x_i x_j} = \beta^2 \sigma_{y_i y_j} = \beta^2 \sigma_{ij} \quad ,$$

making an obvious simplification in notation, whence

$$V_n = \mathrm{Var}(L) = (\Sigma \sigma_i^2 w_i^2 + \Sigma\Sigma' \sigma_{ij} w_i w_j)\beta^2$$

$$= \text{minimum subject to (B.7).} \qquad \text{(B.8)}$$

This is a constrained minimum problem for variation in the unknown w_i, and is equivalent to finding the (unconstrained) minimum of*

$$G_1 = (\Sigma \sigma_i^2 w_i^2 + \Sigma\Sigma' \sigma_{ij} w_i w_j)\beta^2 + \lambda_1 (\Sigma w_i - 1) + \mu_1 [\Sigma (Ey_i) w_i - y_P],$$

where λ_1, μ_1 are the Lagrange multipliers. Since $\beta^2 > 0$ is constant, though unknown, this is the same as minimizing

$$G = \frac{G_1}{\beta^2} = \Sigma \sigma_i^2 w_i^2 + \Sigma\Sigma' \sigma_{ij} w_i w_j + \lambda (\Sigma w_i - 1)$$

$$+ \mu [\Sigma (Ey_i) w_i - y_P] \quad ,$$

* The temporary notation μ, μ_1 should not be confused with the symbols for moments.

where $\lambda = \lambda_1/\beta^2$, $\mu = \mu_1/\beta^2$. Setting the derivatives with respect to w_k, $k = 1, 2, \ldots, n$, equal to 0 and dividing by 2, we have

$$\sigma_k^2 w_k + \sum_{\substack{i=1 \\ (i \neq k)}}^{n}{}' \sigma_{ik} w_i + \lambda + \mu E y_k = 0, \; k = 1, 2, \ldots, n. \quad (B.9)$$

These latter are n linear equations which, with the two in (B.7), form a simultaneous system of (n+2) equations in the (n+2) unknowns w_1, w_2, ..., w_n, λ, μ. The values of λ and μ are useful as a check, since if (B.9) is multiplied by w_k and summed, the result, in view of (B.7) which the w_i's satisfy and (B.8) is

$$V_{n,min} + \lambda + \mu y_p = 0 \quad ,$$

that is, we should have

$$V_{n,min} = -\lambda - \mu y_p \quad .$$

The minimum value $V_{n,min}$ will be denoted by Q_n.

Before solving the set (B.7), (B.9) it is necessary to determine the coefficients in these linear equations. The values of Ey_k are tabulated, as already mentioned. The variances and covariances σ_k^2, σ_{ik} involve complicated integrals. The author has been successful in expressing these integrals

in terms of simpler ones already tabulated (reference 12). The results are, for the variances,*

$$\sigma_i^2 = E(y_i^2) - (Ey_i)^2$$

$$E(y_i^2) = \frac{n!}{(i-1)!(n-i)!} \sum_{r=0}^{n-i} (-1) C_r^{n-i} g_2(i+r),$$

$$i = 1, 2, \ldots, n,$$

where

$$g_2(i+r) = \frac{1}{i+r} \left[\frac{\pi^2}{6} + (\gamma + \log \overline{i+r})^2 \right] ,$$

and γ = Euler's constant = .57721 56649...; and for the covariances,

$$\sigma_{ij} = E(y_i y_j) - (Ey_i)(Ey_j) ,$$

$$E(y_i y_i) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r=0}^{j-i-1} \sum_{s=0}^{n-j} (-1)^{r+s}$$

$$\cdot C_r^{j-i-1} C_s^{n-j} \phi(i+r, j-i-r+s),$$

$$i < j \; ; \; i, j = 1, 2, \ldots, n,$$

where the function ϕ is defined by

$$2tu\phi(t,u) = (u-t) g_2(t+u) + t^2 [g_1(t)]^2 - 2L(1+\frac{u}{t}) + \frac{\pi^2}{6} ,$$

in which g_2 is the same function as before,

$g_1(t) = \frac{1}{t}(\gamma + \log t)$, and

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* All logarithms are natural logarithms, to the base e.

$$L(1+x) = \int_1^{1+x} \frac{\log w}{w-1}\, dw = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} x^n$$

$$= \tfrac{1}{2}(\log x)^2 + \frac{\pi^2}{6} - L\left(1+\frac{1}{x}\right)$$

is Spence's integral, which has been most extensively tabulated (to 12 places) in reference 16. The function g_1 also occurs in an expression for the means:

$$E(y_i) = \frac{n!}{(i-1)!\,(n-i)!} \sum_{r=0}^{n-1} (-1)^r C_r^{\,n-1} g_1(i+r) \quad .$$

The above formulas have been evaluated as far as n = 6 and the results are listed in Table II. The values in the table are believed to be accurate to the number of places shown. Those for the means agree (to within a unit in the 7^{th} place) to the 7 places to which the means have previously been tabulated.

Table II thus provides the coefficients in the system of equations (B.7), (B.9) in the w_i and λ, μ. The right-hand sides of these (n+2) equations are 1, y_P, 0, ..., 0 and the solutions w_i, λ, μ are linear combinations of these with numerical coefficients which involve only σ_i^2, σ_{ij}, and Ey_i, but not y_P. Hence the solutions are all of the form

$$w_i = a_i + b_i y_P \quad , \quad i = 1, 2, \ldots, n$$

$$\lambda = c_1 + d_1 y_P$$

$$\mu = c_2 + d_2 y_P \quad .$$

Substituting these w_i in (B.8) we have an expression of the form

$$Q_n = V_{n,min} = (A_n y_P^2 + B_n y_P + C_n)\beta^2 \quad . \qquad (B.10)$$

The quantities a_i, b_i for the weights w_i, and the coefficients A_n, B_n, C_n, of Q_n, are given in Table I for n = 2 to 6. The solution of the system of equations became increasingly lengthy for increasing values of n, with correspondingly diminishing accuracy, so that the computations were discontinued beyond n = 6. The procedures for handling samples larger than n = 6 are explained in the text of this report.

APPENDIX C

SHORT-CUT METHOD FOR VERY LARGE SAMPLES

(Relates to Section 4.4 of text)

If we have a sample of several hundred or more extreme observations, as may sometimes be the case (e.g. reference 18, where a sample of 485 extremes was analyzed) it is possible to select just three out of all the observations and from them obtain useful estimators.

This technique is based on a method used by F. Mosteller (reference 13) for samples from the normal distribution. If the n sample values from a (continuous) population whose density is f(x) when arranged in ascending order are denoted by the order statistics $x_1, x_2, \ldots, x_n$, and n is very large, the application of Mosteller's method involves taking the observations whose ranks are λn, μn, νn, where $0 < \lambda < \mu < \nu < 1$ with λ, μ, ν suitably determined, and choosing a and b so that*

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* When (as will generally be the case) the ranks λn, μn, νn are not integers, they will be defined to be the nearest integers to these quantities.

$$\hat{\xi} = ax_{\mu n} + b(x_{\nu n} - x_{\lambda n}) \qquad (C.1)$$

is an (asymptotically) unbiased estimator of the parameter $\xi_p = u + \beta y_p$. (The reason for choosing this particular form is discussed below).

The mean and variance of the estimator $\hat{\xi}$ in (C.1) are computed from the corresponding moments of order statistics of the form $x_{\lambda n}$, with n very large and λ a proper fraction not too near 0 or 1. Under these circumstances the theorem used by Mosteller states that in the limit, as n increases indefinitely,

(i) $x_{\lambda n}$ becomes normally distributed, with mean and variance

$$E(x_{\lambda n}) = t_\lambda \qquad (C.2)$$

$$\sigma^2(x_{\lambda n}) = \frac{\lambda(1-\lambda)}{n[f(t_\lambda)]^2}, \qquad (C.3)$$

where t_λ is defined by $\lambda = \int_{-\infty}^{t_\lambda} f(x)dx$; and

(ii) the covariance of any two order statistics $x_{\lambda n}$ and $x_{\mu n}$, $\lambda < \mu$, is given by

$$\operatorname{cov}(x_{\lambda n}, x_{\mu n}) = \frac{\lambda(1-\mu)}{nf(t_\lambda)f(t_\mu)}, \qquad (C.4)$$

where t_μ is defined similarly to t_λ.

For $\hat{\xi}$ in (C.1) to be unbiased in the case where f(x) is the extreme value distribution,

$$E(\hat{\xi}) = \xi_P \equiv u + \beta y_P \qquad (C.5)$$

must be an identity in u, β. We first note that from previous discussion in the text (see Section 4.1), the parameter ξ_P is precisely the abscissa of the ordinate which cuts off the area P to the left. Hence we have simply

$$t_\lambda = \xi_\lambda = u + \beta y_\lambda \ . \qquad (C.6)$$

Equations (C.1), (C.2), and (C.5) then give

$$at_\mu + b(t_\nu - t_\lambda) = u + \beta y_P$$

or

$$a(u + \beta y_\mu) + b(y_\nu - y_\lambda)\beta \equiv u + y_P\beta \ ,$$

from which, on equating coefficients of u and β,

$$a = 1, \ b = \frac{y_P - y_\mu}{y_\nu - y_\lambda} \ . \qquad (C.7)$$

In principle, the fractions λ, μ, ν might be determined so as to minimize the variance of $\hat{\xi}$ and thus make its efficiency a maximum, but this would require very extensive computation which would not be warranted on account of the limited importance of efficiency when the available sample is very large. (For example, a 50 percent efficient estimator with a sample of

1,000 gives results equivalent to using a sample of 500 — still a ~~sufficiently~~ very large sample.*) Instead we consider estimators of ξ_p of the form

$$\hat{\xi} = \hat{u} + y_p\hat{\beta} \quad , \qquad (C.8)$$

where $\hat{u}$, $\hat{\beta}$ are estimators of the two parameters u, β, that involve the fewest possible number of order statistics x_{sn} without undue sacrifice in efficiency as computed for idenfinitely large samples.. The aim is to find, with a minimum amount of computation, separate unbiased estimators $\hat{u}$, $\hat{\beta}$, of the parameters u, β, each of which has minimum variance or best efficiency in some sense, in the hope that the linear combination (C.8), which will also be unbiased, will turn out to have efficiency which is not unreasonably small. This is a heuristic method, since the fact that $\hat{u}$ and $\hat{B}$ are efficient does not imply that their

* These considerations assume that the sample of data is already at hand, perhaps by a survey already made, such as the Thunderstorm Project mentioned in reference 18. Of course, if it is a question of planning for the securing of data, it is desirable to use as efficient an estimator as possible, but in that case, the investigation will rarely be sufficiently extensive to provide samples large enough for the method described in this Appendix to be applicable.

combination $\hat{u} + y_p\hat{\beta}$ is efficient. Much better estimators probably exist, but we are content to obtain just one of reasonable efficiency.

It turns out that the modal parameter u can be estimated by a single order statistic. E. J. Gumbel has shown (reference 4, equation (50)) that the value of u for which $x_{\mu n}$ best (i.e., with least variance or most efficiency) estimates u is $\mu = .20319$. For simplicity, we therefore replace u in (C.8) by

$$\hat{u} = x_{.20n} \quad . \tag{C.9}$$

The scale parameter β requires at least two order statistics, or rather their difference $x_{\nu n} - x_{\lambda n}$, for estimation, multiplied by a suitable unbiasing factor which will become absorbed in the expression for b in (C.7). A considerable number of trials indicates that the pair of values $\lambda = .03$, $\nu = .85$ gives an estimate of β with efficiency probably close to the maximum, if not actually maximum. Since we are not seeking very precise results, this pair of values is adopted here. Thus (C.1), in view of (C.7), becomes

$$\hat{\xi} = x_{.20n} + 0.3256(y_p+0.4759)(x_{.85n}-x_{.03n}). \tag{C.10}$$

The variance of this estimator is obtained from the

rule

$$\mathrm{Var}\left(\sum_{i=1}^{m} a_i x_i\right) = \sum_{i=1}^{m} a_i^2 \sigma_{x_i}^2 + 2 \cdot \sum_{\substack{i,j=1 \\ i<j}}^{m} a_i a_j \,\mathrm{cov}(x_i, x_j),$$

which after simplification gives

$$\sigma^2(\hat{\xi}) = 8.6916\, d^2 - 0.0681\, d + 1.5442 \,, \quad \text{(C.11)}$$

where

$$d = 0.3256\, y_P + 0.1549 \,.$$

Since $\hat{\xi}$ is unbiased, a measure of its efficiency may be obtained by dividing its variance into the Cramér-Rao lower-bound, Q_{LB} (see equation (4.19) and accompanying text; numerical values are given in the Q_o column of Table III, Part A. The results are as follows, for several values of P of interest:

P	Efficiency of $\hat{\xi}$
.95	.645
.99	.649
.999	.652
1(limiting value)	.660

Thus, this large-sample method of estimation is slightly less than two-thirds efficient. However, as noted above, such apparently low efficiency need not be a serious matter in practice.

For convenience, a summary of the method described above is given here.

Summary of Large-Sample Procedures

1. Arrange all n observations (assumed to be independent and from the same extreme-value distribution) in order of increasing size, and then rank them from 1 to n.
2. By hand or mechanical sorting, select the three observations x_r whose ranks are the nearest integers to .03n, .20n, and .85n. Denote these by $x_{.03n}$, $x_{.20n}$, $x_{.85n}$.
3. Compute the predicted values $\hat{\xi}$, for various probability levels P, by formula (C.10).
4. For each P compute the variance from formula (C.11).
5. Take the square root of the variance to obtain the standard deviation. This gives the half-width of the 68-percent confidence band, since for large samples the distribution of $\hat{\xi}$ approaches normality. Similarly, twice the standard deviation determines the 95 percent confidence band, and 2.58 standard deviations determines the 99 percent band.
6. Obtain the efficiencies by dividing the variance into the Cramér-Rao lower bound Q_0 in Table III, Part A.

APPENDIX D

ANALYSIS OF CONFIDENCE INTERVALS IN ORDER-STATISTICS METHOD AND METHOD OF MOMENTS OF GUMBEL

(Related to Section 5, Rule 5, and Section 6.2 of text)

1. Confidence intervals in order-statistics method (based on normality assumption).

In the text, Section 5, Rule 5, the confidence intervals given for various confidence levels in the proposed method are obtained by laying off a certain number of standard deviations, computed for the estimator $\hat{\xi}_P$, on either side of the estimated value given by the fitted line. If this is done for different values of P and the ends joined, as in Figure 4, a confidence band is obtained. The number of standard deviations given in the method — one for a confidence level of 68 percent, two for a level of 95 percent — is based on the assumption that the estimator $\hat{\xi}_P$ is normally distributed. The purpose of this section is to investigate this assumption more closely.

It will be recalled that the estimator $\hat{\xi}_P$ is obtained by splitting the sample into a number of equal groups with perhaps a remainder of different size (see text in connection with equations (4.22), (4.25) and (4.26)). Then $\hat{\xi}_P$ can be written as a function

(4.26)

$$\hat{\xi}_p = t\,\overline{T} + t' \cdot T',$$

where $\overline{T}$ is the average of a certain linear function of the sample variables (equation (4.22)) taken over the k subgroups, T' is another linear function, and t, t' are constants. Thus $\hat{\xi}_p$ is the sum of two parts: (1) an average of k independent random variables (tT_i)* all with the same distribution, and (2) a single variable $(t'T')$ with a somewhat different distribution. By the Central Limit Theorem in probability (reference 1, page 215), according to which the average of a number of random variables having the same distribution (with first two moments existing) is asymptotically normal as the number of variables increases indefinitely, the first part is approximately normal for large k. In fact, extensive experience has shown that a normal distribution is often a remarkably close approximation even if the number of variables k is under 10. Furthermore, the first two moments (actually all) of each variable T_i certainly exist — in fact the proposed method is based upon

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* These variables are independent because the subgroups were assumed to be formed independently.

their computed values. Hence, it is safe to say that for k = 10 or more the first part is very closely normal. The second part (t'T') is a variable which has the same general character as T_i (a weighted sum of order statistics [equation (4.25)]) and hence is believed not to impair significantly the approximate normality of $\hat{\xi}_p$. Its influence is likely to be small, especially if the number of other variables, k, is large.

For samples as large as 100, k = 16 if broken into subgroups of 6, or k = 20 if broken into subgroups of 5. Since these values of k are considerably larger than 10, the preceding discussion shows that it is quite safe to assume normality for $\hat{\xi}_p$ for samples of 100 or more, so that the corresponding multiples of the standard deviation given above are sufficiently accurate in such cases. In fact, it is likely that the normal approximation remains good for practical purposes down to samples of 50 or 60, becoming, of course, worse as sample size decreases still further. However, in the absence of knowledge about the exact distribution of the order-statistics estimator $\hat{\xi}_p$ for smaller samples, the normal approximati[illegible]

apparently the only simple one available for determing confidence limits. It may be noted that approximate methods are also involved in determination of confidence limits in the Gumbel method. This point is further discussed in the following section.

A comparison of the confidence intervals (or bands) for predicted extremes in the order-statistics method and in the Gumbel method is of interest and is presented in Table VIII, columns 2 and 4, 5 and 7, 8 and 10. It is to be noted that for prediction probability P not beyond .99 the method of order-statistics gives an appreciably narrower band for sample sizes 20 and 30, and a significantly wider one for samples of 10. On the other hand, for values of P much beyond .99, the order statistics band widens very rapidly, as compared to a constant width for the band in Gumbel's method. However, there appears to be some question as to the theoretical validity of the Gumbel confidence-band width for large values of P. This point is also considered in the following section.

2. Confidence intervals for largest extremes in Gumbel method.

2.1 Gumbel's derivation of confidence intervals.

The purpose of this section is to inquire into the theoretical validity of the confidence intervals (or confidence band) given for extreme predictions in Gumbel's method.

In the Gumbel method the 68-percent confidence-interval half-width for the largest in a sample of n extremes and for all larger predicted values* is, in Gumbel's notation (Table VII, Part B, Section IV)

$$\Delta_{x,n} = \frac{1.141}{\alpha} = 1.141\beta; \ \beta = 1/\alpha , \qquad (D.1)$$

where β is the scale parameter (or rather, an estimate of it) of the extreme-value distribution from which the observations are assumed to come. To obtain the confidence interval for a given prediction probability $P \geqq n/(n+1)$, the value $\Delta_{x,n}$ is added to, and subtracted from, the estimate given by Gumbel, denoted by him by x (Table VII, Part B, Section III)

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* That is, for all values of P beyond n/(n+1), which is the probability assigned to the largest value in the sample, x_n. For smaller P, the confidence interval is given by a different method with which we shall not be concerned inasmuch as the primary interest is in large values of P corresponding to extreme predictions.

and in this report by $\hat{\xi}_G$ Gumbel's (68 percent) confidence interval for predictions beyond the largest observed extreme x_n is thus given by

$$\hat{\xi}_G \pm 1.141\beta, \qquad (D.2)$$

where β is the scale parameter (or an estimate thereof) of the extreme-value population from which the observed extremes x have been assumed to come:*

$$F(x) = \Phi(y) = \exp(-e^{-y}), \; y = (x-u)/\beta \; . \qquad (D.3)$$

The multipler 1.141 used for the 68 percent confidence band is obtained by setting C = .68 and solving for y the equation

$$\Phi(y) - \Phi(-y) = C \qquad (D.4)$$

which is parameter-free, and gives y(C) = y(.68) = 1.14073 (reference 5, page 6). Thus

$$y = -1.14073 \text{ to } y = 1.14073 \qquad (D.5)$$

is the interval for the reduced variate that cuts off (or corresponds to) a central area of .68 under the extreme-value density curve shown in Figure 2. The corresponding interval that cuts off the same

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* From the theory of extreme values the distribution of the largest of the observed values, x_n, in a sample of n extremes, is exactly an extreme-value distribution that has the same scale parameter β.

area under the original ("unreduced") x-distribution thus has width given by the values (D.5) multiplied by the scale factor β, since

$$x = u + y\beta \quad .$$

The half-width is therefore 1.14073 β, i.e., (D.1).

The following discussion indicates that this method of obtaining confidence intervals is inaccurate in two respects: (1) the confidence interval is of constant instead of increasing width for large P; (2) the scale parameter used is inappropriate.

2.2 Constant width of confidence interval

The above method of Gumbel of obtaining confidence intervals (D.2) treats the estimator $\hat{\xi}_G$ as though it has an extreme-value distribution with the same scale parameter β as in the population underlying the observed extremes x_m (including the largest extreme x_n). This assumption cannot be considered strictly valid since it implies that the confidence width remains constant for all large values of P, as (D.1) does not involve P. In other words, this asserts that from a sample of 20 observations or even 10, for example, we can make

statements about events that will occur with probability 1 in a million or billion yet have the <u>same</u> uncertainty of only $\Delta_{x,n}$ in our estimate for x as for predictions about events with probability, say, 1 in 100. It does not seem reasonable that a limited sample can tell anything at all meaningful about such extremely rare events, let alone predict them with the same amount of uncertainty no matter what the probability of occurrence.

This lack of agreement with common sense indicates that the Gumbel estimator $\hat{\xi}_G = \hat{u} + y_P\hat{\beta}$ cannot be treated, for all large values of P, as if it has an extreme-value distribution with constant scale parameter.

Besides these common-sense considerations, there is another reason why the extreme-value distribution is not appropriate for the Gumbel estimator, at least for large samples of data. The estimator is a sample characteristic of the form

$$\hat{\xi}_G = \bar{x} + k_P\, s_x \quad , \qquad (D.6)$$

where k is a constant for given values of P and n. The appropriate distribution of such an expression is given by a general limit theorem in probability (reference 1, page 367) to the effect that under broad

conditions any sample characteristic based on moments (such as $\hat{\xi}_G$) is, for large values of n, approximately normally distributed. Thus for large values of n the Gumbel estimator (D.6) should be considered to be approximately normal, with variance given by an expression which increases as k_P^2. Moreover, this would yield a confidence band that diverges with increasing P, avoiding the difficulty of the parallel curves mentioned above.

2.3 Inappropriate scale parameter

Little is known about the exact distribution of the Gumbel estimator $\hat{\xi}_G$, particularly for small sample sizes. Yet even if it were an extreme-value distribution (of the form (D.3)), it would seem that its scale parameter would not be β, but a certain multiple of it, B_P, found below. This multiple may be determined by considering the relation between the variance of the distribution (assumed extreme-value) of $\hat{\xi}_G$ and the scale parameter β_1 of this distribution:

$$\sigma^2(\hat{\xi}_G) = \frac{\pi^2}{6} \beta_1^2 \quad . \qquad (D.7)$$

But we have an approximate expression for the left side, namely, (E.6) in Appendix E below. This is of the form

$$\sigma^2(\hat{\xi}_G) = q(y_P)\ \beta^2 \qquad (D.8)$$

where β is the scale parameter of the original (extreme-value) x-distribution, and $q(y_P)$ is a quadratic expression in the probability factor y_P with coefficients involving the quantities $\sigma^2(s)$ and $\mathrm{cov}(\overline{y}, s)$, whose computation by empirical sampling is indicated in Appendix E; $q(y_P)$ may be regarded as a known value, S_P, depending on P. Hence

$$\sigma^2(\hat{\xi}_G) = S_P^2\ \beta^2 \quad . \qquad (D.9)$$

Substituting in (D.7) gives

$$\beta_1 = (\frac{\sqrt{6}}{\pi}\ S_P)\ \beta = B_P\beta \ , \qquad (D.10)$$

which defines the multiple B_P. Thus the confidence-interval half-width (D.1) must be replaced by

$$\Delta' = 1.141\ B_P\beta \quad , \qquad (D.11)$$

where now Δ' is no longer constant with P, but on account of B_P, actually increases very rapidly for large values of y_P corresponding to values of P near 1. Thus we obtain a modified confidence band whose divergence states that the amount of uncertainty increases without limit as we attemp

estimate increasingly improbable events. This also avoids the conflict with common sense mentioned in Section 2.2.

The actual values of B_P are of interest and are given in the following table for several important values of P and for the three sample sizes for which they were computed in Appendix E.

$$B_P = \frac{\sqrt{6}}{\pi} s_P = \frac{\sqrt{6}}{\pi} [\sigma^2(\xi_G)/\beta^2]^{\frac{1}{2}}$$

P	n = 10	n = 20	n = 30
.95	.749	.560	.458
.99	1.093	.825	.673
.999	1.593	1.208	.986

In this table the values of B_P less than 1 indicate that the modified confidence band (equation (D.11)) is better (i.e. narrower) than the Gumbel confidence band and vice versa for the values of B_P greater than 1. Thus, the modified band is indicated to be considerably better in the region P = .95 to .99 for samples of 20 and 30. For samples of 10, the advantage is less at P = .95 and becomes reversed in favor of the original Gumbel confidence band at P = .99 .

The above comparison remains exactly the same for any other confidence level, it being merely necessary to replace 1.41 in equations (D.1) and (D.11) by the corresponding value y(C) determined from (D.4). Thus, for the 95 percent level, y(.95) = 3.06685 (reference 6, Lecture 3, Table 3.1). At each level the confidence intervals of the two methods are affected in the same ratio by such multipliers, that is, their ratio to each other remains B_P, regardless of confidence level C.

3. Comparison of confidence intervals in the Gumbel method and the method of order statistics

Table VIII shows the actual confidence intervals (in terms of the scale parameter β) for the two levels C = .68 and C = .95 for the Gumbel method and as modified by the factor B_P, and also compares these (where applicable) with the intervals given by the order-statistics method. Except for samples of 10, for which the Gumbel interval is apt to be narrower, the modification denoted by B_P, discussed in the previous section, reduces the interval-width for P = .99 (and less) by significant amounts — by about one-sixth or more for samples of 20 (columns 5, 6) and by

about one-third or more for samples of 30 (columns 8, 9). These results are of course implied by the values of B_p, given in the preceding section. Also, as noted in Section 1 above, the order-statistics confidence interval is narrower than the (unmodified) Gumbel interval in many cases, for P not beyond .99, and sample size not below 20. However, it increases beyond the constant Gumbel width for larger probabilities, in agreement with theoretical requirements. At P = .99 or less, there are two additional features to be noted. (1) With increasing confidence level, the numerical factor in the Gumbel interval $\Delta_{x,n}$ increases faster in either the modified interval Δ' or in the order-statistics interval (denoted by Δ_o in Table VIII), so that both the modified method and the order-statistics method reduce the confidence interval of the Gumbel method by constantly increasing percentages as the confidence level increases. For example, for P = .99 and for samples of 20 the order-statistics interval is about 11 percent narrower than the Gumbel interval for a confidence level of 68 percent and about 30 percent narrower for a level of 95 percent (columns 5, 7). (2) Similarly, the percentage

reduction increases with sample size. Thus, for P = .99 and a confidence level of 68 percent, the reductions are 11 percent for samples of 20, and 29 percent for samples of 30 (columns 8, 10).

APPENDIX E

DETAILS OF THEORETICAL COMPARISON BETWEEN ORDER-STATISTICS ESTIMATOR AND MOMENT ESTIMATOR OF GUMBEL

(Related to Section 6.1 of text)

Since the order-statistics estimator has been fully discussed in the text, the remaining problem in making the above comparison is, essentially, to develop the characteristics of the Gumbel estimator.

The method of moments of Gumbel in present use provides the following estimators for the parameters u, β (reference 18, page 11, equations (26), (27); also reference 5, page 10, equation (29), but read $(-\bar{y}_n/\alpha)$ for $(+\bar{y}_n/\alpha)$):

$$\hat{u} = \bar{x} - \frac{\bar{y}_n}{\sigma_n} s_x , \quad \hat{\beta} = \frac{s_x}{\sigma_n} , \tag{E.1}$$

where $\bar{x}$, s_x are the mean and standard deviation of the given sample of size n; $\bar{y}_n$ is a certain computed quantity, depending on the sample size n, which approaches Euler's constant $\gamma = .5772...$ from below as n becomes infinite; and σ_n is another computed quantity, depending on n, which approaches $\pi/\sqrt{6} = 1.28255...$ from below as n becomes infinite.

For sufficiently large samples the quantities $\bar{y}_n$ and σ_n may be replaced by their limiting values*. This gives the somewhat simpler estimators, for computation purposes

$$\hat{u}' = \bar{x} - \frac{\sqrt{6}}{\pi} \gamma s_x, \quad \hat{\beta}' = \frac{\sqrt{6}}{\pi} s_x \quad . \tag{E.2}$$

It is shown below that the net effect of this simplification is to diminish the bias and to greatly understate the relative efficiency of the order-statistics estimator to the Gumbel estimator. Since the asymptotic form (E.2) involves simpler notation and is occasionally used in practice, it has seemed desirable to present this case in detail below (Section 1) and also in the main text. The corresponding results for the original form (E.1) are indicated in Section 2 below and tabulated in Table VI.

1. Comparison with simplified Gumbel estimator

From the estimators (E.2) the following estimator of ξ_p can be built up, which will be denoted by $\hat{\xi}_G$:

* This has been done, e.g., in reference 3, pp. 181 ff. and 188, and in reference 18, p. 10.

$$\hat{\xi}_G = \hat{u}' + \hat{\beta}' y_P = \bar{x} + (y_P - \gamma) \frac{\sqrt{6}}{\pi} s_x \quad . \qquad (E.3)$$

This is a function of the n sample values $x_1, x_2, \ldots, x_n$ and it is desired to find its mean and variance, and thence its bias and efficiency.

The mean is

$$E(\hat{\xi}_G) = u + \gamma\beta + (y_P - \gamma) \frac{\sqrt{6}}{\pi} E(s)\beta \quad ,$$

which can be rearranged to give

$$E(\hat{\xi}_G) = \xi_P + [\frac{\sqrt{6}}{\pi} E(s) - 1](y_P - \gamma)\beta \quad , \qquad (E.4)$$

where $\xi_P = u + y_P\beta$, $E(\bar{x}) = u + \gamma\beta$, and $E(s)$ is the expected value of the sample standard deviation, s, when the sample is from the "reduced" extreme-value distribution $\exp(-e^{-y})$. Equation (E.4) shows that the Gumbel estimator is biased,* (unless $E(s) = \pi/\sqrt{6}$ for all sample sizes, which seems highly unlikely), with bias

* An unbiased estimator analogous to $\hat{\xi}_G$ is

$$\hat{\xi}_o = \bar{x} + (y_P - \gamma) s_x / E(s),$$

for, as in equation (E.4)

$$E(\hat{\xi}_o) = u + \gamma\beta + (y_P - \gamma) E(s)\beta/E(s)$$

$$= u + \beta y_P = \xi_P \quad .$$

However, this estimator could not be used in an actual problem since E(s) is not known. Computation of this quantity was one of the aims of the IBM computing procedures discussed in the text.

$$b(\hat{\xi}_G) = E(\hat{\xi}_G) - \xi_P = [\frac{\sqrt{6}}{\pi} E(s)-1]\ (y_P-\gamma)\beta\ . \qquad (E.5)$$

<u>The variance</u> of the estimator $\hat{\xi}_G$ is

$$\sigma^2(\hat{\xi}_G) = \sigma^2_{\bar{x}} + \frac{6}{\pi^2}(y_P-\gamma)^2\sigma^2(s_x) + 2(y_P-\gamma)\frac{\sqrt{6}}{\pi}\operatorname{cov}(\bar{x},s_x)$$

$$= [\frac{\pi^2}{6n} + \frac{6}{\pi^2}(y_P-\gamma)^2\sigma^2(s) + 2(y_P-\gamma)\frac{\sqrt{6}}{\pi}\operatorname{cov}(\bar{y},s)]\beta^2, \qquad (E.6)$$

where $\sigma^2_{\bar{x}} = \frac{\pi^2}{6n}$; $\sigma^2(s)$ is the variance of the sample standard deviation for samples from the reduced distribution $\exp(-e^{-y})$; and $\operatorname{cov}(\bar{y},s)$ is the covariance of the mean and standard deviation in such samples.

The efficiency of $\hat{\xi}_G$ could be evaluated by suitable generalization of equation (4.19) to biased estimators. The variance Q_n in the denominator would be replaced by the MSE (mean square error).* The numerator would have to be replaced by a complicated expression which, for unbiased estimators, would reduce to Q_{LB}. Instead of evaluating efficiency for the biased estimator $\hat{\xi}_G$, therefore, the discussion will be greatly simplified by limiting it to relative efficiency. The <u>relative efficiency</u> of one estimator

* For discussion of mean square error see equation (4.13) and accompanying text.

(T_1) to another (T_2) is defined as the ratio of mean square errors,

$$R\ (T_1, T_2) = \frac{\text{MSE}\ (T_2)}{\text{MSE}\ (T_1)} \quad . \tag{E.7}$$

This ratio has been used as an index of comparison of two estimators (e.g. reference 7). Thus, the relative efficiency of the order-statistics estimator $\hat{\xi}_P$ to the Gumbel estimator $\hat{\xi}_G$ is, by (4.13) and the fact that the former estimator is unbiased,

$$R(\hat{\xi}_P, \hat{\xi}_G) = \frac{\text{MSE}\ (\hat{\xi}_G)}{\text{MSE}\ (\hat{\xi}_P)} = \frac{\sigma^2(\hat{\xi}_G) + (\text{bias}(\hat{\xi}_G))^2}{\sigma^2(\hat{\xi}_P)}$$

$$= \frac{(\text{E.6}) + (\text{E.5})^2}{(1/k)Q_m} \quad , \tag{E.8}$$

where k is the number of subgroups of size m into which the sample of n is partitioned** (equation (4.23), assuming there is no remainder subgroup), and the expressions needed for the numerator are given by the equation numbers indicated.

The key quantities needed in the calculation of relative efficiencies are, from equations (E.5) and (E.6), E(s), $\sigma^2(s)$, and $\text{cov}(\bar{y}, s)$. For general

** Thus, n = 10 = 2 x 5 gives k = 2, m = 5; n = 20 = 4 x 5 gives k = 4, m = 5; n = 30=5 x 6 gives k = 5, m = 6.

sample size n, their exact values are given by multiple integrals whose evaluation would apparently require a prohibitive amount of labor. Instead, the following method of empirical sampling was used with the aid of IBM calculating and tabulating equipment.

The universe of (reduced) extreme values $\Phi(y) = \exp(-e^{-y})$ was approximated by constructing a population of 12,000 suitable random numbers and punching each number on an IBM punch card. These were then mechanically separated into 1200 random samples of size n = 10 and for each sample the mean $\bar{y}$, standard deviation s, and their product $\bar{y}s$ were obtained. This was equivalent to having a "population" of 1200 means, one of 1200 standard deviations, and one of 1200 products of the mean and standard deviation. It was then assumed that the arithmetic mean of each of the three populations would be a close approximation to the mathematical expectations (averages) of the desired quantities, so that these approximations could be taken as estimates of the moments E(s), $E(\bar{y}s)$. From these values, and the relation

$$E(s^2) = \frac{n-1}{n}\sigma_y^2 = \frac{n-1}{n}\frac{\pi^2}{6} ,$$

the variance

$$\sigma^2(s) = E(s^2) - [E(s)]^2 = \frac{n-1}{n}\frac{\pi^2}{6} - [E(s)]^2$$

was computed, and also the covariance

$$\text{cov}(\bar{y},s) = E(\bar{y}s) - E(\bar{y})E(s) = E(\bar{y}s) - \gamma E(s).$$

The five quantities $E(\bar{y})$, $\sigma^2(\bar{y})$, $E(s)$, $\sigma^2(s)$, and $\text{cov}(\bar{y},s)$ are shown in Table IX, together with the corresponding theoretical values that can be readily calculated.

In actual use this procedure was modified somewhat, since only one value of each of the desired quantities would be produced by the 12,000 cards and 1200 samples. This single value would be subject to the fluctuations of random sampling and would be difficult to rely on in making inferences. This difficulty was met by breaking the "population" of 1200 samples into 12 sets of 100 samples and obtaining 12 values of each of the desired moments instead of only one. These 12 values, although each was based on fewer samples, served to furnish an idea of how the single value based on 1200 samples was affected by sampling variation. Such analysis has provided a far firmer basis for judgement of relative efficiency.

The above procedure resulted in moments calculated for samples of size n = 10. In like manner, 600 random samples of size n = 20 were drawn, after starting afresh by putting all 12,000 cards together, but this time only 6 instead of 12 sets of 100 samples were available, resulting in 6 values of the desired quantities for comparison. Finally, the 12,000 cards were reprocessed to yield 400 samples of size n = 30, giving 4 values each based on a set of 100 samples.

The resulting sets of 12, 6, and 4 values each were substituted in the appropriate formulas (E.5), (E.6), (E.8) in order to obtain the relative efficiency of the order-statistics estimator to the (simplified) Gumbel estimator. These formulas, all of which depend upon y_P, were evaluated at the probability level P = .95. All these results are summarized in Table V which shows the values of the bias, mean square error, and relative efficiency calculated for each set of 100 samples of sizes 10, 20, and 30, together with the corresponding average values obtained from all 1200 samples combined.

For ease of comparison, the relative efficiencies are also charted in Figure 6.

These results constitute the basis of the statement in the text that at the probability level P = .95, for samples of 20 and 30, the proposed method has greater efficiency than the Gumbel method using the simplified estimator, while for samples of 10 the efficiencies are about the same.

2. Comparison with original Gumbel estimator

The estimator corresponding to $\hat{\xi}_G$ in (E.3), built up from the estimators (E.1) is

$$\hat{\xi}_{G,n} = \hat{u} + \hat{\beta} y_P = \bar{x} + k_n s_x \quad , \tag{E.9}$$

where

$$k_n = \frac{\bar{y}_n - y_P}{\sigma_n} = b_{n,P} \cdot d \quad . \tag{E.10}$$

Here

$$d = (y_P - \gamma)\sqrt{6}/\pi \quad ;$$

and

$$b_{n,P} = \frac{\pi/\sqrt{6}}{\sigma_n} \cdot \frac{y_P - \bar{y}_n}{y_P - \gamma} \tag{E.11}$$

is the conversion factor for passing from the multiplier, d, of s_x in (E.3) to k_n in (E.9). It is apparent from the discussion at the beginning of this Appendix that for infinitely large n, $b_{n,P} = 1$, so that (E.9) includes the asymptotic case. For finite n, however, $\sigma_n < \pi/\sqrt{6}$, $\bar{y}_n < \gamma$. Hence, $b_{n,P}$, being a product of two factors each greater than 1, may considerably exceed 1, so that the multiplier $b_{n,P}$ in (E.10) and (E.11) becomes appreciably larger than the multiplier in (E.3). Thus, for samples of 10, 20, 30, computation shows that, for P = .95,

for example,

$$k_{10} = 1.397 \quad d$$

$$k_{20} = 1.234 \quad d$$

$$k_{30} = 1.173 \quad d \quad . \qquad (E.12)$$

The bias of $\hat{\xi}_{G,n}$ is, in manner similar to Section 1, in view of (E.10),

$$b(\hat{\xi}_{G,n}) = [-(y_P-\gamma) + k_n E(s)]\beta$$

$$= (y_P-\gamma)\ [\tfrac{\sqrt{6}}{\pi} E(s) b_{n,P} - 1]\beta \quad . \qquad (E.13)$$

Table VI (columns 2 and 3) indicates that the presence of the factor $b_{n,P}$ converts the small negative biases into larger positive ones.

For the variance we have from (E.9), analogously to Section 1,

$$\sigma^2(\hat{\xi}_{G,n}) = [k_n^2\sigma^2(s) + 2k_n \operatorname{cov}(\overline{y},s) + \frac{\pi^2}{6n}]\beta^2 . \qquad (E.14)$$

The corresponding expression (E.6) may be written

$$\sigma^2(\hat{\xi}_G) = [d^2\sigma^2(s) + 2d \operatorname{cov}(\overline{y},s) + \frac{\pi^2}{6n}]\beta^2 \quad . \qquad (E.15)$$

Comparison of these two expressions shows, since $\operatorname{cov}(\overline{y},s)$ was found to be positive, that replacement of d by the larger value k_n considerably increases the variance of the Gumbel estimator. Values of the variance for the original and simplified estimators are listed in columns 4 and 5 of Table VI.

Comparison of these columns indicates that the variance of the original estimator can become more than half again as large as the variance of the simplified estimators, depending on sample size and probability P. The effect is most marked for the lower levels of P and smaller sample sizes and disappears as shown when both these factors increase.

The result of the above increases in bias and variance is to greatly increase the mean square error (columns 6 and 7, Table VI), and thus to increase the relative efficiency of the order-statistics estimator (columns 8, 9). As a result the order-statistics estimator is up to twice as efficient as the original Gumbel estimator even for samples as small as 10. This tremendous increase in efficiency falls off slowly, as shown, when sample size increases. For fixed sample size the efficiency increases for large values of P. These differences in efficiency are sufficiently large to completely outweigh any fluctuations of random sampling attributable to the empirical sampling method of evaluation used.

It must be concluded, therefore, that the original Gumbel estimator is both more biased and much less efficient than its simplified form. As a result,

comparison of the order-statistics estimator with the original Gumbel estimator gives very conservative results and greatly understates the actual improvement in efficiency of the proposed method over the method in present use.

ESTIMATION OF EXTREMES
(For instructions see Section 5)

Worksheet 1 - Determination of Estimators

Source: NACA - Langley — Sample III

Computer: J. L.
Date: 5/29/52

Record No.	+Δn Observed extremes
1	.75g
2	.90
3	1.08
4	1.20
5	1.38
6	.81
7	.80
8	.75
9	.90
10	1.20
11	.88
12	1.08
13	1.15
14	1.00
15	1.31
16	1.43
17	.98
18	1.02
19	1.01
20	.93
21	1.15
22	.75
23	1.16
Sum	23.62

I. SUBGROUP SIZES AND PROPORTIONALITY FACTORS:

$n = 23 = km + m'$ $= 3 \times 6 + 5$ $\quad t = km/n = 0.78261 \quad t^2/k = 0.20416 \quad t' = m'/n = 0.21739 \quad t'^2 = 0.04726$

$k = 3 \quad m = 6 \quad m' = 5$

IIA. MAIN SUBGROUPS:

Weights a_i, b_i (from Table I)

i =	1	2	3	4	5	6	Check sum
a_i =	.35545	.22549	.16562	.12105	.08352	.04887	1
b_i =	-.45928	-.03599	.07319	.12673	.14953	.14581	- .00001

Observations x_i in increasing order from i = 1 to i = m

Subgroup No.	x_1	x_2	x_3	x_4	x_5	x_6	Check sum	$\Sigma a_i x_i$	$\Sigma b_i x_i$
1	.75	.81	.90	1.08	1.20	1.38	6.12	.89669	.20978
2	.75	.80	.88	.90	1.08	1.20	5.61	.85052	.14168
3	.98	1.00	1.02	1.15	1.31	1.43	6.89	1.06127	.13870
4									
5									
.									
.									
k									
Sum	2.48	2.61	2.80	3.13	3.59	4.01	18.62	2.80848	.49016

$$\bar{T} = \Sigma a_i x_i/k + (\Sigma b_i x_i/k) y_p = .93616 + .16339\, y_p$$

IIB. REMAINDER SUBGROUP:

Weights a_i', b_i' (from Table I)

i =	1	2	3	4	5	6	Check sum
a_i =	.41893	.24628	.16761	.10882	.05835		.99999
b_i =	-.50313	.00654	.13045	.18166	.18448		0

Observations x_i' in increasing order from i = 1 to i = m'

x_1'	x_2'	x_3'	x_4'	x_5'	x_6'	Check sum	$\Sigma a_i' x_i'$	$\Sigma b_i' x_i'$
.75	.93	1.01	1.15	1.16		5.00	.90535	.18340

$$T' = \Sigma a_i' x_i' + (\Sigma b_i' x_i') y_p = .90535 + .18340\, y_p$$

III. ESTIMATORS:

$$\hat{\xi}_p = t\,\bar{T} + t'T' = .92946 + .16774\, y_p$$

$$u = .92946, \quad \beta = .16774$$

ESTIMATION OF EXTREMES

Worksheet 2 - Predicted values, confidence band, efficiency, plotting positions

P	y_P	PREDICTED VALUES $\hat{\xi}_P = .92946 + .16774 y_P$	$Q_m = Q_6$ (from Table III)	$Q_{m'} = Q_5$ (from Table III)	$Var(\hat{\xi}_P) = \frac{t^2}{k} Q_m + t'^2 Q_{m'}$	68%-CONFIDENCE BAND HALF-WIDTH $\sigma(\hat{\xi}_P) = \sqrt{Var(\hat{\xi}_P)}$	$Q_{LB} = Q_o/n$ (Q_o from Table III)	EFFICIENCY $E = \frac{Q_{LB}}{Var(\hat{\xi}_P)}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
			$t^2/k = .20416$	$t'^2 = .04726$		$\beta = .16774$		
Estimate of u: .36788	0	.92946 g	.19117 β^2	.23140 β^2	.04997 β^2	.0375	.04820 β^2	.965
.50	0.36651	.99094 g	.23189 β^2	.27870 β^2	.06051 β^2	.0413	.05994 β^2	.991
.90	2.25037	1.30694 g	1.00065 β^2	1.22831 β^2	.26234 β^2	.0859	.23235 β^2	.886
.95	2.97020	1.42768 g	1.54171 β^2	1.90349 β^2	.40471 β^2	.1067	.34777 β^2	.859
.99	4.60016	1.70109 g	3.27230 β^2	4.07062 β^2	.86045 β^2	.1556	.71035 β^2	.826
.999	6.90726	2.08808 g	6.92044 β^2	8.65173 β^2	1.82176 β^2	.2264	1.46364 β^2	.803
Estimate of β: 1	- - -	- - -	$0.13196 y_P^2 \beta^2$	$0.16665 y_P^2 \beta^2$	$0.03482 y_P^2 \beta^2$	- - -	$0.02643 y_P^2 \beta^2$	.759

PLOTTING POSITIONS

Observed extremes in increasing rank from 1 to n = 23

Rank r	Observed Extreme	Plotting Position r/(n+1)	Rank r	Observed Extreme	Plotting Position r/(n+1)	Rank r	Observed Extreme	Plotting Position r/(n+1)
1	.75	.0417	11	1.00	.4583	21	1.31	.8750
2	.75	.0833	12	1.01	.5000	22	1.38	.9167
3	.75	.1250	13	1.02	.5417	23	1.43	.9583
4	.80	.1667	14	1.08	.5833			
5	.81	.2083	15	1.08	.6250	Sum	23.62	
6	.88	.2500	16	1.15	.6667			
7	.90	.2917	17	1.15	.7083			
8	.90	.3333	18	1.16	.7500			
9	.93	.3750	19	1.20	.7917			
10	.98	.4167	20	1.20	.8333			

TABLE I

Weights for minimum-variance, unbiased, linear order-statistics estimator

$L = \hat{\xi}_P = \sum_{i=1}^{n} (a_i + b_i y_P) x_i$, of percentage points $\xi_P = u + \beta y_P$, and variance

$Var(\hat{\xi}_P) = Q_n = (A_n y_P^2 + B_n y_P + C_n)\beta^2$, for sample size $n = 2$ to 6, $x_1 \leq x_2 \leq \cdots \leq x_n$

(Discussed in Section 4.3)

n		WEIGHTS, $a_i + b_i y_P$, of x_i					
		x_1	x_2	x_3	x_4	x_5	x_6
2	a_i	+0.91637	+0.08363				
	b_i	-0.72135	+0.72135				
	Q_2	$(0.71186\ y_P^2 - 0.12864\ y_P + 0.65955)\beta^2$					
3	a_i	+0.65632	+0.25571	+0.08797			
	b_i	-0.63054	+0.25582	+0.37473			
	Q_3	$(0.34472\ y_P^2 + 0.04954\ y_P + 0.40286)\beta^2$					
4	a_i	+0.51100	+0.26394	+0.15368	+0.07138		
	b_i	-0.55862	+0.08590	+0.22392	+0.24880		
	Q_4	$(0.22528\ y_P^2 + 0.06938\ y_P + 0.29346)\beta^2$					
5	a_i	+0.41893	+0.24628	+0.16761	+0.10882	+0.05835	
	b_i	-0.50313	+0.00653	+0.13045	+0.18166	+0.18448	
	Q_5	$(0.16665\ y_P^2 + 0.06798\ y_P + 0.23140)\beta^2$					
6	a_i	+0.35545	+0.22549	+0.16562	+0.12105	+0.08352	+0.04887
	b_i	-0.45928	-0.03599	+0.07319	+0.12673	+0.14953	+0.14581
	Q_6	$(0.13196\ y_P^2 + 0.06275\ y_P + 0.19117)\beta^2$					

TABLE II

Means, variances, and covariances of order statistics y_i in samples of n from the reduced extreme-value distribution $F(y) = \exp(-e^{-y})$, $n = 2$ to 6, $y_1 \leq y_2 \leq \dots \leq y_n$

(Discussed in Section 4.3 and Appendix B)

n	i	MEANS $E y_i$	VARIANCES AND COVARIANCES, $\sigma_{ij} = \sigma_{ji}$					
			j = 1	j = 2	j = 3	j = 4	j = 5	j = 6
2	1	-0.11593 152	0.68402 804	0.48045 301				
	2	+1.27036 285		1.64493 407				
3	1	-0.40361 359	0.44849 796	0.30137 144	0.24375 810			
	2	+0.45943 263		0.65852 235	0.54629 438			
	3	+1.67582 795			1.64493 407			
4	1	-0.57351 263	0.34402 417	0.22455 344	0.17903 454	0.15388 918		
	2	+0.10608 352		0.41553 113	0.33720 966	0.29271 188		
	3	+0.81278 175			0.65180 236	0.57432 356		
	4	+1.96351 003				1.64493 407		
5	1	-0.69016 715	0.28486 447	0.18202 536	0.14358 737	0.12257 865	0.10901 329	
	2	-0.10689 454		0.30849 748	0.24676 731	0.21226 644	0.18967 383	
	3	+0.42555 061			0.40598 292	0.35267 072	0.31716 095	
	4	+1.07093 582				0.64907 319	0.58991 519	
	5	+2.18665 358					1.64493 407	
6	1	-0.77729 368	0.24658 20	0.15496 74	0.12121 61	0.10291 64	0.09116 19	0.08285 42
	2	-0.25453 448		0.24854 56	0.19670 62	0.16806 28	0.14945 32	0.13619 10
	3	+0.18838 534			0.29761 59	0.25616 60	0.22887 90	0.20925 46
	4	+0.66271 588				0.40185 52	0.36145 55	0.33204 51
	5	+1.27504 579					0.64769 96	0.59985 67
	6	+2.36897 513						1.64493 41

TABLE III

Variances and efficiencies of minimum-variance, unbiased, linear order-statistics estimator, $L = \hat{\xi}_p$, for selected probability levels, P, for sample size n = 2 to 6

(Discussed in Section 4.3; variances based on Table I)

PART A: VARIANCES (Units of β^2)

P	y_P	Q_o*	Q_2	Q_3	Q_4	Q_5	Q_6
.36788**	0	1.10866	0.65955	0.40286	0.29346	0.23140	0.19117
.40	0.08742	1.15825	0.65374	0.40983	0.30125	0.23861	0.19766
.50	0.36651	1.37873	0.70802	0.46732	0.34915	0.27870	0.23189
.60	0.67173	1.72827	0.89434	0.59168	0.44172	0.35225	0.29286
.70	1.03093	2.28472	1.28350	0.82030	0.60442	0.47859	0.39611
.80	1.49994	3.24743	2.06814	1.25271	0.90437	0.70829	0.58218
.90	2.25037	5.34410	3.97502	2.26002	1.59046	1.22831	1.00065
.95	2.97020	7.99866	6.55752	3.59108	2.48700	1.90349	1.54171
.975	3.67625	11.21444	9.80724	5.24369	3.59317	2.73352	2.20527
.99	4.60016	16.33798	15.13171	7.92536	5.37994	4.07062	3.27230
.999	6.90726	33.66365	33.73386	17.19133	11.52099	8.65174	6.92044
1***	∞	$0.60793\ y_P^2$	$0.71186\ y_P^2$	$0.34471\ y_P^2$	$0.22528\ y_P^2$	$0.16665\ y_P^2$	$0.13196\ y_P^2$

* Cramér-Rao lower bound is $Q_{LB} = Q_o / n$, where $Q_o = (0.60793\ y_P^2 + 0.51404\ y_P + 1.10866)\beta^2$, n is sample size.

** These lines give the variances and efficiencies for the order-statistics estimator of the parameter u.

*** The variances for P = 1 are all infinite. Expressing them by means of the dominant term in Q_n permits finding their ratios to obtain the efficiencies. Also the coefficients in y_P^2 in this line are the variances for the order-statistics estimator of the parameter β.

TABLE III, continued

Variances and efficiencies of minimum-variance, unbiased, linear order-statistics estimator, $L = \hat{\xi}_P$, for selected probability levels, P, for sample size n = 2 to 6

(Discussed in Section 4.3)

PART B: EFFICIENCIES

P	y_P	$E_2 = \frac{1}{2} Q_o/Q_2$	$E_3 = \frac{1}{3} Q_o/Q_3$	$E_4 = \frac{1}{4} Q_o/Q_4$	$E_5 = \frac{1}{5} Q_o/Q_5$	$E_6 = \frac{1}{6} Q_o/Q_6$
.36788 *	0	0.8405	0.9173	0.9445	0.9582	0.9666
.40	0.08742	0.8859	0.9421	0.9612	0.9708	0.9766
.50	0.36651	0.9737	0.9834	0.9872	0.9894	0.9909
.60	0.67173	0.9662	0.9737	0.9781	0.9813	0.9836
.70	1.03093	0.8900	0.9284	0.9450	0.9548	0.9613
.80	1.49994	0.7851	0.8641	0.8977	0.9170	0.9297
.90	2.25037	0.6722	0.7882	0.8400	0.8702	0.8901
.95	2.97020	0.6099	0.7425	0.8040	0.8404	0.8647
.975	3.67625	0.5717	0.7129	0.7803	0.8205	0.8475
.99	4.60016	0.5399	0.6872	0.7592	0.8027	0.8321
.999	6.90726	0.4990	0.6527	0.7305	0.7782	0.8107
1 **	∞	0.4270	0.5879	0.6746	0.7296	0.7678

* These lines give the variances and efficiencies for the order-statistics estimator of the parameter u.

** Limiting efficiency as P approaches 1. These are also the efficiencies for the estimator of the parameter β.

TABLE IV

Efficiency of order-statistics estimators for various sample sizes $n = km + m'$ partitioned into subgroups as indicated for P = .99, P = 1

(Discussed in Section 4.4)

n	km + m'	EFFICIENCY P = .99 (percent)	EFFICIENCY P = 1 (percent)	n	km + m'	EFFICIENCY P = .99 (percent)	EFFICIENCY P = 1 (percent)
2 or k·2		54.0	42.7	21	3x6 + 3	80.8	73.6
3 or k·3		68.7	58.8	22	3x6 + 4	81.8	74.9
4 or k·4		75.9	67.5	23	3x6 + 5	82.6	75.9
5 or k·5		80.3	73.0	24	4x6	83.2	76.8
6 or k·6		83.2	76.8	25	5x5	80.3	73.0
7[a]	1x5 + 2	70.5	60.7	26	4x6 + 2	79.9	72.3
8[b]	1x6 + 2	73.3	63.8	27	4x6 + 3	81.3	74.3
9	1x6 + 3	77.7	69.7	28	4x6 + 4	81.9	75.3
10	2x5	80.3	73.0	29	4x6 + 5	82.7	76.1
				30	5x6	83.2	76.8
11	1x6 + 5	81.9	75.0	31	5x5 + 6	80.8	73.7
12	2x6	83.2	76.8	32	5x6 + 2	80.5	73.1
13	2x5 + 3	77.3	69.1	33	5x6 + 3	81.6	74.7
14[c]	2x6 + 2	74.7	66.7	34	5x6 + 4	82.3	75.6
15	3x5	80.3	73.0	35[d]	7x5	80.3	73.0
16	2x6 + 4	81.3	74.2	36	6x6	83.2	76.8
17	2x6 + 5	82.3	75.6	37	7x5 + 2	78.2	70.3
18	3x6	83.2	76.8	38	6x6 + 2	80.9	73.7
19	3x5 + 4	79.3	71.7	39	6x6 + 3	81.9	75.0
20	4x5	80.3	73.0	40[e]	8x5	80.3	73.0
				61	11x5 + 6	80.6	73.3

[a] If partition is 7 = 1x4 + 3, then efficiencies are:
P = .99, eff. = 72.7%; P = 1, eff. = 63.4%

[b] If partition is 8 = 2x4, then efficiencies are:
P = .99, eff. = 75.9%; P = 1, eff. = 67.5%

[c] If partition is 14 = 2x5 + 4, then efficiencies are:
P = .99, eff. = 79.1%; P = 1, eff. = 71.3%

[d] If partition is 35 = 5x6 + 5, then efficiencies are:
P = .99, eff. = 82.8%; P = 1, eff. = 76.2%

[e] If partition is 40 = 6x6 + 4, then efficiencies are:
P = .99, eff. = 82.4%; P = 1, eff. = 75.7%

TABLE V

Biases, mean square errors, and relative efficiencies of proposed order-statistics estimator $\hat{\xi}_P$ to Gumbel estimator $\hat{\xi}_G$, based on empirical sampling results obtained from N samples, for $P = .95$ and sample size $n = 10, 20, 30$

(Discussed in Appendix E; values of R shown in Figure 6)

Set No. (100 samples each)	BIAS (Average value for each set of 100 samples)			MEAN SQUARE ERROR (MSE) (Average value for each set of 100 samples)			RELATIVE EFFICIENCY* R = MSE/Q		
	n = 10	n = 20	n = 30	n = 10	n = 20	n = 30	n = 10 $Q=.95174\beta^2$	n = 20 $Q=.47587\beta^2$	n = 30 $Q=.30834\beta^2$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	units of β			units of β^2					
	N=1200	N=600	N = 400	N=1200	N = 600	N = 400	N =1200	N = 600	N = 400
1	-.25961	-.08075	-.03771	.93875	.56707	.32762	.986	1.192	1.062
2	-.26375	-.05465	-.06069	1.13608	.39516	.25458	1.194	.830	.826
3	-.16007	-.17221	-.08280	.85747	.52061	.39850	.901	1.094	1.292
4	-.34090	-.05595	-.09250	.80648	.59028	.41688	.847	1.240	1.352
5	-.16118	-.20845		.81749	.58173		.859	1.222	
6	-.15951	-.11583		1.13818	.52853		1.196	1.111	
7	-.10497			.88266			.927		
8	-.15613			.92989			.977		
9	-.21475			.87389			.918		
10	-.18451			1.22356			1.286		
11	-.29129			1.00460			1.056		
12	-.32256			1.02068			1.072		
Average	-.21827	-.11464	-.06842	.96914	.53056	.34940	1.018	1.115	1.133
Proportion of sets favorable to the proposed estimator (R > 1):							5 out of 12	5 out of 6	3 out of 4

* For explanation of Q, see equation (E.8) in Appendix E and accompanying discussion.

TABLE VI

Bias and efficiency characteristics of original Gumbel estimator $\hat{\xi}_{G,n}$, and simplified (asymptotic) form $\hat{\xi}_G$, for sample size n = 10, 20, 30, and n infinite, for P = .95, .99, 1

(Discussed in Appendix E)

P	BIAS		VARIANCE		MEAN SQUARE ERROR (MSE)		RELATIVE EFFICIENCY of order-statistics estimator to Gumbel estimator,* R = MSE/Q	
	$\hat{\xi}_{G,n}$	$\hat{\xi}_G$	$\hat{\xi}_{G,n}$	$\hat{\xi}_G$	$\hat{\xi}_{G,n}$	$\hat{\xi}_G$	$\hat{\xi}_{G,n}$	$\hat{\xi}_G$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	units of β		units of β^2		units of β^2			
			n = 10 (computed from empirical sampling results)					
.95	0.64	-0.22	1.48	0.92	1.89	0.97	1.99	1.02
.99	1.02	-0.37	3.32	1.97	4.36	2.10	2.14	1.03
1**	$0.23y_P$	$-0.09y_P$	$0.15y_P^2$	$0.08y_P^2$	$0.20y_P^2$	$0.09y_P^2$	2.38	1.06
			n = 20 (computed from empirical sampling results)					
.95	0.42	-0.11	0.69	0.52	0.87	0.53	1.83	1.11
.99	0.66	-0.19	1.56	1.12	1.99	1.16	1.96	1.14
1**	$0.15y_P$	$-0.05y_P$	$0.07y_P^2$	$0.05y_P^2$	$0.09y_P^2$	$0.05y_P^2$	2.18	1.19
			n = 30 (computed from empirical sampling results)					
.95	0.33	-0.07	0.43	0.34	0.54	0.35	1.76	1.13
.99	0.53	-0.12	0.96	0.75	1.24	0.76	1.89	1.16
1**	$0.12y_P$	$-0.03y_P$	$0.04y_P^2$	$0.03y_P^2$	$0.06y_P^2$	$0.03y_P^2$	2.12	1.21
			n infinite (computed from theory)					
.95	0	0	0	0	0	0	1.237	1.237
.99	0	0	0	0	0	0	1.290	1.290
1	0	0	0	0	0	0	1.389	1.389

* For values of Q, see Table V, headings for cols. 8, 9, 10.

** For P = 1, all quantities except relative efficiency are infinite, for finite sample size. Expressing them in terms of y_P (which is also infinite) permits comparison for very large values of P.

TABLE VII - PART A
PROBABILITIES OF EXTREMES
(Discussed in Section 6.2)

Extremes x	Frequency p	Mean and Standard Deviation x·p	x²·p	Cumulative Frequency m	Plotting Positions m/(N+1)
.75				1	0.0417
.75				2	0.0833
.75				3	0.1250
.80				4	0.1667
.81				5	0.2083
.88				6	0.2500
.90				7	0.2916
.90				8	0.3333
.93				9	0.3750
.98				10	0.4167
1.00				11	0.4583
1.01				12	0.5000
1.02				13	0.5416
1.08				14	0.5833
1.08				15	0.6250
1.15				16	0.6667
1.15				17	0.7083
1.16				18	0.7500
1.20				19	0.7917
1.20				20	0.8333
1.31				21	0.8750
1.38				22	0.9167
1.43				23	0.9583

Sums: N= 23 2362 Σ(xp) 251326 Σ(x²p) Arbitrary Mean: 0

Computer: WRK Date 5/12/52 Data: NACA-Langley Sample III

TABLE VII - PART B
PROBABILITIES OF EXTREMES
(Discussed in Section 6.2)

For Evaluating the Probability of Extreme Values by the Method Developed by Dr. E. J. Gumbel.

Environmental Protection Section, Research & Development Branch, Military Planning Division, Office of The Quartermaster General.

I. Mean and Standard Deviation:

N = 23 $\Sigma(xp)$ = 23.62 $\Sigma(x^2p)$ = 25.1326

$\sqrt{N}$ = 4.79583 Mean: $\overline{x'}$ = 1.0270 $\overline{x^2}$ = 1.0927

Arbitrary Mean: x_0 = 0 $(\overline{x'})^2$ = - 1.0547

True Mean $\bar{x}$ = 1.0270 $(s_x)^2$ = 0.0380

$N/(N-1)$ = _____ Standard Deviation: s_x = 0.1950

II. Parameters:

σ_N = 1.0811 $\bar{y}_N$ = 0.5283

$1/a = s_x/\sigma_N$ = 0.1804 $\bar{y}_N\,(1/a)$ = 0.0953

$1/(a\sqrt{N}) = (1/a)/\sqrt{N}$ = 0.0376 $u = \bar{x} \mp \bar{y}_N(1/a)$ = 0.9317 = mode

NOTE: Upper sign used for maxima, lower sign for minima.

III. Line of Expected Extremes:

$x = u \pm (1/a)y$ = 0.9317 $\pm$ 0.1804 y

y:	-2.00	0.00	3.00	5.00	2.25	4.60
y(1/a):	-.3608	0.00	.5412	.9020	.4059	.8298
x:	.5709	= u = .9317	1.4729	1.8337	x_{10}	x_{100}

NOTE: Values x_{10} and x_{100} are for return periods of 10 and 100

IV. Half-width of 0.68269 Confidence Band, $\sigma_{x,m} = \sigma_{y,m}\sqrt{N}/(a\sqrt{N}) = (\sigma_{y,m}\sqrt{N})\left[(1/a)/\sqrt{N}\right]$:

$\Phi(x)$:	.150	.200	.300	.400	.500	.600	.700	.800	.850
$\sigma_{y,m}\sqrt{N}$:	1.255	1.243	1.268	1.337	1.443	1.598	1.835	2.241	2.585
$\sigma_{x,m}$:	.0472	.0467	.0477	.0503	.0543	.0601	.0690	.0843	.0972

For largest value, $\Delta_{x,N} = 1.141\,(1/a)$ = 0.2058

For next-to-largest value, $\Delta_{x,N-1} = .759\,[N/(N-1)]\,(1/a)$ = 0.1369

V. Expected Extreme, in T periods (years, etc.): $x_T = x_{10} + Z_T(x_{100} - x_{10})$: $x_{100} - x_{10}$ = _____

T	Z_T	$Z_T(x_{100}-x_{10})$	x_T	T	Z_T	$Z_T(x_{100}-x_{10})$	x_T	T	Z_T	$Z_T(x_{100}-x_{10})$	x_T
15	.180			60	.781			140	1.144		
20	.306			70	.847			150	1.173		
25	.404			80	.905			200	1.296		
30	.483			90	.955			300	1.469		
35	.549			100	1.000			400	1.592		
40	.607			110	1.041			500	1.687		
45	.658			120	1.078			750	1.859		
50	.703			130	1.112			1000	1.990		

Place: _____

C-7081(2)
1 Jan. 1951

Data : NACA – Langley Sample No. III

Computer: WRK Date: 5/12/52

TABLE VIII

Comparison of confidence-interval half-widths for extreme predictions given by Gumbel method, by modified method, and by order-statistics method, for samples of n = 10, 20, 30, and for confidence levels of 68 percent and 95 percent

(Discussed in Appendix D)

P	n = 10			n = 20			n = 30		
	Gumbel method	Modified method [a]	Order-statistics method [b,c]	Gumbel method	Modified method [a]	Order-statistics method [b,c]	Gumbel method	Modified method [a]	Order-statistics method [b,c]
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	68 percent confidence level								
	$\Delta = 1.141\ \beta$	$\Delta' = 1.141\ B_p\beta$	$\Delta_o = \sqrt{Q_5/2}$	$\Delta = 1.141\ \beta$	$\Delta' = 1.141\ B_p\beta$	$\Delta_o = \sqrt{Q_5/4}$	$\Delta = 1.141\ \beta$ [d]	$\Delta' = 1.141\ B_p\beta$ [d]	$\Delta_o = \sqrt{Q_6/5}$
.95	1.141 β	0.855 β	0.976 β	1.141 β	0.639 β	0.690 β			0.555 β
.99	1.141 β	1.247 β	1.427 β	1.141 β	0.941 β	1.009 β	1.141 β	0.768 β	0.809 β
.999	1.141 β	1.817 β	2.080 β	1.141 β	1.308 β	1.471 β	1.141 β	1.124 β	1.176 β
	95 percent confidence level								
	$\Delta = 3.067\ \beta$	$\Delta' = 3.067\ B_p\beta$	$\Delta_o = 2\sqrt{Q_5/2}$	$\Delta = 3.067\ \beta$	$\Delta' = 3.067\ B_p\beta$	$\Delta_o = 2\sqrt{Q_5/4}$	$\Delta = 3.067\ \beta$ [d]	$\Delta' = 3.067\ B_p\beta$ [d]	$\Delta_o = 2\sqrt{Q_6/5}$
.95	3.067 β	2.297 β	1.970 β	3.067 β	1.717 β	1.454 β			1.181 β
.99	3.067 β	3.353 β	2.899 β	3.067 β	2.530 β	2.151 β	3.067 β	2.065 β	1.742 β
.999	3.067 β	4.884 β	4.245 β	3.067 β	3.707 β	3.159 β	3.067 β	3.022 β	2.554 β

[a] Values of $B_p = [(6/\pi^2)\sigma^2(\ _G)]^{\frac{1}{2}}$ are based on empirical sampling methods. See Appendix E.

[b] For explanation of the quantity of the form $(1/k)Q_m$ appearing in Δ_o, see equation (E.8) in Appendix E and accompanying discussion.

[c] Based on assumption of normality for order-statistics estimator. For discussion, see Appendix B, Section 1.

[d] Applies only for $P \geq n/(n+1)$ = .909, 952, .968 for n = 10, 20, 30, respectively. See footnote to page ?? (Appendix D).

TABLE IX

Empirical sampling values of first and second moments of sample mean $\bar{y}$ and standard deviation s for samples of n = 10, 20, 30, compared with the corresponding theoretical value where obtainable

(Discussed in Appendix E)

Estimate of	n = 10 (1200 samples)		n = 20 (600 samples)		n = 30 (400 samples)	
	Empirical values	Theoretical values	Empirical values	Theoretical values	Empirical values	Theoretical values
$E(y)$	0.5698	0.5772	0.5698	0.5772	0.5698	0.5772
$\sigma^2(\bar{y})$	0.1663	0.1645	0.0884	0.0822	0.0535	0.0548
$E(s)$	1.1656		1.2211		1.2459	
$\sigma^2(s)$	0.1321		0.0775		0.0513	
$\sigma(\bar{y}, s)$	0.0800		0.0438		0.0297	

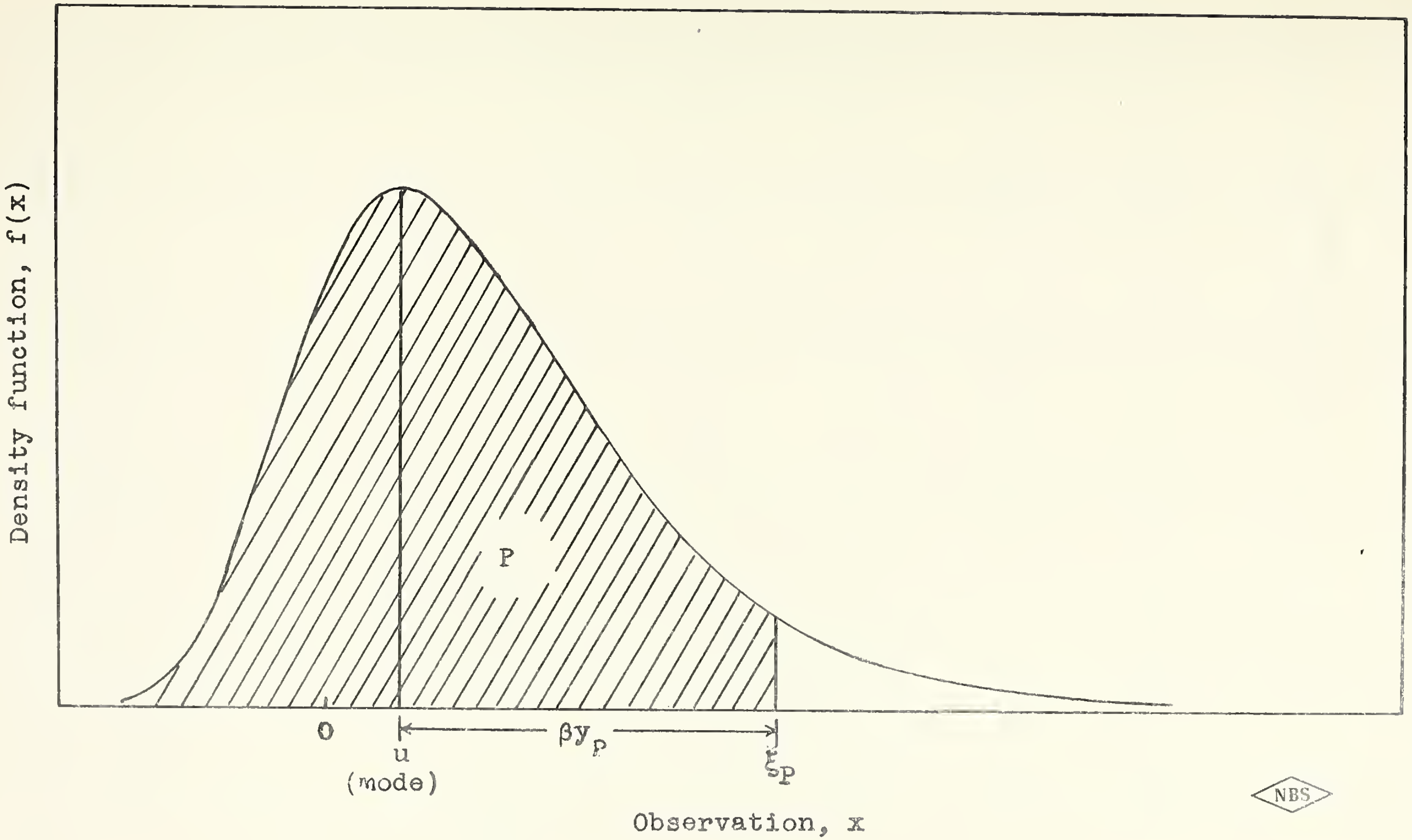

Figure 1 - General form of extreme-value distribution (density function, f(x)) showing relationship of parameter ξ_P to other parameters:

$$\xi_P = u + \beta y_P.$$

(Adapted from Reference 18, Figure 2)

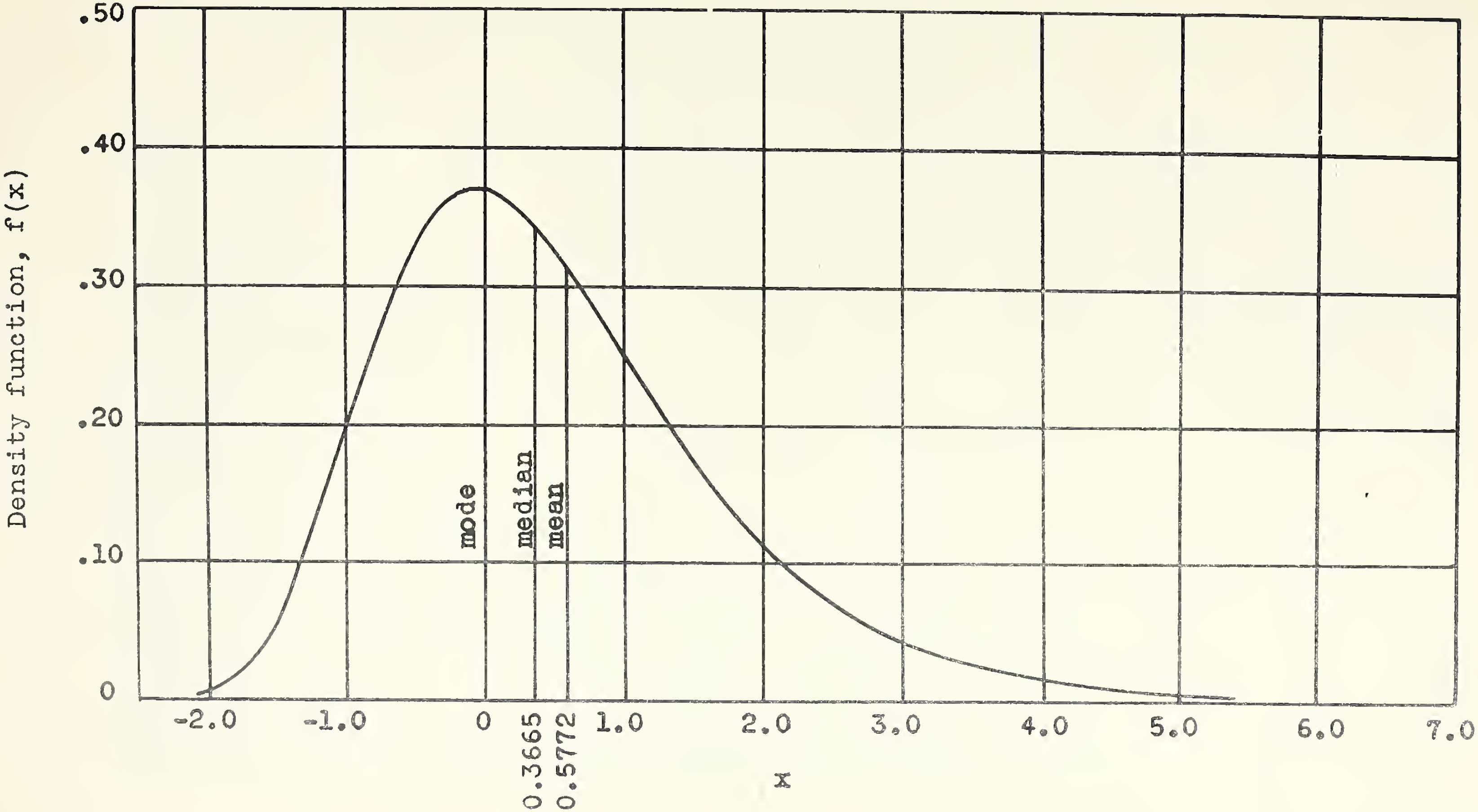

Figure 2 - Density function f(x) for extreme-value distribution with parameters $\beta = 1$, $u = 0$: $f(x) = \exp(-x-e^{-x})$.

(Adapted from reference 18, Figure 2)

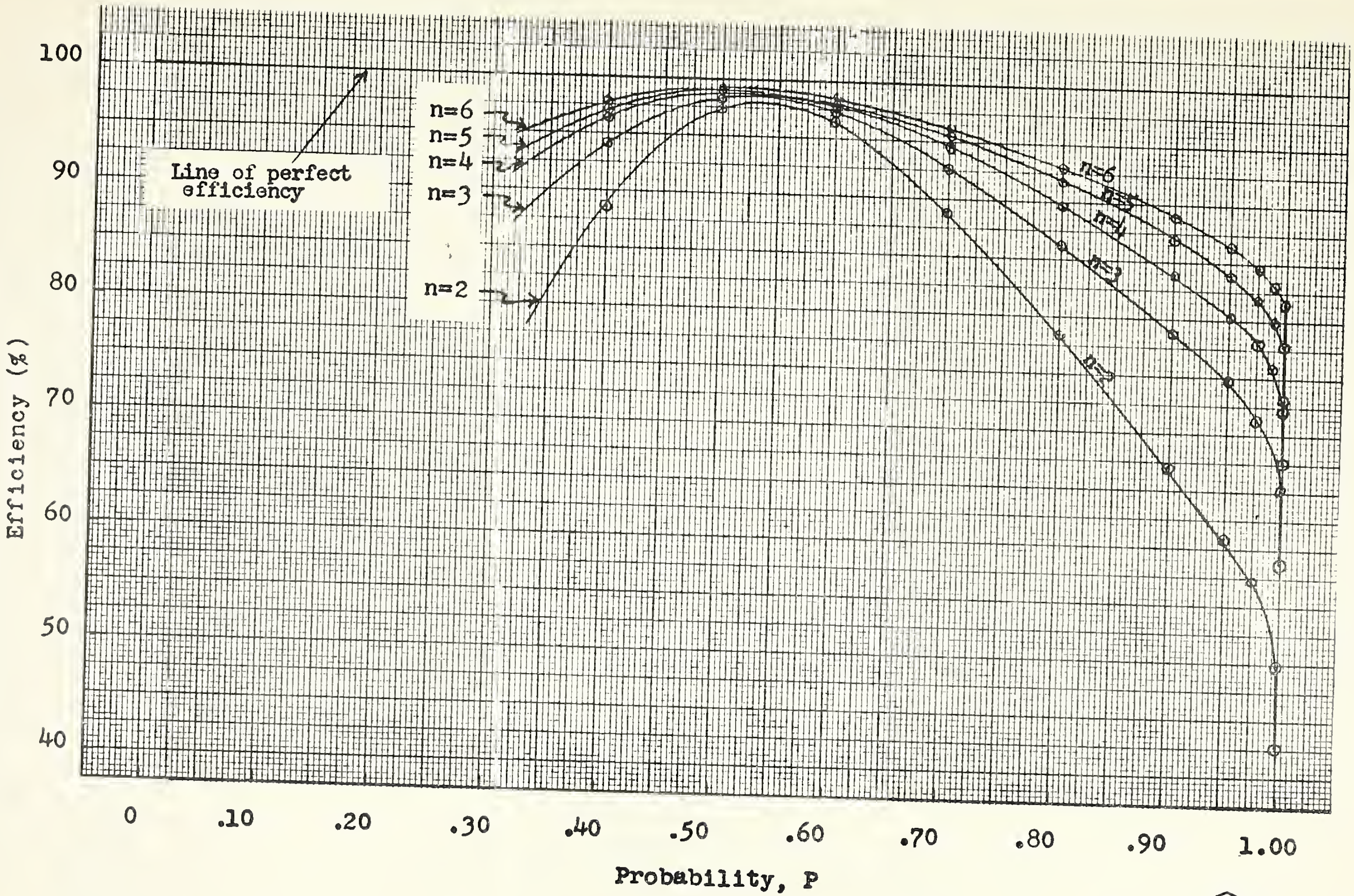

Figure 3. - Comparison of efficiencies of order-statistics estimator $\hat{\xi}_P$ for samples of sizes 2, 3, 4, 5, 6, or for samples of any size if broken into equal subgroups of 2 to 6. (Data from Table III, Part B.)

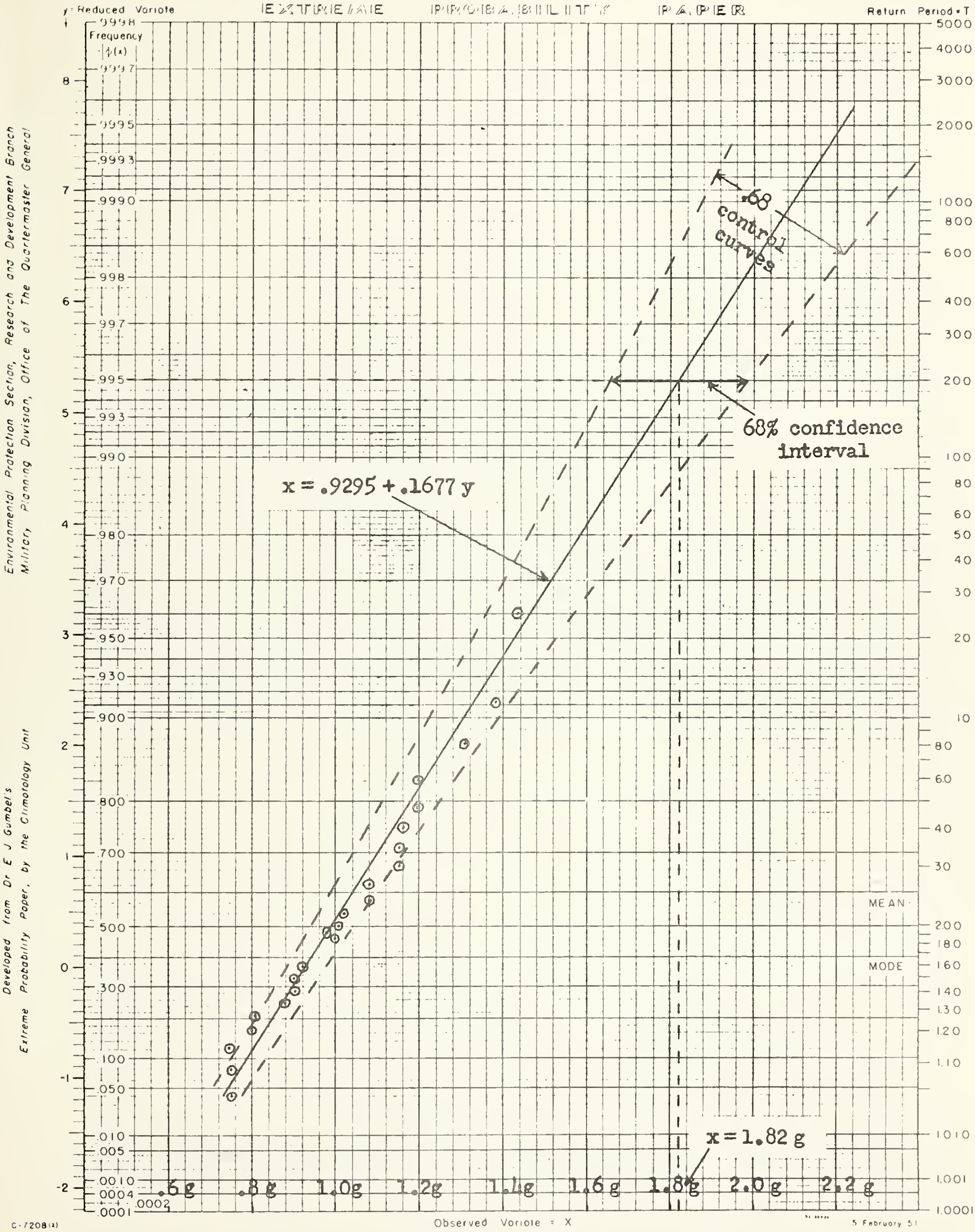

Figure 4. - Graphical analysis of a sample of 23 maximum acceleration increments by method of order statistics. (Data from Worksheet 2)

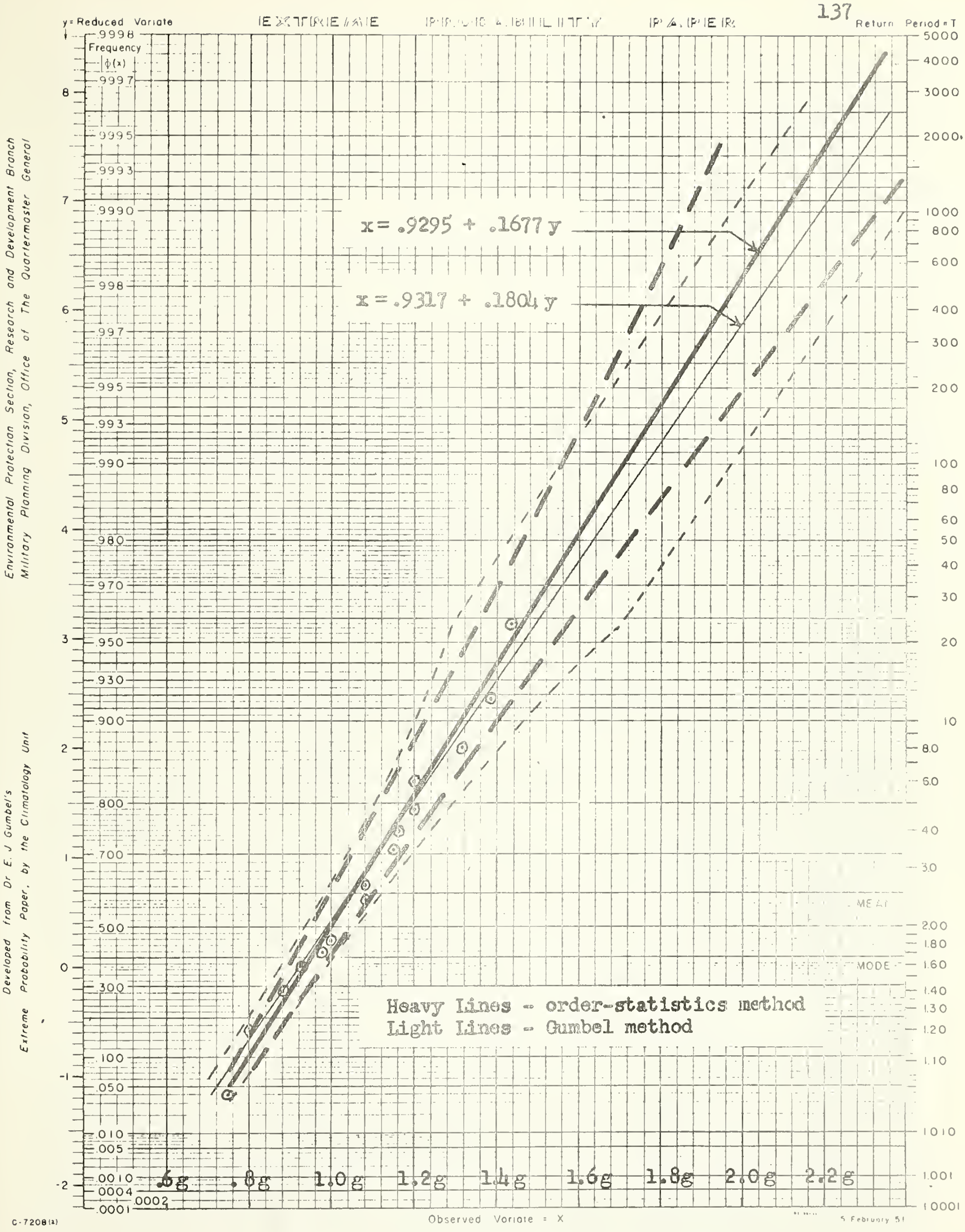

Figure 5. - Comparison of order-statistics and Gumbel methods of analyzing a sample of 23 maximum acceleration increments, showing 68-percent control curves. (8 observations at lower end omitted to avoid crowding.)
(Data from Worksheet 2 and Table VII, Part B)

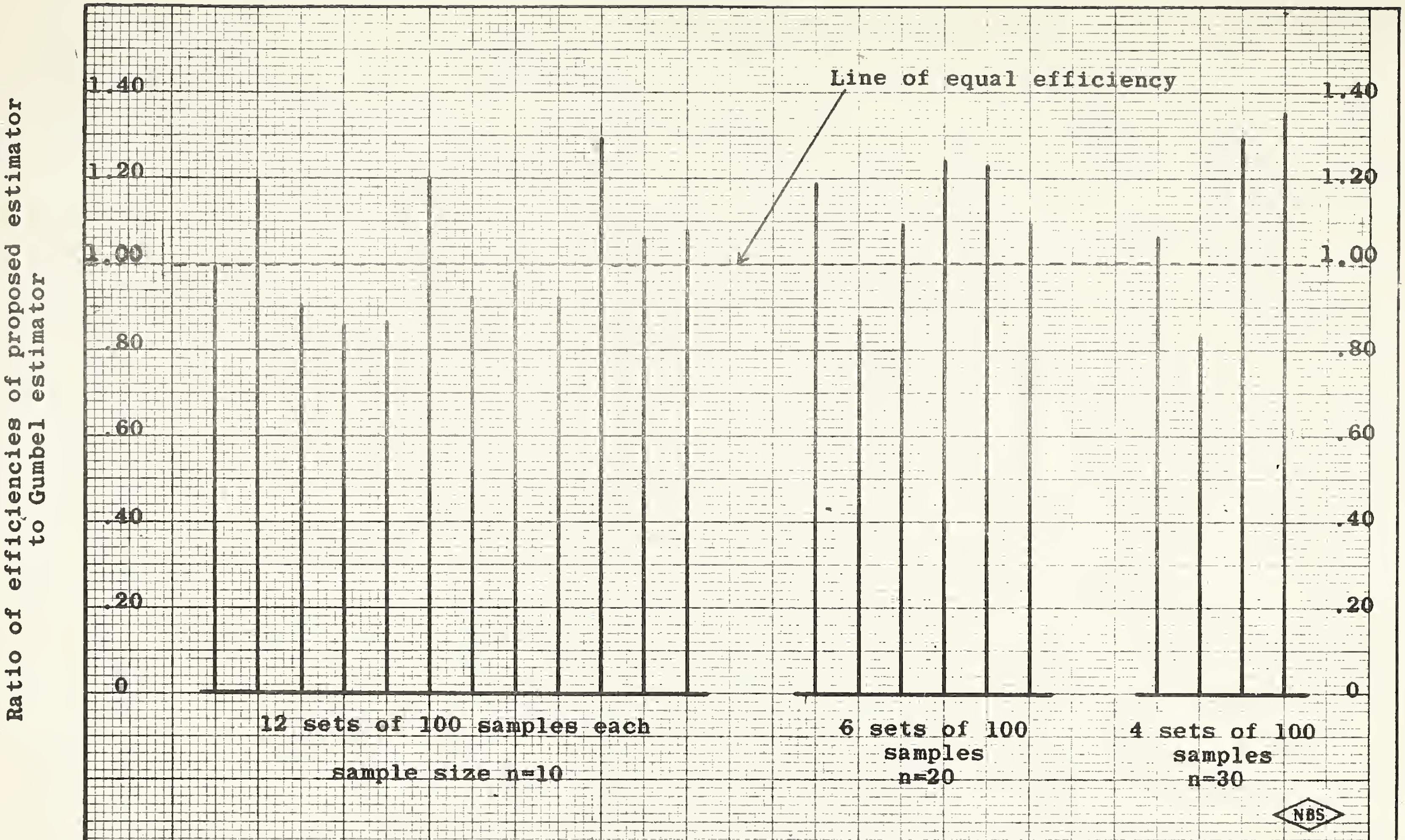

Figure 6.-Comparison of empirical sampling values of relative efficiencies of proposed order-statistics estimator to Gumbel estimator, for P=.95 and sample sizes n=10,20,30. (Data from Table V, cols. 8,9,10)

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