NATIONAL BUREAU OF STANDARDS REPORT

2116

THE VERIFICATION OF AN HYPOTHESIS

CONCERNING THE NORMALITY OF DISTRIBUTIONS BY SMALL SAMPLES

by

A. A. Petrov

Translated from the Russian by Curtis D. Benster

Editor: D. Teichroew

NBS

U. S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

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*DOKLADY AKADEMIĬ NAUK SSSR 1951. vol. LXXVI, No. 3, pp. 355-358

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THE VERIFICATION OF AN HYPOTHESIS

CONCERNING THE NORMALITY OF DISTRIBUTIONS BY SMALL SAMPLES* ** *** ****

by

A. A. Petrov

1. Let there be given an aggregate of N samples of size n

$$x_{11}, \dots, x_{1n},$$
$$\dots\dots$$
$$\dots\dots$$
$$x_{N1}, \dots, x_{Nn}.$$

The question is raised: is the assumption admissible that the i-th sample (for any i) has been obtained by a random selection from an infinite aggregate having a cumulative distribution function $F(a_i x + b_i)$, where F is a given function, the same for all samples, and the parameters a_i and b_i may be different for different samples and are unknown to us. (This assumption will henceforth be called hypothesis F.) In particular, is the assumption admissible that each of our samples has been taken from a normally distributed aggregate with mathematical expectations and dispersions different for the different samples.

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For verification of the stated hypothesis it is natural to consider some functions or other, $\eta(x_1, x_2, \cdots, x_n)$, the distributions of which, computed on the assumption that $x_1, \cdots, x_n$ are independent and subject to the distribution function $F(ax + b)$, do not depend on a and b. One can consider, for instance, the quantities

$$\eta' = \frac{x_{max} - \bar{x}}{s}, \quad \eta'' = \frac{x_{min} - \bar{x}}{s},$$

whose distribution in the case of a normal law F has been studied by N. V. Smirnov [1]. Another method, starting from a distribution, first considered by Thompson [2], of the quantities $\eta_i = \frac{x_i - \bar{x}}{s}$, has been proposed by Arley and Buch [3,4]. The method proposed below has the advantage of not requiring the computation of the [root] mean squares s. In addition, it gives n - 2 separate curves for comparison and may therefore be a more powerful means of discriminating the types of distribution.

2. Let X be a variate having a distribution of continuous type [5] with distribution function F and probability density $f = F'$ and let

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(1) See [1].

(2) See [2].

(3) See [3].

(4) See [4].

(5) See [5].

$x_1' \leq x_2' \leq \ldots \leq x_n'$ be a sequence of n independent observations of the variate X, arranged in increasing order. Let us consider the ratio

$$\xi_k = \frac{x_k' - x_1'}{x_n' - x_1'} \quad (\text{where } 1 < k < n) \quad .$$

These quantities are invariant with respect to the choice of the scale and the origin of coordinates, and their distribution functions will therefore be identical for all distribution functions F(ax + b) differing from F(x) only by the values of the parameters a and b.

It can be shown that the distribution function of the variate ξ_k is

$$F_k(t) = \frac{n!}{(k-2)!(n-k-1)!} \iiint_G [F(y) - F(x)]^{k-2}[F(z) - F(y)]^{n-k-1} \times \tag{1}$$

$$\times f(x)\, f(y)\, f(z)\, dxdydz \quad ,$$

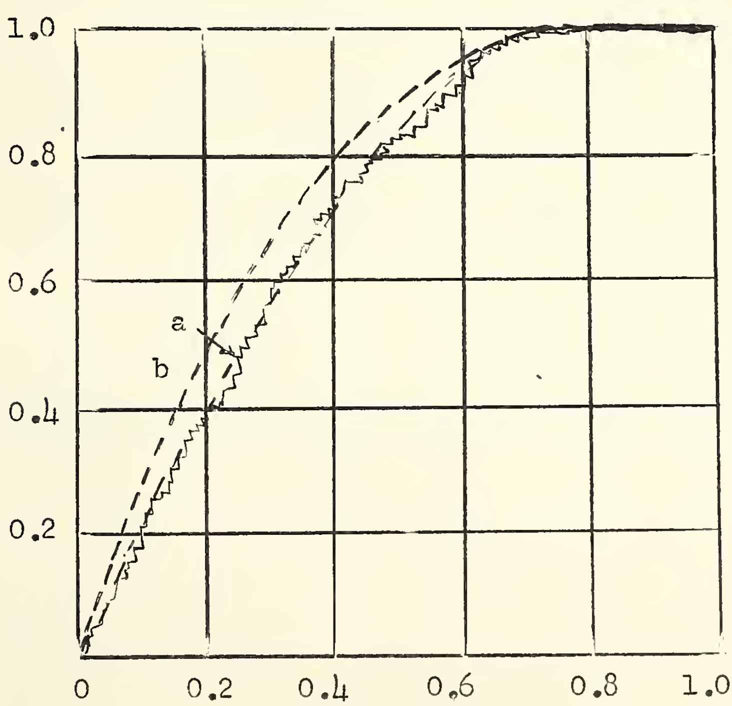

Fig. 1. $F_2(t)$

where the integration is carried out over the region G defined by the inequalities

$$x \leq y \leq z, \quad \frac{y - x}{z - x} < t \quad .$$

3. Let the quantity X be distributed uniformly on the segment (0, 1). Then, on the strength of formula (1),

$$F_k(t) = \frac{n!}{(k-2)!\,(n-k-1)!} \iiint_{G^*} (y - x)^{k-2}(z - y)^{n-k-1} dxdydz \quad , \tag{2}$$

where the integration is carried out over the region G^* defined by the inequalities

$$0 \leq x \leq y \leq z \leq 1, \quad \frac{y - x}{z - x} < t \quad .$$

From the foregoing we obtain, in the case n = 5, for example:

$$F_2(t) = 1 - (1 - t)^3, \quad F_3(t) = t^2(3 - 2t), \quad F_4(t) = t^3 \quad .$$

4. Let the quantity X have a continuous and strictly monotonic distribution function F. By means of the substitution

$X' = F(X)$ the distribution F is reduced to a uniform one, and the distribution function $F_k(t)$ is defined by integral (2) over the region

$$0 \leq x \leq y \leq z \leq 1, \quad \frac{F^{-1}(y) - F^{-1}(x)}{F^{-1}(z) - F^{-1}(x)} < t \quad ,$$

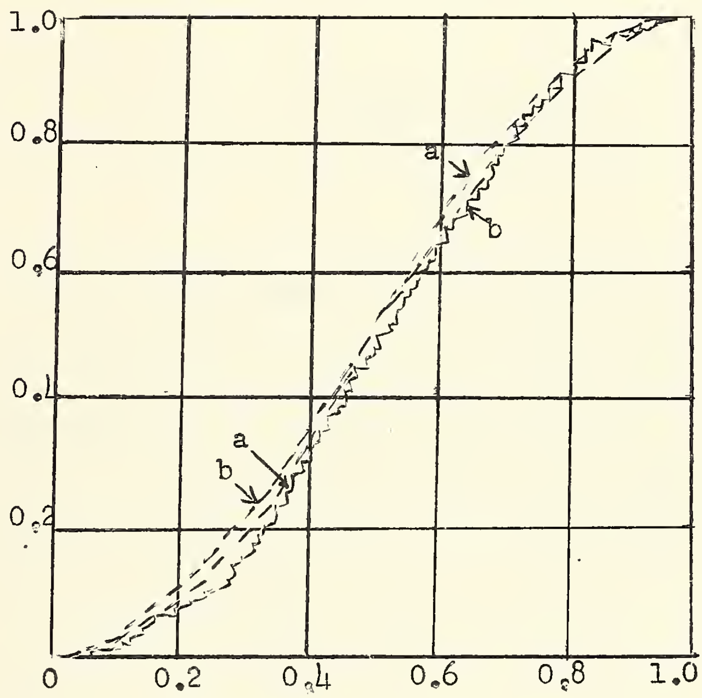

Fig. 2. $F_3(t)$

where F^{-1} is the function inverse to F. Carrying out the integration in expression (2) with respect to the variable x, we

arrive at the following formula, which is suitable for numerical integration:

$$(3) \qquad F_k(t) = \frac{n!}{(k-1)!\,(n-k-1)!} \int_0^1 dz \int_0^z (z-y)^{n-k-1}(y-x_0)^{k-1}\,dy \ ,$$

where

$$x_0 = x_0(y,\ z,\ t) = F\left[\frac{F^{-1}(y) - tF^{-1}(z)}{1-t}\right] \ .$$

5. The recommended method of verifying the admissibility of hypothesis F consists in the following. Each of our N samples gives an observed value of each of the quantities ξ_k $[1<k<n]$. By these N observed values, empirical distribution functions are constructed for the quantities ξ_k . By means of Kolmogorov's criterion[1], a comparison is made between the empirical distribution functions obtained and the theoretical, which have been computed previously by formula (1) or (3). If significant deviations are thereby revealed, be it for even one k, the hypothesis F under test is rejected. In case no such deviations are discovered for any k at all, the data contained in our samples agree well with the hypothesis F.

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(1) See [6].

6. For the practical application of the method described, the functions F_k corresponding to the given value of n and the hypothesis under test, F, must be known. The computations were conducted for $n = 5$ and for two hypotheses--the normal and the uniform distributions.

On the accompanying graphs are exhibited the functions F_2, F_3, F_4, computed for uniform (b) and normal (a) distributions by formulas (2) and (3), and the empirical distribution functions of the quantities ξ_2, ξ_3, ξ_4, constructed in accordance with a sample of 200 groups of 5 taken from tables of random numbers subject to the normal law[1]. In all three cases the empirical distribution functions agree well with the hypothesis of a normal distribution. On comparison with the hypothesis of uniform distribution in the two cases (ξ_2 and ξ_4) we obtain significant deviations. These deviations correspond in the first case to a level of significance of 0.06% and in the second, to a level of significance of 2%.

In conclusion I utilize this opportunity to express to A. N. Kolmogorov my heartfelt gratitude for having posed the problem and for direction during the fulfilment of this work.

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(1) See [7].

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Fig. 3. $F_4(t)$ on next page

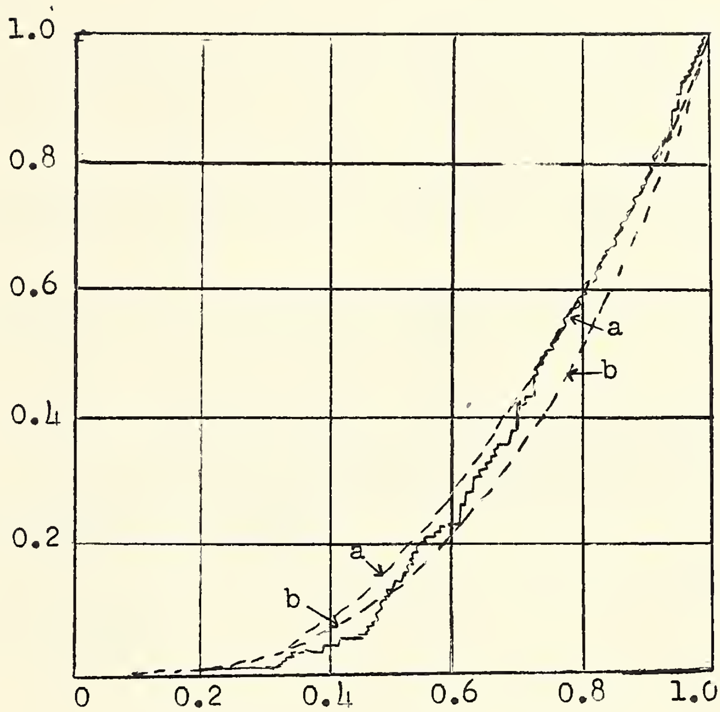

Fig. 3. $F_4(t)$

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THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

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The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

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