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CHARACTERISTIC ROOTS OF QUATERNION MATRICES

by

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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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Characteristic Roots of Quaternion Matrices

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CHARACTERISTIC ROOTS OF QUATERNION MATRICES

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In two recent publications [1]. [2] it was shown that for matrices of (real) quaternion elements an eigenvalue theory can be developed similar to that for complex numbers. If A is such a matrix then quaternion elements and quaternion vectors x can be found such that

$$Ax = x\lambda \ .$$

This note contains some remarks to supplement the results obtained in [1] and [2]. It is, however, self-contained.

For complex matrices it is known that the transposed matrix has the same roots, but different vectors. Let A be a complex matrix and x a vector which corresponds to one of its characteristic roots λ ; let μ be any root of A' and y a characteristic vector which corresponds to it; then it is known that

$$\lambda y'x = \mu y'x \ .$$

From this one concludes that either $\lambda = \mu$ or $y'x = 0$.

In the quaternion case the following generalization holds:

Theorem 1. Let A be a quaternion matrix, λ one of its characteristic roots, x a corresponding vector. Let $\bar{A}'$ be the transposed and conjugate matrix and μ one of its roots with y as a corresponding vector. Then

$$\bar{\lambda}\ \bar{x}'y = \bar{x}'y\ \mu .$$

Proof. The conjugate matrix had to be introduced since the rule $(AB)' = B'A'$ does not hold for quaternion matrices in general. However

$$(\overline{AB})' = \bar{B}'\bar{A}' .$$

By assumption

$$Ax = x\lambda \text{ and } \bar{A}'y = y\mu .$$

Hence

$$\bar{y}'Ax = \bar{y}'x\ \lambda .$$

and

$$\bar{x}'\bar{A}'y = \bar{x}'y\mu .$$

Since these quantities are scalars they are not altered by taking the transpose. Since

$$(\bar{x}'\bar{A}'y) = (\overline{\bar{y}'Ax})'$$

we have

$$\bar{x}'\bar{A}'y = \overline{\bar{y}'Ax}$$

and therefore

$$\bar{x}'y\ \mu = \overline{\bar{y}'x\ \lambda} .$$

i.e.,

$$\bar{\lambda}\ \overline{\bar{y}'x} = \bar{x}'y\ \mu .$$

or

$$\bar{\lambda}\ \bar{x}'y = \bar{x}'y\ \mu .$$

Consider next the case of hermitian matrices when $A = \bar{A}'$. We then have

Theorem 2. A hermitian matrix has only real characteristic roots.

Proof. Let λ be a characteristic root of the hermitian matrix A with x as a corresponding vector. Since λ is also a root of $\bar{A}'$ (= A) with vector x we have by Theorem 1:

$$\bar{\lambda} \cdot \bar{x}'x = \bar{x}'x \, \lambda .$$

Since $\bar{x}'x$ is a real number $\neq 0$ we have

$$\lambda = \bar{\lambda} .$$

In [1] it was shown that the 2x2 quaternion matrix (a_{ik}) has the root $\lambda = a_{21}x_1 + a_{22}$ when x_1 is a solution of $x_1 a_{21} x_1 = a_{11} x_1 - x_1 a_{22} + a_{12}$ and (1,0) is a corresponding vector. Assume further that $a_{ik} = \overline{a_{ki}}$. It can be verified easily that λ is real. By assumption a_{22} is real. Next we show that $\alpha = a_{21}x_1$ is real. For this purpose multiply the conjugate of $x_1 a_{21} x_1$ by x_1 on the right. The result is:

$$\bar{x}_1 a_{12} \, x_1 \bar{x}_1 = (a_{11} - a_{22}) \bar{x}_1 x_1 + a_{21} x_1 .$$

Since $(a_{11} - a_{22})\bar{x}_1 x_1$ is real we see that α has the property that

$$\bar{\alpha} \, r + r_1 = \alpha$$

when r and r_1 are real numbers, r > 0. Hence

$$\alpha \, r + r_1 = \bar{\alpha}$$

and $(\alpha - \bar{\alpha}) r = -(\alpha - \bar{\alpha})$.

Since r > 0 we have $\alpha = \bar{\alpha}$.

Next we consider F, the field of values of the nxn quaternion matrix A, i.e., the set of quaternions

$\sum \bar{x}_i a_{ik} x_k$ where $(x_1, \ldots, x_n)$ is a quaternion vector with $\sum x_i \bar{x}_i = 1$. For this set of quaternions the following theorem holds:

Theorem 3. F is unaltered if the matrix A is replaced by $S^{-1}AS$ where $S = (S_{ik})$ is a unitary matrix. It contains all the characteristic roots. If α is a quaternion in F and ρ an arbitrary quaternion then $\rho^{-1}\alpha\rho$ is also in F. The field of values is bounded and closed.

Proof. Let S be unitary; then $S^{-1} = \bar{S}$. The element in the i-th row and k-th column of $S^{-1}AS$ is then $\sum \bar{S}_{ji} a_{j\ell} S_{\ell k}$ and the field of values of the matrix $S^{-1}AS$ is therefore the set of numbers $\sum \bar{x}_i \bar{S}_{ji} a_{j\ell} S_{\ell k} x_k$ where $\sum x_i \bar{x}_i = 1$. This can be shown to be a set of numbers in the set of values of the matrix A, namely the values corresponding to the vectors $\sum S_{\ell k} x_k$, $\ell = 1, \ldots, n$. Since S is unitary these vectors are unit vectors for $\sum \bar{x}_k \bar{S}_{\ell k} S_{\ell k} x_k = 1$. Since with S also S^{-1} is unitary it follows that conversely the field of values of $S^{-1}AS$ is contained in the field of values of A. This proves the first part of theorem 3.

The next part is proved in the following way. Let λ be any characteristic root of A and x a corresponding vector. Let $x = (x_1, \ldots, x_n)$. We may assume x a unit vector since otherwise every component could be divided by the real scalar $(\sum x_i \bar{x}_i)^{\frac{1}{2}}$ and a unit vector would

thus be obtained. Consider then the numbers in $\hat{F}$ obtained from that particular vector x. It is $\sum \bar{x}_i a_{ik} x_k = \sum x_i \bar{x}_i \lambda = \lambda$. Hence λ lies in F.

The next assertion is proved by considering $\sum \bar{x}_i a_{ik} x_k = \alpha$ for some vector x. Let be an arbitrary quaternion and norm $\rho = r^2$. Consider then the vector $x_1 \rho/r, \ldots x_n \rho/r$ instead of $x_1, \ldots, x_n$. It is still a unit vector. The number in F which corresponds to it is $\rho^{-1} \alpha \rho$. Hence $\rho^{-1} \alpha \rho$ is also in F.

That F is bounded and closed follows from the fact that it is a continuous mapping of the set of vectors $x_1, \ldots, x_n$ with $\sum x_i \bar{x}_i = 1$, hence of the 4 n- dimensional unit sphere.

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[1] H. C. Lee, Eigenvalues and canonical forms of matrices with quaternion coefficients, Proc. Royal Irish Academy LII, 253-260 (1949).

[2] J. Brenner, Matrices of Quaternions, to appear in Pacific Journal of Mathematics

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THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

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