NATIONAL BUREAU OF STANDARDS REPORT

2008

FUNDAMENTAL PROBLEMS

in the

MATHEMATICAL THEORY OF DIFFRACTION

(Steady State Processes)

by

V. D. Kupradze

Onti 1935

Translated from the Russian by Curtis D. Benster

October 6, 1952

Editor: Gertrude Blanch

U. S. DEPARTMENT OF COMMERCE

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V. D. Kupradze

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Editor: Gertrude Blanch

*Published 1935 in Moscow and Leningrad.

**This translation was carried out under the auspices of the Department of Mathematics, University of California, Los Angeles. It was edited by Gertrude Blanch, National Bureau of Standards, Los Angeles.

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The translator has had the assistance of Mr. Jacob Chaitkin, Instructor in Scientific and Technical Russian, California Institute of Technology, in construing a refractory but crucial reading, and to him extends his appreciation.

Curtis D. Benster

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EDITOR'S PREFACE

Kupradze's text came to the attention of the writer during a study of a pamphlet by M. D. Haskind[1] in which Mathieu functions are used to solve a problem in subsonic compressible flow. Haskind states that Kupradze's expressions for the Mathieu-Hankel functions are better than those of Bruno Sieger[2]. Since specialists in the field have heretofore considered Sieger's expansions the best available, the writer began a search for Kupradze's text. No copy seemed to be available in this country,[3] until Professor A. M. Ostrowski of Basle, Switzerland, kindly loaned his personal copy to the National Bureau of Standards. A photostat of the Russian original is now available at the National Bureau of Standards, Los Angeles, and the translation of the text by C. D. Benster is presented here.

The writer read critically the pages relating to Mathieu-function theory -- pages 34 through 69 of the translation. Misprints and errors found in this section have been corrected in the translation. No attempt was made to read the remainder of the text with comparable attention to detail, although obvious errors found by the translator, the proofreaders, and the writer were corrected in the translation. A list of errors discovered is given on pages 159-160 for the benefit

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[1] Haskind, M. D. Oscillations of a wing in a subsonic gas flow. Translated from the Russian by the Graduate Division of Applied Mathematics, Brown University, Translation A9-T-22.

[2] Sieger, B. Die Beugung einer ebenen elektrischen Welle an einem Schirm von elliptischem Querschnitt, Ann. d. Physik, s. 4, 27(1908), 626-664. Diss. Würzburg.

[3] After the translation was ready for distribution, it was discovered that a copy of Kupradze's (Russian) text is available in the Physical Sciences Library, Brown University, Providence, R. I.

of those who may want to read the original Russian text. Readers who have occasion to examine some sections of the text for accuracy of detail are requested to communicate with the writer regarding any other errors they discover.

Regarding Kupradze's expansions of the Mathieu-Hankel functions -- the chief interest of the writer in this text -- it is not at all clear that his expansions are better or simpler than those of Bruno Sieger. It is possible that there may be regions of the complex plane where Kupradze's expansions converge somewhat better than those of Sieger -- especially for the functions of high order, and relatively small argument. Conversely it is believed that Sieger's expansions are superior over other regions. A further report on the relative merits of the various expansions now known will be made after experimenting with them.

Kupradze claims that his expansions are valid over the entire complex ξ-plane, but that those of Sieger are not. This is incorrect. Sieger's expansions, too, converge for all finite values of ξ, and the proof is essentially the same as that given by Kupradze for his expansions. In fact, for the functions of order zero, Kupradze's expansions reduce to Sieger's, except for the normalization factor.

Kupradze's functions suffer from one serious defect. He adopted the normalization suggested by Whittaker[1]. To be specific let

$$ce_{2n}(x, q_1) = \sum_{s=0}^{\infty} A_{2s}^{(2n)} \cos 2sx$$

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[1]Whittaker, E. T. and G. N. Watson. A Course in Modern Analysis, Chapt. XIX, Cambridge, The University Press; American edition, 1946.

be a solution of the even periodic Mathieu function of period π and order 2n. It is known that when q_1 is equal to zero, this periodic solution reduces to a multiple of cos 2nx. For this reason Whittaker suggested that the function be so normalized that $A_{2n}^{(2n)}$ be equal to unity for all orders n. Such a normalization can lead to serious difficulty. For Goldstein[1] proved in 1927 that every coefficient $A_{2s}^{(2n)}$, for n and s different from zero, must pass through zero for at least one value of q_1. For such a value of q_1, therefore, setting $A_{2n}^{(2n)}$ equal to unity means making all other coefficients of the series infinite! It seems that although Kupradze was well acquainted with the work of Ince on Mathieu functions, he did not know of Goldstein's contribution.

Kupradze's expressions seem worse than Sieger's in the region where the asymptotic expansions give good results. Consider, for example, large positive values of ξ. An examination of his expression for $Ze_{2n}(\xi, q_1)$ in equation (51) on page 67 of the translation shows that usually the term involving $a_{2n}^{(2n)}$ will be the dominant one in the approximation to $Ze_{2n}(\xi, q_1)$. In the region where this coefficient is zero or close to it, adjacent terms of the expansion would have to be considered. In contrast, the dominant term in the expansion for large ξ in Sieger's expansions is the first one, and the coefficient associated with this term is always different from zero theoretically. It would seem that there must be fairly large regions where Sieger's expansions are not only simpler, but more rapidly convergent.

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[1]Goldstein, S. Mathieu Functions. Camb. Phil. Soc. Trans. 23 (1927) pp. 303-336.

Nevertheless the translator's contribution is very much worth-while, if only to lay at rest unwarranted claims; and moreover, Kupradze's expansions are different from Sieger's and from that viewpoint worth having.

The writer wishes to express appreciation for the unusually able and intelligent work on the part of the translator. He studied doubtful constructions with a diligence that one does not ordinarily expect, and he made every effort to present, in good English idiom, a faithful translation of the Russian original. He also proofread part of the final manuscript. Thanks are also due to Mr. Eugene Levin and Mr. William Keating, who proofread part of the manuscript. Mr. Levin noted some errors in the chapters he read, and those were corrected. The list of credits would be incomplete without mentioning the careful work of Vendla H. Gordanier and Betty Schoenberg, who typed the manuscript.

Gertrude Blanch

October 6, 1952.

CHAPTER I

THE FUNDAMENTAL EQUATIONS

§1. Electromagnetic diffraction: the problem stated. Fundamental equations. According to the Maxwell electromagnetic theory, a light disturbance at a given point in space is describable by the pair of vectors $\mathfrak{E}$ and $\mathfrak{H}$, which are related by the equations

$$\left.\begin{aligned} \frac{\partial \mathfrak{H}}{\partial t} &= -\,c \operatorname{rot} \mathfrak{E} \;; &\quad \varepsilon \frac{\partial \mathfrak{E}}{\partial t} + \sigma \mathfrak{E} &= c \operatorname{rot} \mathfrak{H} \;; \\ \operatorname{div} \mathfrak{E} &= 0 \;; & \operatorname{div} \mathfrak{H} &= 0 \;, \end{aligned}\right\} \tag{1}$$

where ε and σ are the dialectric constant and the conductivity of the medium, and c is the velocity of light in a vacuum.

Taking $\mathfrak{E}$ and $\mathfrak{H}$ as pure periodic functions of the time:

$$\mathfrak{E} = \mathfrak{R}(\bar{E}e^{i\omega t}) \;; \qquad \mathfrak{H} = \mathfrak{R}(\bar{H}e^{i\omega t}) \tag{2}$$

equations (1) may be given the following form:

$$\left.\begin{aligned} c \operatorname{rot} \bar{E} &= -\, i\omega \bar{H} \;; &\quad c \operatorname{rot} \bar{H} &= (i\omega\varepsilon + \sigma)\bar{E} \;; \\ \operatorname{div} \bar{E} &= 0 \;; & \operatorname{div} \bar{H} &= 0 \;. \end{aligned}\right\} \tag{3}$$

Eliminating $\bar{E}$ and $\bar{H}$ successively from these equations, we obtain for the determination of these components the Helmholtz vibrational equations:

$$\left.\begin{aligned} \Delta\bar{E} + k^2\bar{E} &= 0 \;; \\ \Delta\bar{H} + k^2\bar{H} &= 0 \;, \end{aligned}\right\} \tag{4}$$

Δ being the Laplace operator and

$$k^2 = \frac{\varepsilon\,\omega^2 - i\omega\sigma}{c^2} \quad . \tag{5}$$

In case $\sigma = 0$, k reduces to the ordinary wave number, which is connected with the wave length λ by the equation:

$$k = \frac{\omega\sqrt{\varepsilon}}{c} = \frac{2\pi}{\lambda} \quad . \tag{6}$$

We shall be mainly concerned below with the study of cases of reflection and diffraction of electromagnetic disturbances where there is no dependence of the system on one of the coordinates, for example z. In this connection

No. 1. $H_z = 0$, $E_z \neq 0$, i.e., the vibrational process is polarized perpendicularly to the z-axis. In this case equations (3) take the form

$$\left.\begin{aligned} &\frac{\partial E_z}{\partial y} = -\frac{i\omega}{c} H_x \ ; \qquad \frac{\partial E_z}{\partial x} = \frac{i\omega}{c} H_y \ , \\ &\Delta E_z + k^2 E_z = 0 \quad . \end{aligned}\right\} \tag{7}$$

No. 2. $E_z = 0$, $H_z \neq 0$, i.e., the vibrational process is polarized parallel to the z-axis. In this case equations (3) take the form:

$$\left.\begin{aligned} &\frac{\partial H_z}{\partial y} = \frac{1}{c}(i\omega\varepsilon + \sigma)E_x \ ; \qquad \frac{\partial H_z}{\partial x} = -\frac{1}{c}(i\omega\varepsilon + \sigma)E_y \ ; \\ &\Delta H_z + k^2 H_z = 0 \quad . \end{aligned}\right\} \tag{8}$$

It is readily seen that the general case may be regarded as the result of superimposing the two indicated independent and formally identical cases.

In addition to the Helmholtz equations that the electric and magnetic vectors satisfy, certain boundary conditions must be satisfied on the surface (or on the contour) of the diffracting obstacle. In the case to be considered here, these conditions, in conformity with the Maxwell theory, are expressible in the continuity of the tangential components of E and H.

Some other boundary conditions will be indicated below.

Lastly, if the mathematical solution of the boundary problem in question is to be equivalent to a physical picture of the diffraction, it is still necessary that the conditions for the solution's existence and uniqueness be formulated. These conditions concern the character at infinity of the solutions sought.

In §3 we shall dwell in detail on an investigation of this question.

§2. Diffraction of elastic waves -- the problem stated. Fundamental equations. In accordance with the general theory of elasticity, the displacement vector $u = \bar{u}(x, y)$, in the plane problem, may be represented in the form:

$$\left.\begin{aligned} &\bar{u} = \operatorname{grad} \bar{\phi} + \operatorname{rot} \bar{\Psi} \; ; \qquad \bar{\Psi} = (0, 0, \Psi) \; ; \\ &\text{or} \\ &u_x = \frac{\partial \phi}{\partial x} + \frac{\partial \Psi}{\partial y} \; ; \qquad u_y = \frac{\partial \phi}{\partial y} - \frac{\partial \Psi}{\partial x} \; . \end{aligned}\right\} \qquad (9)$$

ϕ and Ψ represent the wave potentials of the longitudinal and transverse vibrations respectively, propagated in the given homogeneous medium with the constant velocities a and $b < \frac{a}{2}$; here the following wave equations hold:

$$\left.\begin{aligned}\Delta\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2} \quad ; \\ \Delta\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2} \quad .\end{aligned}\right\} \qquad (10)$$

In the simplest diffraction problem or in the simplest external boundary problem of the theory of elasticity, which may be called the generalized external Neumann problem, the law of propagation of the displacement u(x, y) has to be determined at an arbitrary point of the plane, an excision or occlusion of arbitrary form being present in it which distorts the fundamental disturbance and generates additional waves.

At the contour of the obstacle values of the displacements, or definite relations of dependence between them, are ordinarily assigned; the picture of the phenomenon as a whole, moreover, ought to be reconstructible in accordance with them.

Finally, for the existence and uniqueness of the solutions, it is necessary to seek supplementary disturbances in the form of special waves having the properties of disturbances propagating to infinity and dying out there. If we call the diffracting contour (γ), (the analytical properties of the curve (γ) will be looked into in detail below), the simplest problem may be formulated thus: Given on (γ):

$$\left[\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}\right]_{(\gamma)} = f(x, y) \quad ;$$

$$\left[\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}\right]_{(\gamma)} = F(x, y) \quad ;$$

find ϕ and ψ at an arbitrary point outside of (γ) so that the supplementary disturbances arising from the meeting of the incident disturbances and the obstacle (γ) have the form of phases which move in a determinate way off to infinity and there die (the Emission principle).

In the boundary problem of the theory of elasticity just considered, the components of the displacement vector were assigned at the contour of the obstacle. In other problems relations connecting the components of the stress tensor with each other are assigned.

The general expression for the stress tensor is given by the formulas:

$$\left.\begin{aligned} p_{xx} &= \rho\left[a^2 \frac{\partial^2\phi}{\partial x^2} + 2b^2 \frac{\partial^2\psi}{\partial x\partial y} + (a^2 - 2b^2)\frac{\partial^2\phi}{\partial y^2} \right] ; \\ p_{xy} &= b^2\rho\left[2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2}\right] = p_{yx} ; \\ p_{yy} &= \rho\left[(a^2 - 2b^2)\frac{\partial^2\phi}{\partial x^2} + a^2\frac{\partial^2\phi}{\partial y^2} - 2b^2\frac{\partial^2\psi}{\partial x\partial y}\right] . \end{aligned}\right\} \quad (11)$$

In spatial problems the formulas for elastic displacements have, as is known, the following form:

$$\left.\begin{aligned} \bar{u} = \text{grad}\,\bar{\phi} + \text{rot}\,\bar{\psi} \; ; &\qquad \bar{\psi} = (A, B, 0) \; ; \\ \Delta\bar{\phi} = \frac{1}{a^2}\frac{\partial^2\bar{\phi}}{\partial t^2} \; ; &\qquad \Delta\bar{\psi} = \frac{1}{b^2}\frac{\partial^2\bar{\psi}}{\partial t^2} \, , \end{aligned}\right\} \quad (12)$$

and for problems with stresses the basic equations are expressible in the form:

$$\left.\begin{aligned} &u = \operatorname{grad} \bar{\phi} + \operatorname{rot} \bar{\Psi} \; ; \quad \bar{\Psi} = (A, B, 0) \; ; \\ &p_{zx} = \rho b^2 \left[2 \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 B}{\partial z^2} + \frac{\partial^2 B}{\partial x^2} - \frac{\partial^2 A}{\partial x \partial y} \right] ; \\ &p_{zy} = \rho b^2 \left[2 \frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 A}{\partial z^2} - \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 B}{\partial x \partial y} \right] ; \\ &p_{zz} = \rho \left[a^2 \frac{\partial^2 \phi}{\partial z^2} - 2b^2 \left(\frac{\partial^2 A}{\partial y \partial z} - \frac{\partial^2 B}{\partial x \partial z} \right) + (a^2 - 2b^2) \left(\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} \right) \right] . \end{aligned}\right\} \quad (13)$$

In all these cases the boundary problem is expressed at the diffracting contour or diffracting surface by the right members of formulas (11), (12), (13), or by definite combinations of them. The general remarks made above concerning the behavior at infinity of the solutions sought must here also be repeated.

§3. The Uniqueness Theorem. The Emission principle. As has already been stated in §1, all problems with which we shall be concerned below lead to one boundary problem or another of the Helmholtz equation

$$\Delta u + k^2 u = f \quad ,$$

where Δ is a two- or three-dimensional Laplace operator and f is a given function of the point, or, for boundary problems, of the system of equations

$$\Delta u + k_1^2 \, u = f_1 \quad ,$$

$$\Delta v + k_2^2 \, v = f_2 \quad .$$

The question arises: Under what conditions can the aforementioned boundary problems have unique solutions?

The elucidation of this question is the more necessary in that the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} + k^2 u = f(x, y) \qquad (14)$$

is, in regions extending to infinity, not uniquely defined with the assignment of the singularity and with the condition of vanishing at infinity, since the infinite region admits proper vibrations for a continuous spectrum of values k.

In contradistinction to harmonic equations, the vibrational equation admits of a solution which is everywhere regular, different from zero, and vanishing at infinity. Physically, this fact corresponds to the existence of standing waves which, being solutions of the vibrational equation, may be superimposed on the sought solution of the given boundary problem without altering the given boundary conditions. These parasitic waves which thus destroy the uniqueness of the solution of the boundary problems, arise as the result of the imposition of the phases propagated to infinity and coming from infinity. It is perfectly obvious physically that the last type of wave must not arise.

These considerations led A. Sommerfeld in the year 1912 to the Emission principle, which is formulated as follows:

Let (γ) be a simple (simply connected) contour enclosing a portion of the infinite region under consideration; let (Σ) be a circumference of sufficiently large radius to contain within itself the contour (γ); let r be the distance from a point outside (γ) to a point on (Σ); let $\frac{\partial r}{\partial n_\Sigma} = \cos(r, n_\Sigma)$, n_Σ being the external

normal. Then the solution of a plane vibrational equation (14) satisfying the conditions

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial n} + iku\right) = 0 \quad , \tag{15}$$

$$\sqrt{r}u_{r\to\infty} = \text{a finite quantity} \quad ,$$

$$u = 0 \quad \text{or} \quad \frac{\partial u}{\partial n} = 0 \text{ on } (\gamma) \tag{16}$$

will be uniquely defined.

This conclusion is obviously equivalent to the assertion that given conditions (15) - (16) and the conditions of regularity everywhere, the equation $\Delta u + k^2 u = 0$ has only a null solution.

Thus the questions relating to the proof of uniqueness in problems leading to the consideration of a vibrational equation in infinite regions are connected in the closest manner with condition (15), which Sommerfeld called the Emission principle or condition.

The author of the present work has given a proof of this principle in a special work[1], which we shall reproduce in basic outlines.

THEOREM. A solution of the wave equation $\frac{\partial^2 u_1}{\partial t^2} = c^2 \Delta u_1$ of the form

$$e^{-int} u(x, y) \quad ,$$

where $u(x, y)$ is the solution of the equation $\Delta u + k^2 u = 0$ $\left(k = \frac{n}{c}\right)$, satisfying condition (15) at infinity and one of conditions (16) on (γ), can only be identically zero for real k.

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[1] The "Emission Principle" of A. Sommerfeld. (Doklady A. N. S.S.S.R., 1934, #2).

We shall prove the following lemma:

A solution of the equation $\Delta u + k^2 u = 0$ satisfying the conditions

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial n} + iku\right) = 0 \quad , \tag{15}$$

$$\lim_{r\to\infty} \int (\sqrt{r}\, u)^2 \, d\psi = 0 \quad , \tag{17}$$

is identically zero.

We shall represent the sought solution in form

$$u = \sum_{n=0}^{\infty} H_n^{(2)}(kr)(a_n \sin n\theta + b_n \cos n\theta) \quad ,$$

where $H_n^{(2)}(kr)$ is the Hankel function of the second type.

Utilizing the asymptotic expressions for these functions, we may write

$$\sqrt{r}\, u = \sum_{n=0}^{\infty} \left[A + B \cdot 0\left(\frac{1}{|kr|}\right)\right](a_n \sin n\theta + b_n \cos n\theta) \quad ,$$

where A and B are definite bounded quantities.

Now, on the basis of property (17) we have, from Parseval's theorem,

$$a_n = 0 \quad , \quad b_n = 0 \quad ,$$

i.e.,

$$u \equiv 0 \quad .$$

Define the essentially positive form

$$E = \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 + \frac{1}{c^2}\left(\frac{\partial u_1}{\partial t}\right)^2\right] \, .$$

Having in view the familiar physical analogy, let us for brevity call E the quasi-energy.

Let us determine the vector of the flow of quasi-energy from the condition

$$\iint_{\Omega} \frac{\partial E}{\partial t}\, d\omega = \int_{(\gamma)} P\bar{n}\, d\sigma ,$$

where Ω is the region between (γ) and (Σ) and $\bar{n}$ is the ort[1] of the normal. On the other hand, we have

$$\frac{\partial E}{\partial t} = \operatorname{grad} \frac{\partial u_1}{\partial t} \operatorname{grad} u_1 + \frac{\partial u_1}{\partial t} \Delta u_1 = \operatorname{div} \left[\frac{\partial u_1}{\partial t} \operatorname{grad} u_1 \right]$$

and by Gauss' formula

$$\iint_{\Omega} \operatorname{div}\left[\frac{\partial u_1}{\partial t} \operatorname{grad} u_1 \right] d\omega = -\int_{(\gamma)} \left[\frac{\partial u_1}{\partial t} \operatorname{grad} u_1 \right] \bar{n}\, d\sigma ,$$

whence

$$P = -\frac{\partial u_1}{\partial t} \operatorname{grad} u_1 .$$

Let

$$u = \phi + i\psi , \qquad (18)$$

---

(1) Heaviside, O., Electromagnetic Theory §121: "Sometimes it is convenient to have the variation of a vector exhibited in terms of the variations of its tensor and unit vector, or its size and ort." (He elsewhere explains that ort is a workmanlike contraction of orientation.) -- Translator's Note.

then

$$u_1 = \phi \cos nt + \psi \sin nt \quad ,$$

$$P = n(\phi \sin nt - \psi \cos nt)(\cos nt \operatorname{grad} \phi + \sin nt \operatorname{grad} \psi)$$

and therefore

$$P\bar{n} = n\,(\phi \sin nt - \psi \cos nt)\left(\frac{\partial \phi}{\partial n} \cos nt + \frac{\partial \psi}{\partial n} \sin nt\right) . \quad (19)$$

If we take into consideration that, in condition (15), $u = \phi + i\,\psi$, we shall obtain

$$\left.\begin{aligned} \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \phi}{\partial n} - k\,\psi\right) &= 0 \quad ; \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \psi}{\partial n} + k\phi\right) &= 0 \quad . \end{aligned}\right\} \quad (20)$$

Utilizing these equations, we may write

$$\begin{aligned} P\bar{n} = -\,kn(\phi \sin nt - \psi \cos nt)^2 + \Big[\Big(\frac{\partial \phi}{\partial n} - k\,\psi\Big) \cos nt \\ + \Big(\frac{\partial \psi}{\partial n} + k\phi\Big) \sin nt\Big] \cdot (\phi \sin nt - \psi \cos nt) \cdot n \quad . \end{aligned} \quad (21)$$

From (16), (18) and (19) it follows that

$$\int_{(\gamma)} P\bar{n}\,d\sigma = 0 \quad ,$$

consequently no change of quasi-energy occurs across (γ) into region Ω or from it.

On the other hand, in accordance with (21):

$$\left.\begin{aligned}\int_{(\Sigma)} P\bar{n}ds &= -kn \int_{(\Sigma)} (\phi \sin nt - \psi \cos nt)^2 ds \\ &+ n \int_0^{2\pi} (\phi \sin nt - \psi \cos nt) \times \\ &\times \left[\left(\frac{\partial \phi}{\partial n} - k\psi\right) \cos nt + \left(\frac{\partial \psi}{\partial n} + k\phi\right) \sin nt\right] rd\alpha \ .\end{aligned}\right\} \quad (22)$$

In view of the fact that as $r \to \infty$ the integrals $\int \sqrt{r}\,\phi\, d\alpha$ and $\int \sqrt{r}\,\psi\, d\alpha$ are bounded, equations (20) show that the second integral summand in the right part of (22) may be made as small as one pleases by taking r sufficiently large.

Hence it follows that if, as $r \to \infty$,

$$-kn \int_{(\Sigma)} (\phi \sin nt - \psi \cos nt)^2 ds = -kn \int_0^{2\pi} r(\phi \sin nt - \psi \cos nt)^2 d\alpha$$

does not tend to zero, then in accordance with the formula

$$\frac{\partial}{\partial t} \int\int_{\Omega} E d\omega = \int_{(\gamma)} P\bar{n} d\sigma \ ,$$

we shall have, from (22),

$$\frac{\partial}{\partial t} \int\int_{\Omega} E d\omega < 0 \ ,$$

and E tends toward negative infinity, which contradicts its essentially positive nature.

Hence it follows that

$$\lim_{r\to\infty} \int_0^{2\pi} r(\phi \sin nt - \psi \cos nt)^2 \, d\alpha = 0 \; ;$$

$$\lim_{r\to\infty} \int \left[\sqrt{r}\, \phi \right]^2 d\alpha = 0 \; ; \quad \lim_{r\to\infty} \int \left[\sqrt{r}\, \psi \right]^2 d\alpha = 0 \; ;$$

$$\lim_{r\to\infty} \int \left[\sqrt{r} \cdot n \right]^2 d\alpha = 0 \; ,$$

but in this case, in conformity with the lemma, $u \equiv 0$, and the fundamental theorem has consequently been proved.

CHAPTER II

THE METHOD OF CURVILINEAR COORDINATES

THE SIMPLEST PROBLEMS

§4. The diffraction of electromagnetic waves near a cylinder. In the following sections we shall give an exposition of some of the elementary classical cases of the diffraction of electromagnetic waves, the solution being by the method of curvilinear coordinates. The essence of this method consists in the introduction of special coordinates such that the diffracting surface is a coordinatal surface. Once having expressed all the equations and boundary conditions in these coordinates, a considerable formal simplification of the former is usually attained. The actual construction of the fundamental functions is marked, however, by great difficulties, and an effective solution is obtainable for a very limited number of special contours and surfaces only.

Let the diffracting contour (γ) be a circumference of radius ρ. In the plane of (γ) in the direction $\theta = \theta_0$ falls a plane, linearly polarized disturbance u_e; together with this, the electrical vector $E_z \neq 0$ and the vibrational process is polarized perpendicularly to the axis z (Chap. I, §1, #1); then

$$u_e = E_z = e^{ikr\cos(\theta-\theta_0)} \quad . \tag{1}$$

Introducing polar coordinates, we transform equation (7) (Chap. I, §1) to:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0 \quad .$$

For fixed r, u is a periodic function of θ with period 2π, and in view of the symmetry, we may seek it in the form of an even Fourier series in θ with coefficients depending on r:

$$u = \sum_{n=0}^{\infty} a_n(r) \cos n(\theta - \theta_0) \quad , \tag{2}$$

and for the coefficients, $a_n(r)$, we as usual obtain the Bessel equation:

$$\frac{d^2 a_n}{dr^2} + \frac{1}{r}\frac{da_n}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) a_n = 0 \quad ,$$

whence

$$a_n = c_n J_n(kr) + c_n' Y_n(kr) \quad .$$

From the form of a_n it follows that u, as determined from (2), must be separated into u_i and u_a (u_i within γ, u_a outside γ), and in consequence of the obvious regularity of u_i, it is necessary to seek it in the form

$$u_i = \sum_{n=0}^{\infty} B_n J_n(k_i r) \cos n(\theta - \theta_0) \quad . \tag{3}$$

As regards u_a, in conformity with the Emission principle, it is necessary to seek it in the form:

$$u_a = e^{ik_a r \cos(\theta - \theta_0)} + \sum_{n=0}^{\infty} A_n H_n^{(2)}(k_a r) \cos n\,(\theta - \theta_0) \quad . \tag{4}$$

Let us recall that the asymptotic representation of

$$H_n^{(2)}(k_a r) = \sqrt{\frac{2}{\pi k_a r}}\, e^{-i\left(k_a r - \frac{2n+1}{4}\pi\right)} \quad ,$$

satisfies the requirements of the Emission principle.

The indices i and a on the k are introduced to mark the physical difference between the media inside and outside of (γ); this difference is indeed the cause of there being differing frequencies. On the boundary (γ), in conformity with the general Maxwell theory, the equalities

$$u_i = u_a \quad ; \quad \frac{\partial u_i}{\partial r} = \frac{\partial u_a}{\partial r} \quad ; \quad [r = \rho]$$

obtain.

Let us insert in these conditions the series cited above, first having represented u_e too in series form:

$$u_e = e^{ik_a r \cos(\theta - \theta_o)} = J_o(k_a r) + 2 \sum_{n=o}^{\infty} i^n J_n(k_a r) \cos n(\theta - \theta_o) \quad ;$$

then for the coefficients A_n, for example, we have

$$A_n = -2i^n \frac{k_i J_n(k_a\rho)J_n(k_i\rho) - k_a J_n'(k_a\rho)J_n(k_i\rho)}{k_i H_n^2(k_a\rho)J_n'(k_i\rho) - k_a H_n'^2(k_a\rho)J_n(k_i\rho)} \quad . \qquad (5)$$

Analogous expressions can be found for B_n as well. A more detailed inquiry reveals that the series thus obtained converge with a rapidity that is suitable in practice only in those cases for which

$$2\rho \ll \frac{4\pi}{k} \quad ,$$

i.e., where the diameter of the circle is small in comparison with the length of the wave. The case considered in this section is

obviously identical with the phenomenon of diffraction near an infinitely long circular cylinder, on condition that the phenomenon be independent of the z-coordinate. It is easily shown that the "proper vibrations" will be decaying ones in the case in hand. Indeed, it is known that the Hankel functions, in contradistinction to the Bessel function, always have complex values, and therefore the transcendental equations

$$k_i J_n'(k_i \rho)\, H_n^{(2)}(k_a \rho) - k_a J_n(k_i \rho)\, H_n'^{2}(k_a \rho) \tag{6}$$

may have complex roots; since ρ is real, k must be complex; hence follows too our statement concerning the decaying of the proper vibrations.

§5. <u>The diffraction of electromagnetic waves near a sphere</u>[1].

In spherical coordinates equations (1) (Chap. I, §1) have the following form:

$$r \sin\theta \left(\frac{i\omega\varepsilon}{c} + \frac{\sigma}{c}\right) E_r = \frac{\partial (H_\phi \sin\theta)}{\partial\theta} - \frac{\partial (H_\theta)}{\partial\phi} , \tag{7}$$

$$r \sin\theta \left(\frac{i\omega\varepsilon}{c} + \frac{\sigma}{c}\right) E_\theta = \frac{\partial (H_r)}{\partial\phi} - \frac{\partial (rH_\phi \sin\theta)}{\partial r} , \tag{8}$$

$$r \left(\frac{i\omega\varepsilon}{c} + \frac{\sigma}{c}\right) E_\phi = \frac{\partial (rH_\theta)}{\partial r} - \frac{\partial (H_r)}{\partial\theta} , \tag{9}$$

$$-\frac{i\omega}{c}\, r \sin\theta H_r = \frac{\partial (E_\phi \sin\theta)}{\partial\theta} - \frac{\partial (E_\theta)}{\partial\phi} , \tag{10}$$

$$-\frac{i\omega}{c}\, r \sin\theta H_\theta = \frac{\partial (E_r)}{\partial\phi} - \frac{\partial (rE_\phi \sin\theta)}{\partial r} , \tag{11}$$

$$-\frac{i\omega}{c}\, rH_\phi = \frac{\partial (rE_\theta)}{\partial r} - \frac{\partial (E_r)}{\partial\theta} . \tag{12}$$

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[1] G. Mie, Ann. d. Phys. 25, 1908; P. Debye, Ann. d. Phys. 38, Handbuch d. Phys. Bd. XX.

In analogy with a plane vibrational process (Chap. I, §1), we shall here consider separately two cases:

$$\left.\begin{array}{ll} 1) \; E_r = 0 \;, & H_r \neq 0 \\ \text{and} & \\ 2) \; H_r = 0 \;, & E_r \neq 0 \;. \end{array}\right\}$$

The general case is the superposition of these two component processes. In case 2) equation (10) acquires the form

$$\frac{\partial (E_\phi \sin\theta)}{\partial\theta} - \frac{\partial E_\theta}{\partial\phi} = 0 \;,$$

which may be satisfied by putting

$$E_\phi = \frac{1}{r\sin\theta}\frac{\partial^2(ru)}{\partial\phi\,\partial r} \;; \quad E_\theta = \frac{1}{r}\frac{\partial^2(ru)}{\partial r\,\partial\theta} \;; \tag{13}$$

now from the system (7), (8), (9) we find all the components of the magnetic vector and E_r.

Substituting in (11) or (12), moreover, we are in both cases led, for u, to the vibrational equation in spherical coordinates:

$$\frac{1}{r}\frac{\partial^2(ru)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial \sin\theta \frac{\partial u}{\partial\theta}}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + k^2u \;. \tag{14}$$

In case 1), in a completely analogous manner, putting

$$H_\phi = \frac{1}{r\sin\theta}\frac{\partial^2(rv)}{\partial r\,\partial\phi} \;; \quad H_\theta = \frac{1}{r}\frac{\partial^2(rv)}{\partial r\,\partial\theta} \;, \tag{15}$$

we determine the other components of the field and obtain for v the vibrational equation (14).

The components of the field are obtained in the general case by superimposing the two processes just considered by means of the following formulas:

$$\left.\begin{aligned}
E_r &= \frac{\partial^2(ru)}{\partial r^2} + k^2 ru \ , \\
E_\phi &= \frac{1}{r\sin\theta}\frac{\partial^2(ru)}{\partial r\partial\phi} + \frac{i\omega}{c}\frac{1}{r}\frac{\partial(rv)}{\partial\theta} \ , \\
E_\theta &= \frac{1}{r}\frac{\partial^2(ru)}{\partial r\partial\theta} - \frac{i\omega}{c}\frac{1}{r\sin\theta}\frac{\partial(rv)}{\partial\theta} \ , \\
H_r &= \frac{\partial^2(rv)}{\partial r^2} + k^2 rv \ , \\
H_\phi &= -\frac{1}{r}\left(\frac{i\omega\varepsilon}{c} + \frac{\sigma}{c}\right)\frac{\partial(ru)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2(rv)}{\partial r\partial\phi} \ , \\
H_\theta &= \left(\frac{i\omega\varepsilon}{c} + \frac{\sigma}{c}\right)\frac{1}{r\sin\theta}\frac{\partial(ru)}{\partial\phi} + \frac{1}{r}\frac{\partial^2(rv)}{\partial r\partial\theta} \ .
\end{aligned}\right\} \qquad (16)$$

The field exterior to the sphere will consist of an incident disturbance, of a given potential u^e and of a diffracted potential u^*. In addition, a refracted disturbance arises within the sphere; it will be described by the potential u'.

In accordance with the general Maxwell theory, the boundary conditions have the following form:

$$E_\theta^a = E_\theta^i \ ; \quad E_\phi^a = E_\phi^i \ ; \quad H_\theta^a = H_\theta^i \ ; \quad H_\phi^a = H_\phi^i \ . \qquad (17)$$

Here the indices a and i indicate the external and internal nature of the components of the field.

The incident disturbance we shall assume to be plane, from the region of positive z perpendicular to the x-axis, moving in a linearly polarized wave:

$$E_x^e = e^{ik_a z} \; ; \; E_y^e = E_z^e = 0 \; ; \; H_y^e = - e^{ik_a z} \; ; \; H_x^e = H_z^e = 0 \quad . \quad (18)$$

Equation (14) will be integrated by separating the variables:

$$u = f(r)\, Y(\phi, \theta) \quad ,$$

where, as is known, $Y(\phi, \theta)$ turn out to be spherical Laplace functions. Expanding them in Fourier series, we obtain

$$Y_n(\phi, \theta) = \sum_{s=0}^{n} P_{n,s}(a_{n,s} \cos s\phi + b_{n,s} \sin s\phi) \quad ,$$

where $P_{n,s}$ are the associated Legendre spherical functions of the s-th order with argument $\cos\theta$:

$$P_{n,s}(x) = \left(\sqrt{1 - x^2}\right)^s \frac{d^s P_n(x)}{dx^s} \quad .$$

It is obvious that $P_{n,s} = 0$ for $s > n$.

As regards $f(r)$, making the substitution $f(r) = (kr)^{-\frac{1}{2}} C(kr)$ in one of the equations obtained from (14) after separating the variables, we obtain for the determination of $C(kr)$ a Bessel equation with characteristic numbers $(n + \frac{1}{2})$, and consequently

$$C_{n+\frac{1}{2}}(kr) = A_n J_{n+\frac{1}{2}}(kr) + B_n Y_{n+\frac{1}{2}}(kr) \quad .$$

Thus we have as one of the solutions of equation (14)

$$u_n = \frac{Y_n(\theta,\phi)C_{n+\frac{1}{2}}(kr)}{\sqrt{k\,r}}$$

and accordingly the following general expression will also be a solution:

$$u = \sum_{n=1}^{\infty}\sum_{s=o}^{n} (kr)^{-\frac{1}{2}} C_{n+\frac{1}{2}}(kr)\, P_{n,s}(\cos\theta)(a_{n,s}\cos s\phi + b_{n,s}\sin s\phi) \quad .(19)$$

Putting this in (16) and taking the Bessel equation for $C_{n+\frac{1}{2}}(kr)$ into account, we obtain for E_r and H_r the formally identical expressions

$$\sum_{n=1}^{\infty}\sum_{s=o}^{n}\left(\frac{r}{k}\right)^{\frac{1}{2}} \frac{n(n+1)}{r^2} C_{n+\frac{1}{2}}(kr)\, P_{n,s}(\cos\theta)(a_{n,s}\cos s\phi + b_{n,s}\sin s\phi) .(20)$$

For the determination of $a_{n,s}$ and $b_{n,s}$ as well as of A_n and B_n, we shall express the components of the incident disturbance in spherical coordinates and afterwards expand E_r^e and H_r^e in series of spherical functions, upon which the following results are obtained:

$$E_r^e = \sum_{n=1}^{\infty} i^{n-1}(2n+1)\sqrt{\frac{\pi}{2}}\,(k_a r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(k_a r)\, P_{n,1}(\cos\theta)\cos\phi \quad ,$$

$$H_r^e = \sum_{n=1}^{\infty} i^{n+1}(2n+1)\sqrt{\frac{\pi}{2}}\,(k_a r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(k_a r)\, P_{n,1}(\cos\theta)\sin\phi \quad .$$

Comparing these expansions with (20) and (19), it is natural to require that in the expression for u^e we have

$$a_{n,s} = b_{n,s} = 0 \quad ,$$

if.

$$s \neq 1 \; ;$$

$$a_{n,1} = 1 \; , \quad b_{n,1} = 0 \; ; \quad A_n = 0 \; , \quad B_n = \frac{1}{k_a} \sqrt{\frac{\pi}{2}} \, i^{n-1} \frac{2n+1}{n(n+1)} \; ,$$

and that in the expression for v^e we have

$$a_{n,s} = b_{n,s} = 0 \; ,$$

if

$$s \neq 1 \; ;$$

$$a_{n,1} = 0 \; , \quad b_{n,1} = 1 \; ; \quad B_n = 0 \; , \quad A_n = \frac{1}{k_a} \sqrt{\frac{\pi}{2}} \, i^{n+1} \frac{2n+1}{n(n+1)} \; ;$$

thus it becomes evident that the magnetic and electric potentials of the incident disturbance have the form

$$ru^e = \frac{1}{k_a^2} \sum_{n=1}^{\infty} i^{n-1} \sqrt{\frac{\pi k_a r}{2}} \; \frac{2n+1}{n(n+1)} \, J_{n+\frac{1}{2}}(k_a r) \, P_{n,1} \, (\cos\theta) \cos\phi \; ,$$

$$rv^e = \frac{1}{k_a^2} \sum_{n=1}^{\infty} i^{n+1} \sqrt{\frac{\pi k_a r}{2}} \; \frac{2n+1}{n(n+1)} \, J_{n+\frac{1}{2}}(k_a r) \, P_{n,1} \, (\cos\theta) \sin\phi \; .$$

Proceeding from the form of these expansions, we seek u^*, v^* and u_i and v_i in the form of analogous series, where for the first two, in accordance with the Emission principle, the Bessel functions are to be replaced by Hankel functions:

$$ru^* = \frac{1}{k_a^2} \sum_{n=1}^{\infty} \alpha_n^* \sqrt{\frac{\pi k_a r}{2}}\, H^{(2)}_{n+\frac{1}{2}}(k_a r)\, P_{n,1}(\cos\theta) \cos\phi \quad ,$$

$$rv^* = \frac{1}{k_a^2} \sum_{n=1}^{\infty} \beta_n^* \sqrt{\frac{\pi k_a r}{2}}\, H^{(2)}_{n+\frac{1}{2}}(k_a r)\, P_{n,1}(\cos\theta) \sin\phi \quad ,$$

$$ru_i = \frac{1}{k_i^2} \sum_{n=1}^{\infty} \alpha_n^i \sqrt{\frac{\pi k_i r}{2}}\, J_{n+\frac{1}{2}}(k_i r)\, P_{n,1}(\cos\theta) \cos\phi \quad ,$$

$$rv_i = \frac{1}{k_i^2} \sum_{n=1}^{\infty} \beta_n^i \sqrt{\frac{\pi k_i r}{2}}\, J_{n+\frac{1}{2}}(k_i r)\, P_{n,1}(\cos\theta) \sin\phi \quad .$$

Making use of (13), (15) and (17) now, we determine the numerical coefficients of these expansions; so, for example, for the first two series we have:

$$\alpha_n^* = -\, i^{n-1} \frac{2n+1}{n(n+1)} \cdot \frac{\psi_n(p)\psi_n'(Np) - N\psi_n'(p)\,\psi_n(Np)}{\zeta_n(p)\,\psi_n'(Np) - N\zeta_n'(p)\,\psi_n(Np)} \quad ,$$

$$\beta_n^* = -\, i^{n-1} \frac{2n+1}{n(n+1)} \cdot \frac{N\,\psi_n(p)\,\psi_n'(Np) - \psi_n'(p)\,\psi_n(Np)}{N\,\zeta_n(p)\,\psi_n'(Np) - \zeta_n'(p)\,\psi_n(Np)} \quad ,$$

$$p = k_o \rho\,, \quad (\rho \text{ is the radius of the sphere}) \quad ;$$

$$N = \frac{k_i}{k_a} = \sqrt{\varepsilon_i - \frac{i\sigma}{\omega}\,\frac{1}{\varepsilon_a}} \quad ;$$

$$\psi_n(p) = \sqrt{\frac{\pi p}{2}}\, J_{n+\frac{1}{2}}(p) \quad ; \quad \zeta_n(p) = \sqrt{\frac{\pi p}{2}}\, H^{(2)}_{n+\frac{1}{2}}(p) \quad .$$

As regards the convergence of the series here obtained, the same remarks must be repeated as were made at the end of the preceding

section. Numerical calculations and detailed bibliographic references on this topic can be found in the Handbuch d. Physik Bd. XX, pp. 307-16. The results found in the Handbuch bearing on the phenomenon of electromagnetic diffraction near a sphere have been basically reproduced here.

§6. Diffraction of elastic waves near a clamped circular cylinder. In an infinite elastic space with Lamé constants λ and μ and density σ a cylindrical excision of diameter 2ρ is given; an incident plane wave of the longitudinal (or transverse) type is considered; it is periodic with the time, propagated in the plane perpendicular to the axis of the cylinder, which we shall adopt as the z-axis. Thus the problem consists in the study of diffraction near a circular excision in the plane xy-plane.

The equations for the basic potentials, in accordance with § 2 (Chap. I) will be

$$\Delta\phi_1 = \frac{\sigma}{\lambda+2\mu}\frac{\partial^2\phi}{\partial t^2} \quad ; \quad \Delta\psi_1 = \frac{\sigma\cdot\partial^2\psi}{\mu\cdot\partial t^2} \quad ;$$

the components of the displacement and the stress are given by formulas (9) and (11), (§2, Chap. I).

Putting

$$\phi_1 = \Re(e^{i\omega t}\phi) \quad ; \quad \psi_1 = \Re(e^{i\omega t}\psi) \qquad (21)$$

for the lagging potentials ϕ and ψ, we obtain

$$\Delta\phi + k_1^2\phi = 0 \quad ; \quad \Delta\psi + k_2^2\psi = 0 \quad , \qquad (22)$$

where

$$k_1^2 = \frac{\sigma \omega^2}{\lambda + 2\mu} = \frac{\omega^2}{a^2} \ ; \quad k_2^2 = \frac{\sigma \omega^2}{\mu} = \frac{\omega^2}{b^2} \ . \tag{23}$$

Let the incident disturbance be given by the potential ϕ^*:

$$\phi^* = e^{ik_1 r \cos\theta} \ .$$

We seek the supplementary potentials in the form

$$\phi_* = \phi^* + \sum_{n=0}^{\infty} H_n^{(2)}(k_1 r)(L_n \cos n\theta + M_n \sin n\theta) \ ,$$

$$\psi = \sum_{n=0}^{\infty} H_n^{(2)}(k_2 r)(P_n \cos n\theta + Q_n \sin n\theta) \ . \tag{24}$$

In accordance with (9), on the circumference $r = \rho$ there must be fulfilled the conditions

$$\frac{\partial \phi_*}{\partial x} + \frac{\partial \psi}{\partial y} = 0 \ ; \quad \frac{\partial \phi_*}{\partial y} - \frac{\partial \psi}{\partial x} = 0$$

or, in polar coordinates,

$$\frac{\partial \phi_*}{\partial \rho} + \frac{1}{\rho}\frac{\partial \psi}{\partial \theta} = 0 \ ; \quad \frac{\partial \psi}{\partial \rho} - \frac{1}{\rho}\frac{\partial \phi_*}{\partial \theta} = 0 \ .$$

Using the expansion

$$\phi^* = J_0(k, r) + 2\sum_{n=0}^{\infty} i^n J_n(k, r) \cos n\theta \ ,$$

the boundary conditions just written, using the series for ϕ_* and ψ in them, will give a system that is linear in L_n, M_n, P_n, Q_n; from this we find

$$P_n = M_n = Q_n = 0 \qquad (n = 0, 1, 2, \ldots$$

$$L_n = \chi_n + i\lambda_n = -2i^n \frac{\frac{n^2}{\rho^2} J_n(k_1\rho) H_n^{(2)}(k_2\rho) - k_1 k_2 J_n'(k_1\rho) H_n'^{(2)}(k_2\rho)}{\Delta},$$

$$Q_n = q_n + i\mu_n = -2i^n \frac{nk_1}{\rho} \frac{J_n'(k_1\rho) H_n^{(2)}(k_1\rho) - J_n(k_1\rho) H_n'^{(2)}(k_1\rho)}{\Delta},$$

$$\Delta = \frac{n^2}{\rho^2}\left[H_n^{(2)}(k_1\rho)\, H_n^{(2)}(k_2\rho)\right] - k_1 k_2 H_n'^{(2)}(k_1\rho)\, H_n'^{(2)}(k_2\rho).$$

Taking into consideration the known formulas for the differentiation of cylindrical functions, it is easily shown that with real k_1 and k_2,

$$\Delta \neq 0 .$$

§7. Diffraction near a circular cylinder with a given stress at the boundary. In this and in succeeding paragraphs we shall use for stress several symbols differing from those of §2, Chap. I.

The vector of the stress for an element on the circle with normal $n = r$ has the following form

$$P^{(r)} = P^{(x)} \cos\theta + P^{(y)} \sin\theta .$$

The conditions of absence of stress (free borders) at the boundary are given notationally thus:

$$P_x^{(r)} = \sigma_x \cos\theta + \tau_{xy} \sin\theta = 0 \;;\; P_y^{(r)} = \tau_{xy} \cos\theta + \sigma_y \sin\theta = 0, \qquad (25)$$

where (σ_x, τ_{xy}) are the components of $P^{(x)}$ and (τ_{xy}, σ_y) are the components of $P^{(y)}$.

Formulas (11), §2, Chap. I, now take the form

$$\left.\begin{aligned}\sigma_x &= -\omega^2\rho\phi_* - 2\mu\frac{\partial^2\phi_*}{\partial y^2} + 2\mu\frac{\partial^2\psi}{\partial x\partial y}\ ;\\ \tau_{xy} &= 2\mu\frac{\partial^2\phi_*}{\partial x\partial y} - \omega^2\rho\psi - 2\mu\frac{\partial^2\psi}{\partial x^2}\ ;\\ \sigma_y &= -\omega^2\rho\phi_* - 2\mu\frac{\partial^2\phi_*}{\partial x^2} - 2\mu\frac{\partial^2\psi}{\partial x\partial y}\ .\end{aligned}\right\} \tag{26}$$

Having expressed the boundary conditions (25) in polar coordinates, and introducing the values of σ_x, τ_{xy}, σ_y from (26) and of ϕ_* and ψ from (24), we obtain, for the coefficients L_n, M_n, P_n, Q_n, the formulas

$$L_n = \frac{D_H(k_2)\left[\frac{4nk_1}{\rho}\, i^nJ_n'(k_1\rho) - \frac{4n}{\rho^2}\, i^nJ_n(k_1\rho)\right] - \cdots \rightarrow}{D_H(k_1)D_H(k_2) - \Delta_H(k_1,k_2)\Delta_H(k_2,k_1)}$$

$$\cdots\rightarrow - \Delta_H(k_1,k_2)\left[\left(2k_2^2 - \frac{4n^2}{\rho^2}\right) i^nJ_n(k_1\rho) + \frac{4k_1}{\rho}\, i^nJ_n'(k_1\rho)\right]\ ;$$

$$Q_n = \frac{D_H(k_1)\left[\left(2k_2^2 - \frac{4n^2}{\rho^2}\right) i^nJ_n(k_1\rho) + \frac{4k_1}{\rho}\, i^nJ_n'(k_1\rho)\right] - \cdots \rightarrow}{D_H(k_1)D_H(k_2) - \Delta_H(k_1,k_2)\Delta_H(k_2,k_1)}$$

$$\cdots\rightarrow - \Delta_H(k_2,k_1)\left[\frac{4nk_1}{\rho}\, i^nJ_n'(k_1\rho) - \frac{4n}{\rho^2}\, i^nJ_n(k_1\rho)\right]\ ;$$

$$M_n = P_n = 0 \qquad (n = 0, 1, 2, \cdots)\ ;$$

$$D_H(k_1) = \frac{2n}{\rho^2} H_n^{(2)}(k_1\rho) - \frac{2k_1 n}{\rho} H_n'^{(2)}(k_1\rho)\ ;$$

$$\Delta_H(k_1,k_2) = \left(\frac{2n^2}{\rho^2} - k_1^2\right) H_n^{(2)}(k_2\rho) - \frac{2k_2}{\rho} H_n'^{(2)}(k_2\rho)\ ;$$

$$\Delta_H(k_2,k_1) = \left(\frac{2n^2}{\rho^2} - k_2^2\right) H_n^{(2)}(k_1\rho) - \frac{2k_1}{\rho} H_n'^{(2)}(k_1\rho)\ .$$

§8. <u>Diffraction near a cylindrical occlusion.</u> In a region of "falling" potential ψ^* there is a circular occlusion of elastic material with constants λ_1, μ_1 and density ρ_1. In this case the boundary conditions express the continuity of the components of the displacements and the stresses:

$$u_x^{(i)} = u_x^{(a)} \quad ; \quad u_y^{(i)} = u_y^{(a)} \quad ;$$
$$P_{x_1}^{(r)} = P_{x_a}^{(r)} \quad ; \quad P_{y_i}^{(r)} = P_{y_a}^{(r)} \quad . \qquad (27)$$

In consequence of the evident regularity of the components of the refracted field, only cylindrical functions of the first type must be entered in the expression for the inner potentials. The external potentials, however, are in accordance with the Emission principle, expressible by means of Hankel functions:

$$\left.\begin{aligned}
\phi_*^{(a)} &= \phi^{(a)*} + \sum_{n=0}^{\infty} H_n^{(2)}(k_1 r)(L_n \cos n\theta + M_n \sin n\theta) \quad ; \\
\psi^{(a)} &= \sum_{n=0}^{\infty} H_n^{(2)}(k_2 r)(P_n \cos n\theta + Q_n \sin n\theta) \quad ; \\
\phi^{(i)} &= \sum_{n=0}^{\infty} J_n(k_3 r)(L_n' \cos n\theta + M_n' \sin n\theta) \quad ; \\
\psi^{(i)} &= \sum_{n=0}^{\infty} J_n(k_4 r)(P_n' \cos n\theta + Q_n' \sin n\theta) \quad ;
\end{aligned}\right\} \qquad (28)$$

both k_1 and k_2 retain their former values, but k_3 and k_4 are defined by the formulas:

$$k_3^2 = \frac{\rho_1 \omega^2}{\lambda_1 + 2\mu_1} = \frac{\omega^2}{a_1^2} \quad ; \quad k_4^2 = \frac{\rho_1 \omega^2}{\mu_1} = \frac{\omega^2}{b_1^2} \quad . \qquad (29)$$

Expressing the boundary conditions in the sought potentials by means of the formulas indicated above, and rewriting everything in polar coordinates, we arrive at the following system of equations for the determination of the unknown coefficients of the several series (28):

$$L_o H_o'^{2}(k_1\rho) - L_o' J_o'(k_3\rho) = -3J_o'(k_1\rho) \quad ;$$

$$L_o\left[k_2^2 H_o^{(2)}(k_1\rho) + \frac{2k_1}{\rho} H_o'^{(2)}(k_1\rho)\right] - L_o'\left[k_4^2 J_o(k_3\rho) + \frac{2k_3}{\rho} J_o'(k_3\rho)\right] =$$

$$= -3k_2^2 J_o(k_1\rho) - \frac{6k_1}{\rho} J_o'(k_1\rho) \quad ;$$

$$k_1 H_n'^{(2)}(k_1\rho) L_n + \frac{n}{\rho} H_n^{(2)}(k_2\rho) Q_n - k_3 J_n'(k_3\rho) L_n' - \frac{n}{\rho} J_n(k_4\rho) Q_n' =$$

$$= -2i^n k_1 J_n'(k_1\rho) \quad ;$$

$$\frac{n}{\rho} H_n^{(2)}(k_1\rho) L_n + k_2 H_n'^{(2)}(k_2\rho) Q_n - \frac{n}{\rho} J_n(k_3\rho) L_n - k_4 J_n'(k_4\rho) Q_n' =$$

$$= -2i^n \frac{n}{\rho} J_n(k_1\rho) \quad ;$$

$$\Delta_H(k_2, k_1) L_n - D_H(k_2) Q_n - \Delta_J(k_4, k_3) L_n' + D_J(k_4) Q_n' =$$

$$= \left(2k_2^2 - \frac{4n^2}{\rho^2}\right) i^n J_n(k_1\rho) + \frac{4k_1}{\rho} i^n J_n'(k_1\rho) \quad ;$$

$$D_H(k_1) L_n - \Delta_H(k_1, k_2) Q_n - D_J(k_3) L_n' + \Delta_J(k_3, k_4) Q_n' =$$

$$= \frac{4nk_1}{\rho} i^n J_n'(k_1\rho) - \frac{4n}{\rho^2} i^n J_n(k_1\rho) \quad ;$$

$$M_n = P_n' = M_n' = P_n' = 0 \qquad (n = 0, 1, 2, \cdots)$$

The symbols adopted here will be clear from the explanations given at the end of the preceding section. The subscript J

accompanying D and Δ indicates the presence of a Bessel function, and H to that of a Hankel function.

The problem of investigating the determinant of this system we shall here leave open.

The rapidity of convergence of the series (28) depends on the smallness of 2ρ in comparison with the length of $\lambda = \frac{2\pi}{k_1}$.

§9. The diffraction of an immovable clamped elliptical cylinder. The problem stated; the differential equations. In the preceding examples circular and spherical functions turned out to be the fundamental functions of the problems there considered. It is clear that this circumstance is the consequence of an appropriate choice of the coordinates in which these problems were studied.

In the case of an ellipse, the most natural functions in which the solution of the problem can be expressed are the functions of an elliptical cylinder—the Mathieu and Mathieu-Hankel functions.

The theory of spherical functions has been sufficiently well developed for the solution of diffraction problems; the theory of functions of an elliptical cylinder, however, requires additional investigation in many of its parts. In the next several sections we shall be preoccupied with this matter.

As regards this problem of diffraction, near a clamped elliptical cylinder, of the wave (we assume the field to be independent of one of the coordinates):

$$\Phi^* = u\, e^{i(k_1 y + \omega t)}, \tag{1}$$

we note that K. Sezawa[1] was concerned with this problem in 1927. He did not, however, obtain a solution, in consequence of an underestimation of some of the principal circumstances connected with the Emission principle.

Let us introduce elliptical coordinates:

$$x = \mathrm{ch}\,\xi \cos\eta \; ; \quad y = \mathrm{sh}\,\xi \sin\eta \; ;$$
$$\xi \geq 0 \, , \qquad -\pi \leq \eta \leq +\pi \, . \tag{2}$$

The entire plane will thereby have been covered with confocal ellipses and hyperbolas of the families

$$\frac{x^2}{\mathrm{ch}^2\,\xi} + \frac{y^2}{\mathrm{sh}^2\,\xi} = 1 \; ; \qquad \frac{x^2}{\cos^2\eta} - \frac{y^2}{\sin^2\eta} = 1 \, . \tag{3}$$

The curves ξ = constant, η = constant are obviously mutually orthogonal. Let us remark that from the equations $r = \sqrt{\mathrm{ch}^2\,\xi - \sin^2\eta}$; $\phi = \mathrm{arctg}\,\eta\ \mathrm{th}\,\xi$, it follows that

$$\left.\begin{array}{c} \lim\limits_{\xi\to\infty}(r - \mathrm{ch}\,\xi) = 0 \; ; \qquad \lim\limits_{\xi\to\infty}(r - \mathrm{sh}\,\xi) = 0 \; ; \\ \lim\limits_{\xi\to\infty}\phi = \eta \, , \end{array}\right\} \tag{4}$$

i.e., that at great distances formulas (2) reduce to the usual expression for polar coordinates, $x = r\cos\phi$; $y = r\sin\phi$. Thus our

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[1] "Scattering of elastic waves and some allied problems" Bull. Earthquaque Res. Inst., vol. IV, 1927, Tokyo. See also V. Kupradze, "Diffrazione delle onde elastiche sopra un contorno ellastico." Atti d. Accad. dei Lincei, Ser. XI, vol. XVIII, f. 3-4.

isothermal net asymptotically approaches the polar net of concentric circles and rays. This circumstance permits an a priori expectation that the asymptotic behavior of the special functions that will appear here will be analogous to the behavior at great distances of the usual cylindrical functions.

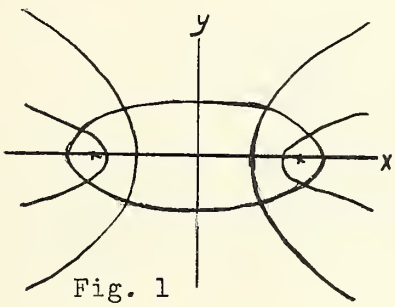

Fig. 1

We shall utilize this circumstance further on during the solution of the problem. If we suppose the solutions of the problem to be periodic with the time, i.e., reckon that

$$\Phi_1 = \Re\, e^{i\omega t}\, \Phi(x, y) \quad , \quad \Psi_1 = \Re\, e^{i\omega t}\, \Psi(x, y) \quad , \tag{5}$$

then in the new coordinates the fundamental wave equations will acquire the form

$$\left.\begin{aligned} \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} + k_1^2 (\mathrm{ch}^2 \xi - \cos^2 \eta)\, \Phi = 0 \quad ; \\ \frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} + k_2^2 (\mathrm{ch}^2 \xi - \cos^2 \eta)\, \Psi = 0 \quad , \end{aligned}\right\} \tag{6}$$

where

$$k_1^2 = \frac{\sigma \omega^2}{\lambda + 2\mu} = \frac{\omega^2}{c_1^2} \quad ; \qquad k_2^2 = \frac{\sigma \omega^2}{\mu} = \frac{\omega^2}{c_2^2} \quad ;$$

σ is the density, λ and μ the Lamé constants.

If, moreover, Φ and Ψ be sought in the form

$$F(\xi)\, G(\eta) \quad , \tag{7}$$

equations (6) will be transformed accordingly:

$$\frac{1}{F(\xi)} \frac{d^2F}{d\xi^2} + k_1^2 \operatorname{ch}^2 \xi = -\frac{1}{G(\eta)} \frac{d^2G}{d\eta^2} + k_1^2 \cos^2 \eta \quad .$$

Hence, equating to an arbitrary constant A, we obtain

$$\frac{d^2F(\xi)}{d\xi^2} + (k_1^2 \operatorname{ch}^2 \xi + A)\, F(\xi) = 0 \quad , \tag{8}$$

$$\frac{d^2G(\eta)}{d\eta^2} - (k_1^2 \cos^2 \eta + A)\, G(\eta) = 0 \quad . \tag{9}$$

Comparing these equations with each other, we see that if $G(\eta)$ is the general solution of equation (9), then

$$G(i\xi) = F(\xi) \tag{10}$$

will be a solution of (8), and, conversely, if $F(\xi)$ is a solution of (8), then

$$G(\eta) = F\left(-\frac{\eta}{i}\right) \tag{11}$$

will be a solution of (9). There is therefore no need of investigating these two equations separately.

We shall be occupied in detail with the equation

$$\frac{d^2G(\eta)}{d\eta^2} - (k_1^2 \cos^2 \eta + A)\, G(\eta) = 0 \quad ,$$

which we shall put in the form

$$\frac{d^2G(\eta)}{d\eta^2} + (a + 16q_1 \cos 2\eta)\, G(\eta) = 0 \quad ,$$

where

$$a = -A - \frac{k_1^2}{2} \; ; \quad q_1 = -\frac{k_1^2}{32} \; . \tag{12}$$

Thus if the general solution of equation (9), $G(\eta)$, be constructed, the general solution of (8), i.e., $F(\xi)$, will be found in conformity with (10), and accordingly the general solution of (6) will be found, of the form

$$F(\xi)\, G(\eta) \; .$$

Physically, however, it is obvious that of all such products only those that are single-valued functions of the point are suitable. For this the function $G(\eta)$ must be periodic, of period 2π. By this condition, in turn, a discrete sequence of values of the parameter A is determined, for which equation (9) admits of such periodic solutions.

As regards the function $F(\xi)$, only such particular solutions are suitable as are amenable to the conditions of the Emission principle for infinitely large ξ.

Thus the mathematical part of the problem before us consists in seeking particular solutions of equations (8) and (9) satisfying the stated conditions.

§10. Mathieu functions. It is known from the general theory of differential equations with periodic coefficients that if periodic solutions of equation (9) are to exist, it is necessary that a definite dependence between a and q_1 be established.

This relation of dependence will be given below; it is, for given q_1, the characteristic Mathieu equation.

In the special case $q_1 = 0$, and $a = n^2$, equation (9) reduces to the following:

$$\frac{d^2G(\eta)}{d\eta^2} + n^2G(\eta) = 0 \quad , \tag{13}$$

with the fundamental system of solutions

$$\sin n\eta \quad , \quad \cos n\eta \quad .$$

We shall adopt a terminology calling such periodic solutions of equation (9) as for $q_1 = 0$ reduce to $\cos n\eta$ and $\sin n\eta$ Mathieu functions of the n th order.

The first of these (the even) are denoted by the symbol $ce_n(\eta , q_1)$, and the second (the odd), by the symbol $se_n(\eta , q_1)$.

In diffraction problems we shall have occasion to meet with all types of Mathieu functions, which are comprehended in four classes represented by Fourier series.

We shall give them a somewhat special notation convenient to our purposes:

$$ce_{2n}(\eta, q_1) = \sum_{s=1}^{s=n} \alpha_s^{(2n)} \cos(2n - 2s)\eta + \cos 2n\eta + \sum_{s=1}^{s=\infty} \beta_s^{(2n)} \cos(2n + 2s)\eta \quad ; \tag{14}$$

$$ce_{2n+1}(\eta , q_1) = \sum_{s=1}^{s=n} \gamma_s^{(2n+1)} \cos(2n - 2s + 1)\eta + \cos(2n + 1)\eta + \sum_{s=1}^{\infty} \delta_s^{(2n+1)} \cos(2n + 2s + 1)\eta \quad ; \tag{15}$$

$$se_{2n}(\eta , q_1) = \sum_{s=1}^{s=n} \bar{\alpha}_s^{(2n)} \sin(2n - 2s)\eta + \sin 2n\eta + \sum_{s=1}^{s=\infty} \bar{\beta}_s^{(2n)} \sin(2n + 2s)\eta \quad ; \tag{14*}$$

$$se_{2n+1}(\eta, q_1) = \sum_{s=1}^{s=n} \bar{\gamma}_s^{(2n+1)} \sin(2n - 2s + 1)\eta$$

$$+ \sin(2n+1)\eta + \sum_{s=1}^{\infty} \bar{\delta}_s^{(2n+1)} \sin(2n + 2s + 1)\eta \quad . \tag{15*}$$

Let us recall at this point that for fixed a equation (9) does not simultaneously admit of two differing periodic solutions, wherein it differs from equation (13), for which, on the contrary, this circumstance obtains.

This property of equation (9), important in principle, long remained unnoticed until it was proved in 1922 by Ince[1]. (Judging by his works, it is apparently even yet unknown to the Japanese seismologist Sezawa.)

Series (14)-(15*) are known to converge over the entire plane, and consequently the coefficients α_n, β_n, γ_n, δ_n diminish more rapidly than the n-th power of any number.

However it is of essential importance to us in diffraction theory to obtain more exact estimates of these coefficients. In the following section we shall give the form of these coefficients and the estimates we need.

§11. Construction of the Mathieu functions. Periodic solutions of equations (9) may be constructed by the method of Frobenius[2], as has been done by Whittaker and Watson[3].

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(1) E. L. Ince. A proof of the impossibility of the coexistence of two Mathieu functions. Proc. Cambr. Philos. Soc., 21, 1922.

(2) Journ. f. Math. LXXVI, 1873, pp. 214-224.

(3) A Course of Modern Analysis. N.Y. Macmillan, 1946. p. 421 ff.

However, for our purposes it will be more convenient to proceed otherwise, thereby considerably simplifying the derivation of the functions we require, as well as of several of their properties.

Let

$$a = (2n)^2 + 8p \quad . \tag{16}$$

Equation (9) then acquires the form

$$\frac{d^2G(\eta)}{d\eta^2} + 4n^2G(\eta) = -8(p + 2q_1 \cos 2\eta)\, G(\eta) \quad . \tag{17}$$

Let us employ expansion (14) in this equation:

$$- \sum_{s=1}^{n} 4(n - s)^2 \alpha_s^{(2n)} \cos(2n - 2s)\eta - 4n^2 \cos 2n\eta -$$

$$- \sum_{s=1}^{\infty} 4(n + s)^2 \beta_s^{(2n)} \cos(2n + 2s)\eta + 4n^2 \sum_{s=1}^{s=n} \alpha_s^{(2n)} \cos(2n - 2s)\eta +$$

$$+ 4n^2 \cos 2n\eta + 4n^2 \sum_{s=1}^{\infty} \beta_s^{(2n)} \cos(2n + 2s)\eta + 8p \sum_{s=1}^{n} \alpha_s^{(2n)} \cos(2n - 2s)\eta +$$

$$+ 8p \cos 2n\eta + 8p \sum_{s=1}^{\infty} \beta_s^{(2n)} \cos(2n + 2s)\eta + 16q_1 \cos 2n\eta \cos 2\eta +$$

$$+ 16q_1 \cos 2\eta \sum_{s=1}^{n} \alpha_s^{(2n)} \cos(2n - 2s)\eta +$$

$$+ 16q_1 \cos 2\eta \sum_{s=1}^{\infty} \beta_s^{(2n)} \cos(2n + 2s)\eta = 0 \quad .$$

Comparing the coefficients of $\cos 2n\eta$, $\cos(2n - 2)\eta$, $\cdots$, $\cos(2n - 2k)\eta$, $\cos(2n + 2)\eta$, etc., first having resolved the product of the cosines:

$$16q_1 \sum_{s=1}^{n} \alpha_s^{(2n)} \cos(2n - 2s)\eta \cos 2\eta = 8q_1 \sum_{s=1}^{n} \Big\{ \alpha_s^{(2n)} [\cos 2(n - s + 1)\eta +$$

$$+ \cos 2(n - s - 1)\eta] \Big\} ;$$

$$16q_1 \sum_{s=1}^{\infty} \beta_s^{(2n)} \cos(2n + 2s)\eta \cos 2\eta = 8q_1 \sum_{s=1}^{\infty} \Big\{ \beta_s^{(2n)} [\cos 2(n + s + 1)\eta +$$

$$+ \cos 2(n + s - 1)\eta] \Big\} ;$$

$$16q_1 \cos 2n\eta \cos 2\eta = 8q_1 [\cos 2(n + 1)\eta + \cos 2(n - 1)\eta] ,$$

we shall obtain the following system of equations:

$$p + q_1 \Big[\alpha_1^{(2n)} + \beta_1^{(2n)} \Big] = 0 ; \tag{18}$$

$$s(s - 2n)\alpha_s^{(2n)} = 2 \Big\{ p\alpha_s^{(2n)} + q_1 \Big[\alpha_{s-1}^{(2n)} + \alpha_{s+1}^{(2n)} \Big] \Big\} , \; (s = 1, 2, \cdots, n^{(1)}) \tag{19}$$

$$s(s + 2n)\beta_s^{(2n)} = 2 \Big\{ p\beta_s^{(2n)} + q_1 \Big[\beta_{s-1}^{(2n)} + \beta_{s+1}^{(2n)} \Big] \Big\} , \; (s = 1, 2, \cdots \infty) \tag{20}$$

where $\alpha_0^{(2n)} = \beta_0^{(2n)} = 1$ and with respect to α it is moreover necessary to keep in mind that s varies from 1 to n.

Let us introduce the notations

$$\sigma_s = - 4q_1 \Big\{ s(s + 2n) - 2p \Big\}^{-1} ; \quad \mu_s = - 4q_1 \Big\{ s(s - 2n) - 2p \Big\}^{-1} .$$

Eliminating

$$\beta_1^{(2n)}, \beta_2^{(2n)}, \cdots, \beta_{s-1}^{(2n)}, \beta_{s+1}^{(2n)}, \cdots$$

from system (20), we obtain

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(1) For s = n, one must write $-n^2 \alpha_n^{(2n)} = 2p\alpha_n^{(2n)} + 2q_1 \alpha_{n-1}^{(2n)}$. See editor's Note 1.

$$\beta_s^{(2n)} \Delta_o = (-)^s \prod_{r=1}^{s} \sigma_r \Delta_s \quad , \qquad (20^*)$$

where Δ_s is an infinite determinant of the von Koch type:

$$\Delta_s = \begin{vmatrix} 1, & \sigma_{s+1}, & 0 & 0, & \cdots \\ \sigma_{s+2}, & 1, & \sigma_{s+2}, & 0, & \cdots \\ 0, & \sigma_{s+3}, & 1, & \sigma_{s+3}, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} .$$

This special determinant is absolutely convergent, since the necessary and sufficient condition of this, consisting in the absolute convergence of the series

$$\sigma_{s+1} + \sigma_{s+2} + \sigma_{s+3} + \cdots \quad ,$$

is fulfilled.

Moreover, Whittaker has shown[1] that

$$\lim_{s \to \infty} \Delta_s = 1 \quad . \qquad (21)$$

It can be similarly shown that

$$\alpha_s^{(2n)} D_o = (-)^s \prod_{r=1}^{s} \mu_r D_s \quad , \qquad (19^*)$$

where

$$D_s = \begin{vmatrix} 1, & \mu_{s+1}, & 0, & 0, & \cdots \\ \mu_{s+2}, & 1, & \mu_{s+2}, & 0, & \cdots \\ 0, & \mu_{s+3}, & 1, & \mu_{s+3}, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

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(1) Ibid., p. 423.

is a finite determinant, since here s varies only up to n. Equation (18) plays the role of the characteristic equation. Let us transcribe it in the following form:

$$p\Delta_o D_o - q_1(\sigma_1\Delta_1 D_o + \mu_1\Delta_o D_1) = 0 \quad . \tag{18*}$$

Multiplying this expression by

$$\prod_{r=1}^{\infty}\left\{1 - \frac{2p}{r(r+2n)}\right\} \cdot \prod_{r=1}^{r=n}\left\{1 - \frac{2p}{r(r-2n)}\right\} ,$$

it is easily turned into an integral function in p and q_1.

Having fixed n, it can be shown that the root p of equation (18*) can be represented as a power series in q_1 commencing with the term q_1^2.

If $|q_1|$ is small enough, we have, for the value of p already found,

$$\Delta_o \neq 0 \quad , \quad D_o \neq 0 \quad ,$$

since for $q_1 = 0$, $\Delta_o = 1$, $D_o = 1$.

In addition, it will be shown below that for small q_1 and with any n, the following inequalities hold:

$$2p \neq r(r + 2n) \quad , \quad 2p \neq r(r - 2n) \quad , \tag{21*}$$

in the latter of which inequalities r runs over integral values up to n only.

The number p thus determined in the function of n and q_1 plays the role of the characteristic number for equation (9). In future it will be essential to know the asymptotic behavior of p for n tending to ∞.

§12. The asymptotic behavior of the proper values of the Mathieu equations. The assumption was made above that $2p \neq r(r + n)$ for any n, and $r = 1, 2, \cdots$. We will now elucidate the conditions under which these inequalities actually hold, and in addition, will study the asymptotic character of the distribution of the spectrum of proper numbers.

Setting $\lambda_n = (2n)^2 + 8p$ together with $a = (2n)^2 + 8p$, the Mathieu equation $\frac{d^2u}{dx^2} + (\lambda_n + 16q_1 \cos 2x)\, u = 0$ is an equation of the Sturm-Liouville type, and to the theory of its proper values (characteristic numbers) one may apply the theory of that type of equations, based, for example, on the familiar variation principle.

It is possible to show in just the same way as is done for non-periodic boundary conditions, that the characteristic numbers corresponding to periodic solutions form an increasing sequence

$$\lambda_0^{(i)} < \lambda_1^{(i)} < \lambda_2^{(i)} \cdots \qquad (i = 1, 2,)$$

The two superscripts are to indicate the evenness (1) or oddness (2) of the corresponding fundamental functions. On the other hand, it was shown above (with an insignificant variation in notation) that to the numbers

$$\lambda_{2n}^{(1)} = (2n)^2 + \phi_{2n}^{(1)}(q_1) , \qquad \lambda_{2n}^{(2)} = (2n)^2 + \phi_{2n}^{(2)}(q_1) ,$$

$$\lambda_{2n+1}^{(1)} = (2n + 1)^2 + \phi_{2n+1}^{(1)}(q_1) , \lambda_{2n+1}^{(2)} = (2n + 1)^2 + \phi_{2n+1}^{(2)}(q_1)$$

there correspond the solutions

$$ce_{2n}(\eta\ ,\ q_1) = \sum_{s=o}^{\infty} a_{2s}^{(2n)} \cos 2s\eta \quad ,$$

$$ce_{2n+1}(\eta\ ,\ q_1) = \sum_{s=o}^{\infty} b_{2s+1}^{(2n+1)} \cos (2s+1)\eta \quad ,$$

$$se_{2n}(\eta\ ,\ q_1) = \sum_{s=o}^{\infty} \bar{a}_{2s}^{(2n)} \sin 2s\eta \quad , \qquad (^*)$$

$$se_{2n+1}(\eta, q_1) = \sum_{s=o}^{\infty} \bar{b}_{2s+1}^{(2n+1)} \sin (2s+1)\eta \quad ,$$

$\phi_{2n}^{(i)}(q_1)$, $\phi_{2n+1}^{(i)}(q_1)$, $(i = 1, 2)$ being here special power series in q_1, reducing to zero for $q_1 = 0$; and a, b, $\bar{a}$, $\bar{b}$ being coefficients found in the manner indicated above.

We shall show that

$$\lambda_{p_1}^{(i)} > \lambda_{p_1-1}^{(j)} \quad , \qquad \begin{pmatrix} i = 1, 2 \\ j = 1, 2 \end{pmatrix} \qquad (^{**})$$

i.e., we shall prove that the classification of the proper numbers in accordance with the principle adopted by us coincides with the classification that emerges from the variation principle.

It is obvious that

$$\lambda_{p_1}^{(i)}(0) > \lambda_{p_1-1}^{(j)}(0) \quad ;$$

now for $q_1 = q_1^* \neq 0$, let

$$\lambda_{p_1}^{(i)}(q_1^*) = \lambda_{p_1-1}^{(j)}(q^*) = \mu \quad .$$

Two cases are then conceivable:

#1. To the characteristic number μ there correspond two different periodic solutions: ce_{p_1} and ce_{p_1-1}, se_{p_1} and se_{p_1-1}, ce_{p_1} and se_{p_1-1}, or lastly se_{p_1} and ce_{p_1-1}, which is impossible according to Ince's theorem mentioned previously.

#2. The characteristic number μ is multiple, and moreover some of the pairs of periodic functions indicated above become equal, which is also impossible, for obvious causes.

Thus supposition (2) is untrue, and inequality (**) is consequently valid.

This examination shows that the classification of the proper values set forth in §8 is also an increasing one, and therefore the asymptotic character obtained for the characteristic numbers on the basis of the variation principle is valid for them too.

In accordance with this principle, the characteristic numbers of the equation

$$u'' + 16q_1 \cos 2x \cdot u + \lambda_n u = 0 \quad ,$$

which give periodic solutions of period 2π, are given by the greatest of the least values of the integral

$$\int_0^{2\pi} (u'^2 - 16q_1 \cos 2z \cdot u^2)\, dz$$

on condition that

$$\int_0^{2\pi} u^2 dz = 1 \quad .$$

It is clear that the sought maximinimum is less than

$$\text{maximin} \int_0^{2\pi} u'^2 \, dz + \max (16 q_1 \cos 2z) \quad ,$$

but

$$\text{maximin} \int_0^{2\pi} u'^2 \, dz$$

are the characteristic numbers of the equation

$$u'' + ku = 0$$

for the interval $(0, 2\pi)$.

Accordingly $\lambda_{p_1}^{(i)} = (p_1)^2$ + a finite quantity, where this latter cannot exceed $|16 \, q_1|$, whence

$$p < 2q_1 \quad ,$$

and if $|q_1| < 1$ under the condition of the problem, we have

$$2p \neq r(r + 2n) \quad , \qquad (r = 1,2,3 \cdots)$$

Q.E.D..

Comparing this result with (16) and (18^*), we conclude that the values of p that are the roots of the equation

$$p\Delta_o D_o - q_1(\sigma_1 \Delta_1 D_o + \mu_1 \Delta_o D_1) = 0 \quad ,$$

satisfy conditions (21^*) and remain bounded as n increases without limit.

Utilizing this property, we shall give several estimates for the coefficients of (20^*):

$$\beta_s^{(2n)} = (-)^s \frac{\Delta_s}{\Delta_o} \prod_{r=1}^{s} \frac{q_1}{r(r+2n)-2p},$$

$$\beta_s^{(2n)} = (-)^s \frac{\Delta_s}{\Delta_o} \prod_{r=1}^{s} \frac{q_1}{r^2\left[1+\frac{2n}{r}-\frac{2p}{r^2}\right]} =$$

$$= \frac{(-)^s\Delta_s}{(s!)^2\Delta_o} \prod_{r=1}^{s_1-1} \frac{q_1}{\left[1+\frac{2n}{r}-\frac{2p}{r^2}\right]} \prod_{r=s_1}^{s} \frac{q_1}{\left[1+\frac{2n}{r}-\frac{2p}{r^2}\right]},$$

where s_1 is so chosen that $2n > \frac{2p}{s_1}$, which is always possible in consequence of the boundedness of p for any n.

Under the last $\prod$ sign it is possible to neglect the positive summand $\frac{2n}{r} - \frac{2p}{r^2}$ in the denominator, and the entire product is less in modulus than $q_1^{s-s_1+1}$.

As regards the first $\prod$ sign, we have there a finite number of finite factors, and the entire product is consequently bounded.

It is clear that for any n this product will be less than some number $Mq_1^{s_1-1}$; finally

$$\left|\beta_s^{(2n)}\right| < \left|\frac{Mq_1^s\Delta_s}{(s!)^2\Delta_o}\right| . \tag{22}$$

In addition to this formula, one can obtain another inequality

$$\beta_s^{(2n)} = (-)^s \frac{\Delta_s}{\Delta_o} \prod_{r=1}^{s} \frac{q_1}{2n\cdot r\left[\frac{r}{2n}+1-\frac{p}{rn}\right]} =$$

$$= \frac{(-)^s \Delta_s}{(2n)^s \cdot s!\Delta_o} \prod_{r=1}^{s_2-1} \cdots \prod_{r=s_2}^{s} \frac{q_1}{\left[\frac{r}{2n}+1-\frac{p}{rn}\right]} ,$$

where s_2 is so chosen that $\frac{s_2}{2} > \frac{p}{s_2}$, which is always possible in consequence of the boundedness of p for any n. Furthermore, reiterating the foregoing reasoning, we obtain

$$\left|\beta_s^{(2n)}\right| < \left|\frac{\mathcal{M}\, q_1^s \Delta_s}{(2n)^s \cdot s!\Delta_o}\right| .$$

Finally

$$\beta_s^{(2n)} = \frac{(-)^s \Delta_s}{\Delta_o} \prod_{r=1}^{s} \frac{q_1}{r(r+2n)\left[1-\frac{2p}{r(r+2n)}\right]} =$$

$$= \frac{(-)^s \Delta_s q_1^s}{\Delta_o} \frac{(2n)!}{s!(s+2n)!} \prod_{r=1}^{s} \frac{1}{1-\frac{2p}{r(r+2n)}} ,$$

and since $2p \neq r(r + 2n)$ for any n, and $r = 1, 2, 3, \cdots$, we have

$$\left|\beta_s^{(2n)}\right| \leq \left|\frac{Hq_1(2n)!\Delta_s}{s!(s+2n)!\Delta_o}\right| . \qquad (22^*)$$

Δ_o, for small q_1, is different from zero, since $\Delta_o(q_1 = 0) = 1$. Inequality (22) is stronger than the two preceeding ones.

Let us pass on to deriving estimates for $\alpha_s^{(2n)}$. From (19*) we have

$$\alpha_s^{(2n)} = (-)^s \frac{D_s}{D_o} \prod_{r=1}^{s} \frac{q_1}{r(r-2n)\left[1-\frac{2p}{r(r-2n)}\right]} = \tag{23}$$

$$= \frac{D_s q_1^s (2n-s-1)!}{D_o s!(2n-1)!} \prod_{r=1}^{s} \frac{1}{1-\frac{2p}{r(r-2n)}} .$$

Having noted that

$$\prod_{r=1}^{s} \frac{1}{1-\frac{2p}{r(r-2n)}} = \prod_{r=1}^{s} \left\{1 + \frac{2p}{r(r-2n)-2p}\right\} ,$$

and utilizing the fact that $\prod_{r=1}^{s} (1 + a_r) < e^u$, where $u = \sum_{r=1}^{s} a_r$, we shall have

$$\prod_{r=1}^{s} \frac{1}{1-\frac{2p}{r(r-2n)}} < e^{\sum_{r=1}^{s} \frac{2p}{r(r-2n)-2p}} .$$

Since $2p \neq r(r - 2n)$, this product remains bounded for any $s \leq n$. Let us recall that in the expression for $\alpha_s^{(2n)}$, s is always less than or equal to n.

We finally have[1]

$$\left|\alpha_s^{(2n)}\right| \leq \left|\frac{Sq_1^s(2n-s-1)!D_s}{D_o \cdot s!(2n-1)!}\right| . \tag{23*}$$

We shall demonstrate below that the ratios $\frac{\Delta_s}{\Delta_o}$ and $\frac{D_s}{D_o}$ are bounded for any s, and therefore, from (22*) and (23*) we obtain, finally:

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[1] In all of these arguments we have considered $p > 0$; in the contrary case our inequality is valid <u>a fortiori</u>.

$$|\alpha_s^{(2n)}| < \frac{\mathfrak{A}\, q_1^s (2n-s-1)!}{s!(2n-1)!}\ , \qquad (23^*)$$

$$|\beta_s^{(2n)}| < \frac{\mathfrak{B}\, q_1^s (2n)!}{s!(s+2n)!}\ , \qquad (22^*)$$

where $\mathfrak{A}$, $\mathfrak{B}$ are finite positive quantities[(1)] not dependent upon s and n.

Let us prove the boundedness of the expression $\frac{\Delta_r}{\Delta_o}$ for any n. Utilizing the Hadamard theorem concerning the upper bound of the modulus of the determinant, we have

$$|\Delta_{r,m}| \bmod \begin{vmatrix} 1, & \sigma_{r+1}, & 0, & 0, & \cdots \\ \sigma_{r+2}, & 1, & \sigma_{r+2}, & 0, & \cdots \\ 0, & \sigma_{r+3}, & 1, & \sigma_{r+3}, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & \cdot & \cdot & \sigma_{r+m}, & 1 \end{vmatrix} <$$

$$< \left| \sqrt{(1+\sigma_{r+1}^2)(1+2\sigma_{r+2}^2)\cdots(1+2\sigma_{r+m}^2)} \right|$$

$$|\Delta_r| = \lim_{m\to\infty} |\Delta_{r,m}| \leq \lim_{m\to\infty} \left| \sqrt{(1+2\sigma_{r+1}^2)\cdots(1+2\sigma_{r+m}^2)} \right| =$$

$$= \lim_{m\to\infty} e^{\frac{1}{2}\sum_{k=1}^{m} \ln(1+2\sigma_{r+k})}\ ;$$

but since $\sum_{k=1}^{\infty} |\ln(1+2\sigma_{r+k}^2)|$ has a common finite upper bound for any n, the boundedness of Δ_r for any r is proved with this. Moreover, since Δ_o for $q_1 = 0$ is equal to 1, Δ_o is for small q_1 different from zero; hence follows the uniform boundedness of the radio $\frac{\Delta_r}{\Delta_o}$, Q.E.D..

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(1) See editor's note 2.

§13. <u>Properties of the Mathieu functions</u>. It can be shown(1) that the functions $ce_{2n}(\eta, q_1)$ and $se_{2n+1}(\eta, q_1)$ are solutions of the equation

$$G(\eta) = \lambda \int_{-\pi}^{+\pi} e^{ik_1 \sin\eta \sin\theta} G(\theta, q_1)\, d\theta \quad . \tag{24}$$

In addition to this, the equation

$$ce_m(\eta, q_1) = \lambda \int_0^{2\pi} e^{ik_1 \cos\eta \cos\theta} ce_m(\theta, q_1) d\theta \tag{25}$$

is valid for all even functions, and

$$se_{2n+1}(\eta, q_1) = \lambda \int_0^{2\pi} \sin(k_1 \sin\eta \sin\theta)\, se_{2n+1}(\theta, q_1)\, d\theta \ .$$

E. Poole(2) has, moreover, obtained an integral equation for the function se_{2n}:

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(1)Whittaker, On the partial differential equations of mathematical physics. Math. Ann. Vol. 57. See also the same author's work in Proc. Int. Congr. of Math., 1912. Whittaker's derivations must be somewhat modified as to form in the following manner. The integral

$$\lambda \int_{-\pi}^{+\pi} e^{ik_1 \sin\eta \sin\theta} G(\theta, q_1)\, d\theta \tag{25*}$$

having been substituted in equation (9) following two integrations by parts, leads to the equation:

$$\left[-e^{ik_1 \sin\eta \sin\theta} \frac{dG}{d\theta}\right]_{-\pi}^{+\pi} + \int_{-\pi}^{+\pi} e^{ik_1 \sin\eta \sin\theta} [G''(\theta) + (a + 16 q_1 \cos 2\eta) G(\theta)] d\theta = 0 \ .$$

If $G(\theta)$ is an even function, $G'(\theta) = 0$ for $\theta = \pm\pi$; if, however, $G(\theta)$ is an odd function, $G'(\pi) = G'(-\pi)$; the integrated function equals zero in both cases, therefore. The expression standing under the integral equals zero in consequence of (9). Thus integral (25*)

(Continued to page 50)

(Footnote (2) also appears on page 50.

$$se_{2n}(\eta, q_1) = \lambda \int_0^{2\pi} \cos\eta \cos\theta \, sh(k_1 \sin\eta \sin\theta) se_{2n}(\theta, q_1) d\theta \ .$$

Integral equations can serve for the effective construction of the Mathieu functions; we shall show this briefly in an example for $ce_o(\eta, q_1)$:

$$ce_o(\eta, q_1) = 1 + \sum_{r=1}^{\infty} \alpha_r^o \cos 2r\eta \ .$$

Substitute this expansion in (25') and put $\eta = \frac{\pi}{2}$:

$$ce_o\left(\frac{\pi}{2}, q_1\right) = \lambda \int_0^{2\pi} ce_o(\theta, q_1)\, d\theta = 2\pi\lambda \ ,$$

$$\lambda = \frac{1}{2\pi} ce_o\left(\frac{\pi}{2}, q_1\right) \ . \qquad (26)$$

Now (25') may be written as follows:

$$ce_o(\eta, q_1) = \frac{1}{2\pi} ce_o\left(\frac{\pi}{2}, q_1\right) \int_0^{2\pi} e^{k_1 \cos\eta \cos\theta} ce_o(\theta, q_1)\, d\theta \ . \qquad (27)$$

But

$$\epsilon^{k_1 \cos\eta \cos\theta} = J_o(ik_1 \cos\theta) + 2 \sum_{n=1}^{\infty} \left[J_n(ik_1 \cos\theta) \cos n\eta\right] i^{-n} \ .$$

Introducing this expression in (27) and comparing the coefficients of $\cos 2n\eta$, we find that

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(1) Footnote continued from page 49:
is a periodic solution of the Mathieu function. It is not possible by the same reasoning, however, to obtain integral equations for ce_{2n+1} and se_{2n+1}, since

$$\int_{-\pi}^{+\pi} e^{ik_1 \sin\eta \sin\theta} se_{2n}(\theta, q_1) d\theta = \int_{-\pi}^{+\pi} e^{ik_1 \sin\eta \sin\theta} ce_{2n+1}(\theta, q_1) d\theta = 0.$$

(2) On certain classes of Mathieu functions. Proc. London Math. Soc., 20, 1921. (Footnote reference on page 49).

$$\alpha_r^{(o)} = (-)^r \frac{ce_o\left(\frac{\pi}{2}, q_1\right)}{\pi} \int_0^{2\pi} J_{2r}(ik_1 \cos \theta)\, ce_o(\theta, q_1)\, d\theta \quad , r \geq 1 \quad .(28)$$

If we now make use of the familiar equality

$$\int_0^{2\pi} \cos^{2m}\theta \cos 2n\theta d\theta = \frac{\pi \cdot (2m)!}{2^{2m-1}(m+n)!(m-n)!} \quad ,$$

we will obtain, from (28)[1]

$$\alpha_r^{(o)} = (-1)^r \left(\frac{k_1^{2r}}{2^{4r-1}(r!)^2} - \frac{r(3r+4)k_1^{2r+4}}{2^{4r+7}(r+1)!(r+1)!} + \cdots \right).$$

The orthogonality properties of a Mathieu function follow from the homogeneous integral equations with symmetric kernels that they satisfy:

$$\left.\begin{aligned} &\int_{-\pi}^{+\pi} ce_m(\eta, q_1)\, se_n(\eta, q_1)\, d\eta = 0 \quad ; \\ &\int_{-\pi}^{+\pi} ce_m(\eta, q_1)\, ce_n(\eta, q_1)\, d\eta = 0 \quad ; \\ &\int_{-\pi}^{+\pi} se_m(\eta, q_1)\, se_n(\eta, q_1)\, d\eta = 0 \quad . \end{aligned}\right\} m \neq n \qquad (29)$$

It has already been pointed out above that equation (9) belongs to the Sturm-Liouville class of functions.

It is known that the proper numbers of this equation, under given boundary conditions, form a countable non-diminishing sequence $\lambda_1, \lambda_2, \cdots, \lambda_n$, and that the corresponding system of characteristic functions forms a complete, orthogonal system.

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(1) P. Humbert, Fonctions de Lamé et Fonctions de Mathieu. Mem. Sci. Mat. 926.

Any continuous function with piecewise-continuous derivatives can be expanded in an absolutely and uniformly convergent series of these characteristic functions.

By means of this proposition, it can be shown, in particular, that

$$\left.\begin{aligned}
ce'_{2s}(\eta\,,\,q_1) &= \sum_{k=o}^{\infty} m_{2k}^{(2s)}\ se_{2k}(\eta\,,\,q_2)\ ,\\
ce'_{2s+1}(\eta\,,\,q_1) &= \sum_{k=o}^{\infty} n_{2k+1}^{(2s+1)}\ se_{2k+1}(\eta\,,\,q_2)\ ,\\
se'_{2s}(\eta\,,\,q_1) &= \sum_{k=o}^{\infty} p_{2k}^{(2s)}\ ce_{2k}(\eta\,,\,q_2)\ ,\\
se'_{2s+1}(\eta\,,\,q_1) &= \sum_{k=o}^{\infty} q_{2k+1}^{(2s+1)}\ ce_{2k+1}(\eta\,,\,q_2)\ ,
\end{aligned}\right\} \qquad (30)$$

where, on the basis of (29), we have

$$\left.\begin{aligned}
m_{2k}^{(2s)} &= \frac{\int_{-\pi}^{+\pi} ce'_{2s}(\eta,q_1)se_{2k}(\eta,\,q_2)d\,\eta}{\int_{-\pi}^{+\pi} se_{2k}^2(\eta,q_2)d\,\eta}\ ,\\
n_{2k+1}^{(2s+1)} &= \frac{\int_{-\pi}^{+\pi} ce'_{2s+1}(\eta,\,q_1)se_{2k+1}(\eta,\,q_2)d\eta}{\int_{-\pi}^{+\pi} se_{2k+1}^2(\eta,\,q_2)d\,\eta}\ ,\\
p_{2k}^{(2s)} &= \frac{\int_{-\pi}^{+\pi} se'_{2s}(\eta,\,q_1)ce_{2k}(\eta,q_2)d\,\eta}{\int_{-\pi}^{+\pi} ce_{2k}^2(\eta\,,\,q_2)\ d\eta}\ ,\\
q_{2k+1}^{(2s+1)} &= \frac{\int_{-\pi}^{+\pi} se'_{2s+1}(\eta,\,q_1)ce_{2k+1}(\eta\,,\,q_2)d\eta}{\int_{-\pi}^{+\pi} ce_{2k+1}^2(\eta,\,q_2)d\eta}\ .
\end{aligned}\right\} \qquad (31)$$

§14. The Mathieu-Hankel functions. To conform with the Emission principle, it is necessary that non-periodic solutions of the Mathieu equation (8), $F(\xi)$ be introduced such as to give a phase outbound to infinity and with an attenuation of $\frac{1}{\sqrt{r}}$.

As a preliminary let us prove the following proposition.

If a function $\Omega(\xi,\eta)$, periodic with respect to η, is a solution of the vibrational equation

$$\Delta^2 \Omega(\xi,\eta) + k_1^2(\mathrm{ch}^2\xi - \cos^2\eta)\,\Omega(\xi,\eta) = 0 \tag{6*}$$

and $G(\eta)$ is a periodic solution of the equation

$$G''(\eta) - (k_1^2 \cos^2\eta + A)G(\eta) = 0 \;, \tag{9*}$$

then the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \Omega(\xi,\eta)\,G(\eta)\,d\eta = F(\xi) \tag{32}$$

is a solution of the Mathieu equation

$$F''(\xi) + (k_1^2\,\mathrm{ch}^2\xi + A)F(\xi) = 0 \;, \tag{8*}$$

where we shall, of course, consider solutions that differ from zero. Indeed, we shall introduce (32) in (8*) and prove the equality

$$\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\partial^2\Omega(\xi,\eta)}{\partial\xi^2} + (k_1^2\,\mathrm{ch}^2\xi + A)\,\Omega(\xi,\eta)\right\}G(\eta)\,d\eta = 0. \tag{33}$$

We rewrite (33) with the aid of (6*):

$$\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\partial^2\Omega}{\partial\eta^2} - (k_1^2\cos^2\eta + A)\,\Omega\right\}G(\eta)\,d\eta = 0 \;. \tag{34}$$

Applying a double integration by parts to $\int_0^{2\pi} \frac{\partial^2\Omega}{\partial\eta^2} G(\eta)\, d\eta$, and taking into consideration that $G(2\pi) = G(0)$, $G'(2\pi) = G'(0)$, we find that

$$\int_0^{2\pi} \frac{\partial^2\Omega(\xi,\eta)}{\partial\eta^2} G(\eta)d\eta = \int_0^{2\pi} \Omega(\xi,\eta) \frac{\partial^2 G(\eta)}{\partial\eta^2} d\eta \quad .$$

Now (34) gives

$$\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{d^2G}{d\eta^2} - (k_1^2\cos^2\eta + A)G(\eta)\right\}\Omega(\xi,\eta) = 0$$

in accordance with (9*), and our statement is proved.

This result gives us the possibility of easily constructing solutions that conform with the requirements of the Emission principle.

It is known that vibrational equation (6) has, for example, such particular solutions periodic with respect to η:

$$H_{2n}^{(2)}(k_1r)\sin 2n\theta \quad , \quad H_{2n}^{(2)}(k_1r)\cos 2n\theta \quad , \quad H_{2n+1}^{(2)}(k_1r)\sin(2n+1)\theta \quad ,$$

$$H_{2n+1}^{(2)}(k_1r)\cos(2n+1)\theta \quad ,$$

where

$$r = \sqrt{ch^2\xi - \sin^2\eta} \quad , \quad \theta = \operatorname{arctg}\{\tan\eta\, th\,\xi\} \quad .$$

On the basis of the lemma proved above, let us set up particular solutions of (8*):

$$\frac{1}{\pi}\int_0^{2\pi} H_{2n}^{(2)}(k_1r)\sin 2n\theta \, se_{2n}(\eta, q_1)\,d\eta = Se_{2n}(\xi, q_1) \quad , \qquad (I)$$

$$\frac{1}{\pi}\int_0^{2\pi} H_{2n+1}^{(2)}(k_1r)\sin(2n+1)\theta \, se_{2n+1}(\eta, q_1)\,d\eta = Se_{2n+1}(\xi, q_1), \qquad (II)$$

$$\frac{1}{\pi}\int_0^{2\pi} H_{2n}^{(2)}(k_1r)\cos 2n\theta \, ce_{2n}(\eta, q_1)\,d\eta = Ze_{2n}(\xi, q_1) \quad , \qquad (III)$$

$$\frac{1}{\pi}\int_0^{2\pi} H_{2n+1}^{(2)}(k_1r)\cos(2n+1)\theta \, ce_{2n+1}(\eta, q_1)\,d\eta = Ze_{2n+1}(\xi, q_1). \qquad (IV)$$

If with a Hankel function of even order we take under the integral a Mathieu function of uneven order, and oppositely, the integral will reduce to zero; this will be immediately apparent if formula (50*) be taken into consideration. In addition, we remark that an identity of the indices of the Hankel function and the Mathieu function under the integrals guarantees that they differ from zero, which will also be evident from the asymptotic estimates that are subjoined.

Let us give asymptotic representations of the constructed functions as $\xi \to \infty$; we shall utilize the asymptotic expression of the Hankel function:

$$H_{2n}^{(2)}(k_1r) = \sqrt{\frac{2}{k_1\pi r}}\left\{e^{-i\left[k_1r - \frac{4n+1}{4}\pi\right]} + O\left(\frac{1}{r}\right)\right\} \quad ,$$

and rewrite (I) in the following form:

$$\frac{1}{\pi}\sqrt{\frac{2}{k_1\pi}}\int_0^{2\pi} e^{i\frac{4n+1}{4}\pi}\left[\frac{e^{-ik_1 r}}{\sqrt{r}}-\frac{e^{-ik_1 \mathrm{ch}\,\xi}}{\sqrt{\mathrm{ch}\,\xi}}\right.$$

$$\left.+\frac{e^{-ik_1 \mathrm{ch}\,\xi}}{\sqrt{\mathrm{ch}\,\xi}}\right]\sin 2n\theta\, se_{2m}(\eta\,,\,q_1)\,d\eta =$$

$$=\frac{1}{\pi}\sqrt{\frac{2}{k_1\pi}}\,e^{i\frac{4n+1}{4}\pi}\left[\int_0^{2\pi}\frac{e^{-ik_1 \mathrm{ch}\,\xi}}{\sqrt{\mathrm{ch}\,\xi}}\sin 2n\eta\, se_{2m}(\eta\,,\,q_1)\,d\eta\,+\right.$$

$$+\int_0^{2\pi}\left\{\frac{e^{-ik_1 r}}{\sqrt{r}}-\frac{e^{-ik_1 \mathrm{ch}\,\xi}}{\sqrt{\mathrm{ch}\,\xi}}\right\}\sin 2n\theta\, se_{2m}(\eta, q_1)\,d\eta\;+$$

$$\left.+\int_0^{2\pi}\frac{e^{-ik_1 \mathrm{ch}\,\xi}}{\sqrt{\mathrm{ch}\,\xi}}(\sin 2n\theta-\sin 2n\,\eta)se_{2m}(\eta\,,\,q_1)\,d\eta\right]+$$

$$+\int_0^{2\pi} 0\left(\frac{1}{r^{3/2}}\right)\sin 2n\theta\; se_{2m}(\eta\,,\,q_1)\,d\eta = Se_{2n}(\xi\,,q_1)\quad .$$

Since as $\xi \to \infty$

$$\mathrm{th}\;\xi \to 1\quad ,$$

we have

$$\lim \theta = \lim_{\xi\to\infty}\;\mathrm{arctg}\;\mathrm{tg}\;\eta\cdot\mathrm{th}\;\xi = \eta\quad ,$$

and

$$\lim_{\xi\to\infty}\;[\sin 2n\theta - \sin 2n\,\eta] = 0\quad ;$$

here the order of this difference for large ξ will not be less than $0\left(\frac{1}{\cosh^2\xi}\right)$; indeed, since

$$\lim \theta = \lim_{\xi \to \infty} \operatorname{arctg} \operatorname{tg} \eta \frac{1-\frac{1}{e^{2\xi}}}{1+\frac{1}{e^{2\xi}}} ,$$

θ will approach η, and accordingly $\sin 2n\theta$ will approach $\sin 2n\,\eta$ as fast as $\frac{1}{e^{2\xi}}$ or $\frac{1}{ch^2\xi}$ approach zero (as $\xi \to \infty$).

Moreover,

$$\frac{e^{-ik_1 r}}{\sqrt{r}} - \frac{e^{-ik_1 ch\,\xi}}{\sqrt{ch\,\xi}} = \frac{e^{-ik_1 ch\xi\sqrt{1-\frac{\sin^2\eta}{ch^2\xi}}} - \left(1-\frac{\sin^2\eta}{ch^2\xi}\right)^{\frac{1}{4}} e^{-ik_1 ch\,\xi}}{\sqrt{ch\,\xi}\sqrt[4]{1-\frac{\sin^2\eta}{ch^2\xi}}} =$$

$$= \frac{\cos\left[k_1 ch\xi\sqrt{1-\frac{\sin^2\eta}{ch^2\xi}}\right] - i\sin\left[k_1 ch\,\xi\sqrt{1-\frac{\sin^2\eta}{ch^2\xi}}\right]}{\sqrt{ch\,\xi}\sqrt[4]{1-\frac{\sin^2\eta}{ch^2\xi}}} -$$

$$- \frac{\cos(k_1 ch\,\xi) - i\sin(k_1 ch\,\xi) + 0\left(\frac{1}{ch^2\xi}\right)}{\sqrt{ch\,\xi}\sqrt[4]{1-\frac{\sin^2\eta}{ch^2\xi}}} =$$

$$= \frac{\left\{\cos\left[k_1 ch\,\xi - 0\left(\frac{1}{ch\,\xi}\right)\right] - \cos(k_1 ch\,\xi)\right\}}{\sqrt{ch\,\xi}\sqrt[4]{1-\frac{\sin^2\eta}{ch^2\xi}}} .$$

$$- \frac{i\left\{\sin\left[k_1 ch\,\xi - 0\left(\frac{1}{ch\,\xi}\right)\right] - \sin(k_1 ch\,\xi)\right\} + 0\left(\frac{1}{ch^2\xi}\right)}{\sqrt{ch\,\xi}\sqrt[4]{1-\frac{\sin^2\eta}{ch^2\xi}}} = \frac{0\left(\frac{1}{ch\,\xi}\right)}{\sqrt{ch\,\xi}\sqrt[4]{1-\frac{\sin^2\eta}{ch^2\xi}}} ;$$

therefore, assuming that $\xi \to \infty$ in the preceding sum of integrals, we may write that for sufficiently large ξ :

$$Se_{2m}(\xi, q_1) = \frac{1}{\pi}\sqrt{\frac{2}{k_1\pi \operatorname{ch}\xi}}\, e^{-i(k_1 \operatorname{ch}\xi - \frac{4n+1}{4}\pi)} \int_0^{2\pi} \sin 2n\eta \, se_{2n}(\eta, q_1) d\eta \; +$$

$$+ \frac{0\left(\frac{1}{\operatorname{ch}\xi}\right)}{\sqrt{\operatorname{ch}\xi}\sqrt[4]{1-\frac{\sin^2\eta}{\operatorname{ch}^2\xi}}} \int_0^{2\pi} \sin 2n\eta \, se_{2n} d\eta + 0\left(\frac{1}{\operatorname{ch}^{3/2}\xi}\right) e^{-ik_1 \operatorname{ch}\xi} \; +$$

$$+ \int_0^{2\pi} 0\left(\frac{1}{r^{3/2}}\right) \sin 2n\eta \, se_{2n}\, \eta d\eta \;,$$

and finally

$$\underset{\xi\to\infty}{Se_{2m}} \cong \sqrt{\frac{2}{k_1\pi r}}\, e^{-i\left(k_1 r - \frac{4n+1}{4}\pi\right)} + 0\left(\frac{1}{\operatorname{ch}^{3/2}\xi}\right) \;, \qquad (35)$$

if, for large ξ, $r = \operatorname{ch}\xi\sqrt{1-\frac{\sin^2\eta}{\operatorname{ch}^2\xi}}$ be written for ch ξ.

Analogous asymptotic estimates may be obtained for the three other Mathieu-Hankel functions.

We may thus affirm that (I), (II), (III), (IV) represent those solutions of equation (8) that satisfy the Emission principle.

Separating (I) and the others by the formula

$$H_{2n}^{(2)} = J_{2n} - iY_{2n} \;, \qquad (36)$$

where Y_{2n} is the Neumann function, we have, for $Se_{2n}(\xi, q_1)$, the complex representation

$$\Re(Se_{2n}) + I(Se_{2n}) \cdot i \quad \text{and so forth.} \tag{37}$$

$\Re(Se_{2n})$ and $I(Se_{2n})$ are two linearly independent solutions of equation (8^*), the coefficients of which have no singularities; accordingly $Se_{2n}(\xi, q_1)$ does not have real roots for ξ, since in the contrary case $\Re(Se_{2n})$ and $I(Se_{2n})$ would have a common real root. It follows from this reasoning that neither does $\frac{d}{d\xi} Se_{2n}(\xi, q_1)$ have a real root of ξ.

§15. Expansion of the Mathieu-Hankel function. Having remarked that

$$r = \frac{1}{2}\sqrt{e^{2\xi} + e^{-2\xi} + 2\cos 2\eta}\,,$$
$$\sin\theta = \frac{sh\,\xi \sin\eta}{r}\,; \quad \cos\theta = \frac{ch\,\xi\cos\eta}{r}\,, \tag{38}$$

let us consider the following solution of (6^*):

$$\begin{aligned} &H_n^{(2)}\left(\frac{k_1}{2}\sqrt{e^{2\xi} + e^{-2\xi} + 2\cos 2\eta}\right)(\cos n\theta + i\sin n\theta) = \\ &= H_n^{(2)}\left(\frac{k_1}{2}\sqrt{e^{2\xi} + e^{-2\xi} + 2\cos 2\eta}\right)\left(\frac{ch\,\xi\cos\eta + i\,sh\,\xi\sin\eta}{r}\right)^n = \\ &= H_n^{(2)}\left(\frac{k_1}{2}\sqrt{e^{2\xi} + e^{-2\xi} + 2\cos 2\eta}\right)\left(\frac{e^{\xi}e^{i\eta} + e^{-\xi}e^{-i\eta}}{2r}\right)^n = \\ &= (-)^n e^{in\eta}\left[-\frac{e^{\xi} + e^{-\xi}e^{-2i\eta}}{\sqrt{e^{2\xi} + e^{-2\xi} + 2\cos 2\eta}}\right]^n H_n^{(2)}\left(\frac{k_1}{2}\sqrt{e^{2\xi} + e^{-2\xi} + 2\cos 2\eta}\right). \end{aligned} \tag{39}$$

Make the substitution

$$2\eta = \pi - \omega$$

and for brevity designate

$$\frac{k_1}{2} e^{\xi} = R \quad , \quad \frac{k_1}{2} e^{-\xi} = \rho \quad .$$

(39) then takes the form

$$H_n^{(2)}\left(\frac{k_1}{2}\sqrt{e^{2\xi} + e^{-2\xi} + 2\cos 2\eta}\right) e^{in\theta} =$$

$$= (-)^n e^{in\eta}\left[\frac{\rho e^{i\omega} - R}{\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}}\right]^n H_n^{(2)}\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega} \quad . \tag{40}$$

Dropping the factor $(-1)^n \exp i\, n\eta$ in the above solution[1] of the vibrational equation, let us study the expansion in a Fourier series of the following:

$$[H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right)] e^{in\phi} \quad .$$

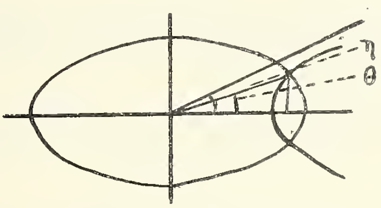

Fig. 2.

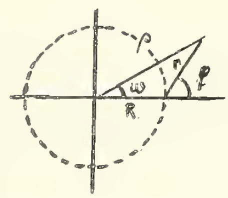

Fig. 3.

It is obvious from Fig. 3 that

$$\sin\phi = \frac{\rho\sin\omega}{\sqrt{R^2+\rho^2-2R\rho\cos\omega}} \quad ,$$

$$\cos\phi = \frac{\rho\cos\omega - R}{\sqrt{R^2+\rho^2-2R\rho\cos\omega}} \quad ,$$

$$e^{in\phi} = \left[\frac{\rho e^{i\omega} - R}{\sqrt{R^2+\rho^2-2R\rho\cos\omega}}\right]^n \quad ,$$

$$e^{-in\phi} = \left[\frac{\rho e^{-i\omega} - R}{\sqrt{R^2+\rho^2-2R\rho\cos\omega}}\right]^n \quad ; \quad .$$

---

[1] See editor's Note 5.

and it is obvious that the function

$$u = H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right) e^{in\phi} ,$$

as a function of ρ and ω, satisfies the vibrational equation

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \omega^2} + u = 0$$

as does the function

$$H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right) e^{-in\phi} =$$

$$= H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right)\left[\frac{\rho e^{-i\omega} - R}{\sqrt{R^2+\rho^2-2R\rho\cos\omega}}\right]^n .$$

Hence we see that the function

$$v = e^{-in\omega}\cdot u = H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right) e^{in\phi}\, e^{-in\omega} =$$

$$= H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right)\left[\frac{\rho - Re^{-i\omega}}{\sqrt{R^2+\rho^2-2R\rho\cos\omega}}\right]^n$$

as a function of R and ω is a solution of the equation

$$\frac{\partial^2 v}{\partial R^2} + \frac{1}{R}\frac{\partial v}{\partial R} + \frac{1}{R^2}\frac{\partial^2 v}{\partial \omega^2} + v = 0 .$$

From these remarks it follows that within the circle $|\rho| < |R|$, u may be represented in the form

$$u = \sum_{s=-\infty}^{s=+\infty} A_s(R) J_s(\rho)\, e^{is\omega} , \qquad (41)$$

and that outside the circle $\rho < |R|$, v may be represented by the expansion

$$v = \sum_{s=-\infty}^{s=+\infty} C_s(R)\, K_s(\rho)\, e^{is\omega} \quad , \qquad (42)$$

where $C_s(R)$ is a certain cylindrical function whose character will be classified below[1]. Hence

$$u = e^{in\omega} \cdot v = \sum_{s=-\infty}^{\infty} C_s(R)\, K_s(\rho) e^{i\omega(n+s)} \quad . \qquad (43)$$

Comparing (43) and (41), we obtain

$$A_s(R) = C_{s-n}(R) \quad .$$

Consequently we have, finally,

$$H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right) e^{in\phi} = \sum_{s=-\infty}^{\infty} C_{s-n}(R) J_s(\rho)\, e^{is\omega} \quad , \qquad (44)$$

$$H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right) e^{-in\phi} = \sum_{s=-\infty}^{\infty} C_{s+n}(R) J_s(\rho)\, e^{is\omega} \quad . \qquad (45)$$

Let us determine the coefficients of these expansions. Employing the general theory of Fourier series, we obtain

$$C_{s-n}(R) J_s(\rho) =$$

$$= \frac{1}{\pi} \int_0^{2\pi} H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right) \left[\frac{\rho e^{i\omega} - R}{\sqrt{R^2+\rho^2-2R\rho\cos\omega}} \right]^n e^{is\omega}\, d\omega =$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left\{ \sqrt{\frac{2}{\pi R}}\, e^{-i\sqrt{R^2+\rho^2-2R\rho\cos\omega} + \frac{2n+1}{4}\pi i} \right\} (-1)^n (e^{is\omega} + \varepsilon)\, d\omega ,$$

---

[1] At this point $K_s(\rho)$ is an unspecified function of ρ, and it should not be identified with the Bessel function $K_s(\rho)$. Editor.

where

$$\varepsilon R^{3/2} \leq M \quad ,$$

M being a finite number.

Furthermore, expanding

$$\sqrt{R^2 + \rho^2 - 2R\rho \cos \omega} = R - \rho \cos \omega + \varepsilon_1 \quad ,$$

we obtain

$$C_{s-n}(R)J_s(\rho) = \sqrt{\frac{2}{\pi R}}\, e^{-iR + \frac{2n+1}{4}\pi i}\, \frac{(-)^n}{\pi} \int_0^{2\pi} e^{i\rho \cos\omega + is\omega} d\omega + \varepsilon_2 \quad ,$$

where

$$\varepsilon_2 = O(R^{-3/2}) \quad .$$

It is known that

$$J_s(x) = \frac{(-i)^s}{2\pi} \int_0^{2\pi} e^{i(x \cos \omega + s\omega)}\, d\omega \quad ;$$

therefore

$$C_{s-n}(R)J_s(\rho) = 2\sqrt{\frac{2}{\pi R}}\, e^{-iR + \frac{2n+1}{4}\pi i}\, (i)^s(-1)^n J_s(\rho) \, ;$$

hence

$$C_{s-n}(R)J_s(\rho) = \sqrt{\frac{2}{\pi R}}\, e^{-iR + \frac{2(s-n)+1}{4}\pi i}\, 2e^{(-\frac{s}{2}+n)\pi i}(-1)^n(i)^s J_s(\rho) \quad .$$

Comparing the asymptotic expansions of both members, we see[1] that

$$C_{s-n}(R) = H^{(2)}_{s-n}(R) \quad .$$

Thus we finally arrive at

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[1] The result can in fact be obtained directly from the addition theorem for Bessel functions. See G. N. Watson's Treatise, Chapt. XI, eq. (6). Editor.

$$H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right) e^{in\phi} = \sum_{s=-\infty}^{\infty} H_{s-n}^{(2)}(R) J_s(\rho)\, e^{is\omega} \quad , \qquad (46)$$

and quite analogously, at

$$H_n^{(2)}\left(\sqrt{R^2 + \rho^2 - 2R\rho\cos\omega}\right) e^{-in\phi} = \sum_{s=-\infty}^{\infty} H_{s+n}^{(2)}(R)\, J_s(\rho)\, e^{is\omega} \ . \qquad (47)$$

Compare (46) with (40):

$$H_n^{(2)}\left(\frac{k_1}{2}\sqrt{e^{2\xi} + e^{-2\xi} + 2\cos 2\eta}\right) e^{in\theta} =$$

$$= (-)^n \sum_{s=-\infty}^{\infty} H_{s-n}^{(2)}(R) J_s(\rho)\, e^{i(s\omega + n\eta)} =$$

$$= (-)^n \sum_{s=-\infty}^{\infty} (-)^s H_{s-n}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_s\left(\frac{k_1}{2} e^{-\xi}\right) e^{(n-2s)\eta i} = \qquad (48)$$

$$= \sum_{s=-\infty}^{\infty} (-)^{n-s} [\cos(n-2s)\eta +$$

$$+ i\sin(n-2s)\eta] H_{s-n}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_s\left(\frac{k_1}{2} e^{-\xi}\right) \ .$$

In case $n = 2k$ we shall have

$$H_{2k}^{(2)}\left(\frac{k_1}{2}\sqrt{e^{2\xi} + e^{-2\xi} + 2\cos 2\eta}\right) e^{2k\theta i} =$$

$$= \sum_{s=-\infty}^{k-1} (-1)^s [\cos 2(k-s)\eta + i\sin 2(k-s)\eta] H_{s-2k}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_s\left(\frac{k_1}{2} e^{-\xi}\right) +$$

$$+ \sum_{s=k}^{\infty} (-1)^{-s} [\cos 2(k-s)\eta + i\sin 2(k-s)\eta] H_{s-2k}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_s\left(\frac{k_1}{2} e^{-\xi}\right) \ .$$

Let us substitute, in the first sum, $s = -r$, in the second, $s = 2k + r$; having made some simple transformations, we obtain:

$$H_{2k}^{(2)}\left(\frac{k_1}{2}\sqrt{e^{2\xi}+e^{-2\xi}+2\cos 2\eta}\right)e^{2k\theta i} =$$

$$= H_k^{(2)}\left(\frac{k_1}{2}e^{\xi}\right) J_k\left(\frac{k_1}{2}e^{-\xi}\right) + \sum_{r=1-k}^{\infty}(-)^r\left\{\left[H_{r+2k}^{(2)}\left(\frac{k_1}{2}e^{\xi}\right)J_r\left(\frac{k_1}{2}e^{-\xi}\right) + \right.\right.$$

$$\left. + H_r^{(2)}\left(\frac{k_1}{2}e^{\xi}\right) J_{r+2k}\left(\frac{k_1}{2}e^{-\xi}\right)\right]\cos 2(k+r)\eta \quad (49)$$

$$+\left[H_{r+2k}^{(2)}\left(\frac{k_1}{2}e^{\xi}\right) J_r\left(\frac{k_1}{2}e^{-\xi}\right)\right.$$

$$\left.\left. - H_r^{(2)}\left(\frac{k_1}{2}e^{\xi}\right) J_{r+2k}\left(\frac{k_1}{2}e^{-\xi}\right)\right] i\sin 2(k+r)\eta\right\} .$$

In case $n = 2k + 1$, we shall obviously have

$$H_{2k+1}^{(2)}\left(\frac{k_1}{2}\sqrt{e^{2\xi}+e^{-2\xi}+2\cos 2\eta}\right)e^{(2k+1)\theta i} =$$

$$= \sum_{r=-k}^{\infty}(-)^r\left\{\left[H_{r+2k+1}^{(2)}\left(\frac{k_1}{2}e^{\xi}\right) J_r\left(\frac{k_1}{2}e^{-\xi}\right) + \right.\right.$$

$$\left. + H_r^{(2)}\left(\frac{k_1}{2}e^{\xi}\right) J_{r+2k+1}\left(\frac{k_1}{2}e^{-\xi}\right)\right]\cos 2(k+r+\tfrac{1}{2})\eta +$$

$$(50)$$

$$+\left[H_{r+2k+1}^{(2)}\left(\frac{k_1}{2}e^{\xi}\right) J_r\left(\frac{k_1}{2}e^{-\xi}\right)\right.$$

$$\left.\left. - H_r^{(2)}\left(\frac{k_1}{2}e^{\xi}\right) J_{r+2k+1}\left(\frac{k_1}{2}e^{-\xi}\right)\right] i\sin 2(k+r+\tfrac{1}{2})\eta\right\} .$$

Replacing η by $-\eta$ in formulas (49) and (50), and combining the two equations thus obtained, afterwards introducing new indices for the summation

$$s = r + k - 1 \quad \text{and} \quad s = r + k ,$$

we are led finally to the following four[1] expansions:

$$H^{(2)}_{2k}\left(\frac{k_1}{2}\sqrt{e^{2\xi}+e^{-2\xi}+2\cos 2\eta}\right)\sin 2k\theta =$$

$$=\sum_{s=0}^{\infty}(-1)^{s+k+1}\left[H^{(2)}_{s+k+1}\left(\frac{k_1}{2}e^{\xi}\right)J_{s-k+1}\left(\frac{k_1}{2}e^{-\xi}\right)-\right.$$

$$\left.-H^{(2)}_{s-k+1}\left(\frac{k_1}{2}e^{\xi}\right)J_{s+k+1}\left(\frac{k_1}{2}e^{-\xi}\right)\right]\sin 2(s+1)\eta \quad ;$$

$$H^{(2)}_{2k}\left(\frac{k_1}{2}\sqrt{e^{2\xi}+e^{-2\xi}+2\cos 2\eta}\right)\cos 2k\theta = H^{(2)}_{k}\left(\frac{k_1}{2}e^{\xi}\right)J_k\left(\frac{k_1}{2}e^{-\xi}\right)+$$

$$+\sum_{s=0}^{\infty}(-1)^{s+k+1}\left[H^{(2)}_{s+k+1}\left(\frac{k_1}{2}e^{\xi}\right)J_{s-k+1}\left(\frac{k_1}{2}e^{-\xi}\right)+\right.$$

$$\left.+H^{(2)}_{s-k+1}\left(\frac{k_1}{2}e^{\xi}\right)J_{s+k+1}\left(\frac{k_1}{2}e^{-\xi}\right)\right]\cos 2(s+1)\eta \quad ;$$

$$H^{(2)}_{2k+1}\left(\frac{k_1}{2}\sqrt{e^{2\xi}+e^{-2\xi}+2\cos 2\eta}\right)\sin(2k+1)\theta = \qquad (50^*)$$

$$=\sum_{s=0}^{\infty}(-)^{s-k}\left[H^{(2)}_{s+k+1}\left(\frac{k_1}{2}e^{\xi}\right)J_{s-k}\left(\frac{k_1}{2}e^{-\xi}\right)-\right.$$

$$\left.-H^{(2)}_{s-k}\left(\frac{k_1}{2}e^{\xi}\right)J_{s+k+1}\left(\frac{k_1}{2}e^{-\xi}\right)\right]\sin 2\left(s+\tfrac{1}{2}\right)\eta \quad ;$$

$$H^{(2)}_{2k+1}\left(\frac{k_1}{2}\sqrt{e^{2\xi}+e^{-2\xi}+2\cos 2\eta}\right)\cos(2k+1)\theta =$$

$$=\sum_{s=0}^{\infty}(-)^{s-k}\left[H^{(2)}_{s+k+1}\left(\frac{k_1}{2}e^{\xi}\right)J_{s-k}\left(\frac{k_1}{2}e^{-\xi}\right)+\right.$$

$$\left.+H^{(2)}_{s-k}\left(\frac{k_1}{2}e^{\xi}\right)J_{s+k+1}\left(\frac{k_1}{2}e^{-\xi}\right)\right]\cos 2\left(s+\tfrac{1}{2}\right)\eta \quad .$$

The uniform and absolute convergence of these series is proved just as for the series $Se_{2m}(\xi, q_1)$.

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[1] See editor's Note 3.

Putting these expansions into formulas (I) - (IV), and using (*) (§ 14) and (§12), respectively, we obtain the fundamental expansions:

$$\left.\begin{aligned}
Se_{2n}(\xi, q_1) &= \sum_{s=1}^{\infty} (-)^{s-n} \bar{a}_{2s}^{(2n)} \Big[H_{s+n}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_{s-n}\left(\frac{k_1}{2} e^{-\xi}\right) \\
&\quad - H_{s-n}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_{s+n}\left(\frac{k_1}{2} e^{-\xi}\right) \Big] ; \\
Se_{2n+1}(\xi, q_1) &= \sum_{s=1}^{\infty} (-)^{s-n-1} \bar{b}_{2s-1}^{(2n+1)} \Big[H_{n+s}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_{s-n-1}\left(\frac{k_1}{2} e^{-\xi}\right) \\
&\quad - H_{s-n-1}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_{s+n}\left(\frac{k_1}{2} e^{-\xi}\right) \Big] ; \\
Ze_{2n}(\xi, q_1) &= \sum_{s=1}^{\infty} (-)^{s-n} a_{2s}^{(2n)} \Big[H_{s+n}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_{s-n}\left(\frac{k_1}{2} e^{-\xi}\right) + \\
&\quad + H_{s-n}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_{s+n}\left(\frac{k_1}{2} e^{-\xi}\right) \Big] + \\
&\quad + 2a_0^{(2n)} H_n^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_n\left(\frac{k_1}{2} e^{-\xi}\right) ; \\
Ze_{2n+1}(\xi, q_1) &= \sum_{s=1}^{\infty} (-)^{s-n-1} b_{2s-1}^{(2n+1)} \Big[H_{n+s}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_{s-n-1}\left(\frac{k_1}{2} e^{-\xi}\right) + \\
&\quad + H_{s-n-1}^{(2)}\left(\frac{k_1}{2} e^{\xi}\right) J_{s+n}\left(\frac{k_1}{2} e^{-\xi}\right) \Big] .
\end{aligned}\right\} \quad (51)$$

These expansions merit attention not only because they are simpler than any expansions yet known of functions analogous to the Mathieu-Hankel functions constructed above, but also because they converge uniformly (and absolutely) for all values of ξ, whereas Heine's expansions converge only for a ξ larger than a certain finite number, and Sieger's[1] expansions, for $\xi > 0$.

[1] As stated in Editor's Preface, Sieger's expansions are also valid over the entire complex ξ-plane.

For proof let us look, for example, at the series for $Se_{2m}(\xi, q_1)$. The expression for the remainder of this series for $s > N$, N being a large number (utilizing the well-known expressions[1] for Bessel functions whose arguments are small in modulus compared with the order). can be obtained as follows:

$$J_s(x) = \frac{x^s}{2^s \cdot s!}(1 + \epsilon_1) \quad ; \quad |\epsilon_1| < \frac{e^{\left|\frac{x^2}{4}\right|} - 1}{s+1} ,$$
$$H_s^{(2)}(x) = i\frac{2^s(s-1)!}{\pi x^s}(1 + \epsilon_2) \quad ; \quad |\epsilon_2| < \frac{e^{\left|\frac{x^2}{4}\right|} - 1}{s-1} , \qquad (52)$$

acquires the following form:

$$|R_N| = \left| \sum_{s=N}^{\infty} (-)^{s-n-1} \bar{a}_{2s}^{(2m)} \left[H_{s+n}^{(2)} J_{s-n} - H_{s-n}^{(2)} J_{s+n} \right] \right| =$$

$$= \sum_{s=N}^{\infty} \frac{\vartheta_1 q_1^{s-n}(2n)!}{(s-m)!(s+n)!} \left[\frac{2^{s+n}(s+n-1)! \left(\frac{k_1}{2} e^{-\xi}\right)^{s-n}}{\pi \left(\frac{k_1}{2} e^{\xi}\right)^{s+n} 2^{s-n}(s-n)!} + \right.$$

$$\left. + \frac{2^{s-n}(s-n-1)! \left(\frac{k_1}{2} e^{-\xi}\right)^{s+n}}{\pi \left(\frac{k_1}{2} e^{\xi}\right)^{s-n} 2^{s+n}(s+n)!} \right] .$$

For each Mathieu-Hankel function of number 2m, N is to be chosen such that

$$N - m > 2n ,$$

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[1] See editor's note 4.

which is always possible, since any finite number, for instance 0, 1, 2, 3, etc.[1], may be taken for n, and 2m is a given fixed number.

Then $(s - n)! > (2n)!$, and simple transformations give

$$|R_n| < \sum_{s=N}^{\infty} \frac{(2n)!(Cq_1^{s-m}}{(s-n)!(s-m)!} ,$$

where c and n are constants independent of ξ ; choosing N suitably, we see that for any ξ

$$|R_N| < \varepsilon ,$$

where ε is any positive number arbitrarily[3] small.

The uniform (and absolute) convergence of all series (51) is thus proved.

§16. Diffraction near an ellipse (Fig. 4). Let us expand the falling potential

$$\Phi^* = \mathfrak{U} e^{i(k_1 y + \omega t)} = \mathfrak{U} e^{i\omega t} \cdot e^{ik_1 \operatorname{sh}\xi \sin\eta}$$

in a generalized Fourier series in terms of the functions of an elliptical cylinder[2]

$$e^{ik_1 \operatorname{sh}\xi \sin\eta} = \sum_{n=0}^{\infty} [B_n ce_n(i\xi, k_1) ce_n(\eta, k_1) + C_n se_n(i\xi, k_1) se_n(\eta, k_1)] .$$

---

(1)Cf. the Heine, Sieger, and other functions exhibited in M.T.O. Strutt's book Lame'sche-Mathieusche Funktionen in Physik, etc., Berlin 1932, pp. 45-48.

(2)Whittaker and Watson, Op. Cit..

(3)Provided all β_s exist. See editor's note 3.

The coefficients B_n and C_n are to be determined on the basis of (29). Noting that

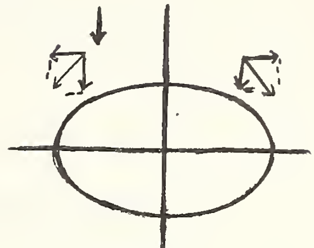

Fig. 4

$$\int_{-\pi}^{+\pi} e^{ik_1 \operatorname{sh}\xi \sin\eta} ce_{2n+1}(\eta, k_1)\, d\eta = 0 \ ,$$

$$\int_{-\pi}^{+\pi} e^{ik_1 \operatorname{sh}\xi \sin\eta} se_{2n}(\eta, k_1)\, d\eta = 0 \ ,$$

the final form of the expansion will be the following:

$$\mathcal{U} e^{i\omega t} \cdot e^{ik_1 \operatorname{sh}\xi \sin\eta} = \tag{53}$$

$$= \mathcal{U} e^{i\omega t} \left[\sum_{s=0}^{\infty} \left\{ B_{2s} ce_{2s}(i\xi, k_1) ce_{2s}(\eta, k_1) + C_{2s+1} se_{2s+1}(i\xi, k_1) se_{2s+1}(\eta, k_1) \right\} \right].$$

This wave, on meeting an obstacle, generates two types of new disturbances, which are characterized by the potentials Φ and Ψ; as has been already pointed out above, they (the disturbances) must be solutions of the equations of elasticity, must satisfy the conditions on the boundary, and must represent a wave with phase outbound to infinity and dying out there.

In order to clarify the form in which the potentials Φ and Ψ should be sought, we will make use of some mechanical properties of the problem in question.

The physical symmetry and homogeneity of all the conditions require that

$$u(x, y) = - u(- x, y) \ ,$$
$$v(x, y) = v(- x, y) \ . \tag{54}$$

We shall endeavor so to construct the potentials as to satisfy these conditions.

For the time being let

$$\Phi(x, y) = \Phi_1(x, y) + \Phi_2(x, y), \quad \Psi(x, y) = \Psi_1(x, y) + \Psi_2(x, y)$$

and together with this, let

$$\begin{aligned} \Phi_1(x, y) &= -\Phi_1(-x, y), \quad \Psi_1(x, y) = -\Psi_1(-x, y), \\ \Phi_2(x, y) &= \Phi_2(-x, y), \quad \Psi_2(x, y) = \Psi_2(-x, y). \end{aligned} \tag{55}$$

Then

$$\begin{aligned} u(x, y) &= \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y} = \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial x} + \frac{\partial \Psi_1}{\partial y} + \frac{\partial \Psi_2}{\partial y}, \\ v(x, y) &= \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x} = \frac{\partial \Phi_1}{\partial y} + \frac{\partial \Phi_2}{\partial y} - \frac{\partial \Psi_1}{\partial x} - \frac{\partial \Psi_2}{\partial x}, \end{aligned} \tag{56}$$

whence

$$\begin{aligned} u(-x, y) &= \left[\frac{\partial}{\partial x}\left\{\Phi_1 + \Phi_2\right\} + \frac{\partial}{\partial y}\left\{\Psi_1 + \Psi_2\right\}\right]_{\substack{x=-x \\ y=y}}, \\ v(-x, y) &= \left[\frac{\partial}{\partial y}\left\{\Phi_1 + \Phi_2\right\} - \frac{\partial}{\partial x}\left\{\Psi_1 + \Psi_2\right\}\right]_{\substack{x=-x \\ y=y}}; \end{aligned} \tag{56*}$$

satisfying the conditions (54) from (56) and (56*), we obtain:

$$\frac{\partial \Phi_1}{\partial x} = -\frac{\partial \Psi_2}{\partial y}; \quad \frac{\partial \Phi_1}{\partial y} = \frac{\partial \Psi_2}{\partial x},$$

and accordingly

$$\Psi_2(x, y) + i\Phi_1(x, y)$$

is an analytic function of the complex variable $z = x + iy$ and Ψ_2 and Φ_1 are thus solutions of the Laplace equation:

$$\Delta^2 \Psi_2 = 0 , \qquad \Delta^2 \Phi_1 = 0 .$$

On the other hand the equality

$$\Delta^2 \Phi_1 + k_1^2 \Phi_1 = 0 ; \quad \Delta^2 \Psi_2 + k_2^2 \Psi_2 = 0$$

must hold, whence

$$\Phi_1(x, y) = \Psi_2(x, y) \equiv 0 ;$$

thus, desiring to satisfy (54), one must put

$\Phi(x, y) = \Phi_2(x, y)$ i.e., an even function of x ,

$\Psi(x, y) = \Psi_1(x, y)$ i.e., an odd function of x .

Now noting that

$$se_{2s}(\eta, k_1) = -se_{2s}(\pi - \eta, k_1) , \quad se_{2s+1}(\eta, k_1) = se_{2s+1}(\pi - \eta, k_1) ,$$

$$ce_{2s}(\eta, k_1) = ce_{2s}(\pi - \eta, k_1) , \quad ce_{2s+1}(\eta, k_1) = -ce_{2s+1}(\pi - \eta, k_1) ,$$

we conclude that Φ and Ψ should be sought in the form

$$\Phi(\xi, \eta) = \sum_{s=0}^{\infty} [a_{2s} Ze_{2s}(\xi, k_1) ce_{2s}(\eta, k_1) + b_{2s+1} Se_{2s+1}(\xi, k_1) se_{2s+1}(\eta, k_1)] , \tag{57}$$

$$\Psi(\xi, \eta) = \sum_{s=0}^{\infty} [c_{2s+1} Ze_{2s+1}(\xi, k_2) ce_{2s+1}(\eta, k_2) + d_{2s} Se_{2s}(\xi, k_2) se_{2s}(\eta, k_2)] . \tag{58}$$

(Here we have not written factors depending upon the time, for the sake of brevity.) One is readily satisfied that the boundary conditions of our problem have, in the new coordinates, the form:

$$\left[\frac{\partial \Phi^*}{\partial \xi} + \frac{\partial \Phi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta}\right]_{\xi=\xi_0} = 0 \quad ; \quad \left[\frac{\partial \Phi^*}{\partial \eta} + \frac{\partial \Phi}{\partial \eta} - \frac{\partial \Psi}{\partial \xi}\right]_{\xi=\xi_0} = 0 \ .$$

Rewriting these conditions by means of (53), (57) and (58)[1]:

$$\sum_{s=0}^{\infty} \bar{B}_{2s} ce'_{2s}(i\xi_0, k_1) ce_{2s}(\eta, k_1) + \bar{C}_{2s+1} se'_{2s+1}(i\xi_0, k_1) se_{2s+1}(\eta, k_1) +$$

$$+ a_{2s} Ze'_{2s}(\xi_0, k_1) ce_{2s}(\eta, k_1) + b_{2s+1} Se'_{2s+1}(\xi_0, k_1) se_{2s+1}(\eta, k_1) +$$

$$+ c_{2s+1} Ze_{2s+1}(\xi_0, k_2) ce'_{2s+1}(\eta, k_2) + d_{2s} Se_{2s}(\xi_0, k_2) se'_{2s}(\eta, k_2) = 0 \ ,$$

$$\sum_{s=0}^{\infty} \bar{B}_{2s} ce_{2s}(i\xi_0, k_1) ce'_{2s}(\eta, k_1) + \bar{C}_{2s+1} se_{2s+1}(i\xi_0, k_1) se'_{2s+1}(\eta, k_1) +$$

$$+ a_{2s} Ze_{2s}(\xi_0, k_1) ce'_{2s}(\eta, k_1) + b_{2s+1} Se_{2s+1}(\xi_0, k_1) se'_{2s+1}(\eta, k_1) -$$

$$- c_{2s+1} Ze'_{2s+1}(\xi_0, k_2) ce_{2s+1}(\eta, k_2) + d_{2s} Se'_{2s}(\xi_0, k_2) se_{2s}(\eta, k_2) = 0 \ .$$

As will be shown below, all the series met with here are absolutely convergent, and a rearrangement of them is permissible.

Using formulas (30), the following formulas are readily obtained:

$$\sum_{s=0}^{\infty} d_{2s} Se_{2s}(\xi_0, k_1) se'_{2s}(\eta, k_2) = \sum_{s=0}^{\infty} ce_{2s}(\eta, k_1) \sum_{r=0}^{\infty} d_{2r} p_{2s}^{(2r)} Se_{2r}(\xi_0, k_2) \ ,$$

$$\sum_{s=0}^{\infty} b_{2s+1} Se_{2s+1}(\xi_0, k_1) se'_{2s+1}(\eta, k_1) =$$

$$= \sum_{s=0}^{\infty} ce_{2s+1}(\eta, k_2) \sum_{r=0}^{\infty} b_{2r+1} q_{2s+1}^{(2r+1)} Se_{2r+1}(\xi_0, k_1) \ ,$$

---

[1] $\bar{B}_{2s} = B_{2s} \cdot i \quad ; \quad \bar{C}_{2s+1} = C_{2s+1} \cdot i \ .$

$$\sum_{s=0}^{\infty} a_{2s} Ze_{2s}(\xi_o, k_1) ce'_{2s}(\eta, k_1) = \sum_{s=0}^{\infty} se_{2s}(\eta, k_2) \sum_{r=0}^{\infty} a_{2r} m_{2s}^{(2r)} Ze_{2r}(\xi_o, k_1) ,$$

$$\sum_{s=0}^{\infty} c_{2s+1} Ze_{2s+1}(\xi_o, k_2) ce'_{2s+1}(\eta, k_2) =$$

$$= \sum_{s=0}^{\infty} se_{2s+1}(\eta, k_1) \sum_{r=0}^{\infty} c_{2r+1} n_{2s+1}^{(2r+1)} Ze_{2r+1}(\xi_o, k_2) .$$

Moreover,

$$\sum_{s=0}^{\infty} \bar{B}_{2s} ce_{2s}(i\xi_o, k_1) ce'_{2s}(k_1, \eta) = \sum_{s=0}^{\infty} B_{2s} ce_{2s}(i\xi_o, k_1) \sum_{k=0}^{\infty} m_{2k}^{(2s)} se_{2k}(\eta, k_2) =$$

$$= \sum_{k=0}^{\infty} se_{2k}(\eta, k_2) \sum_{s=0}^{\infty} B_{2s} m_{2k}^{(2s)} ce_{2s}(i\xi_o, k_1) =$$

$$= \sum_{s=0}^{\infty} se_{2s}(\eta, k_2) \sum_{r=0}^{\infty} B_{2r} m_{2s}^{(2r)} ce_{2r}(i\xi_o, k_1) ,$$

and, analogously,

$$\sum_{s=0}^{\infty} \bar{C}_{2s+1} se_{2s+1}(i\xi_o, k_1) se'_{2s+1}(\eta, k_1) =$$

$$= \sum_{s=0}^{\infty} ce_{2s+1}(\eta, k_2) \sum_{r=0}^{\infty} C_{2r+1} q_{2s+1}^{(2r+1)} ce_{2r+1}(i\xi_o, k_1) .$$

Employing these equations in the preceding boundary conditions, collecting the coefficients of identical Mathieu functions and equating them (the sum) to zero, we obtain:

$$\left.\begin{aligned}
&\bar{B}_{2s}ce'_{2s}(i\xi_o, k_1) + a_{2s}Ze'_{2s}(\xi_o, k_1) + \\
&\quad + \sum_{r=o}^{\infty} d_{2r}p_{2s}^{(2r)}Se_{2r}(\xi_o, k_2) = 0 \quad ; \\
&\bar{C}_{2s+1}se'_{2s+1}(i\xi_o, k_1) + b_{2s+1}Se'_{2s+1}(\xi_o, k_1) + \\
&\quad + \sum_{r=o}^{\infty} c_{2r+1}m_{2s+1}^{(2r+1)}Ze_{2r+1}(\xi_o, k_2) = 0 \quad ; \\
&\sum_{r=o}^{\infty} \bar{B}_{2r}m_{2s}^{(2r)}ce_{2r}(i\xi_o, k_1) + \qquad (s = 0, 1, 2, \cdots,) \\
&\quad + \sum_{r=o}^{\infty} a_{2r}m_{2s}^{(2r)}Ze_{2r}(\xi_o, k_1) - d_{2s}Se'_{2s}(\xi_o, k_2) = 0 \quad ; \\
&\sum_{r=o}^{\infty} \bar{C}_{2r+1}q_{2s+1}^{(2r+1)}se_{2r+1}(i\xi_o, k_1) + \\
&\quad + \sum_{r=o}^{\infty} b_{2s+1}q_{2s+1}^{(2r+1)}Se_{2r+1}(\xi_o, k_1) - c_{2s+1}Ze'_{2s+1}(\xi_o, k_2) = 0 \quad .
\end{aligned}\right\} \quad (59)$$

§17. Proof of the uniqueness. Let us employ the notations:

$$\left.\begin{aligned}
&-\bar{B}_{2s}ce'_{2s}(i\xi_o, k_1) = h_{2s} \;, \quad \sum_{r=o}^{\infty} \bar{B}_{2r}ce_{2r}(i\xi_o, k_1)m_{2s}^{(2r)} = \chi_{2s} \;; \\
&a_{2s}Ze'_{2s}(\xi_o, k_1) = x_{2s} \;, \quad d_{2s}Se'_{2s}(\xi_o, k_2) = y_{2s} \;,
\end{aligned}\right\} \quad (60)$$

and in the first and third of equations (59) let us withdraw from under the Σ sign the summands for r = s (the main term), noting this with a ' on the Σ:

$$\left.\begin{aligned} & a_{2s} Ze'_{2s}(\xi_{o}, k_1) + p_{2s}^{(2s)} d_{2s} Se_{2s}(\xi_{o}, k_2) = \\ & = h_{2s} - \sum_{r=o}^{\infty}{}' p_{2s}^{(2r)} d_{2r} Se_{2r}(\xi_{o}, k_2) \quad ; \\ & - a_{2s} m_{2s}^{(2s)} Ze_{2s}(\xi_{o}, k_1) + d_{2s} Se'_{2s}(\xi_{o}, k_2) = \\ & = \chi_{2s} + \sum_{r=o}^{\infty}{}' m_{2s}^{(2r)} a_{2r} Ze_{2r}(\xi_{o}, k_1) \quad ; \end{aligned}\right\} \qquad (61)$$

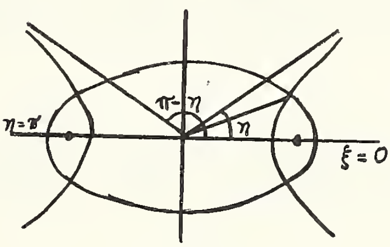

Fig. 5.

Hence we have

$$\left.\begin{aligned} & x_{2s} \frac{Ze'_{2s}(\xi_{o}, k_1)}{Ze_{2s}(\xi_{o}, k_1)} + p_{2s}^{(2s)} y_{2s} = \\ & = h_{2s} - \sum_{r=o}^{\infty}{}' p_{2s}^{(2r)} d_{2r} Se_{2r}(\xi_{o}, k_2) \quad ; \\ & - x_{2s} m_{2s}^{(2s)} + \frac{Se'_{2s}(\xi_{o}, k_2)}{Se_{2s}(\xi_{o}, k_2)} y_{2s} = \\ & = \chi_{2s} + \sum_{r=o}^{\infty}{}' m_{2s}^{(2r)} a_{2r} Ze_{2r}(\xi_{o}, k_1) \quad . \end{aligned}\right\} \qquad (62)$$

Let

$$\Delta_{2s} = \frac{Ze_{2s}'(\xi_o,k_1) \cdot Se_{2s}'(\xi_o,k_2)}{Ze_{2s}(\xi_o,k_1) \cdot Se_{2s}(\xi_o,k_2)} + m_{2s}^{(2s)} p_{2s}^{(2s)} \quad . \qquad (63)$$

Then from (62), with the aid of the notations (60), we obtain

$$x_{2s} = \frac{1}{\Delta_{2s}} \Bigg[h_{2s} \frac{Se_{2s}'}{Se_{2s}} - \frac{Se_{2s}'}{Se_{2s}} \sum_{r=o}^{\infty}{}' \, p_{2s}^{(2r)} \, y_{2r} -$$

$$- \chi_{2s} p_{2s}^{(2s)} - p_{2s}^{(2s)} \sum_{r=o}^{\infty}{}' \, m_{2s}^{(2r)} x_{2r} \Bigg] ;$$

$$y_{2s} = \frac{1}{\Delta_{2s}} \Bigg[\chi_{2s} \frac{Ze_{2s}'}{Ze_{2s}} + \frac{Ze_{2s}'}{Ze_{2s}} \sum_{r=o}^{\infty}{}' \, m_{2s}^{(2r)} x_{2r} +$$

$$+ h_{2s} m_{2s}^{(2s)} - m_{2s}^{(2r)} \sum_{r=o}^{\infty}{}' \, p_{2s}^{(2r)} \, y_{2r} \Bigg] .$$

Let us moreover designate $x_{2s} = z_{2s}$, $y_{2s} = z_{2s+1}$; then we have the system

$$\left.\begin{aligned}
&z_{2s} + \frac{1}{\Delta_{2s}}\left[p_{2s}^{(2s)} \sum_{r=o}^{\infty}{}' m_{2s}^{(2r)} z_{2r} + \frac{Se'_{2s}}{Se_{2s}} \sum_{r=o}^{\infty}{}' p_{2s}^{(2s)} z_{2r+1}\right] = \\
&\qquad = \frac{1}{\Delta_{2s}}\left[h_{2s}\frac{Se'_{2s}}{Se_{2s}} - \chi_{2s}p_{2s}^{(2s)}\right] ; \\
&z_{2s+1} - \frac{1}{\Delta_{2s}}\left[\frac{Ze'_{2s}}{Ze_{2s}} \sum_{r=o}^{\infty}{}' m_{2s}^{(2r)} z_{2r} - m_{2s}^{(2s)} \sum_{r=o}^{\infty}{}' p_{2s}^{(2r)} z_{2r+1}\right] = \\
&\qquad = \frac{1}{\Delta_{2s}}\left[\chi_{2s}\frac{Ze'_{2s}}{Ze_{2s}} + h_{2s}m_{2s}^{(2s)}\right] .
\end{aligned}\right\}(s=0,1,2,\cdots)(64)$$

We shall prove that

$$0 < \epsilon_1 < \left\{1 - \sum_{i=o}^{\infty}{}' |\lambda_{ik}|\right\} < \epsilon_2 ;\ \epsilon_1,\ \epsilon_2 > 0,\ (k = 0,1,2,\ \cdots\ i-1,\ i+1,\cdots),$$

λ_{ik} being the coefficients in the k-th equation, ϵ_1, ϵ_2 being finite numbers independent of k.

Copying in somewhat more detail, it must be shown that

$$\left.\begin{aligned}
&0 < \epsilon_1 < \left\{1 - \frac{1}{|\Delta_{2s}|}\left[|p_{2s}^{(2s)}| \sum_{r=o}^{\infty}{}' |m_{2s}^{(2r)}| + \right.\right. \\
&\qquad \left.\left. + \left|\frac{Se'_{2s}}{Se_{2s}}\right| \sum_{r=o}^{\infty}{}' |p_{2s}^{(2r)}| \right]\right\} < \epsilon_2 ; \\
&0 < \epsilon_1 < \left\{1 - \frac{1}{|\Delta_{2s}|}\left[|m_{2s}^{(2s)}| \sum_{r=o}^{\infty}{}' |p_{2s}^{(2r)}| + \right.\right. \\
&\qquad \left.\left. + \left|\frac{Ze'_{2s}}{Ze_{2s}}\right| \sum_{r=o}^{\infty}{}' |m_{2s}^{(2r)}| \right]\right\} < \epsilon_2 .
\end{aligned}\right\} \qquad (65)$$

In these inequalities $m_{2s}^{(2r)}$, $p_{2s}^{(2r)}$ have values determinable from formulas (31).

Comparing the Mathieu coefficients from formula (*) with their expressions from formulas (14)-(15*), we find

$$a_{2k}^{(2r)} = \begin{cases} \alpha_{r-k}^{(2r)} & r > k, \\ 1 & r = k, \\ \beta_{k-r}^{(2r)} & r < k; \end{cases} \quad \bar{a}_{2k}^{(2r)} = \begin{cases} \bar{\alpha}_{r-k}^{(2r)} & r > k, \\ 1 & r = k, \\ \bar{\beta}_{k-r}^{(2r)} & r < k. \end{cases} \tag{66}$$

Therefore by formulas (22*) and (23*) we obtain the inequalities:

$$\text{for } r > k: \quad |a_{2k}^{(2r)}| = |a_{r-k}^{(2r)}(q_1)| < \frac{\mathfrak{A}_1 q_1^{r-k}(2r+k-r-1)!}{(r-k)!(2r-1)!} = \frac{\mathfrak{A}_1 q_1^{r-k}(r+k-1)!}{(r-k)!(2r-1)!}; \tag{67}$$

$$\text{for } r < k: \quad |a_{2k}^{(2r)}| = |\beta_{k-r}^{(2r)}(q_1)| < \frac{\mathfrak{B}_1 q_1^{k-r}(2r)!}{(k-r)!(2r+k-r)!} = \frac{\mathfrak{B}_1 q_1^{k-r}(2r)!}{(k-r)!(k+r)!}; \tag{68}$$

$$\int_{-\pi}^{+\pi} ce'_{2r}(\eta, q_1) se_{2s}(\eta, q_2) d\eta = -\sum_{k=o}^{\infty} 2k\, a_{2k}^{(2r)}(q_1)\bar{a}_{2k}^{(2s)}(q_2);$$

$$\int_{-\pi}^{+\pi} se_{2s}^2(\eta, q_2) d\eta = \sum_{k=o}^{\infty} [\bar{a}_{2k}^{(2s)}(q_2)]^2;$$

$$m_{2s}^{(2r)} = -\frac{\sum_{k=o}^{\infty} 2k\, a_{2k}^{(2r)}(q_1)\bar{a}_{2k}^{(2s)}(q_2)}{\sum_{k=o}^{\infty} [\bar{a}_{2k}^{(2s)}(q_2)]^2}; \quad p_{2s}^{(2r)} = \frac{\sum_{k=o}^{\infty} 2k\, \bar{a}_{2k}^{(2r)}(q_1) a_{2k}^{(2s)}(q_2)}{\sum_{k=o}^{\infty} [a_{2k}^{(2s)}(q_2)]^2}.$$

Let $r < s$; then

$$|m_{2s}^{(2r)}| < \left|\sum_{k=o}^{\infty} 2ka_{2k}^{(2r)}(q_1)\ \bar{a}_{2k}^{(2s)}(q_2)\right| =$$

$$= \left|\sum_{k=1}^{r} 2ka_{2k}^{(2r)}\bar{a}_{2k}^{(2s)} + \sum_{k=r+1}^{s} 2ka_{2k}^{(2r)}\bar{a}_{2k}^{(2s)} + \sum_{k=s+1}^{\infty} 2ka_{2k}^{(2r)}\bar{a}_{2k}^{(2s)}\right| . \tag{69}$$

Let $r > s$; then

$$|m_{2s}^{(2r)}| < \left|\sum_{k=o}^{s} 2ka_{2k}^{(2r)}\bar{a}_{2k}^{(2s)} + \sum_{k=s+1}^{r} 2ka_{2k}^{(2r)}\bar{a}_{2k}^{(2s)} + \sum_{k=r+1}^{\infty} 2ka_{2k}^{(2r)}\bar{a}_{2k}^{(2s)}\right| . \tag{70}$$

Let us by turns employ formulas (67) and (68) for the estimation of sums (69) and (70).

#1. $r < s$:

$$\sum_{k=o}^{r} 2k\,|a_{2k}^{(2r)}\bar{a}_{2k}^{(2s)}| < \sum_{k=o}^{r} 2k\,\frac{\mathfrak{A}_1 q_1^{r-k}(r+k-1)!\,\mathfrak{A}_2 q_2^{s-k}(s+k-1)!}{(r-k)!(2r-1)!(s-k)!(2s-1)!} <$$

$$< \sum_{k=o}^{r} 2k\,\frac{\mathfrak{A}\,q_1^{r+s-2k}(r+k-1)!(s+k-1)!}{(r-k)!(s-k)!(2r-1)!(2s-1)!} , \tag{71}$$

on the assumption that $\mathfrak{A} > \mathfrak{A}_1\ \mathfrak{A}_2\ q_1 > q_2$;

$$\sum_{k=r+1}^{s} 2k\,|a_{2k}^{(2r)}\bar{a}_{2k}^{(2s)}| < \sum_{k=r+1}^{s} \frac{2k\,\mathfrak{B}\,q_1^{s-r}(2r)!(s+k-1)!}{(k-r)!(k+r)!(s-k)!(2s-1)!} , \tag{72}$$

where $\mathfrak{B} > \mathfrak{A}_1\ \mathfrak{B}_1$;

$$\sum_{k=s+1}^{\infty} 2k\,|a_{2k}^{(2r)}\bar{a}_{2k}^{(2s)}| < \sum 2k\,\frac{Aq^{2k-r-s}(2r)!(2s)!}{(k-r)!(k-s)!(k+r)!(k+s)!} ;\quad A > \mathfrak{B}_1\mathfrak{B}_2 . \tag{73}$$

#2. $r > s$:

$$\sum_{k=0}^{s} 2k\,|a_{2k}^{(2r)}\bar{a}_{2k}^{(2s)}| < \sum_{k=0}^{s} 2k\,\frac{\mathfrak{A}q_1^{r+s-2k}(r+k-1)!(s+k-1)!}{(r-k)!(2r-1)!(s-k)!(2s-1)!}\;; \tag{74}$$

$$\sum_{k=s+1}^{r} 2k\,|a_{2k}^{(2r)}\bar{a}_{2k}^{(2s)}| < \sum_{k=s+1}^{r} \frac{2k\mathfrak{B}_1^{r-s}(2s)!(r+k-1)!}{(r-k)!(2r-1)!(k-s)!(2s-1)!} \tag{75}$$

$$\sum_{k=r+1}^{\infty} 2k\,|a_{2k}^{(2r)}\bar{a}_{2k}^{(2s)}| < \sum_{k=r+1}^{\infty} 2k\,\frac{Aq_1^{2k-r-s}(2r)!(2s)!}{(k-r)!(k-s)!(k+r)!(k+s)!}\;. \tag{76}$$

In inequalities (65), which are to be proved, it is necessary to estimate expressions of the form

$$\sum_{r=0}^{\infty}{}' m_{2s}^{(2r)} \quad . \qquad (r = 0, 1, 2, \cdots s-1, s+1, \cdots)$$

Let us distinguish two cases, $r < s$ and $r > s$:

$$\sum_{r=0}^{\infty}{}' m_{2s}^{(2r)} = \sum_{r=0}^{s-1} m_{2s}^{(2r)} + \sum_{r=s+1}^{\infty} m_{2s}^{(2r)} \quad . \tag{77}$$

In estimating the sum

$$\sum_{r=0}^{s-1} m_{2s}^{(2r)}$$

we shall use inequalities (71)-(73), and in estimating

$$\sum_{r=s+1}^{\infty} m_{2s}^{(2r)}$$

we shall use the inequalities (74)-(73).

The following sum must thus be estimated

$$\sum_{r=0}^{\infty}{}' |m_{2s}^{(2r)}| < \sum_{r=0}^{s-1} \sum_{k=0}^{r} \frac{2k\mathfrak{A} q_1^{r+s-2k}(r+k-1)!(s+k-1)!}{(r-k)!(s-k)!(2r-1)!(2s-1)!} +$$

$$+ \sum_{r=0}^{s-1} \sum_{k=r+1} \frac{2k\mathfrak{B} q_1^{s-r}(2r)!(s+k-1)!}{(k-r)!(k+r)!(s-k)!(2s-1)!} +$$

$$+ \sum_{r=0}^{s-1} \sum_{k=s+1}^{\infty} \frac{2kA q_1^{2-k-r-s}(2r)!(2s)!}{(k-r)!(k-s)!(k+r)!(k+s)!} +$$

$$+ \sum_{r=s+1}^{\infty} \sum_{k=0}^{s} \frac{2k\mathfrak{A} q_1^{r+s-2k}(r+k-1)!(s+k-1)!}{(r-k)!(s-k)!(2r-1)!(2s-1)!} + \qquad (78)$$

$$+ \sum_{r=s+1}^{\infty} \sum_{k=s+1}^{r} \frac{2k\mathfrak{B} q_1^{r-s}(2s)!(r+k+1)!}{(r-k)!(2r-1)!(k-s)!(2s-1)!} +$$

$$+ \sum_{r=s+1}^{\infty} \sum_{k=r+1}^{\infty} \frac{2kA q_1^{2k-r-s}(2r)!(2s)!}{(k-r)!(k-s)!(k+r)!(k+s)!} \; .$$

The double sums in the right part of (78), for $|q_1|$ less than unity, are easily estimated; without exhibiting all the intermediate transformations, we shall show that

$$\sum_{r=0}^{\infty}{}' |m_{2s}^{(2r)}| < \frac{q_1}{1-q_1} - \frac{(M+2\mathfrak{B}\mathfrak{D}_1+4Ae+2\mathfrak{A}\mathfrak{D}_3+2\mathfrak{B}\mathfrak{D}_4)}{h} \; , \qquad (79)$$

the constants being

$$M = \mathfrak{A} \sum_{k=0}^{r} \frac{1}{(r-k)!(s-k)!} \; ,$$

$$\mathfrak{D}_1 \geq \lim_{\gamma\to\infty} \frac{r+\gamma}{\gamma!} \frac{2\gamma-1}{2r+1} \; ,$$

$$\mathfrak{D}_3 \geq \sum_{k=0}^{s} \frac{k}{(r-k)!(s-k)!} \; ,$$

$$\mathfrak{D}_4 \geq \lim_{\gamma\to\infty} \frac{s(s+\gamma)}{\gamma!} (2\gamma - 1) \; .$$

In like manner it is shown that

$$\sum_{r=0}^{\infty}{}' p_{2s}^{(2r)} < \frac{q_1}{1-q_1} \cdot g \quad , \tag{80}$$

where g is a constant bounded number independent of r and s.

Using (65), (79) and (80), we write

$$\left|\frac{p_{2s}^{(2s)}}{\Delta_{2s}}\right| \sum_{r=0}^{\infty}{}' |m_{2s}^{(2r)}| < \frac{|hq_1|}{(1-q_1)\left|m_{2s}^{(2s)} + \frac{1}{p_{2s}^{(2s)}} \frac{Ze'_{2s}Se'_{2s}}{Ze_{2s}Se_{2s}}\right|} \quad ,$$

$$\left|\frac{1}{\Delta_{2s}} \frac{Se'_{2s}}{Se_{2s}}\right| \sum_{r=0}^{\infty}{}' |p_{2s}^{(2r)}| < \frac{gq_1}{(1-q_1)\left|\frac{Ze'_{2s}}{Ze_{2s}} + \frac{m_{2s}^{(2s)}p_{2s}^{(2s)}Se_{2s}}{Se'_{2s}}\right| Se_{2s}} \quad . \tag{81}$$

It will be proved below that the denominators of the right members of the latter inequalities are greater than 1 for any s.

Therefore (65) will be valid if q_1 be chosen so that the inequality

$$\left[1 - \frac{(h+g)q_1}{1-q_1}\right] < \varepsilon_2 \quad , \qquad \varepsilon_2 > 0 \quad .$$

holds, i.e.,

$$q_1 < \frac{1}{1+g+h} \quad . \tag{82}$$

Let us pass to the investigation of Δ_{2s}.

We apply the formulas for the differentiation of cylindrical functions:

$$\frac{d}{dx} C_s(x) = -\frac{s}{x} C_s(x) + C_{s-1}(x) \quad ; \quad \frac{d}{dx} C_s(x) = \frac{s}{x} C_s(x) - C_{s+1}(x) \quad .$$

From (51) we find

$$\mathrm{Se}_{2m}(\xi, q_1) = \sum_{s=1}^{\infty} (-)^{s-n-2m} a_{2s}^{(2m)} \left\{ \left[2nH_{s-n}^{(2)} J_{s+n} + 2nH_{s+n}^{(2)} J_{s-n} \right] + \right.$$

$$+ \frac{k_1}{2} e^{\xi} \left[H_{s+n+1}^{(2)} J_{s-n} - H_{s-n-1}^{(2)} J_{s+n} \right] - \tag{83}$$

$$\left. - \frac{k_1}{2} e^{-\xi} \left[H_{s+n}^{(2)} J_{s-n-1} - H_{s-n}^{(2)} J_{s+n-1} \right] \right\} .$$

(For the sake of brevity, here and henceforth the arguments of $H^{(2)}$ and J are not cited explicitly, but it is throughout understood that $\frac{1}{2}k_1 e^{\xi}$ stands with $H^{(2)}$, and $\frac{1}{2}k_1 e^{-\xi}$ stands with J.)

The uniform (and absolute) convergence of the series (83) is proved just as for series (51), from which it nowise differs in essence.

Series similar to (83) may be written for the three other Mathieu-Hankel functions too.

In future it will be necessary to estimate a lower bound for the modulus of the quantity

$$\mathrm{Se}_{2m}(\xi , q_1) .$$

We have

$$\mathrm{Se}_{2m}(\xi , q_1) = (-)^{m-n-1} (H_{m+n}^{(2)} J_{m-n} - H_{m-n}^{(2)} J_{m+n}) \times$$

$$\times \left[1 + \sum_{s=1}^{\infty}{}' (-)^{s-m} a_{2s}^{(2m)} \frac{H_{s+n}^{(2)} J_{s-n} - H_{s-n}^{(2)} J_{s+n}}{H_{n+m}^{(2)} J_{m-n} - H_{m-n}^{(2)} J_{m+n}} \right] . \tag{84}$$

For $\xi = \xi_o$, where ξ_o is a real positive number or zero, the expression

$$H^{(2)}_{m+n}\left(\frac{k_1}{2} e^{\xi}\right) J_{m-n}\left(\frac{k_1}{2} e^{-\xi}\right) - H^{(2)}_{m-n}\left(\frac{k_1}{2} e^{\xi}\right) J_{m+n}\left(\frac{k_1}{2} e^{-\xi}\right) =$$

$$= J_{m-n}\left(\frac{k_1}{2} e^{-\xi}\right) H^{(2)}_{m-n}\left(\frac{k_1}{2} e^{\xi}\right) \left[\frac{H^{(2)}_{m+n}}{H^{(2)}_{m-n}} - \frac{J_{m+n}}{J_{m-n}} \right]$$

cannot be equal to zero for $n \neq 0$; indeed, since $\xi_o \geq 0$ and $k_1 < 1$, we have

$$0 < \frac{k_1}{2} e^{-\xi_o} < 2,4,$$

and since in the interval (0, 2.45) none of the Bessel functions has a root,

$$J_{m-n}\left(\frac{k_1}{2} e^{-\xi_o}\right) \neq 0 \quad ;$$

moreover

$$H_{m-n}\left(\frac{k_1}{2} e^{\xi_o}\right) \neq 0 \quad ,$$

since the Hankel functions do not have real roots. Therefore even the difference

$$\left(\frac{H^{(2)}_{m+n}}{H^{(2)}_{m-n}} - \frac{J_{m+n}}{J_{m-n}} \right) \neq 0$$

for real ξ.

The convergence of the series

$$\sum_{s=1}^{\infty} \left| \bar{\alpha}^{(2m)}_{2s} \left(H^{(2)}_{s+n} J_{s-n} - H^{(2)}_{s-n} J_{s+n} \right) \right|$$

was shown above. Accordingly the series

$$Q \equiv \sum_{s=1}^{\infty} \alpha_{2s}^{(2m)} \frac{H^{(2)}_{s+n} J_{s-n} - H^{(2)}_{s-n} J_{s+n}}{H^{(2)}_{m+n} J_{m-n} - H^{(2)}_{m-n} J_{m+n}} .$$

also converges absolutely together with it. Let us denote by M_1 the ratio of the largest in modulus of its denominators to $q_1^{|s-m|}$; then, using the inequalities for $|\bar{\alpha}_{2s}^{(2m)}|$, we shall have[1]

$$|Q_m| < M_1 \sum_{s=1}^{\infty}{}' q_1^{|s-m|} < \frac{2M_1 q_1}{1-q_1} .$$

Let

$$q_1 < \frac{1}{1+2M_1} . \qquad (85)$$

Then, from (84)

$$|Se_{2m}(\xi , q_1)| > |H^{(2)}_{m+n} J_{m-n} - H_{m-n} J_{m+n}| \left[1 - \frac{2M_1 q_1}{1-q_1} \right] . \qquad (86)$$

Because they are completely analogous, we shall not write out here estimates like (86) for the three remaining functions M - H.

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[1] Editor's note: The above is a literal translation of the Russian. The author's meaning is perhaps better carried as follows:

Let T_s denote any of the denominators in (84g); i.e.,

$$T_s = H^{(2)}_{m+n} J_{m-n} - H^{(2)}_{m-n} J_{m+n} .$$

Form the sequence

$$\frac{q_1^{|s-m|}}{T_s} = P_s ,$$

and let M_1 be the largest in modulus of all $|P_s|$.

In consequence of the absolute convergence of series (83), the series

$$q_1\left(H^{(2)}_{m+n}\,J_{m-n} - H^{(2)}_{m-n}\,J_{m+n}\right)\sum_{s=1}^{\infty}\left|\frac{\bar{a}^{(2m)}_{2s}}{(H^{(2)}_{m+n}J_{m-n}-H_{m-n}J_{m+n})}\times\right.$$

$$\times\left[2n\left(H^{(2)}_{s-n}\,J_{s+n} + H^{(2)}_{s+n}\,J_{s-n}\right) - \frac{k_1}{2}\,e^{\xi}\left(H^{(2)}_{s+n-1}\,J_{s-n} - H^{(2)}_{s-n-1}\,J_{s+n}\right) -\right.$$

$$\left.\left. - \frac{k_1}{2}\,e^{-\xi}\left(H^{(2)}_{s+n}\,J_{s-n-1} - H^{(2)}_{s-n}\,J_{s+n-1}\right)\right]\right| \tag{87}$$

converges absolutely.

On the other hand, we remark that this series depends on the number m only through the coefficients of the Mathieu function, $\bar{a}^{(2m)}_{2s}$, and it is evident from formulas (66), (23^*) and (22^*) that beginning with some finite value, they cannot increase with increasing m; accordingly a positive number M_2, dependent on m, can be found, such that

$$\left|\frac{d}{d\xi}\,Se_{2m}(\xi\,,\,q_1)\right| < q_1 M_2\left|H^{(2)}_{m+n}\,J_{m-n} - H^{(2)}_{m-n}\,J_{m+n}\right| \quad . \tag{87^*}$$

Analogous inequalities also hold for the three other Mathieu-Hankel functions.

Let us choose $|q_1|$ so that

$$M_2|q_1| < 1 \quad . \tag{88}$$

Finally, it is evident from (47) that

$$m^{(2s)}_{2s} = -\frac{2s+A}{1+B}\quad ;\quad p^{(2s)}_{2s} = \frac{2s+A_1}{1+B_1}\quad ; \tag{89}$$

A, B, A_1, B_1 are quantities which, given a choice of q_1 greater in modulus than 1, are dependent on s through the Mathieu coefficients, and accordingly they do not increase together with s, beginning with a certain finite value of s. Taking into consideration now (86), (87*) and (89), let us consider, beginning with a certain s sufficiently large, the expression

$$\omega_1 = m_{2s}^{(2s)} + \frac{1}{p_{2s}^{(2s)}} \frac{Ze'_{2s}Se'_{2s}}{Ze_{2s}Se_{2s}} .$$

If $\frac{1}{p_{2s}^{(2s)}} \frac{Ze'_{2s}Se'_{2s}}{Ze_{2s}Se_{2s}}$ is a negative quantity, then $|\omega_1| > 1$; if, however, it is a positive quantity, then

$$|\omega_1| > \left| -\frac{2s+A}{1+B} + \frac{1}{\frac{2s+A_1}{1+B_1}\left[1-\frac{2M_1q_1}{1-q_1}\right]} \right| > 1 .$$

One may obviously show, analogously, that

$$\left| \frac{Ze'_{2s}}{Ze_{2s}} + \frac{m_{2s}^{(2s)} p_{2s}^{(2s)} Se_{2s}}{Se'_{2s}} \right| > 1 .$$

These inequalities prove the validity of inequalities (81).

Inequalities (65) provide proof of the uniqueness of the solutions of system (64), given all the assumptions made regarding $|q_1|$ [see formulas (82), (85), (88)]. (It is obvious that these assumptions reduce to one inequality that bounds $|q_1|$ from above.)

Indeed, let system (64) have two different systems of finite solutions; then the corresponding homogeneous system will have solutions different from zero; we shall show the impossibility of the

last assumption if inequality (65) is to be observed. Whereupon the uniqueness of the solutions of the nonhomogeneous system (64) will be proved.

Assuming the existence of non-zero bounded solutions of the homogeneous system corresponding to (64), the cases must be considered where the solutions have a limiting point (exact upper bound).

Let x^* be the exact upper bound of the system of solutions; furthermore, let

$$x_n = x^* - \frac{\varepsilon_1}{N}$$

be the n-th root, in number, of the homogeneous system; let us consider the n-th equation

$$x^* - \frac{\varepsilon_1}{N} + \sum_{r=o}^{\infty}{}' \lambda_{in} x_i = 0$$

and let us demonstrate that such an equality is impossible. Indeed,

$$\left| \sum_{i=o}^{\infty}{}' \lambda_{in} x_i \right| \leq \sum_{i=o}^{\infty}{}' |\lambda_{in}| |x_i| < x^* \sum_{i=o}^{\infty}{}' |\lambda_{in}| \; .$$

But since

$$|x^*| > |x^*| \sum_{i=o}^{\infty}{}' |\lambda_{in}|$$

to a finite number, N may always be chosen so that

$$|x^* - \frac{\varepsilon_1}{N} > x^* \sum_{i=o}^{\infty}{}' |\lambda_{in}| \; ;$$

and the more so will

$$|x^* - \frac{\varepsilon_1}{N}| > | \sum_{i=o}^{\infty} \lambda_{in} x_i | \quad ,$$

and consequently the equality

$$x^* - \frac{\varepsilon_1}{N} + \sum_{i=o}^{\infty\,\prime} \lambda_{in} x_i = 0$$

is impossible and our assumption falls.

We are thus led to the conclusion that the homogeneous system cannot have solutions different from zero, i.e., system (59) has a unique system of finite solutions.

§18. <u>Proof of the existence and solution of system (64).</u> Although system (64) may be solved by the method of Schmidt[1], for numerical computation it is more convenient to employ the method of successive approximations.

To show the possibility of its employment, let us subject system (64) to a preliminary transformation, writing it in the following form:

$$\sum_{r=o}^{\infty} \lambda_{k,r} z_r = \sigma_k \quad . \qquad (k = 0, 1, 2, \cdots \infty) \qquad \text{(64bis)}$$

Together with this, we employ the symbols:

$$\left.\begin{aligned} \lambda_{2i,2j} &= \frac{p_{2i}^{(2j)}}{\Delta_{2i}} m_{2i}^{(2j)} \ ; \qquad \lambda_{2i,2j+1} = \frac{1}{\Delta_{2i}} \frac{Se'_{2i}}{Se_{2i}} p_{2i}^{(2j)} \ ; \\ \lambda_{2i+1,2j} &= - \frac{1}{\Delta_{2i}} \frac{Ze'_{2i}}{Ce'_{2i}} m_{2i}^{(2j)} \ ; \quad \lambda_{2i+1,2j+1} = \frac{m_{2i}^{(2i)}}{\Delta_{2i}} p_{2i}^{(2j)} \quad \text{for } i \neq j; \end{aligned}\right\} (90^*)$$

furthermore

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(1) Riesz. Equations lineares, etc., Paris, 1926.

$$\lambda_{2i,2i} = 1, \quad \lambda_{2i,2i+1} = 0 \quad ,$$

$$\sigma_{2i} = \frac{1}{\Delta_{2i}}\left[h_{2i}\frac{Se'_{2i}}{Se_{2i}} - \chi_{2i}\, p_{2i}^{(2i)}\right] ,$$

$$\sigma_{2i+1} = \frac{1}{\Delta_{2i}}\left[\chi_{2i}\frac{Ze'_{2i}}{Ze_{2i}} + h_{2i}\, m_{2i}^{(2i)}\right] .$$

Let us estimate the order of diminution of the coefficients[(1)]

$$\left.\begin{aligned} \sigma_{2i} &= \frac{1}{\Delta_{2i}}\left[h_{2i}\frac{Se'_{2i}}{Se_{2i}} - \chi_{2i}\, p_{2i}^{2i}\right] ; \\ \sigma_{2i+1} &= \frac{1}{\Delta_{2i}}\left[\chi_{2i}\frac{Ze'_{2i}}{Ze_{2i}} + h_{2i}\, m_{2i}^{2i}\right] ; \\ h_{2i} &= - B_{2i}\, ce'_{2i}(i\xi_o, k_1) ; \\ \chi_{2i} &= \sum_{r=o}^{\infty} B_{2r}\, ce_{2r}(i\xi_o, k_1)\, m_{2i}^{2r} . \end{aligned}\right\} \quad (90)$$

Apart from this, from the expansion of $e^{ik_1 sh\xi \sin\eta}$ in §8 we find

$$B_{2i} = \frac{\int_o^{2\pi} e^{ik_1 sh\xi \sin\eta}\, ce_{2i}(\eta, k_1)\, d\eta}{ce_{2i}(i\xi_o, k_1)\int_o^{2\pi}[ce_{2i}(\eta, k_1)]^2 d\eta} ,$$

whence

$$|B_{2i}| < \left|\int_o^{2\pi} e^{ik_1 sh\xi \sin\eta}\, ce_{2i}(\eta, k_1)\, d\eta\right| ;$$

---

(1) Superscripts have been dropped for the sake of brevity.

integrating by parts, we put

$$e^{ik_1 \operatorname{sh}\xi \sin\eta} = u \; ; \quad ce_{2i}(\eta, k_1)d\eta = dv \; ,$$

$$du = ik_1 \operatorname{sh}\xi \cos\eta\, e^{ik_1 \operatorname{sh}\xi \sin\eta} d\eta \; ; \qquad v = \int ce_{2i}(\eta, k_1)d\eta =$$

$$= \int \sum_{k=o}^{\infty} a_{2k}^{2i} \cos 2k\eta\, d\eta = \sum_{k=o}^{\infty} d_{2k}^{2i} \int \cos 2k\eta\, d\eta = + \sum_{k=o}^{\infty} \frac{a_{2k}^{2i}}{2k} \sin 2k\eta \; ;$$

whence

$$v = \frac{A(\eta)}{2i} \; ,$$

where $A(\eta) = + \sum_{k=1}^{\infty} \frac{2i\alpha_{2k}^{2i}}{2k} \sin 2k\eta$. We shall prove that $A(\eta)$ is a number bounded above for any η of the interval $(0, 2\pi)$. Indeed,

$$|A(\eta)| < \sum_{k=o}^{\infty} \frac{i}{k} \left| a_{2k}^{2i} \sin 2k\eta \right| < \sum_{k=1}^{\infty} \frac{i}{k} \left| a_{2k}^{2i} \right| < 1 +$$

$$+ \sum_{k=1}^{i-1} \frac{\mathcal{U} q_1^{i-k}(i+k-1)!}{(i-k)!(2i-1)!} \frac{i}{k} + \sum_{k=i+1}^{\infty} \frac{\mathcal{B}_1 q_1^{k-i}(2i)!}{(k-i)!(k+i)!} \frac{i}{k} < 1 +$$

$$+ \sum_{k=1}^{i-1} \frac{\mathcal{U} q_1^{i-k}}{k(i-k)!} + \sum_{k=i+1}^{\infty} \frac{\mathcal{B}_1 q_1^{k-i}}{k(k-i)!} < 1 + \sum_{k=1}^{\infty}{}' \frac{E q_1^{|i-k|}}{k|i-k|!} \; , \; (E = \text{const})$$

which proves our statement.

Returning to the partial integration, we have

$$B_{2i} = \left[e^{ik_1 \operatorname{sh}\xi \sin\eta} \frac{A(\eta)}{2i} \right]_0^{2\pi} - \frac{1}{2i} \int_0^{2\pi} A(\eta) ik_1 \operatorname{sh}\xi \cos\eta\, e^{ik_1 \operatorname{sh}\xi \sin\eta} d\eta \; .$$

Let us integrate once again by parts, putting

$$ik_1 \, sh\,\xi \cos\eta \, e^{ik_1 sh\,\xi \sin\eta} = u \;, \quad A(\eta)d\eta = dv \;,$$

$$du = \left[- k_1^2 \, sh^2\xi \cos^2\eta \, e^{ik_1 sh\,\xi \sin\eta} - ik_1 \, sh\,\xi \sin\eta e^{ik_1 sh\,\xi \sin\eta} \right] d\eta \;,$$

$$v = \int A(\eta)d\eta = + \sum_{k=1}^{\infty} \frac{i}{k} \, a_{2k}^{2i} \int \sin 2k\eta \, d\eta = - \sum_{k=1}^{\infty} \frac{i}{2k^2} \, a_{2k}^{2i} \cos 2k\eta \;,$$

whence

$$v = \frac{H(\eta)}{2i} \;,$$

where

$$H(\eta) = - \sum_{k=1}^{\infty} \frac{i^2}{k^2} \, a_{2k}^{2i} \cos 2k\eta \;.$$

We shall prove that $H(\eta)$ is a number bounded above for any η of the interval $(0, 2\pi)$. Indeed,

$$|H(\eta)| < \sum_{k=1}^{\infty} \frac{i^2}{k^2} \left| a_{2k}^{2i} \right| < 1 + \sum_{k=1}^{i-1} \frac{\mathcal{U} q_1^{i-k}(i+k-1)!}{(i-k)!(2i-1)!} \left(\frac{i}{k} \right)^2 +$$

$$+ \sum_{k=i+1}^{\infty} \frac{\mathfrak{B}_1 q_1^{k-i}(2i)!}{(k-i)!(k+i)!} \left(\frac{i}{k} \right)^2 < 1 + \sum_{k=1}^{\infty} E q_1^{|i-k|} \;,$$

Q.E.D..

Returning to the partial integration, we have

$$B_{2i} = - \frac{1}{2i} \left[ik_1 \, sh\,\xi \cos\eta \, e^{ik_1 sh\,\xi \sin\eta} \frac{H(\eta)}{2i} \right]_0^{2\pi} - \frac{1}{(2i)^2} \int_0^{2\pi} (k_1^2 sh^2\xi \cos^2\eta +$$

$$+ ik_1 \, sh\,\xi \sin\eta) e^{ik_1 sh\,\xi \sin\eta} H(\eta) \, d\eta \;.$$

Integrate once again by parts, putting

$$(k_1\ \mathrm{sh}^2\xi \cos^2\eta + ik_1\ \mathrm{sh}\,\xi \sin\eta)e^{ik_1 \mathrm{sh}\,\xi \sin\eta} = u\ , \quad H(\eta)d\eta = dv\ ,$$

$$du = [(-2k_1^2\ \mathrm{sh}^2\xi \cos\eta \sin\eta + ik_1\ \mathrm{sh}\,\xi \cos\eta) +$$

$$+ ik_1^3\ \mathrm{sh}^3\xi \cos^3\eta - k_1^2\ \mathrm{sh}^2\xi \sin\eta \cos\eta] =$$

$$= [ik_1^3\ \mathrm{sh}^3\xi \cos^3\eta - 3k_1^2\ \mathrm{sh}^2\xi \cos\eta \sin\eta + ik_1\ \mathrm{sh}\,\xi \cos\eta]e^{ik_1 \mathrm{sh}\,\xi \sin\eta}\ ,$$

$$v = \int H(\eta)d\eta = -\sum_{k=1}^{\infty}\left(\frac{i}{k}\right)^2 a_{2k}^{2i}\int \cos 2k\eta\, d\eta = -\sum_{k=1}^{\infty}\frac{i^2}{2k^3}a_{2k}^{2i}\sin 2k\eta\ ,$$

whence

$$v = \frac{\Omega(\eta)}{2i}\ ,$$

where

$$\Omega(\eta) = -\sum_{k=1}^{\infty}\left(\frac{i}{k}\right)^3 a_{2k}^{2i}\cos 2k\eta\ .$$

We shall prove that $\Omega(\eta)$ is a number bounded above for any η of the interval $(0, 2\pi)$. Indeed,

$$|\Omega(\eta)| < \sum_{k=1}^{i-1}\left(\frac{i}{k}\right)^3 \frac{\mathfrak{A}\, q_1^{i-k_1}(i+k_1-1)!}{(i-k)!(2i-1)!} + \sum_{k=i+1}^{\infty}\left(\frac{i}{k}\right)^3 \frac{\mathfrak{B}_1 q_1^{k_1-1}(2i)!}{(k-i)!(k-i)!} +$$

$$+ 1 < 1 + \sum_{k=i}^{\infty} E q_1^{|i-k|}\ ,$$

Q.E.D..

Again returning to the integration by parts, we have

$$B_{2i} = -\frac{1}{(2i)^2}\left[(k_1^2\, sh^2\xi \cos^2\eta + ik_1\, sh\,\xi \sin\eta)e^{ik_1 sh\,\xi \sin\eta}\,\frac{\Omega(\eta)}{2i}\right]_0^{2\pi} +$$

$$+\frac{1}{(2i)^3}\int_0^{2\pi}[ik_1^3\, sh^3\xi \cos^3\eta - 3k_1^2\, sh^2\xi \sin\eta\cos\eta +$$

$$+ ik_1\, sh\,\xi\cos\eta]e^{ik_1\, sh\,\xi\sin\eta}\Omega(\eta)d\eta \quad .$$

Let us apply a final integration by parts, putting

$$[ik_1^3\, sh^3\xi\cos^3\eta - 3k_1^2\, sh^2\xi\sin\eta\cos\eta + ik_1\, sh\,\xi\cos\eta]e^{ik_1 sh\,\xi\sin\eta} = u \; ,$$

$$\Omega(\eta)d\eta = dv \; ,$$

$$du = [-6ik_1^3\, sh^3\xi\cos^2\eta\sin\eta - 4k_1^2\, sh^2\xi\cos^2\eta +$$

$$+ 3k_1^2\, sh^2\xi\sin^2\eta - ik_1\, sh\,\xi\sin\eta - k_1^4\, sh^4\xi\cos^4\eta]e^{ik_1 sh\,\xi\sin\eta} \; ,$$

$$v = \int\Omega(\eta)d\eta = -\sum_{k=1}^{\infty}\left(\frac{i}{k}\right)a_{2k}^{2i}\int\sin 2k\eta = \sum_{k=1}^{\infty}\frac{i^3}{2k^4}a_{2k}^{2i}\cos 2k\eta \; ;$$

whence

$$v = \frac{\Xi(\eta)}{2i} \; ,$$

where

$$(\Xi) = \sum_{k=1}^{\infty}\left(\frac{i}{k}\right)^4 a_{2k}^{2i}\cos 2k\eta \; .$$

We shall prove that $\Xi(\eta)$ is a number bounded above for any η ; indeed,

$$|\Xi(\eta)| < \sum_{k=1}^{i-1} \frac{\mathfrak{A} q_1^{i-k}(i+k-1)!}{(i-k)!(2i-1)!}\left(\frac{i}{k}\right)^4 + \sum_{k=i+1}^{\infty} \frac{\mathfrak{B}_1 q_1^{k-i}(2i)!}{(k-i)!(k-i)!}\left(\frac{i}{k}\right)^4 +$$

$$+ 1 < 1 + \sum_{k=1}^{\infty} E q_1^{|i-k|} ,$$

which proves our statement. Consequently

$$B_{2i} = \frac{1}{(2i)^3}\Big[\quad \Big]_o^{2\pi} + \frac{1}{(2i)^4}\int_o^{2\pi} [6ik_1^3 \operatorname{sh}^3\xi \cos^2\eta \sin\eta - 4k_1^2 \operatorname{sh}^2\xi \cos^2\eta +$$

$$+ 3k_1^2 \operatorname{sh}^2\xi \sin^2\eta - ik_1 \operatorname{sh}\xi \sin\eta - k_1^4 \operatorname{sh}^4\xi \cos^4\eta] e^{ik_1 \operatorname{sh}\xi \sin\eta}\, \Xi(\eta) d\eta ,$$

whence we obtain the following inequality:

$$|B_{2i}| < \frac{x_1}{(2i)^4} , \tag{91}$$

where x_1 is a constant dependent on ξ and k_1, and is independent of i.

On the basis of (88), we have from (87):

$$\left.\begin{aligned} |h_{2i}| &< \frac{x_1}{(2i)^2} \frac{|Ce'_{2i}(i\xi_o,k_1)|}{(2i)^2} < \frac{1}{(2i)^2} ; \\ |\gamma_{2i}| &< \frac{x_1}{(2i)^3} \sum_{r=o}^{\infty} \frac{|m_{2i}^{2r} Ce_{2r}(i\xi_o,k_1)|}{(2i)} = \frac{1}{(2i)^3} . \end{aligned}\right\}$$

On the basis of (89), from (86) we obtain the estimate we have been seeking for the free coefficients of system (64bis):

$$\left.\begin{aligned} |\sigma_{2i}| &< \frac{1}{|\Delta_{2i}|}\left[\left|h_{2i}\frac{Se'_{2i}}{Se_{2i}}\right| + \left|e_{2i}p_{2i}^{2i}\right|\right] < \frac{1}{(2i)^4} ; \\ |\sigma_{2i+1}| &< \frac{1}{|\Delta_{2i}|}\left[\left|e_{2i}\frac{Ze'_{2i}}{Ze_{2i}}\right| + \left|h_{2i}m_{2i}^{2i}\right|\right] < \frac{1}{(2i)^4} . \end{aligned}\right\} \tag{92}$$

An analysis that is quite like this one shows that $|\sigma_1|$ and $|\sigma_o| < \frac{1}{2^4}$; in the course of this we use the properties established above that

$$\Delta_{2i} = 0\left(\frac{1}{(2i)^2}\right) \quad ; \quad p_{2i}^{2i} = 0(2i) \quad ; \quad m_{2i}^{2i} = 0(2i) \quad ;$$

$$\frac{Se_{2i}'}{Se_{2i}} \; , \quad \frac{Ze_{2i}'}{Ze_{2i}}$$

are finite numbers for all i.

Let us now pass on to the construction of the solution of (64bis) by the method of successive approximations.

Let us adopt, for approximate values of the roots, the following quantities:

$$z_k = \sigma_k \quad , \quad z_k^{(i+1)} = \sigma_k - \sum_{s=o}^{\infty}{}' \lambda_{ks} z_s^{(i)} \quad , \quad (i = 1, 2, \cdots)$$

the prime on Σ indicating the omission of the value $s = k$.

Then

$$z_k^{(1)} = \sigma_k \quad ; \quad z_k^{(i+1)} - z_k^{(1)} = - \sum_{s=o}^{\infty}{}' \lambda_{ks}[z_k^{(i)} - z_s^{(i-1)}] \quad ,$$

and consequently

$$z_k = z_k^{(1)} + \sum_{k=1}^{\infty}[z_k^{(i+1)} - z_k^{(i)}] = z_k^{(i)} - \sum_{i=1}^{\infty} \sum_{s=o}^{\infty}{}' \lambda_{ks}[z_s^{(i)} - z_s^{(i-1)}], \quad (93)$$

or

$$z_k = z_k^{(1)} + [z_k^{(2)} - z_k^{(1)}] + [z_k^{(3)} - z_k^{(2)}] + [z_k^{(4)} - z_k^{(3)}] + \cdots$$

Thus

$$\left.\begin{aligned}
&z_k^{(1)} = c_k \; ; \\
&z_k^{(2)} - z_k^{(1)} = -[\lambda_{k_o} z_o^{(1)} + \lambda_{k_1} z_1^{(1)} + \cdots + \lambda_{kk-1} z_{k-1}^{(1)} + \\
&\qquad + \lambda_{kk+1} z_{k+1}^{(1)} + \cdots] \; ; \\
&z_k^{(3)} - z_k^{(2)} = -[\lambda_{k_o}(z_o^{(2)} - z_o^{(1)} + \lambda_{k_1}(z_1^{(2)} - z_1^{(1)} + \\
&\qquad + \lambda_{kk-1}\left(z_{k-1}^{(2)} - z_{k-1}^{(1)}\right) + \cdots + \lambda_{kk+1}\left(z_{k+1}^{(2)} - z_{k+1}^{(1)}\right) + \cdots] \; ; \\
&z_k^{(4)} - z_k^{(3)} = -[\lambda_{k_o}(z_o^{(3)} - z_o^{(2)}) + \lambda_{k_1}(z_1^{(3)} - z_1^{(2)}) + \\
&\qquad + \cdots \lambda_{kk-1}(z_{k-1}^{(3)} - z_{k-1}^{(2)} + \lambda_{kk+1}\left(z_{k+1}^{(3)} - z_{k+1}^{(2)}\right) + \cdots] \; , \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \text{(93bis)}$$

whence, since

$$z_o^{(1)} < \frac{1}{2^4} \; , \quad \text{and } z_k^{(1)} < \frac{1}{(2_k)^4} \text{ for } k \geq 1 \; ,$$

we have

$$[z_k^{(2)} - z_k^{(1)}] < \sum_{s=o}^{\infty}{}' |\lambda_{ks} z_s^{(1)}| < \frac{1}{2^4} \sum_{s=o}^{\infty}{}' |\lambda_{ks}| < \frac{1}{2^4 P} \; ; \; P < 1 : P = \text{const.}$$

P is a constant and does not depend on k, therefore

$$[z_k^{(3)} - z_k^{(1)}] < \frac{1}{2^4 P} \sum_{s=o}^{\infty}{}' |\lambda_{ks}| < \frac{1}{2^4 P^2} \; ,$$

$$[z_k^{(4)} - z_k^{(3)}] < \frac{1}{2^4 P^2} \sum_{s=o}^{\infty}{}' |\lambda_{ks}| < \frac{1}{2^4 P^3} \text{ and so forth} \; ;$$

consequently

$$z_k < \frac{1}{2^4} \frac{P}{P-1} \quad . \tag{94}$$

We shall now show that z_k, determined from the absolutely convergent double series (93), is actually the solution of (64bis).

To accomplish this let us add the equations of (93bis), effecting the summation by columns in the right member, this being possible in consequence of the absolute convergence of the double series (94) obtained:

$$\begin{aligned} z_k = \sigma_k - \sum_{s=0}^{\infty}{}' \lambda_{ks} \left\{ z_s^{(1)} + (z_s^{(2)} - z_s^{(1)}) + (z_s^{(3)} - z_s^{(2)}) + \cdots = \right. \\ = \sigma_k - \sum_{s=0}^{\infty} \lambda_{ks} z_s \quad , \end{aligned} \tag{95}$$

or, since $a_{kk} = 1$, from (95):

$$\sum_{s=0}^{\infty}{}' \lambda_{ks} z_s = \sigma_k \quad , \qquad (k = 0, 1, 2, \cdots)$$

i.e., we obtain system (64bis); this proves, then, that series (93) actually gives the solution of system (64bis). Now if we rewrite system (93) and (93bis) in the coefficients λ_{ks} and σ_s, we shall finally have

$$\left. \begin{aligned} z_k = \sigma_k - \sum_{s=0}^{\infty}{}' \lambda_{ks} \sigma_s + \sum_{s=0}^{\infty}{}' \sum_{r=0}^{\infty}{}' \lambda_{ks} \lambda_{sr} \sigma_r - \\ - \sum_{s=0}^{\infty}{}' \sum_{r=0}^{\infty}{}' \sum_{p=0}^{\infty}{}' \lambda_{ks} \lambda_{sr} \lambda_{rp} \sigma_p + \cdots \end{aligned} \quad (k = 0, 1, 2, \cdots) \right\} \tag{96}$$

This series does give the solution of system (64bis); the coefficients λ_{ij} are given by formula (90^*).

Reverting now to formula (45), we find a_{2s} and d_{2s} which figure in the Fourier series of the reflected potentials:

$$a_{2s} = \frac{z_{2s}}{Ze_{2s}(k_1, \xi_o)} \; ; \qquad d_{2s} = \frac{z_{2s+1}}{Se_{2s}(k_2, \xi_o)} \; . \tag{97}$$

In obtaining formulas (97), we have proceeded from the first and third equations of system (59); it is scarcely necessary to emphasize that a quite analogous consideration of the second and fourth equations of system (59) gives

$$c_{2s+1} = \frac{z^*_{2s+1}}{Ze_{2s+1}(\xi_o, k_3)} \; ; \qquad b_{2s+1} = \frac{z^*_{2s}}{Se_{2s+1}(\xi_o, k_1)} \tag{98}$$

where z^*_k is determined by a series analogous to (96), and the modulus of z^*_k is estimated by an inequality analogous to (94).

Entering (97) and (98) in (57) and (58), we obtain

$$\Phi(\xi, \eta) = \tag{99}$$

$$= \sum_{s=o}^{\infty} \frac{Ze_{2s}(\xi, k_1)}{Ze_{2s}(\xi_o, k_1)} z_{2s} ce_{2s}(\eta, k_1) + \frac{Se_{2s+1}(\xi, k_1)}{Se_{2s+1}(\xi_o, k_1)} z^*_{2s} se_{2s+1}(\eta, k_1) \; ,$$

$$\Psi(\xi, \eta) = \tag{100}$$

$$= \sum_{s=o}^{\infty} \frac{Ze_{2s+1}(\xi, k_2)}{Ze_{2s+1}(\xi_o, k_2)} z^*_{2s+1} ce_{2s+1}(\eta, k_2) + \frac{Se_{2s}(\xi, k_2)}{Se_{2s}(\xi_o, k_2)} z_{2s+1} se_{2s}(\eta, k_2) \; .$$

The exact evaluation of series (96) will probably indicate a z_k of such an order as regards k as will guarantee the double differentiability of series (99) and (100) with respect to ξ and η.

We shall leave this question open for the time being. However, if the application of Fourier's method to the diffraction problem be considered as possible a priori, the question solves itself, since the existence and uniqueness of the solution have been established by us given the assumptions made regarding the smallness of q_1 [see inequalities (82), (85), (88)]. On the other hand, since by (12)

$$|q_1| = \frac{k_1^2}{32}; \quad k_1 = \frac{\omega}{c},$$

where ω is the frequency, c_1 the speed of propagation of the incident disturbance; the smallness requirement on $|q_1|$ is equivalent to the smallness of ω or to large values of c_1; these two tokens are, as we know, properties of long waves. Thus the picture of the phenomenon of diffraction obtained by us above holds for long waves where they bend around an elliptical contour.

§19. The diffraction of elastic waves near a segment of finite length. The fundamental functions of the problem that were constructed in the preceding section, and all their expansions, as has been shown above, preserve their definite meaning for $\xi = 0$.

On the other hand, setting $\xi = 0$ in our formulas, we shall by that fact be considering the phenomenon of diffraction near a focal

segment of the family of ellipses and hyperbolas considered above. This is also the problem of diffraction near a segment of finite length.

It is wholly obvious that a complete solution of this problem is given by the formulas

$$\Phi(\xi,\eta) = \sum_{s=0}^{\infty} \frac{Ze_{2s}(\xi, k_1)}{Ze_{2s}(0, k_1)} z_{2s}\, ce_{2s}(\eta, k_1) + \frac{Se_{2s+1}(\xi, k_1)}{Se_{2s+1}(0, k_1)} z_{2s}\, se_{2s+1}(\eta, k_1) \quad ,$$

$$\Psi(\xi,\eta) = \sum_{s=0}^{\infty} \frac{Ze_{2s+1}(\xi, k_2)}{Ze_{2s+1}(0, k_2)} z^{*}_{2s+1}\, ce_{2s+1}(\eta, k_2) + \frac{Se_{2s}(\xi, k_2)}{Se_{2s}(0, k_2)} z_{2s+1}\, se_{2s}(\eta, k_2) \quad ,$$

which are obtained from (99) and (100) by replacing ξ_0 by 0 in them.

CHAPTER III

METHODS OF CONSTRUCTION

§20. <u>General observations</u>. While the method of curvilinear coordinates may theoretically be applied to a wide class of contours and surfaces, the methods expounded in this chapter are distinguished in this respect by a more particular character; these methods, however, possess a far-reaching effectiveness and offer the possibility of an exact calculation not only of the qualitative but also of the quantitative side of phenomena.

On the other hand, the ideas lying at the root of the methods of construction expounded here are still far from having been exhausted, and will doubtless serve as the source of numerous generalizations. Some of these will be indicated below.

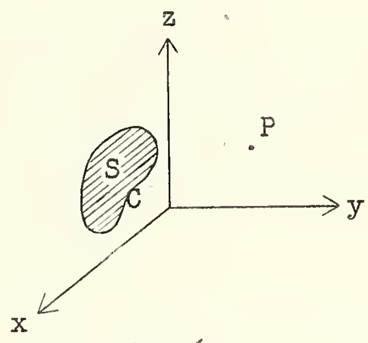

Fig. 6

We shall begin with an exposition of the idea of Sommerfeld, who, using the Cauchy integral over multi-sheeted surfaces, constructs multiple-valued solutions of the vibrational equation that have the necessary properties at infinity and acquire the required boundary values.

Imagine in space a point source of electromagnetic disturbances that are distorted by the presence of a screen S with contour C that occupies part of the plane xz (Fig. 6). It is moreover assumed that the screen has an infinitely great conductivity.

This implies that the components E_x and E_z are equal to zero on the screen. At the point P the sought E and H will have a simple pole; at infinity the Emission Principle must be observed.

Using Thomson's well-known "Reflection Principle," Sommerfeld constructs a point P' representing the mirror image in the screen of the point P.

Then if u(x, y, z, P) is a solution of the equation

$$\Delta u + k^2 u = 0 \quad , \tag{1}$$

having a simple pole at the point P, the function

$$v(x, y, z, P, P') = u(x, y, z, P) - u(x, y, z, P')$$

will satisfy the required "boundary" condition on the screen. However, this function has, apart from the point P, still another polarity at the point P', which does not correspond to any real equivalent in the physical picture of the phenomenon.

Let us therefore conceive of our space as "double-sheeted," and for which the screen S represents a "branching surface"; let us make an incision along the line C and so connect the two "regions" of our "double-sheeted" space that the different semispaces are mutually connected along the screen.

The function v(x, y, z, P, P'), single-valued in such a space, will represent the sought solution, since by now the pole P' is located not in the physical but in the mathematical space.

Nevertheless in order to have the opportunity of applying effective methods of the theory of the functions of a complex variable, it is necessary to shift the investigation to the plane case.

We shall replace the "sheeted space" introduced above by an ordi-

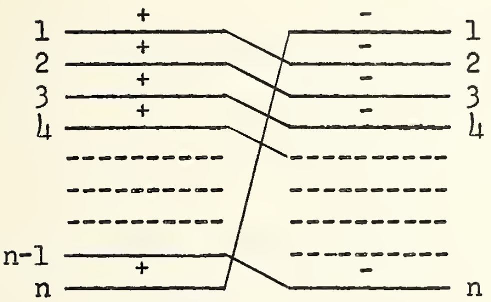

Fig. 7

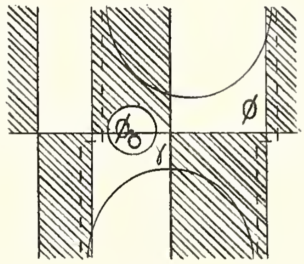

Fig. 8

nary double-sheeted Riemann surface; moreover, we shall for generality begin with a consideration of an n-sheeted Riemann surface and assume that the sought n-valued solution has an infinite number of branch points.

Making an incision from the origin of coordinates to ∞ and "stitching" the layers as is shown in Fig. 7, we introduce polar coordinates (r, ϕ) in the Riemann surface constructed, ϕ running through the values $0 \leq \phi \leq 2n\pi$; thus in the k-th layer we have

$$2(k - 1)\pi \leq \phi \leq 2k\pi \quad .$$

In the plane of the complex variable $z = r + i\phi$ let us consider the function

$$f_n(z) = \log \left[e^{\frac{iz}{n}} - e^{\frac{i\phi_o}{n}} \right] ,$$

where ϕ_o is a point on the real axis. Then we have by the Cauchy formula

$$e^{ikr \cos (\phi - \phi_o)} = \frac{1}{2\pi i} \int_{(\gamma)} e^{ikr \cos (z - \phi)} f_1'(z) dz \quad , \qquad (2)$$

where γ is a circle with center ϕ_o, since the left member is

exactly equal to the residue of the function under the integral sign at its sole singular point $z = \phi_o$. It is quite obvious that formula (2) remains valid with a continuous deformation of (γ) into a curve consisting, in the drawing adduced, of two branches $C(\phi)$ going off to infinity and of the two vertical dotted lines adjoining these branches. Clearly a deformation like this must be realized in such a way that the integral does not lose its sense at infinitely distant parts of the contour. But as is readily seen, in the cross-hatched portions of the drawing,

$$R[e^{ikr \cos (\phi-\phi_o)}] < 0$$

and at infinity the contour $C(\phi)$ lies wholly within the cross-hatched region. In consequence of the periodicity of the integrand function (the width of each cross-hatched strip equals π), and of the contrary direction to be followed in integrating along the dotted vertical lines, they are mutually cancelled, and we have

$$e^{ikr \cos (\phi-\phi_o)} = \frac{1}{2\pi i} \int_{G(\phi)} e^{ikr \cos (r-\phi)} f_1'(z)dz = u_1(r, \phi, \phi_o) .$$

Now let us consider the function

$$u_n(r, \phi, \phi_o) = \frac{1}{2\pi n} \int_{G(\phi)} e^{ikr \cos (z-\phi)} f_n'(z)dz \quad ; \qquad (3)$$

u_n is a solution of the vibrational equation, as one may easily satisfy oneself by executing the differentiation under the integral sign, which differentiation is legitimate in this case. The period of this function is equal to $2n\pi$ and to make this function uniform (i.e., single-valued) we must utilize the n-sheeted Riemann surface.

We shall in addition show that $u_n(r, \phi, \phi_o)$ has the following two properties:

1) $u_n(r, \phi, \phi_o) \to K\, u_1(r, \phi, \phi_o)$ as $r \to \infty$ and $|\phi-\phi_o| \leq \alpha < \pi$,

2) $u_n(r, \phi, \phi_o) \to K\, 0$ as $r \to \infty$ and $\pi \leq \beta \leq |\phi - \phi_o|$.

For proof of these statements we shall investigate the first and k-th sheets, and by means of a continuous deformation we shall modify $C(\phi)$ and $C[\phi + 2(k-1)\pi]$ in such a way that they lie wholly in cross-hatched regions (Fig. 9). In the first sheet we shall of course avoid the singular point $z = \phi_o$.

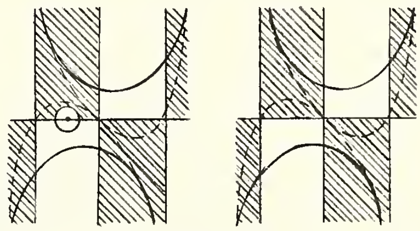

Fig. 9

Having accomplished the aforementioned deformation, and passing to the limit as $r \to \infty$, we see that since

$$R\left\{e^{ikr \cos (\phi-\phi_o)}\right\} < 0 \quad ,$$

in the cross-hatched region $u_n(r, \phi, \phi_o)$ in case 1 (i.e., in the first leaf) gives $u_1(r, \phi, \phi_o)$, and in case 2 (i.e., in all the other sheets) gives zero. Both our statements are thus proved. We shall show that generally

$$u_1 = \sum_{s=1}^{n} u_n^{(s)} \quad , \tag{4}$$

where $u_n^{(s)}$ is the value of u_n at the point (r, ϕ) in the s-th sheet.

Designating by $\int_{(s+1)}$ the integral along the contour $C(\phi+2s\pi)$, we shall have

$$u_1 = \sum_{s=1}^{n} u_n^{(s)}$$

$$= \sum_{s=1}^{n} \int_{(s)} e^{ikr\cos(z-\phi)} f_n'(z)dz = \int_{\Gamma} e^{ikr\cos(z-\phi)} f_n'(z)dz\ , \qquad (4^*)$$

where Γ is composed of the curves $C(\phi)$, $C(\phi+2\pi), \cdots, C[\phi+2(n-1)\pi]$. Closing this curve with two dotted vertical lines (Fig. 10) arrayed

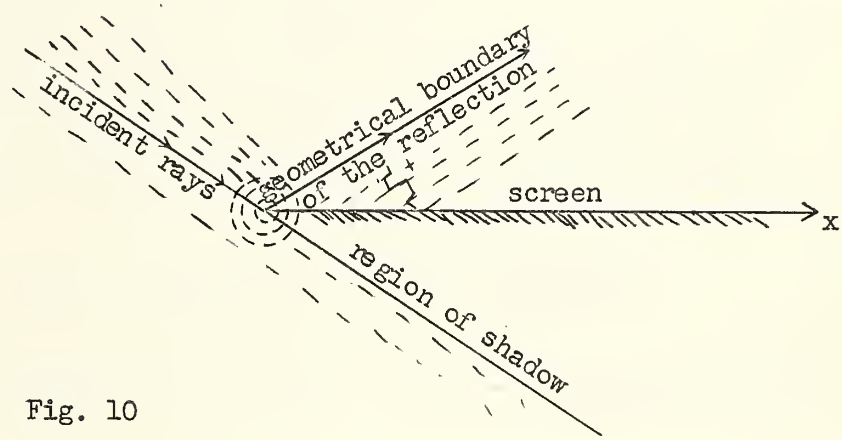

Fig. 10

at a distance of $2n\pi$ from each other, and taking into account the periodicity of the integrand function, of period $2n\pi$, we satisfy ourselves of the validity of equation (4^*). Summarizing the inquiry, we are led to the conclusion that the function defined by (3) is an n-valued solution of the vibrational equation having at infinity the character of a plane wave incident along the direction $\phi = \phi_0$; it describes the condition of an undisturbed electromagnetic field in an n-sheeted Riemann surface.

§21. Diffraction near a semi-infinite segment $y = 0,\ x > 0$. As an example of the Sommerfeld method, let us consider the cases of diffraction near a segment $y = 0,\ x > 0$.

Here in accordance with §1, Chap. 1, we distinguish two cases, which in polar coordinates give the following two conditions on the screen $y = 0,\ x > 0$.

$$1)\quad u = 0\ , \qquad 2)\quad r\frac{\partial u}{\partial r} = 0\ .$$

For the solution of both of these problems, it is enough to construct a double-valued solution ($n = 2$) of the vibrational equation and get its "mirror image" in the screen.

Let us consider a plane wave incident at an angle $\phi = \phi_0$. The double-valued solution that we seek will then, in conformity with what has been said in the preceding paragraph, have the following form:

$$u_2(r,\ \phi,\ \phi_0) = \frac{1}{4\pi i}\int_{C(\phi)} e^{ikr\cos(z-\phi)} f_2'(z)dz\ ,$$

and the complete solution of problems 1 and 2 will be given by the functions

$$1)\quad v(r,\ \phi,\ \phi_0,\ -\phi_0) = u_2(r,\ \phi,\ \phi_0) - u_2(r,\ \phi,\ -\phi_0)\ , \tag{5}$$

$$2)\quad v^*(r,\ \phi,\ \phi_0,\ -\phi_0) = u_2(r,\ \phi,\ \phi_0) + u_2(r,\ \phi,\ -\phi_0)\ . \tag{6}$$

In order to satisfy ourselves that the functions v and v^* do indeed satisfy the necessary boundary conditions, let us set

$$\frac{\partial}{\partial r}\left\{\frac{u_2^{(1)}-u_2^{(2)}}{u_1}\right\} = T\ ;$$

$$T = -\frac{k}{2\pi}\, e^{-2ikr\cos^2[\frac{1}{2}(\phi-\phi_0)]}\int_{C(\phi)}\sin\frac{z+\phi_0-2\phi}{2}\, e^{2ikr\cos^2[\frac{1}{2}(z-\phi)]}dz\ .$$

Make the substitution $z = \phi - \pi + i\xi$; $z = \phi + \pi + i\xi$ for $-\infty < \xi < +\infty$; carrying out the integration we obtain

$$\frac{\partial}{\partial r}\left\{\frac{u_2^{(1)} - u_2^{(2)}}{u_1}\right\} = \frac{2}{\sqrt{-i\pi}}\ \frac{\partial}{\partial r}\int_0^{\sqrt{2kr}\cos[\frac{1}{2}(\phi-\phi_0)]} e^{-i\xi^2} d\xi .$$

On the other hand

$$\frac{u_2^{(1)} + u_2^{(2)}}{u_1} = 1 ,$$

and accordingly

$$u_2^{(1)} = \frac{1}{2}u_1\left\{1 + \frac{2}{\sqrt{-i\pi}}\int_0^{\sqrt{2kr}\cos[\frac{1}{2}(\phi-\phi_0)]} e^{-i\xi^2} d\zeta ,\right\}$$

$$u_2^{(2)} = \frac{1}{2}u_1\left\{1 + \frac{2}{\sqrt{-i\pi}}\int_0^{-\sqrt{2kr}\cos[\frac{1}{2}(\phi-\phi_0)]} e^{-i\xi^2} d\zeta .\right\}$$

If we take into account the fact that in the first sheet ϕ varies over the interval $0 < \phi < 2\pi$, and in the second $-2\pi < \phi < 0$, and that

$$\int_{-\infty}^{0} e^{-i\zeta^2} d\zeta = \frac{1}{2}\sqrt{-i\pi} ,$$

we shall at last have

$$u_2(r,\phi,\phi_0) = u_2^{(1)} = u_2^{(2)} = u_1 e^{i(\frac{1}{4}\pi)}\pi^{-\frac{1}{2}}\int_{-\infty}^{\sqrt{2kr}\cos[\frac{1}{2}(\phi-\phi_0)]} e^{-i\xi^2} d\xi ,$$

where

$$u_1 = e^{i\,kr\cos(\phi-\phi_0)} .$$

Substituting this result in (5) and (6) we are convinced directly that for $\phi = 0$ the required boundary conditions are in fact satisfied.

§22. <u>Diffraction near an angle</u>. Let two half-planes intersect in an angle $\frac{n}{m}\pi$ that is rational in π; in a plane perpendicular to one of them let us introduce polar coordinates such that the straight lines $\phi = 0$ and $\phi = \frac{n}{m}\pi$ constitute the sides of the angle

in question.

We shall call the region $0 < \phi < \frac{n}{m}\pi$ the physical region B; in it we shall study the phenomenon in question. On the straight lines $\phi = 0$, $\phi = \frac{n}{m}\pi$ there must be observed the boundary conditions:

1) $u = 0$ in case 1, Chap. 1, §1.

2) $\frac{\partial u}{\partial n} = 0$ in case 2, Chap. 1, §1.

The solution of both problems is accomplished by a generalized process of "mirror images" whose basic idea was described

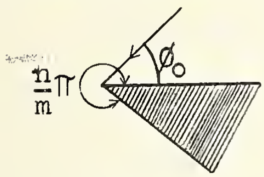

Fig. 11

in the preceding section. Having reflected the physical region B in the line $\phi = 0$, we obtain the region B, in which $0 < \phi < \frac{2n}{m}\pi$; uniting m similar regions, we obtain an angle of $2n\pi$, which we afterwards reflect in the n-sheeted Riemann surface.

We shall now show that the complete solution of the problem may be compounded of the n-valued solutions (3) of the following form

$$\begin{aligned}\Omega_n(r, \phi, \phi_o) \equiv\ & u_n(r,\phi,\phi_o) \mp u_n(r,\phi-\phi_o) + u_n[r,\phi,\tfrac{2(m-1)}{m}n\pi+\phi_o] \\ & \mp u_n(r, \phi, \tfrac{2n}{m}\pi - \phi_o) + u_n[r, \phi, \tfrac{2(m-2)}{m}n\pi + \phi_o] \\ & \mp u_n(r, \phi, \tfrac{4n}{m}\pi - \phi_o) + u_n[r, \phi, \tfrac{2n}{m}\pi + \phi_o] \\ & \mp u_n[r, \phi, \tfrac{2(m-1)}{m}n\pi - \phi_o] \ .\end{aligned} \tag{7}$$

Obviously $\Omega_n(r, \phi, \phi_o)$ as a linear combination of u_n is a solution of the vibrational equation; besides this, we know from the foregoing that at sufficiently great distances from the angle

this solution is a plane wave incident at an angle ϕ_o.

We shall show now that $\Omega_n(r, \phi, \phi_o)$ satisfies boundary conditions 1 and 2.

Indeed, we know from the preceding section that expressions of the form

$$u_n(r, \phi, \phi_o) \mp u_n(r, \phi, -\phi_o) \text{ and } u_n(r, \phi, \phi_o) \mp u_n(r, \phi, \frac{2\pi n}{m} - \phi_o)$$

satisfy conditions 1 and 2 respectively on lines $\phi = 0$ and $\phi = \frac{n}{m}\pi$; moreover, $u_n(r, \phi, \phi_o) = u_n(r, \phi, \phi_o + 2n\pi)$.

Having taken these remarks into account, we are satisfied without further ado that $\Omega(r, \phi, \phi_o)$ actually satisfies conditions 1 and 2 at both boundaries of the diffracting region.

§23. The method of parabolic coordinates.[1] Let the diffracting screen occupy half of the plane xz ($x > 0$).

The wave falling upon the screen normally is given by the potential

$$\Phi = F(ct + y) \quad .$$

We seek a solution of the equation

$$0 = \frac{\partial^2\Phi}{\partial t^2} + c^2\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) ,$$

satisfying on both sides of the screen the condition $\frac{\partial\Phi}{\partial y} = 0$.

Obviously the function

$$\chi = \frac{\partial\Phi}{\partial x} ,$$

which is a solution of the equation $\frac{\partial^2\Phi}{\partial t^2} = c^2\Delta\Phi$ must reduce to zero

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[1]Proceedings of the London Math. Soc., vol. 8, 1910, p. 422 ff.

in the negative part of the plane xz, at the same time as the normal derivative $\frac{\partial x}{\partial y}$ must be equal to zero on both sides of the screen.

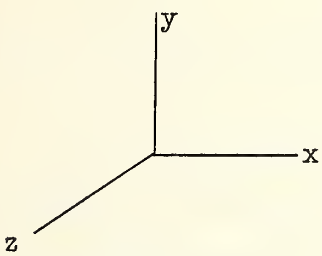

Fig. 12

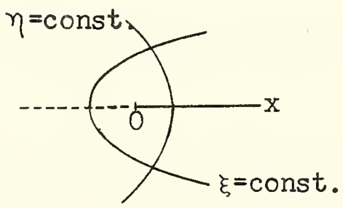

Fig. 13

Employing polar coordinates

$$x = r\cos\theta\,, \quad y = r\sin\theta\,,$$

we see that these conditions are satisfied by the function

$$\chi = f(ct - r)r^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta\ .$$

Now let us introduce parabolic coordinates:

$$\xi = r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta\ ; \quad \eta = r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta\ ,$$

then

$$x = \xi^2 - \eta^2\ ; \quad y = 2\xi\eta\ ; \quad r = \xi^2 + \eta^2\ .$$

The curves ξ = constant, η = constant, form a confocal system of parabolas.

The curve $\eta = 0$ corresponds to the section of the screen. In the new variables we have

$$\frac{\partial\Phi}{\partial x} = \frac{1}{2r}\left(\xi\frac{\partial\Phi}{\partial\xi} - \eta\frac{\partial\Phi}{\partial\eta}\right); \quad \frac{\partial\Phi}{\partial y} = \frac{1}{2r}\left(\eta\frac{\partial\Phi}{\partial\xi} + \xi\frac{\partial\Phi}{\partial\eta}\right)$$

whence, since $\chi = \frac{\partial\Phi}{\partial x}$ and $\chi = f(ct - r)r^{-\frac{1}{2}}\cos\frac{1}{2}\theta$, we have

$$\xi\frac{\partial\Phi}{\partial\xi} - \eta\frac{\partial\Phi}{\partial\eta} = 2\xi f(ct - \xi^2 - \eta^2)\ .$$

Integrating this equation while choosing the supplementary function so that for $\eta = 0$, $\frac{\partial \Phi}{\partial \eta} = 0$ too, we have:

$$\Phi = \int_0^\infty f(ct + y - \zeta^2)d\zeta - \int_0^\infty f(ct - y - \zeta^2)d\zeta + \frac{1}{2}F(ct + y) + \frac{1}{2}F(ct - y).$$

It is evident that for large negative values of x the Φ that is found must reduce to $F(ct + y)$; but when θ is almost equal to π and r has a large value, the upper limits of the integrals just written will be $+\infty$ and $-\infty$, and therefore under these conditions we have:

$$\Phi = \int_0^\infty f(ct + y - \zeta^2)d\zeta - \int_0^\infty f(ct - y - \zeta^2)d\zeta + \frac{1}{2}F(ct + y) + \frac{1}{2}F(ct - y),$$

and this expression for Φ can coincide with $F(ct + y)$ only on condition that

$$\int_0^\infty f(y - \zeta^2)d\zeta = \frac{1}{2}F(y) \; ; \quad -\infty < y < +\infty \; . \tag{8}$$

Thus the solution of the problem reduces to this integral equation of the first type.

Let us consider the case when

$$F(y) = e^{iky} \; ; \tag{9}$$

since

$$\int_0^\infty e^{-ik\xi^2} d\xi = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{k}} e^{-\frac{1}{4}i\pi} ,$$

the solution of the problem will be

$$f(y) = \sqrt{\frac{k}{\pi}} e^{\frac{1}{4}i\pi} e^{iky} \; .$$

This, the simplest case, offers us the possibility of solving the more general problem as well.

Using the Fourier integral, we have:

$$F(y) = \frac{1}{\pi} \int_0^\infty dk \int_{-\infty}^{+\infty} F(\alpha) e^{ik(y-\alpha)} d\alpha \quad ,$$

whence, on the basis of the foregoing, we have

$$f(y) = \frac{1}{\sqrt{\pi^3}} \int_0^\infty \sqrt{k}\, dk \int_{-\infty}^{+\infty} F(\alpha) e^{i[k(y-\alpha)+\frac{1}{4}\pi]} d\alpha \quad . \qquad (10)$$

Equation (8) may easily be converted into Abel's equation; indeed, having made the substitution

$$\zeta = \sqrt{y - \beta} \quad ,$$

we obtain

$$F(y) = \int_{-\infty}^{y} \frac{f(\beta)}{\sqrt{y - \beta}} d\beta \,, \quad -\infty < y < \infty \quad ,$$

and, using the usual method for finding the solution of Abel's equation, we find that

$$f(\beta) = \frac{1}{\pi} \frac{d}{d\beta} \int_{-\infty}^{\beta} \frac{F(t)}{\sqrt{\beta - t}} dt \quad . \qquad (11)$$

If $F(t) = 0$ and $\lim_{t\to-\infty} \sqrt{\beta - t}F(t)$ exists, we have, from (11),

$$f(\beta) = \frac{1}{\pi} \frac{d}{d\beta} \lim_{t\to-\infty} \sqrt{\beta - t}\, F(t) + \frac{1}{\pi} \int_{-\infty}^{\beta} \frac{F'(t)}{\sqrt{\beta - t}} dt \quad ,$$

and in particular, if $F(t) = \sqrt{-t}\, G(t)$ and $G(-\infty)$ is a finite number different from zero, then

$$\lim_{t\to-\infty} \sqrt{\beta - t}\, F(t) = G(-\infty) \quad ,$$

and we at last have

$$f(\beta) = \frac{1}{\pi} \int_{-\infty}^{\beta} \frac{F'(t)}{\sqrt{\beta - t}} dt \quad . \qquad (12)$$

CHAPTER IV

THE INTEGRAL EQUATION METHOD

§24. General remarks. Auxiliary formulas. All the methods discussed in the preceding sections for solving the boundary problems connected with the Helmholtz equation have a limited sphere of application; the actual solution of the problem requires in each separate case a choice of the special coordinates giving the simplest notation for the given problem, and the construction and investigation of the special class of functions with which it is possible to represent fully the solution of the problem. We have already seen, in §9-19, with what sort of difficulties one is obliged to cope; for many practically important curves and surfaces these difficulties are still greater, and indeed for arbitrary contours, despite that they may be analytic and without angular points, the difficulties are actually insurmountable, the relevant realms of mathematical analysis being in their current state.

As regards the methods of construction described in Chap. III, they may be extended to open polyhedral regions with comparative ease, but for closed regions of arbitrary form they are inapplicable without considerable modification, or even a modification of them in principle.

The method of integral equations has a considerable advantage in respect of generality over the methods referred to above. This circumstance is connected primarily with the well-known generality, a generality in principle, that always attends the integral equations

of all problems of mathematical physics. On the other hand, in the given case we have to do with a generalization of the familiar method of Fredholm for the solution of the harmonic problems of Dirichlet and Neumann, which are solved in integral equations in the most general form.

This analogy is particularly complete in the case of problems in electromagnetic diffraction, i.e., in the case where a single wave equation is under study.

The integral equations that arise in this connection have regular kernels and the investigation of their solutions will reduce analogously to a similar investigation in the theory of Laplace equations.

The matter is somewhat more complicated in problems of the diffraction of elastic waves, i.e., in the case where there is a system of wave equations.

Even in the simplest case, when the boundary conditions give the value of the displacement at the boundary, the integral equations have singular kernels, and the applicability of the Fredholm theory requires an additional preliminary analysis, which reduces to the regularization of the singular system of equations obtained.

In the course of this we shall use the following assumption:

If $M(x, y)$, $N(x, y)$ and $f(x)$ are holomorphic functions of the real variables x and y, periodic, of period γ with respect to each of the variables and such that the first two do not have other singularities with respect to y except a simple pole $y = x$, i.e., in the neighborhood of $y = x$ they have the form

$$M(x, y) = M_0(x, y) + \frac{M_1(x)}{y - x} ,$$

$$N(x, y) = N_0(x, y) + \frac{N_1(x)}{y - x} ,$$

where $M_0(x, y)$, $N_0(x, y)$, $M_1(x)$, $N_1(x)$ are holomorphic functions of x and y, then the following equation holds:

$$\text{V.p.} \int_0^{\gamma} N(x,y)dy \quad \text{V.p.} \int_0^{\gamma} M(y,z)f(z)dz = -\pi^2 N_1(x)M_1(x)f(x) + \int_0^{\gamma} f(z)dz. \quad \text{V.p.} \int_0^{\gamma} N(x,y)M(y,z)dy \quad . \tag{I}$$

V.p. (Valeur principale) indicates that the integral has meaning only in the sense of Cauchy's "principal value."

This formula allows us to permute the order of integration in a double integral with Cauchy's "principal value."

In particular, if one of the functions is everywhere regular, the product $M_1(x)N_1(x) = 0$, and instead of (I) we have:

$$\int_0^{\gamma} N(x,y)dy \quad \text{V.p.} \int_0^{\gamma} M(y,z)f(z)dz = \int_0^{\gamma} f(z)dz. \quad \text{V.p.} \int_0^{\gamma} N(x,y)M(y,z)dy$$

or

$$\text{V.p.} \int_0^{\gamma} N(x,y)dy \int_0^{\gamma} M(y,z)f(z)dz = \int_0^{\gamma} f(z)dz \quad \text{V.p.} \int_0^{\gamma} N(x,y)M(y,z)dy \quad .$$

A proof of equation (I) can be found in an article of G. Bertrand: "La theorie des Marées et les equations integrales." Ann. Ec. Norm., (3), XL-1923, pp. 201-247.

§ 25. Generalized logarithmic potentials of a simple and of a double layer. Let us recall the fundamental results of the classical theory of logarithmic potential.

Considering the attracting masses as distributed, with density

$\mu(s)$, along a given singly- or multiply-connected closed contour (γ), which has a tangent that changes its direction continuously everywhere, we have for the potential of such a contour layer:

$$v = \int_{(\gamma)} \mu(s) \lg \frac{1}{r} ds \quad ;$$
$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2} \; ; \quad \xi = \xi(s), \quad \eta = \eta(s) \quad . \tag{1}$$

This potential of a simple layer is, as is known, everywhere continuous, including even passage across the boundary (γ). Its tangential derivative is also everywhere continuous, on condition of the existence of continous first derivatives of the functions $\mu(s)$, but the normal derivative has a jump of $+2\pi\mu(\sigma)$ on passage across the boundary at point $P(\sigma)$.

Together with this, the following relations hold between the limiting values of the normal derivatives, as the point of action (i.e., the point at which the potential itself is being considered) tends toward a point on the contour from within (i) and from outside (e), and their value on the contour,

$$\left(\frac{\partial u}{\partial n_\sigma}\right)_i = -\pi\mu(\sigma) + \int_{(\gamma)} \mu(s) \frac{\cos(n_\sigma, r)}{r} ds \quad ;$$
$$\left(\frac{\partial u}{\partial n_\sigma}\right)_e = +\pi\mu(\sigma) + \int_{(\gamma)} \mu(s) \frac{\cos(n_\sigma, r)}{r} ds \quad . \tag{2}$$

Here n_σ denotes the direction of the positive (inward) normal to (γ) at the point $s = \sigma$.

The logarithmic potential of a double layer, i.e., the distribution of the magnetic moments with density $\nu(s)$ along (γ), is defined by the integral

$$w = \int_{(\gamma)} \nu(s) \frac{\cos(n_s, r)}{r} ds \quad , \tag{3}$$

$$\frac{\cos(n_s,r)}{r} = \frac{\partial}{\partial n}\left(\lg \frac{1}{r}\right) = \frac{\partial \phi}{\partial s} \quad , \tag{4}$$

where $d\phi$ is the element of angle under which the arc ds is seen from the point at which the potential is being considered. The potential of a double layer is no longer everywhere a continuous function of the point, since on passing across the boundary of the magnetic masses it has a jump equal to $2\pi\nu(\sigma)$, defined by the following formulas:

$$\begin{aligned} (u_\sigma)_i &= \pi\nu(\sigma) + \int_{(\gamma)} \nu(s) \frac{\cos(n_s,r)}{r} ds \quad , \\ (u_\sigma)_e &= -\pi\nu(\sigma) + \int_{(\gamma)} \nu(s) \frac{\cos(n_s,r)}{r} ds \quad . \end{aligned} \tag{5}$$

The normal derivative is continuous; the tangential derivative, however, also has a jump equal to $2\pi\nu(\sigma)$, defined by the formulas

$$\begin{aligned} \left[\left(\frac{\partial u}{\partial s}\right)_\sigma\right]_i &= \pi\left[\left(\frac{\partial \nu}{\partial s}\right)_\sigma\right]_\gamma + \frac{\partial}{\partial s}\left[\int_{(\gamma)} \nu(s) \frac{\cos(n_s,r)}{r} ds\right] \quad , \\ \left[\left(\frac{\partial u}{\partial s}\right)_\sigma\right]_e &= -\pi\left[\left(\frac{\partial \nu}{\partial s}\right)_\sigma\right]_\gamma + \frac{\partial}{\partial s}\left[\int_{(\gamma)} \nu(s) \frac{\cos(n_s,r)}{r} ds\right] \quad , \end{aligned} \tag{6}$$

here $\left[\left(\frac{\partial u}{\partial s}\right)_\sigma\right]_i$, etc., signify that we have in view the limiting value of the tangential derivative of the function $\nu(s)$ at the point σ from within, etc.

We shall now construct potentials analogous to those just now adduced, and playing the same role in the theory of the equation $\Delta u + k^2 u = 0$ as do the harmonic potentials adduced above in the theory of the equation $\Delta u = 0$.

For clarity we shall explain once more all the most important symbols.

Let (γ) be a continuous, closed (for simplicity : simply-connected) curve

$$x = \xi(s) \quad , \quad y = \eta(s) \quad , \quad 0 \leq s \leq \chi \quad ,$$

having no singular points.

Furthermore, let $\mu(s)$ and $\nu(s)$ be functions that, together with their first derivatives, are continuous along (γ); let $P(x,y)$ be an arbitrary point of the plane; let $M(\xi, \eta)$ be a variable point on (γ); let n_s and n_σ be normals to (γ) at the points where the arc equals s and σ respectively; finally,[1] let

$$H_0^{(2)} J(kr) = J_0(kr) - iY(kr)$$

be the Hankel function of the second type of the zero-th order.

Let us define the functions

$$V(x,y) = \int_0^{\chi} \mu(s) \left[\frac{1}{2i} H_0^2(kr) \right] ds \tag{7}$$

and

$$W(x,y) = \int_0^{\chi} \nu(s) \left[\frac{\partial}{\partial n_s} \frac{1}{2i} H_0^2(kr) \right] ds \; ; \tag{8}$$

because of the evident analogy of (7) and (8) to (1) and (3) respectively, we shall call (7) the generalized potential of a simple layer and (8) the generalized potential of a double layer. However for brevity of diction we shall drop the word "generalized" where this will not occasion misunderstanding.

Since the special function $H_0^2(kr)$ is a solution of the equation $\Delta u + k^2 u = 0$ everywhere outside the attracting masses, the integrals written above will naturally also be solutions of this same equation. In order to satisfy ourselves of this, let us

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[1]In the rest of this chapter $H_0^{(2)}(y)$ is indicated by the somewhat unorthodox, but more compact, symbol $H_0^2(y)$. For the sake of simplicity, the author's notation is retained. Editor.

seek a solution of the vibrational equation in the form $u = u(kr)$,

$$r = \sqrt{r_1^2 - 2r_1r_2 \cos(\theta - \theta_o) + r_2^2} \quad .$$

From the equation $\Delta u + k^2u = 0$ we obtain

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + k^2u = 0$$

or the Bessel equation

$$\frac{d^2u}{dy^2} + \frac{1}{y}\frac{du}{dy} + \left(1 - \frac{n^2}{y^2}\right)u = 0$$

for the case $n = 0$ with argument $y = kr$. The Hankel function that we used above has the form:

$$H_o^2(y) = J_o(y) - iY_o(y) = \frac{2i}{\pi}\int_o^{\infty} e^{-iy \cos i\phi}\, d\phi \quad .$$

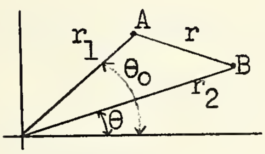

Fig. 14

Proceeding from this representation of the Hankel function, it is easily shown that the generalized potentials (7) and (8) at sufficiently great distances satisfy the Emission principle.

By means of the familiar formula

$$H_o^2(kr) = J_o(kr) - \frac{2i}{\pi} Y_o(kr) \lg (kr) - \frac{2i}{\pi} \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} J_{2p}(kr)$$

the generalized potential of a simple layer, (7), may be written in the following form:

V(x,y)

$$= \frac{1}{\pi} \int_0^{\chi} \mu(s) Y_o(kr) \lg \frac{1}{kr}\, ds + \frac{1}{\pi} \int_0^{\chi} \mu(s) \left[\frac{\pi}{2i} J_o(kr) - \sum_{p=1}^{\infty} \frac{(-)^{p+1}}{p} J_{2p}(kr) \right] ds$$

$$= \frac{1}{\pi} \int_0^{\chi} \mu(s) \lg \frac{1}{kr}\, ds + \frac{1}{\pi} \int_0^{\chi} \mu(s) [Y_o(kr) - 1] \lg \frac{1}{kr}\, ds$$

$$+ \frac{1}{2i} \int_0^{\chi} \mu(s) \left[J_o(kr) - \frac{2i}{\pi} \sum_{p=1}^{\infty} \frac{(-)^{p+1}}{p} J_{2p}(kr) \right] ds \; .$$

Comparing this expression for V(x, y) with (1), and taking into account the continuity of all the summands, except the first, on passing across (γ), we obtain from formula (2):

$$\left. \begin{aligned} \left(\frac{\partial V}{\partial n_\sigma}\right)_i &= -\mu(\sigma) + \int_{(\gamma)} \mu(s) \frac{\partial}{\partial n_\sigma} \left[\frac{1}{2i} H_o^2(kr) \right] ds \; ; \\ \left(\frac{\partial V}{\partial n_\sigma}\right)_e &= \mu(\sigma) + \int_{(\gamma)} \mu(s) \frac{\partial}{\partial n_\sigma} \left[\frac{1}{2i} H_o^2(kr) \right] ds \; , \end{aligned} \right\} \qquad (9)$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the normal to the arc (γ).

The generalized potential of a double layer, determined from (8), also has properties analogous to those expressed in formulas (5) and (6). Indeed, we know that

$$\frac{\partial}{\partial n_s} H_o^{(2)}(kr) = -k \cos(r, n_s) H_1^2(kr) = \frac{2i}{\pi} \frac{\cos(n_s, r)}{r} - k \cos(r, n_s)$$

$$\times \left\{ J_o(kr) - \frac{2i}{\pi} J_1(kr) \lg \frac{kr}{2} \right. \qquad (10)$$

$$\left. - \frac{i}{\pi} \sum_{p=o}^{\infty} \frac{(-)^p \left(\frac{kr}{2}\right)^{1+2p}}{p!(p+1)!} [\Psi(p+1) + \Psi(p+2)] \right\} \; .$$

Substituting this expression for the kernel under the integral into (8) and taking into consideration the continuity of all the

summands, except the first, on passing the boundary (γ), we have (3), (5) and (6):

$$\left.\begin{aligned} (W_\sigma)_i &= \nu(\sigma) + \int_{(\gamma)} \nu(s) \frac{\partial}{\partial n_s}\left[\frac{1}{2i} H_o^2(kr_{s\sigma})\right] ds \,; \\ (W_\sigma)_e &= -\nu(\sigma) + \int_{(\gamma)} \nu(s) \frac{\partial}{\partial n_s}\left[\frac{1}{2i} H_o^2(kr_{s\sigma})\right] ds \,, \end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned} \left[\left(\frac{\partial W}{\partial s}\right)_\sigma\right]_i &= \left[\left(\frac{\partial V}{\partial s}\right)_\sigma\right]_\gamma + \int_{(\gamma)} \nu(s) \frac{\partial}{\partial s_s}\left[\frac{1}{2i} H_o^2(kr_{s\sigma})\right] ds \,; \\ \left[\left(\frac{\partial W}{\partial s}\right)_\sigma\right]_e &= -\left[\left(\frac{\partial V}{\partial s}\right)_\sigma\right]_\gamma + \int_{(\gamma)} \nu(s) \frac{\partial}{\partial s_s}\left[\frac{1}{2i} H_o^2(kr_{s\sigma})\right] ds \,; \end{aligned}\right\} \quad (12)$$

in these formulas $\frac{\partial}{\partial s}$ everywhere denotes differentiation along the tangent and

$$r_{s\sigma} = \sqrt{[\xi(s) - \xi(\sigma)]^2 + [\eta(s) - \eta(\sigma)]^2} \;.$$

In addition to this, because of the continuity of the normal derivative of the potential of the double layer,

$$\frac{\partial W}{\partial n_i} = \frac{\partial W}{\partial n_e} = \frac{\partial W}{\partial n_\gamma} \;.$$

Lastly, it is not hard to verify by direct computation that the generalized potentials (7) and (8) asymptotically satisfy the Emission principle, because of the special selection of the integral kernels.

§26. Fundamental boundary problems and integral equations. The first fundamental problem -- the generalized Dirichlet problem -- as has already been noted in other places of this book, is

formulated thus:

To be found, a function $u(x, y)$, single-valued over the plane (xy), that satisfies the equation

$$\Delta u + k^2 u = 0 \quad , \tag{13}$$

the boundary condition along (γ)

$$au_e + bu_i = \omega(s) \quad , \tag{14}$$

(where a and b are given constants, and $\omega(s)$ is a function given along (γ)), and the Emission principle at infinity.

As regards the boundary condition (14) we shall remark that for $a = 0$, $b = 1$, we shall have the internal problem (D_i) of Dirichlet, and for $a = 1$, $b = 0$, the external problem (D_e) of Dirichlet.

As in ordinary potential theory, the problem reduces to finding a distribution (the law of distribution) of radiating masses such that the potentials defined by us satisfy all the conditions set above.

We have already seen that the functions defined by (7) and (8) satisfy the first and the last conditions.

It remains to choose the arbitrary functions $\mu(s)$ and $\nu(s)$ so as to satisfy the second condition as well.

The first fundamental boundary problem may be solved in the general formulation set above, by endeavoring to find the solution in the form of the generalized potential of a double layer, i.e., in the form of a function defined by (8).

Substituting it in (14) and taking (11) into account, we

obtain for a = 0, b = 1,

$$(D_i) \qquad \nu(\sigma) + \int_0^{\gamma} \nu(s)K(s, \sigma, k)ds = \omega(\sigma) \quad ;$$

for a = 1, b = 0,

$$(D_e) \qquad \nu(\sigma) - \int_0^{\gamma} \nu(s)K(s, \sigma, k)ds = - \omega(\sigma) \quad ,$$

where

$$\frac{1}{2i}\frac{\partial}{\partial n_s} H_o^2(kr) = - \frac{1}{2i} kH_1^2(kr)\frac{\partial r}{\partial n_s} = - \frac{1}{2i} kH_1^2(kr) \cos \phi = K(s,\sigma,k) \ ,$$

and ϕ is the angle between the direction of the inner normal at the point s and the segment r from the point s to the point s = σ.

The second fundamental boundary problem, the generalized Neumann problem, is formulated just as the first problem is, but instead of boundary condition (14), a condition is set here that relates to the value of the normal derivatives at the contour, to wit:

$$a \frac{\partial u}{\partial n_e} + b \frac{\partial u}{\partial n_i} = \tau(s) \quad , \qquad (15)$$

where a and b are assigned constants, and $\tau(s)$ is a point function given along (γ).

In solving this problem it is convenient to use the generalized potential of the simple layer.

Introducing into (15) in place of u_e and u_i their values from (7) and taking (9) into consideration, we have: for a = 0, b = 1,

$$(N_i) \qquad \mu(\sigma) - \int_0^{\gamma} \mu(s)K(\sigma, s, k)ds = - \tau(\sigma) \quad ,$$

and for a = 1, b = 0,

$$(N_e) \qquad \mu(\sigma) + \int_0^{\gamma} \mu(s)K(\sigma, s, k)ds = \tau(\sigma) \quad ,$$

where, as is obvious,

$$\frac{1}{2i}\frac{\partial}{\partial n_\sigma} H_o^2(kr) = -\frac{1}{2i} kH_1^2(kr) \cos \psi = K(\sigma, s, k) \quad ,$$

and ψ is the angle between the inner normal at the point $s = \sigma$ and the segment $r(s, \sigma)$.

Apart from the boundary conditions of the Dirichlet and Neumann problems, there may be still others, to which one physical problem or another leads. So, for example, the problems of diffraction considered in §4 lead to the following conditions on the boundary:

$$u_i = u_e + u^* \; , \quad \frac{\partial u_i}{\partial n} = \frac{\partial u_e}{\partial n} + \frac{\partial u^*}{\partial n} \; ;$$

here, as there, u_i is the sought potential within (γ), which in this case confines a material medium of conductivity different from zero, in consequence of which there are also obtained refracted waves; u_e is the potential outside (γ), and u^* is the fundamental, given potential outside of (γ), which upon meeting (γ) generates the diffracted and refracted rays.

In solving such a problem it is natural to seek the potentials within and outside of (γ) in the form of generalized potentials of the simple and double layers.

Let

$$u_i = \int_0^{\chi} \mu(s) \left[\frac{1}{2i} H_o^2(kr)\right] ds \quad ; \tag{16}$$

$$u_e = \int_0^{\chi} \nu(s) \frac{\partial}{\partial n_s}\left[\frac{1}{2i} H_o^2(kr)\right] ds \quad . \tag{17}$$

Then we obtain from the boundary conditions a system of Fredholm integral equations:

$$\int_0^{\chi} \mu(s)L(s,\sigma,k)ds = -\nu(\sigma) + \int_0^{\chi}\nu(s)K(s,\sigma,k)ds + u^*(\sigma) \quad ,$$

$$-\mu(\sigma) + \int_0^{\chi} \mu(\sigma)K(\sigma,s,k)ds = \int_0^{\chi} \nu(s) \frac{\partial^2}{\partial n_\sigma \partial n_s}\left[\frac{1}{2i} H_0^2(kr)\right] ds + \frac{\partial u^*}{\partial n_\sigma} . \qquad (18)$$

Here $L(s, \sigma, k)$ denotes $\frac{1}{2i} H_0^2(kr)$, and $\frac{\partial^2}{\partial n_\sigma \partial n_s}$ denotes a successive differentiation along the normals at the points $s = \sigma$ and $s = s$.

It is scarcely necessary to show that a number of other boundary problems as well, connected with the Helmholtz equation and important from a practical point of view, lead to the Fredholm equation; for example, the problem of the stationary thermal state in which a definite combination of the sought function and its normal derivative is given on the contour, a problem of tidal theory where the boundary condition relates the value of the normal derivative on the contour to the value of the tangential derivative, etc.

§ 27. Investigation of the existence and uniqueness of the solutions. Let us rewrite the integral equations (D_e) and (N_e), employing the parameter λ in them:

$$(D_e) \qquad \nu(\sigma) + \lambda\int_0^{\chi} \nu(s)K(s, \sigma, k)ds = -\omega(\sigma) \quad ,$$

$$(N_e) \qquad \mu(\sigma) + \lambda\int_0^{\chi} \mu(s)K(\sigma, s, k)ds = \tau(\sigma) \quad .$$

Here one must keep in mind the fact that the physical problems of diffraction under our consideration do not lead to equations (D_e) and (N_e) strictly, but to particular cases of them, namely those for which λ equals -1 and +1 respectively. We shall prove the following fundamental theorems:

Theorem 1. The necessary and sufficient condition that $\lambda = -1$ be a characteristic number of equation (D_e) is the equality of the frequency k to one of the proper frequencies of the internal Neumann problem

$$\Delta u + k^2 u = 0 \; ; \quad \left[\frac{\partial u}{\partial n_i}\right]_\gamma = 0 \; . \tag{19}$$

The rank of the characteristic number $\lambda = -1$ is equal to the multiplicity of k, and the fundamental functions of the equation

$$(D_e^o) \qquad \nu(\sigma) - \int_0^\gamma \nu(s) K(s, \sigma, k) ds = 0$$

are the contour values of the fundamental solutions of the internal Neumann problem (19).

Theorem 2. The necessary and sufficient condition that $\lambda = +1$ be a characteristic number of equation (N_e) is the equality of the frequency k to one of the proper frequencies of the internal Dirichlet problem

$$\Delta u + k^2 u = 0 \; , \quad [u_i]_\gamma = 0 \; . \tag{20}$$

The rank of the characteristic number $\lambda = +1$ is equal to the multiplicity of k, and the fundamental functions of the equation

$$(N_e^o) \qquad \mu(\sigma) + \int_0^\gamma \mu(s) K(\sigma, s, k) ds = 0$$

are the contour values of the normal derivatives of the fundamental solutions of the internal Dirichlet problem (20).

Proof of the necessity. The necessity of the conditions formulated in Theorems 1 and 2 follows from the following proposition:

If k is not a characteristic number of the internal homogeneous Neumann problem

$$\Delta u + k^2 u = 0 \ , \quad \left[\frac{\partial u}{\partial n_i}\right]_\gamma = 0 \ , \tag{19}$$

then $\lambda = -1$ cannot be a characteristic number of the equation

$$(D_e) \qquad \nu(\sigma) + \lambda\int_0^{\chi} \nu(s)K(s, \sigma, k)ds = -\omega(\sigma) \ .$$

Indeed, assuming the contrary, we are led to the conclusion that the equation associated with (D_e^o),

$$(N_i^o) \qquad \nu(\sigma) - \int_0^{\chi} \nu(s)K(\sigma, s, k)ds = 0$$

must have non-zero solutions.

Comparing this equation with equation (N_i), we see that with the aid of the solutions (N_i^o) just indicated, we shall construct the generalized potentials of the simple layers, which will be solutions of the internal Neumann problem (19). But since, according to the condition, k is not a characteristic number, all solutions of (19) are zero within (γ), and by the continuity of the potential of the simple layer the solutions obtained by us will also have zero limiting values for an external approach to the points of the contour. But on the other hand, the potentials of simple layers, constructed by us in the role of solutions, satisfy the Emission principle; and as we know from §3, Chap. I, such a solution is unique, whence it follows that our solutions are identically zero.

Denoting these solutions by u_r $(r = 1, 2, \cdots, n)$ we have

$$u_r \equiv 0 \quad \text{everywhere,}$$

and

$$\left(\frac{\partial u_r}{\partial n}\right)_e - \left(\frac{\partial u_r}{\partial n}\right)_i = 2\mu(\sigma) = 0 \ ,$$

whence it follows that

$$\mu(\sigma) \equiv 0 \quad ,$$

i.e., all solutions (N_i^0) are zero.

From this it follows that the assumption made was incorrect, and the proposition is proved in full.

Obviously a parallel proposition, relating to Theorem 2, may be proved analogously:

If k is not a characteristic number of the internal homogeneous Dirichlet problem:

$$\Delta u + k^2 u = 0 \ , \quad [u_i]_\gamma = 0 \quad , \tag{20}$$

then $\lambda = +1$ cannot be a characteristic number of the integral equation

(N_e) $$\mu(\sigma) + \lambda \int_0^\gamma \mu(s) K(\sigma, s, k) ds = + \tau(\sigma) \quad .$$

The proof of the sufficiency obviously reduces to a proof of the following proposition:

If $k = k^*$ is a characteristic number of the internal homogeneous Neumann problem, (19), then $\lambda = -1$ is a characteristic number of the integral equation

$$\nu(\sigma) + \lambda \int_0^\gamma \nu(s) K(s, \sigma, k^*) ds = - \omega(\sigma) \quad ,$$

and the fundamental functions of the corresponding homogeneous equation (D_e^0) are contour values of the solutions of (19).

Let $k = k^*$ be an n-tuple characteristic number of problem (19), and u_r $(r = 1, 2, \cdots, n)$ the corresponding fundamental functions of the problem.

Let μ_p $(p = 1, 2, \cdots, m)$ be linearly independent fundamental functions of the integral equation

$$(N_i^o) \qquad \mu(\sigma) - \int_o^{\gamma} \mu(s)K(\sigma, s, k^*)ds = 0 \quad .$$

We shall show that $m \not> n$. Assuming the contrary, let us define the potentials:

$$u_p^* = \int_o^{\gamma} \mu_p(s) \left[\frac{1}{2i} H_o^2(kr) \right] ds \qquad (p = 1, 2, \cdots, m) \quad . \tag{21}$$

These u_p^* are obviously solutions of problem (19), but since the latter has for $k = k^*$ only n linearly independent solutions, a relation of dependence exists,

$$\sum_{p=1}^{m} a_p u_p^* = 0$$

or

$$\frac{1}{2i} \int_o^{\gamma} H_o^2(kr) \sum_{p=1}^{m} a_p \mu_p(s) ds = 0 \quad .$$

Employing with this potential of a simple layer the reasoning that we used in the proof of the necessity, we obtain

$$\sum_{p=1}^{m} a_p \mu_p(s) = 0 \quad ,$$

which contradicts the assumption concerning the linear independence of the solutions of equation (N_i^o). Hence follows the incorrectness of the assumption that we made, $m > n$, and our proposition is proved.

We shall now prove that $m \not< n$.

Let the fundamental functions of the problem (19), u_j ($j = 1, 2, \cdots, n$) have on (γ) the respective values[1] $+2h_j(s)$.

Let us define by means of these latter the following n potentials

- - - - - - - - - - - - - -

[1]The linear independence of these values is readily shown.

of double layers:

$$W_j = \int_0^{\chi} h_j(s) \frac{\partial}{\partial n_s}\left[\frac{1}{2i} H_0^2(kr)\right] ds , \quad (j = 1, 2, \cdots, n) ;$$

and on the basis of (11) we write

$$[W_j(\sigma)]_i - [W_j(\sigma)]_e = 2h_j(\sigma) , \quad \left(\frac{\partial W_j}{\partial n}\right)_e = \left(\frac{\partial W_j}{\partial n}\right)_i .$$

Let us now introduce into the reasoning the functions $u^{(j)}$, defined as follows:

$$u^{(j)} \begin{cases} = u_j \text{ within } (\gamma) \\ = 0 \text{ outside } (\gamma) \end{cases}$$

and define the difference

$$\chi_j = W_j - u^{(j)} ;$$

obviously

$$[\chi_j(\sigma)]_e - [\chi_j(\sigma)]_i = \{[W_j(\sigma)]_e - [W_j(\sigma)]_i\} + \{[u^{(j)}(\sigma)]_i - [u^{(j)}(\sigma)]_e\}$$

$$= -2h_j(\sigma) + 2h_j(\sigma) = 0 ;$$

$$\left(\frac{\partial \chi_j}{\partial n}\right)_e - \left(\frac{\partial \chi_j}{\partial n}\right)_i = \left(\frac{\partial u^{(j)}}{\partial n}\right)_i - \left(\frac{\partial u^{(j)}}{\partial n}\right)_e = 0 ,$$

since $\left(\frac{\partial u^{(j)}}{\partial n}\right)_i$ are the limiting values of the normal derivatives of u_j and accordingly equal 0 in accordance with (19).

On the other hand,

$$\chi_j = W_j - u^{(j)}$$

satisfies the Emission principle and by the uniqueness theorem (§3, Chap. I) we have

$$W_j = u^{(j)} .$$

Consequently

$$W_j = \int_0^{\gamma} h_j(s) \frac{\partial}{\partial n_s} \left[\frac{1}{2i} H_0^2(k^* r) \right] ds = \begin{matrix} u_j \text{ within } (\gamma) \\ 0 \text{ outside } (\gamma) \end{matrix} \quad ; \tag{22}$$

in view of the discontinuity of the potential of a double layer in passing across (γ), we shall have, from (22),

$$h_j(\sigma) - \int_0^{\gamma} h_j(s) K(s, \sigma, k^*) ds = 0 \qquad (j = 1, 2, \cdots, n) \quad ; \tag{23}$$

but this is just equation (D_e^0), which thus has, for $k = k^*$, n linearly independent solutions; on the other hand, (23) shows that the fundamental functions of equation (D_e^0) are, accurate to a constant coefficient, contour values of the fundamental solutions of problem (19).

Passing on to an investigation of the characteristic numbers of equation (N_e), we shall prove the following proposition:

If $k = k_*^*$ is a characteristic number of the internal homogeneous Dirichlet problem

$$\Delta u + k^2 u = 0 \qquad [u_i]_\gamma = 0 \quad , \tag{20}$$

then $\lambda = +1$ is a characteristic number of the integral equation

$$(N_e) \qquad \mu(\sigma) + \lambda \int_0^{\gamma} \mu(s) K(\sigma, s, k_*^*) ds = \tau(\sigma) \quad .$$

Let $k = k_*^*$ be a n-tuple characteristic number of problem (20) and u_r $(r = 1, 2, \cdots, n)$ the fundamental functions corresponding to it. Furthermore let ν_p $(p = 1, 2, \cdots, m)$ be linearly independent solutions of the equation

$$\nu(\sigma) + \int_0^{\gamma} \nu(s) K(s, \sigma, k_*^*) ds = 0 \quad .$$

We shall show that $m \not> n$; assuming the contrary, let us define the potentials

$$u_p^* = \int_0^{\gamma} \nu_p(s) \frac{\partial}{\partial n_s}\left[\frac{1}{2i} H_0^2(k_*^* r)\right] ds \quad .$$

Obviously a relation of linear dependence must exist between these m solutions of problem (20) in consequence of the assumption that $m > n$; hence, as above, follows a linear dependence between the $\nu_p(s)$, which contradicts the assumption concerning their linear independence, and accordingly $m \not> n$.

We shall now prove that $m \not< n$. Let there be $2r_r(s)$, ($r = 1, 2, \cdots, n$) contour values of the normal derivates of u_r, ($r = 1, 2, \cdots, n$); let us define the n potentials of simple layers:

$$V_j = \int_0^{\gamma} r_j(s) \left[\frac{1}{2i} H_0^2(k_*^* r)\right] ds \,. \quad (j = 1, 2, \cdots, n)$$

On the basis of formulas (9) we have

$$\left(\frac{\partial V_j}{\partial n}\right)_e - \left(\frac{\partial V_j}{\partial n}\right)_i = 2r_j(\sigma), \quad (V_j)_e = (V_j) \quad .$$

Now, as above, let us introduce into the reasoning the functions $u^{(j)}$, defined as follows:

$$u^{(j)} \begin{cases} = u_j \text{ within } (\gamma) \\ = 0 \text{ outside } (\gamma) \end{cases} \qquad (j = 1, 2, \cdots, n)$$

and define the difference

$$\chi_j = V_j - u^{(j)} \quad .$$

Then

$$(\chi_j)_e - (\chi_j)_i = [(V_j)_e - (V_j)_i] + [(u^{(j)})_i - (u^{(j)})_e] = 0 \quad ,$$

$$\left(\frac{\partial \chi_i}{\partial n}\right)_e - \left(\frac{\partial \chi_j}{\partial n}\right)_i = \left[\left(\frac{\partial V_j}{\partial n}\right)_e - \left(\frac{\partial V_j}{\partial n}\right)_i\right] + \left[\left(\frac{\partial u^{(j)}}{\partial n}\right)_i - \left(\frac{\partial u^{(j)}}{\partial n}\right)_e\right]$$

$$= 2r_j(\sigma) - 2r_j(\sigma) = 0 \quad ,$$

and therfore, by the uniqueness theorem,

$$V_j = u^{(j)} \quad .$$

We finally arrive at the conclusion that

$$V_j = \int_0^{\gamma} r_j(s) \left[\frac{1}{2i} H_0^2(k_*^* r) \right] ds = \begin{cases} u_j \text{ within } (\gamma) \\ 0 \text{ outside } (\gamma) \end{cases} \quad (j = 1, 2, \cdots, n)$$

In passing across (γ) we shall have, in view of the discontinuity of the normal derivative of the potential of a simple layer,

$$r_j(\sigma) + \int_0^{\gamma} r_j(s) K(\sigma, s, k_*^*) ds = 0 \quad . \qquad (24)$$

But this is just equation (N_e^o), which, for $k = k_*^*$ thus has n linearly independent solutions.

On the other hand, it is evident from (24) that the fundamental functions of equation (N_e^o) are, accurate to a constant coefficient, contour values of the normal derivatives of the fundamental solutions of problem (20), which thus proves the proposition enunciated above.

Combining the results obtained, we have a complete proof of Theorems 1 and 2.

§ 28. Exceptional cases. Let us once again give a reformulation of the gist of the results obtained above:

a) The problem of finding a solution of the vibrational equation

$$\Delta u + k^2 u = 0 \quad ,$$

such as to satisfy the Emission principle at distances infinitely remote from the contour, and such as to satisfy on the contour the boundary condition $u_i = \omega(s)$ or $u_e = \omega(s)$ leads to the consideration

of the Fredholm integral equations

$$(D_i) \qquad \nu(\sigma) + \int_0^{\chi} \nu(s)K(s, \sigma, k)ds = \omega(\sigma) \quad ,$$

$$(D_e) \qquad \nu(\sigma) - \int_0^{\chi} \nu(s)K(s, \sigma, k)ds = -\omega(\sigma) \quad ;$$

moreover, if k is different from the characteristic numbers of the problem

$$\Delta u + k^2 u = 0 \ , \quad \left[\frac{\partial u}{\partial n_i}\right]_\gamma = 0 \quad , \tag{19}$$

then equation (D_e) has a unique, continuous solution, which is found with the aid of the Fredholm resolvent;

b) Problems with the contour conditions $\left(\frac{\partial u}{\partial n_i}\right) = \tau(s)$ or $\left(\frac{\partial u}{\partial n_e}\right) = \tau(s)$ lead to the integral equations

$$(N_i) \qquad \mu(\sigma) - \int_0^{\chi} \mu(s)K(\sigma, s, k)ds = - \tau(\sigma) \quad ,$$

$$(N_e) \qquad \mu(\sigma) + \int_0^{\chi} \mu(s)K(\sigma, s, k)ds = \tau(\sigma) \quad ;$$

here, if k is different from the characteristic numbers of the problem

$$\Delta u + k^2 u = 0 \ , \quad [u_i]_\gamma = 0 \quad , \tag{20}$$

then equation (N_e) has a unique, continuous solution, which is constructible with the aid of the Fredholm resolvent.

Thus problems concerning the diffraction of incident disturbances with frequencies different from the frequencies of the proper vibrations of the diffracting contour have been solved in full.

In this section we shall indicate the solution of those problems in particular for the case of the frequencies excluded from the general theory set forth above.

A. <u>The Dirichlet problem</u>. Let $k = k^*$ be an n-tuple characteristic number of problem (19). In this case the associated homogeneous equations

$$\mu(\sigma) - \int_0^{\chi} \mu(s)K(\sigma, s, k^*)ds = 0 \quad ,$$

$$\nu(\sigma) - \int_0^{\chi} \nu(s)K(\sigma, s, k^*)ds = 0$$

have, in accordance with Theorem 1, the n linearly independent solutions μ_j^o and ν_j^o $(j = 1, 2, \cdots, n)$, which we may consider to have been rendered bi-orthonormal without restricting the generality.

In this case equation (D_e) has a definite solution only on condition that

$$\int_0^{\chi} \omega(\sigma)\mu_j^o(\sigma)d\sigma = 0 \quad . \tag{24a}$$

For the observance of this condition we shall introduce a new function, defined in the interval $(0, \chi)$:

$$\overline{\omega(\sigma)} = -\omega(\sigma) + \sum_{j=1}^{n} a_j\nu_j^o(\sigma) \quad , \tag{25}$$

where

$$a_j = \int_0^{\chi} \omega(\sigma)\mu_j^o(\sigma)d\sigma \quad .$$

Then condition (24a) will be fulfilled for the function $\overline{\omega(\sigma)}$ and the integral equation

$$(\bar{D}_e) \qquad \overline{\nu(\sigma)} - \int_0^{\chi} \overline{\nu(s)}K(s, \sigma, k^*)ds = -\omega(\sigma) + \sum_{j=1}^{n} a_j\nu_j^o(\sigma)$$

has for its general solution

$$\overline{\nu(\sigma)} = \overline{\omega(\sigma)} + \int_0^{\chi} \Gamma(s, \sigma, k^*)\,\overline{\omega(s)}ds + \sum_{j=1}^{n} c_j\nu_j^o(\sigma) \quad ,$$

where $\Gamma(s, \sigma, k^*)$ is the suitably modified Fredholm resolvent for

the equation $(\overline{D}_e)$, and $\sum_{j=1}^{n} c_j \nu_j^o(\sigma)$ is the sum of the n linearly independent solutions of the homogeneous integral equation (D_e^o).

Putting this value of $\overline{\nu(\sigma)}$ in (8) and taking into account that

$$\int_o^{\gamma} \nu_j^o(s) \frac{\partial}{\partial n_s} \left[\frac{1}{2i} H_o^2(k^* r) \right] ds$$

is equal to zero outside of (γ), in accordance with (22) and (23) $[\nu_j^o = h_j]$, we will have a single-valued solution $\overline{W(x,y)}$ completely determined by its value on (γ), to wit $\overline{\omega(\sigma)}$.

Now let us note that the solutions of problem (19) may be expressed thus:

$$\omega_j(x, y) = \frac{1}{2i} \int_o^{\gamma} \mu_j^o(s) H_o^2(k^* r) ds, \qquad (j = 1, 2, \cdots, n)$$

and, as was proved in Theorem 1, the limiting values of these functions on the contour are linearly expressible in terms of the ν_j^o.

Let $\|d_{jk}\|$ be the matrix inverse to this linear transformation.

We shall show that the sought solution of the equation $\Delta u + k^{*2} u = 0$, with boundary condition $[u_e]_\gamma = \omega(s)$, which solution satisfies the Emission principle at infinity, will be:

$$u(x, y) = - \overline{W(x, y)} + \sum_{j=1}^{n} \sum_{k=1}^{n} a_j d_{jk} \omega_k(x, y) \quad , \tag{26}$$

where

$$\overline{W(x, y)} = \frac{1}{2i} \int_o^{\gamma} \overline{\nu(s)} \frac{\partial}{\partial n} H_o^2(k^* r) ds \quad ,$$

$\overline{\nu(s)}$ being any solution of equation $(\overline{D}_e)$, and

$$r = \sqrt{[x - \xi(s)]^2 + [y - \eta(s)]^2} \quad .$$

In order to prove this statement it is sufficient to show that

the function defined by (26) fulfills the necessary boundary condition, since it is perfectly obvious that it satisfies both the other conditions directly.

From (26) we shall have, for points on (γ),

$$u(\sigma) = \omega(\sigma) - \sum_{j=1}^{n} a_j \nu_j^o(\sigma) + \sum_{j=1}^{n} a_j \nu_j^o(\sigma) = \omega(\sigma) \quad ,$$

Q.E.D.

The second fundamental boundary problem may also be subjected to analogous reasoning in the case of the exceptional values for k.

Let $k = k_*^*$ be an m-tuple characteristic number of problem (20); then the solution of the equation $\Delta v + k_*^{*2} v = 0$ with boundary condition $\left(\frac{\partial v}{\partial n_e}\right) = \tau(s)$, which solution satisfies the Emission principle at infinity, will have the analogous form:

$$v(x, y) = \overline{v(x, y)} - \sum_{j=1}^{m} \sum_{k=1}^{m} b_j r_{jk} \pi_k(x, y) \quad , \tag{27}$$

where

$$\overline{v(x, y)} = \frac{1}{2i} \int_0^{\gamma} H_0^2(k_*^* r)\, \overline{\mu(s)}\, ds \quad ,$$

$\overline{\mu(s)}$ is any solution of the equation

$$\overline{\mu(\sigma)} + \int_0^{\gamma} \overline{\mu(s)} K(\sigma, s, k_*^*) ds = \tau(\sigma) - \sum_{j=1}^{m} b_j \nu_j^{(1)}(\sigma) \quad ,$$

$\nu_j^{(1)}$ and $\mu_j^{(1)}$ are bi-orthonormed solutions of equations

$$\mu(\sigma) + \int_0^{\gamma} \mu(s) K(\sigma, s, k_*^*) ds = 0 \quad ,$$

$$\nu(\sigma) + \int_0^{\gamma} \nu(s) K(s, \sigma, k_*^*) ds = 0 \quad ,$$

and b_j and π_k are defined by the formulas:

$$b_j = \int_0^{\gamma} \tau(\sigma) \nu_j^{(1)}(\sigma) d\sigma \quad ,$$

$$\pi_k(x, y) = \frac{1}{2i} \int_0^{\gamma} \nu_k^{(1)}(s) \frac{\partial}{\partial n_s} H_0^2(k_*^* r) ds \quad .$$

The constants r_{jk} form a matrix inverse to the matrix of the linear transformation which, in accordance with Theorem 2, connects the limiting values of $\Pi_k(x, y)$ with $\nu_k^{(1)}(\sigma)$.

Formulas (26) and (27) solve in full both the fundamental boundary problems for the exceptional values of k.

§ 29. The integral equations of the stationary theory of elasticity. The problem of the diffraction of established elastic disturbances near an immobile, walled-up obstacle of form (γ), as has already been remarked in §2, Chap. I, leads to the following boundary problem of a system of vibrational equations.

To find: solutions of the system

$$\begin{aligned} \Delta\Phi + k_1^2\Phi &= 0 \quad , \\ \Delta\Psi + k_2^2\Psi &= 0 \quad , \end{aligned} \qquad (28)$$

such as to be regular everywhere outside of (γ) and such as to satisfy the conditions on (γ):

$$\begin{cases} \left(\dfrac{\partial\Phi}{\partial n} + \dfrac{\partial\Psi}{\partial s}\right)_e = - \dfrac{\partial\Phi^*}{\partial n_e} \;; \\ \left(\dfrac{\partial\Phi}{\partial s} - \dfrac{\partial\Psi}{\partial n}\right)_e = - \dfrac{\partial\Phi^*}{\partial s} \;. \end{cases} \qquad (29)$$

(where Φ^* are the values on (γ) of the assigned fundamental "falling potential") and such as to satisfy at infinity the Emission condition:

$$\left.\begin{aligned} \lim_{\infty} \Phi &= \lim_{r\to\infty}\sqrt{r}\left(\frac{\partial\Phi}{\partial r} + ik_1\Phi\right) = 0 \;; \\ \lim_{\infty} \Psi &= \lim_{r\to\infty}\sqrt{r}\left(\frac{\partial\Psi}{\partial r} + ik_2\Psi\right) = 0 \;; \\ r^2 &= (x-\xi)^2 + (y-\eta)^2 \;. \end{aligned}\right\} \qquad (30)$$

If the potentials that are sought be constructed in the form of generalized potentials of simple layers

$$\Phi(x, y) = \int_{\gamma} \rho(s) H_0^2(k_1 \sqrt{(\xi(s) - x)^2 + (\eta(s) - y)^2}) ds \quad , \quad (31)$$

$$\Psi(x, y) = \int_{\gamma} \mu(s) H_0^2(k_2 \sqrt{(\xi(s) - x)^2 + (\eta(s) - y)^2}) ds \quad , \quad (32)$$

then on the basis of (9) and (11) it is immediately obvious that conditions (29) will be fulfilled if the "density" $\rho(s)$ and $\mu(s)$ be found from the equations

$$\left.\begin{aligned} &\int_0^{\chi} \rho(s) \frac{\partial}{\partial n_\sigma} H_0^2[k_1 r(s,\sigma)] ds + 2i\rho(\sigma) \\ &\qquad + \text{V.p.} \int_0^{\chi} \mu(s) \frac{\partial}{\partial s_\sigma} H_0^2[k_2 r(s,\sigma)] ds = f(\sigma); \quad (*) \\ &\text{V.p.} \int_0^{\chi} \rho(s) \frac{\partial}{\partial s_\sigma} H_0^2[k_1 r(s,\sigma)] ds - 2i\mu(\sigma) \\ &\qquad - \int_0^{\chi} \mu(s) \frac{\partial}{\partial n_\sigma} H_0^2[k_2 r(s,\sigma)] ds = \phi(\sigma); \quad (**) \end{aligned}\right\} (33)$$

in these equations $\frac{\partial}{\partial s_\sigma}$ denotes a differentiation along the tangent at the point $s = \sigma$, and $f(s)$ and $\phi(s)$ respectively equal

$$f(s) = -2i \frac{\partial \Phi^*(s)}{\partial n} ;$$

$$\phi(s) = -2i \frac{\partial \Phi^*(s)}{\partial s} .$$

V.p. in front of the integral signifies that the integral has meaning only in the sense of Cauchy's "principal value" ("Valeur principal").

Let us now occupy ourselves with an investigation of the character of the singularity that figures in equation (33).

If we take into consideration the fact that

$$H_o^2[kr(s,\sigma)] = \Big\{- \frac{2i}{\pi} J_o(kr) \lg \frac{kr}{2}$$

$$+ J_o(kr)\left(1 - \frac{2i}{\pi} C\right) - \frac{2i}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n kr^{2n} \sum_{s=1}^{n} \frac{1}{s}}{2^{2n}(n!)^2}\Big\} , \qquad (34)$$

it is obvious that

$$\frac{\partial}{\partial s_\sigma} H_o^2[kr(s,\sigma)] = \omega(s,\sigma,k) - \frac{2i}{\pi}\frac{1}{r}\frac{\partial r}{\partial s_\sigma} = \omega(s,\sigma,k) - \frac{2i}{\pi}\frac{\sin\psi}{r} , \qquad (35)$$

where

$$\omega(s, \sigma, k) = \Big\{- \frac{2i}{\pi} J_o'(kr)k \lg \frac{kr}{2} - \frac{2i}{\pi}[J_o(kr) - 1]\frac{1}{r}$$

$$+ \left(1 - \frac{2i}{\pi} C\right)kJ_o'(kr) - \frac{2i}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n 2n(kr)^{2n-1}k \sum_{s=1}^{n} \frac{1}{s}}{2^{2n}(n!)^2}\Big\} \frac{\partial r}{\partial s_\sigma} , \qquad (35^*)$$

and

$$\frac{\partial r}{\partial s_\sigma} = \cos(r, s_\sigma) = \sin(n_\sigma, r) = \sin\psi \quad .$$

Since for $s = \sigma$ the angle $(n_\sigma, r) \to \frac{\pi}{2}$, and $r \to 0$, we see that the kernel $\frac{\partial}{\partial s_\sigma} H_o^2(kr)$ at the point $s = \sigma$ has a pole of the first order with residue equal to $- \frac{2i}{\pi}$.

In expressions (33) the differentiation along the tangent under the integral signs has as yet only a formal character.

In order to show the legitimacy of these operations, we write

$$\frac{\partial}{\partial s_\sigma} \int_0^{\chi} \mu(s)H_o^2(kr)ds = \frac{\partial}{\partial s_\sigma} \int_0^{\sigma-\varepsilon} \mu(s)H_o^2[kr(s,\sigma)]ds$$

$$+ \frac{\partial}{\partial s_\sigma} \int_{\sigma+\varepsilon}^{\chi} \mu(s)H_o^2[kr(s,\sigma)]ds + \frac{\partial}{\partial s_\sigma} \int_{\sigma-\varepsilon}^{\sigma+\varepsilon} \mu(s)H_o^2[kr(s,\sigma)]ds \quad .$$

In the first two integrals, the differentiation can obviously

be actually accomplished; we then obtain

$$\frac{\partial}{\partial s_\sigma}\int_0^{\chi}\mu(s)H_0^2[kr(s,\sigma)]ds \tag{36}$$

$$= \int_0^{\chi}\mu(s)\frac{\partial}{\partial s_\sigma}H_0^2(kr)ds + H_0^2[kr(\sigma-\varepsilon,\sigma)]\mu(\sigma-\varepsilon) + \int_0^{\chi}\mu(s)\frac{\partial}{\partial s_\sigma}H_0^2(kr)ds$$

$$- H_0^2[kr(\sigma+\varepsilon,\sigma)\mu(\sigma+\varepsilon)] + \frac{\partial}{\partial s_\sigma}\int_{\sigma-\varepsilon}^{\sigma+\varepsilon}\mu(s)H_0^2 ds \ .$$

To compute the integrals figuring in this equation we utilize an expansion of the square of the chord in powers of the increment of arc (Fig. 15):

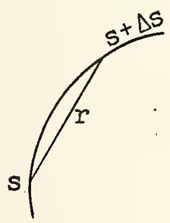

Fig. 15

$$[r(s,s+\Delta s)]^2 = (\Delta s)^2 - \frac{(\Delta s)^4}{12\,R^2} + \frac{(\Delta s)^5}{12\,R^3}\frac{dR}{ds} - \dots$$

R is the radius of curvature at the point s, hence

$$2\ \lg\frac{kr}{2} = 2\ \lg\frac{k\Delta s}{2} - \frac{(\Delta s)^2}{12\,R^2}(1 + a\Delta s) \quad ,$$

where a is a bounded number, or

$$\lg\frac{kr}{2} = \frac{1}{2}\lg\frac{(k\Delta s)^2}{4} - \frac{(\Delta s)^2}{24\,R^2}(1 + a\Delta s) \quad . \tag{37}$$

On the basis of this dependence of the chord upon the increment of arc, one is directly convinced of the validity of the equation

$$H_0^2[kr(\sigma-\varepsilon,\sigma)]\mu(\sigma-\varepsilon) - H_0^2[kr(\sigma+\varepsilon,\sigma)]u(\sigma+\varepsilon) \qquad {}^1$$

$$= \frac{2i}{\pi}[\mu(\sigma+\varepsilon)-\mu(\sigma-\varepsilon)]\lg\frac{k\varepsilon}{2} + \frac{i\varepsilon^2}{12\pi R^2}[(1-A\varepsilon)\mu(\sigma-\varepsilon)-(1+A\varepsilon)\mu(\sigma+\varepsilon)]$$

$$+\Big\{-\frac{2i}{\pi}(J_0[kr(\sigma,\sigma-\varepsilon)]-1)\lg\frac{kr(\sigma,\sigma-\varepsilon)}{2} + J_0(kr(\sigma,\sigma-\varepsilon))(1-\frac{2i}{\pi}C)$$

$$-\frac{2i}{\pi}\left[\left(\frac{kr(\sigma,\sigma-\varepsilon)}{2}\right)^2-\dots\right]\mu(\sigma-\varepsilon) + \frac{2i}{\pi}[J_0(kr(\sigma,\sigma+\varepsilon))-1]\lg\frac{kr(\sigma,\sigma+\varepsilon)}{2}$$

$$- J_0(kr(\sigma,\sigma+\varepsilon))(1-\frac{2i}{\pi}C) + \frac{2i}{\pi}\left[\left(\frac{kr(\sigma,\sigma+\varepsilon)}{2}\right)^2+\dots\right]\mu(\sigma+\varepsilon)\Big\} \tag{38}$$

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[1] We designate Δs by ε for simplicity.

As $\varepsilon \to 0$, r tends to zero and the expression standing in braces, as a regular function, tends to zero; moreover, if $\mu(s)$ has a derivative, then

$$[\mu_{\varepsilon\to 0}(s+\varepsilon) - \mu(s-\varepsilon)] \lg \frac{k\varepsilon}{2}$$

tends to zero as $\varepsilon \lg_{\varepsilon\to 0} \varepsilon$, and accordingly the whole expression (38) thereby tends to zero.

It remains to consider the last summand in the right part of (36).

For this let us make the substitution

$$s = \sigma + t \quad ,$$

then

$$\frac{\partial}{\partial s_\sigma} \int_{\sigma-\varepsilon}^{\sigma+\varepsilon} \mu(s)H_o^2(kr(s,\sigma))ds = \frac{\partial}{\partial s_\sigma} \int_{-\varepsilon}^{+\varepsilon} \mu(\sigma+t)H_o^2[kr(\sigma,\sigma+t)]dt$$

$$= \int_{-\varepsilon}^{+\varepsilon} \mu(\sigma+t) \frac{\partial}{\partial s_\sigma} H_o^2[kr(\sigma,\sigma+t)]dt$$

$$+ \int_{-\varepsilon}^{+\varepsilon} H_o^2[kr(\sigma,\sigma+t)] \frac{\partial}{\partial s_\sigma} \mu(\sigma+t)dt \quad .$$

Having taken into consideration the equality

$$\lg \frac{kr}{2} = \frac{1}{2} \lg \frac{k^2t^2}{2} - \frac{t^2}{24 R^2} (1 + at)$$

after elementary transformations with the aid of (34) and (35), employing the first theorem of the mean, it follows that as $\varepsilon \to 0$ the integrals in hand tend to zero.

We thus have, finally,

$$\frac{\partial}{\partial s_\sigma} \int_0^\chi \mu(s)H_o^2(kr)ds = \text{V.p.} \int_0^\chi \mu(s) \frac{\partial}{\partial s_\sigma} H_o^2(kr(s,\sigma))ds$$

$$= \text{V.p.} \int_0^\chi \left[\omega(s,\sigma,k) - \frac{2i}{\pi}\frac{\sin\psi}{r}\right] \mu(s)\, ds \quad .$$

§30. Reduction to a regular system. Each of the equations of system (33) contains an integral with a singular kernel; as is obvious from the section preceding, the character of these singularities is such as we have already encountered in §24. In addition, all the functions figuring in system (33) are periodic with respect to s and σ, of period χ, and are holomorphic when s and σ vary along the real axis, except for the term $-\frac{2i}{\pi}\frac{\sin\phi}{r}$, which has a simple pole at the point $s = \sigma$, with residue equal to $-\frac{2i}{\pi}$.

From this it follows that to regularize system (33) we must utilize formula (1) from §24, Chap. IV.

From system (33) we find:

$$\rho(s) = \frac{1}{2i} f(s) - \frac{1}{2i} \text{ V.p.} \left\{ \int_0^\chi \rho(z) \frac{\partial}{\partial n_s} H_0^2[k_1 r(z,s)]dz + \int_0^\chi \mu(z) \frac{\partial}{\partial s_s} H_0^2[k_2 r(z,s)]dz \right\} ; \tag{39}$$

$$\mu(s) = -\frac{1}{2i} \phi(s) + \frac{1}{2i} \text{ V.p.} \left\{ \int_0^\chi \rho(z) \frac{\partial}{\partial s_s} H_0^2[k_1 r(z,s)]dz - \int_0^\chi \mu(z) \frac{\partial}{\partial n_s} H_0^2[k_2 r(z,s)]dz \right\} . \tag{40}$$

Substituting (39) in (**) of system (33), and (40) in (*), we obtain

$$\int_0^\chi \rho(s) \frac{\partial}{\partial n_\sigma} H_0^2[k_1 r(s,\sigma)]ds + 2i\rho(\sigma) + \text{V.p.} \int_0^\chi \left\{ -\frac{1}{2i}\phi(s) + \frac{1}{2i} \text{ V.p.} \int_0^\chi \rho(z) H_0^2[k_1 r(z,s)]dz - \frac{1}{2i} \text{ V.p.} \right.$$

$$\left. \times \int_0^\chi \mu(z) \frac{\partial}{\partial n_s} H_0^2[k_2 r(z,s)]dz \right\} \frac{\partial}{\partial s_\sigma} H_0^2[k_2 r(s,\sigma)]ds = f(\sigma) . \tag{33 bis}$$

$$\int_0^{\chi} \mu(s) \frac{\partial}{\partial n_\sigma} H_0^2[k_2 r(s,\sigma)]ds + 2i\mu(\sigma) - \text{V.p.} \int_0^{\chi} \Big\{ \frac{1}{2i} \cdot f(s) - \frac{1}{2i}$$

$$\times \text{V.p.} \int_0^{\chi} \rho(z) \frac{\partial}{\partial n_s} H_0^2[k_1 r(z,s)]dz - \frac{1}{2i} \text{V.p.} \int_0^{\chi} \mu(z)$$

$$\times \frac{\partial}{\partial s_s} H_0^2[k_2 r(z,s)]dz \Big\} \frac{\partial}{\partial s_\sigma} H_0^2[k_1 r(s,\sigma)]ds = - \phi(\sigma) \quad .$$

Let us apply formula 1, § 24, Chap. IV to the double integrals encountered in these formulas:

$$\left.\begin{aligned}
&\int_0^{\chi} \rho(s) \frac{\partial}{\partial n_\sigma} H_0^2[k_1 r(s,\sigma)]ds + 2i\rho(\sigma) - \frac{1}{2i} \text{V.p.} \\
&\quad \times \int_0^{\chi} \phi(s) \frac{\partial}{\partial s_\sigma} H_0^2[k_2 r(s,\sigma)]ds - 2i\rho(\sigma) + \frac{1}{2i} \int_0^{\chi} \rho(z)dz \ \text{V.p.} \\
&\quad \times \int_0^{\chi} \frac{\partial}{\partial s_s} H_0^2[k_1 r(z,s)] \frac{\partial}{\partial s_\sigma} H_0^2[k_2 r(s,\sigma)]ds - \frac{1}{2i} \int_0^{\chi} \mu(z)dz \ \text{V.p.} \\
&\quad \times \int_0^{\chi} \frac{\partial}{\partial n_s} H_0^2[k_2 r(r,s)] \frac{\partial}{\partial s_\sigma} H_0^2(k_2 r)ds = f(\sigma) \quad ; \\
&\int_0^{\chi} \mu(s) \frac{\partial}{\partial n_\sigma} H_0^2[k_2 r(s,\sigma)]ds + 2i\mu(\sigma) - \frac{1}{2i} \text{V.p.} \\
&\quad \times \int_0^{\chi} f(s) \frac{\partial}{\partial s_\sigma} H_0^2[k_1 r(s,\sigma)]ds - 2i\mu(\sigma) + \frac{1}{2i} \int_0^{\chi} \mu(z)dz \ \text{V.p.} \\
&\quad \times \int_0^{\chi} \frac{\partial}{\partial s_s} H_0^2(k_2 r) \frac{\partial}{\partial s_\sigma} H_0^2(k_1 r)ds + \frac{1}{2i} \int_0^{\chi} \rho(z)dz \ \text{V.p.} \\
&\quad \times \int_0^{\chi} \frac{\partial}{\partial n_s} H_0^2(k_1 r) \frac{\partial}{\partial s_\sigma} H_0^2(k_1 r)ds = - \phi(\sigma) \quad .
\end{aligned}\right\} \quad (41)$$

Now introduce the designations

$$\left.\begin{aligned}
&\frac{\partial}{\partial n_\sigma} H_0^2[k_1 r(z,\sigma)] + \frac{V.p}{2i} \int_0^\chi \frac{\partial}{\partial s_s} H_0^2[k_1 r(z,s)] \frac{\partial}{\partial s_\sigma} H_0^2[k_2 r(s,\sigma)]ds \\
&\qquad = K(z,\sigma,k_1,k_2); \\
&\frac{1}{2i} \text{V.p.} \int_0^\chi \frac{\partial}{\partial n_s} H_0^2[k_2 r(z,s)] \frac{\partial}{\partial s_\sigma} H_0^2[k_2 r(s,\sigma)]ds = L(z,\sigma,k_2) ; \\
&f(\sigma) + \frac{1}{2i} \text{V.p.} \int_0^\chi \phi(s) \frac{\partial}{\partial s_\sigma} H_0^2[k_2 r(s,\sigma)]ds = F_1(\sigma) ; \\
&\frac{\partial}{\partial n_\sigma} H_0^2[k_2 r(z,\sigma)] + \frac{1}{2i} \text{V.p.} \int_0^\chi \frac{\partial}{\partial s_s} H_0^2[k_2 r(z, s)] \frac{\partial}{\partial s_\sigma} H_0^2[k_1 r(s,\sigma)]ds \\
&\qquad = K(z,\sigma,k_2,k_1); \\
&\frac{V.p}{2i} \int_0^\chi \frac{\partial}{\partial n_s} H_0^2[k_1 r(z,s)] \frac{\partial}{\partial s_\sigma} H_0^2[k_1 r(z,\sigma)]ds = L(z,\sigma,k_1); \\
&-\phi(\sigma) + \frac{1}{2i} \text{V.p.} \int_0^\chi f(s) \frac{\partial}{\partial s_\sigma} H_0^2[k_1 r(s,\sigma)]ds = F_2(\sigma) .
\end{aligned}\right\} \quad (42)$$

System (41) then acquires the following form:

$$\left.\begin{aligned}
&\int_0^\chi \rho(z)K(z,\sigma,k_1,k_2)dz - \int_0^\chi \mu(z)L(z,\sigma,k_2)dz = F_1(\sigma); \\
&\int_0^\chi \mu(z)K(z,\sigma,k_2,k_1)dz + \int_0^\chi \rho(z)L(z,\sigma,k_1)dz = F_2(\sigma).
\end{aligned}\right\} \quad (43)$$

We thus reduce the singular system (33) to the system (43) of equations of the first kind, which are now regular, as will be shown below.

We obviously must now show, on the one hand, the equivalence of system (43) and (33), and on the other hand, we must satisfy ourselves that system (43) does in fact contain no kind of singularity.

§ 31. Proof of the equivalence of systems (33) and (43). Obviously any solution of system (33) is also a solution of (43) too. We shall show that, conversely, any solution of system (43) is

also a solution of system (33).

For this let us introduce into the investigation the two functions

$$\psi(s) = \rho(s) - \frac{1}{2i} f(s) + \frac{1}{2i} \int_0^{\chi} \rho(z) \frac{\partial}{\partial n_s} H_0^2[k_1 r(z, s)]dz$$
$$+ \text{V.p.} \frac{1}{2i} \int_0^{\chi} \mu(z) \frac{\partial}{\partial s_s} H_0^2[k_2 r(z,s)]dz ,$$

$$\chi(s) = \mu(s) + \frac{1}{2i} \phi(s) + \frac{1}{2i} \int_0^{\chi} \Gamma(z) \frac{\partial}{\partial n_s} H_0^2[k_2 r(z,s)]dz$$
$$- \text{V.p.} \frac{1}{2i} \int_0^{\chi} \rho(z) \frac{\partial}{\partial s_s} H_0^2[k_1 r(z,s)]dz .$$

Moreover, from (33 bis) we find

$$\rho(s) = \frac{1}{2i} f(s) - \frac{1}{2i} \int_0^{\chi} \rho(z) \frac{\partial}{\partial n_s} H_0^2[k_1 r(z,s)]dz + \text{V.p.}\left(\frac{1}{2i}\right)^2$$
$$\times \int_0^{\chi} \phi(z) \frac{\partial}{\partial s_s} H_0^2[k_2 r(z,s)]dz - \left(\frac{1}{2i}\right)^2 \text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_s} H_0^2[k_2 r(z,s)]dz$$
$$\times \text{V.p.} \int_0^{\chi} \rho(t) \frac{\partial}{\partial s_z} H_0^2[k_1 r(t,z)]dt + \left(\frac{1}{2i}\right)^2 \text{V.p.}$$
$$\times \int_0^{\chi} \frac{\partial}{\partial s_s} H_0^2[k_2 r(z,s)]dz \cdot \int_0^{\chi} \mu(t) \frac{\partial}{\partial n_z} H_0^2[k_2 r(t,z)]dt ,$$

$$\mu(s) = -\frac{1}{2i} \phi(s) - \frac{1}{2i} \int_0^{\chi} \mu(z) \frac{\partial}{\partial n_s} H_0^2[k_2 r(z,s)]dz + \left(\frac{1}{2i}\right)^2 \text{V.p.}$$
$$\times \int_0^{\chi} f(z) \frac{\partial}{\partial s_s} H_0^2[k_1 r(z,s)]dz - \frac{1}{2i} \text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_s} H_0^2[k_1 r(z,s)]dz$$
$$\times \text{V.p.} \int_0^{\chi} \rho(t) \frac{\partial}{\partial n_z} H_0^2[k_1 r(t,z)]dt - \left(\frac{1}{2i}\right)^2 \text{V.p.}$$
$$\times \int_0^{\chi} \frac{\partial}{\partial s_s} H_0^2[k_1 r(z,s)]dz \ \text{V.p.} \int_0^{\chi} \mu(t) \frac{\partial}{\partial s_z} H_0^2[k_2 r(t,z)]dt .$$

Let us put these expressions for ρ and μ into $\psi(s)$ and $\chi(s)$; carrying out the obvious reductions and simplifications, we obtain, for $\psi(s)$:

$$\psi(s) = -\frac{1}{2i}\, \text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_s} H_o^2[k_2 r(z,s)]dz \;\text{V.p.} \int_0^{\chi} \rho(t)\frac{\partial}{\partial s_z} H_o^2[k_1 r(t,z)]dt$$

$$- \frac{1}{8i}\, \text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_s} H_o^2[k_2 r(s,z)]dz \;\text{V.p.} \int_0^{\chi} f(z)\frac{\partial}{\partial s_z} H_o^2[k_1 r(z,t)]dt$$

$$+ \frac{1}{8i}\, \text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_s} H_o^2[k_2 r(s,z)]dz \;\text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_z} H_o^2[k_1 r(z,t)]dt$$

$$\times \text{V.p.} \int_0^{\chi} \rho(y)\frac{\partial}{\partial n_t} H_o^2[k_1 r(t,y)]dy + \frac{1}{8i}\, \text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_s} H_o^2[k_2 r(s,z)]dz$$

$$\times \text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_z} H_o^2[k_1 r(z,t)]dt \;\text{V.p.} \int_0^{\chi} \mu(y)\frac{\partial}{\partial s_t} H_o^2[k_2 r(t,y)]dy \quad .$$

Changing suitably the designation of the variables of integration, we obtain, as the result of elementary transformations:

$$\psi(s) = \frac{1}{4}\, \text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_s} H_o^2[k_2 r(s,t)]dt \;\text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_t} H_o^2[k_1 r(t,z)]\psi(z)dz,$$

and, once again applying here the equation [(I), § 24] for the successive integrals with principal values, we finally obtain

$$\int_0^{\chi} \psi(z)R(s,z)dz = 0 \quad , \tag{44}$$

where

$$R(s,z) = \text{V.p.} \int_0^{\chi} \frac{\partial}{\partial s_s} H_o^2[k_2 r(s,t)] \frac{\partial}{\partial s_z} H_o^2[k_1 r(t,z)]dt \quad . \tag{45}$$

Integral equation (44) must be satisfied, on the one hand, for all values of s and z, and on the other hand, for any values of k_1 and k_2.

Taking into account the properties of R(s,z) and $\psi(z)$, we obtain from (44) that

$$\psi(z) \equiv 0 \quad . \tag{46}$$

Repeating all the foregoing reasoning for $\chi(s)$, we show analogously that here too,

$$\chi(z) \equiv 0 \quad . \tag{47}$$

Equations (46) and (47) prove the equivalence of systems (33) and (43).

§32. Investigation of system (43). It has been shown above that

$$\frac{\partial}{\partial s_\sigma} H_0^2[kr(s,\sigma)] = \omega(s,\sigma,k) - \frac{2i}{\pi}\frac{\sin\psi}{r} ,$$

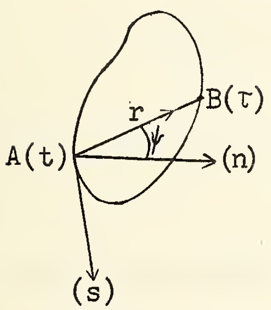

Fig. 16

where $\omega(s,\sigma,k)$ is a regular function determined from formula (35) (§29).

Using this formula, we shall show below that system (43) contains no kind of singularity.

Let

$$\xi = f(t) \ , \qquad \eta = \phi(t)$$

be the equation of the curve (γ), and

$$f(t) = f(t+\chi_1) \ ; \quad \phi(t) = \phi(t+\chi_1) \quad . \tag{48}$$

In particular, the variable <u>t</u> may coincide with the arc <u>s</u>, reckoned from an arbitrary fixed point on (γ). For generality, however, we shall not assume this at present.

Imagine that point A, with coordinates $\xi(t)$, $\eta(t)$, is attracted by the mass $\rho(\tau)$, concentrated at the point B with coordinates $\xi(\tau)$, $\eta(\tau)$ (Fig. 16). As regards the attraction, we have everywhere in view a gravitational law such as the generalized logarithmic potentials constructed in §25 will correspond to. In accordance with this, the attraction exerted on point A by the element ds at the point B is expressed by the quantity

$$\rho(\tau)H_o^2[kr(t,\tau)]ds \quad ,$$

and the tangential component of this force of attraction at point A will be

$$\left[\omega(t,\tau,k)ds - \frac{2i}{\pi}\frac{\sin\psi}{r}ds\right] = \left[\omega(t,\tau,k)ds - \frac{2i}{\pi}\frac{\cos(r,s)}{r}ds\right]\rho(\tau) \ .$$

Let us express these functions in terms of t and τ:

$$r = \sqrt{[\xi(\tau)-\xi(t)]^2 + [\eta(\tau)-\eta(t)]^2} \ ; \quad ds = \sqrt{\xi'^2(\tau)+\eta'^2(\tau)}d\tau \ ,$$

$$\cos(r,s) = \frac{\xi'(t)[\xi(\tau)-\xi(t)] + \eta'(t)[\eta(\tau)-\eta(t)]}{\sqrt{[\xi(\tau)-\xi(t)]^2 + [\eta(\tau)-\eta(t)]^2}\cdot\sqrt{\xi'^2(t)+\eta'^2(t)}}$$

and consequently

$$\rho(\tau)\frac{\cos(r,s)}{r}ds$$
$$= \frac{\xi'(t)[\xi(\tau)-\xi(t)] + \eta'(t)[\eta(\tau)-\eta(t)]\sqrt{\xi'^2(\tau)+\eta'^2(\tau)}}{[\xi(\tau)-\xi(t)]^2 + [\eta(\tau)-\eta(t)]^2\sqrt{\xi'^2(t)+\eta'^2(t)}}\rho(\tau)d\tau. \quad (49)$$

We shall assume below that $\xi = f(t)$ and $\eta = \phi(t)$ are holomorphic functions of t and that $f'(t)$ and $\phi'(t)$ are not simultaneously equal to zero.

As a function of τ the denominator of the fraction (49) reduces to zero only for $\tau = t$, and in the neighborhood of the point t

$$\tau = t + \Delta t \quad .$$

We write the expansions:

$$\xi(\tau) = \sum_{n=o}^{\infty}\frac{(\Delta t)^n\xi^{(n)}(t)}{n!} \ ; \qquad \xi'(\tau) = \sum_{n=o}^{\infty}\frac{(\Delta t)^n\xi^{(n+1)}(t)}{n!} \ ,$$

$$\eta(\tau) = \sum_{n=o}^{\infty}\frac{(\Delta t)^n\eta^{(n)}(t)}{n!} \ ; \qquad \eta'(\tau) = \sum_{n=o}^{\infty}\frac{(\Delta t)^n\eta^{(n+1)}(t)}{n!} \ ,$$

by means of which the following expression for $\frac{\cos(r,s)}{r}ds$ is readily obtained:

$$\frac{\left[(\Delta t)(\xi'^2+\eta'^2) + \frac{(\Delta t)^2}{2!}(\xi'\xi''+\eta'\eta'') + \cdots\right]\sqrt{\xi'^2+\eta'^2+2\Delta t(\xi'\xi''+\eta'\eta'')+\cdots}}{[(\Delta t)^2(\xi'^2+\eta'^2) + (\Delta t)^3(\xi'\xi''+\eta'\eta'') +\cdots]\sqrt{\xi'^2+\eta'^2}};$$

separating out the $(\Delta t)(\xi'^2+\eta'^2)$, after elementary transformations we obtain

$$\frac{\cos(r,s)}{r}ds = \frac{1}{\tau-t} + \frac{1}{2}\frac{\xi'\xi'' + \eta'\eta''}{\xi'^2 + \eta'^2} + \cdots \tag{50}$$

and since

$$\xi'^2(\tau) + \eta'^2(\tau) = \xi'^2 + \eta'^2 + 2(\Delta t)(\xi'\xi'' + \eta'\eta'') + \cdots,$$

$$\frac{1}{ds} = \frac{1}{\sqrt{\xi'^2(\tau)+\eta'^2(\tau)}} = 1 - \frac{(\Delta t)}{\xi'^2+\eta'^2}(\xi'\xi'' + \eta'\eta'') + \cdots,$$

$$\frac{\cos(r,s)}{r} = \frac{(\xi'^2+\eta'^2)^{-1/2}}{\tau - t} - \frac{1}{2}\frac{1}{(\xi'^2+\eta'^2)^{3/2}}(\xi'\xi''+\eta'\eta'') + \cdots \quad . \tag{51}$$

In the right parts of (50) and (51) let us for brevity denote the sum of the regular summands by Ξ_1 and Ξ_2; then

$$\frac{\cos(r,s)}{r}ds = \frac{1}{\tau-t} + \Xi_1, \tag{50}$$

(51) $$\frac{\cos(r,s)}{r} = \frac{(\xi'^2+\eta'^2)^{-1/2}}{\tau - t} + \Xi_2 .$$

These formulas permit us to establish the character of the integral kernels of system (43).

In accordance with (42) and (50)-(51), we shall have

$$K(z,\sigma,k_1,k_2) = \frac{\partial}{\partial n_\sigma} H_0^2[k_1 r(z,\sigma)] + \frac{1}{2i} \text{ V.p.}$$
$$\times \int_0^{\chi} \left\{ \frac{4}{\pi^2(z-s)(s-\sigma)} + \frac{2i}{\pi(s-z)} \Xi_2' + \frac{2i}{\pi(\sigma-s)} \Xi_1' + E(z,s,\sigma) \right\} ds$$
$$= \frac{2}{i\pi^2} \frac{1}{z-\sigma} \lg \left[1 - \frac{\chi(z-\sigma)}{\sigma(z-\chi)} \right] + \Omega_1(k_1,k_2,z,\sigma) \quad , \tag{52}$$

where the values Ξ_2' and Ξ_1' and E are obvious from the notation adopted above and used in the transformations of the expression, and

$$\Omega_1(k_1,k_2,z,\sigma)$$
$$= \frac{\partial}{\partial n_\sigma} H_0^2[k_1 r(z,\sigma)] + \frac{1}{\pi} \text{ V.p. } \int_0^{\chi} \left[\frac{1}{s-z} \Xi_2' + \frac{1}{\sigma-s} \Xi_1' + E(z,s,\sigma) \right] ds$$

is a regular function for $z = \sigma$.

Thus we see that the functions $K(z,\sigma,k_1,k_2)$ and $K(z,\sigma,k_2,k_1)$ are regular functions; as regards the functions $L(z,\sigma,k_2)$ and $L(z,\sigma,k_1)$, their regularity for $z = \sigma$ is obvious from the integral representations (42).

Hence it follows that system (43) actually has no kind of singularity; let us rewrite this system in the following form:

$$\left.\begin{aligned} &\frac{2}{\pi^2 i} \int_0^{\chi} \left\{ \frac{\rho(z)}{z-\sigma} \lg \left[1 - \frac{\chi(z-\sigma)}{\sigma(z-\chi)} \right] + \Omega_1(k_1,k_2,z,\sigma)\rho(z) \right\} dz \\ &\qquad - \int_0^{\chi} \mu(z) L(z,\sigma,k_2) dz = F_1(\sigma), \\ &\frac{2}{\pi^2 i} \int_0^{\chi} \left\{ \frac{\mu(z)}{z-\sigma} \lg \left[1 - \frac{\chi(z-\sigma)}{\sigma(z-\chi)} \right] + \Omega_2(k_1,k_2,z,\sigma)\mu(z) \right\} dz \\ &\qquad + \int_0^{\chi} \rho(z) L(z,\sigma,k_1) dz = F_2(\sigma). \end{aligned}\right\} \tag{53}$$

§ 33. Reduction to a regular Fredholm system. Let us define the function, regular for $z = \sigma$,

$$\left[\frac{1}{z-\sigma}\lg\frac{z(\gamma-\sigma)}{\sigma(\gamma-z)} + \frac{\gamma}{\sigma(\sigma-\gamma)}\right] = R(z,\sigma) \quad .$$

Differentiating system (53) with respect to σ term-by-term, (one can readily satisfy oneself of the legitimacy of this operation),[1] we obtain:

$$\frac{2}{\pi^2 i}\ \text{V.p.}\int_0^{\gamma}\frac{\rho(z)}{z-\sigma}[R(z,\sigma) + \Omega_1'(k_1,k_2,z,\sigma)]dz - \int_0^{\gamma}\mu(z)L'(z,\sigma,k_2)dz = F_1'(\sigma),$$

$$\frac{2}{\pi^2 i}\ \text{V.p.}\int_0^{\gamma}\frac{\mu(z)}{z-\sigma}[R(z,\sigma) + \Omega_2'(k_1,k_2,z,\sigma)]dz + \int_0^{\gamma}\rho(z)L'(z,\sigma,k_1)dz = F_2'(\sigma);$$

the sign ' in these equations denoting differentiation with respect to σ. Lastly, employing the notations

$$\left.\begin{aligned}\frac{2}{\pi^2 i}[R(z,\sigma) + \Omega_1'(k_1,k_2,z,\sigma)] &= M(z,\sigma)\ ;\\ \frac{2}{\pi^2 i}[R(z,\sigma) + \Omega_2'(k_1,k_2,z,\sigma)] &= N(z,\sigma)\ ,\end{aligned}\right\} \tag{54}$$

we have

$$\left.\begin{aligned}\text{V.p.}\int_0^{\gamma}\frac{M(z,\sigma)}{z-\sigma}\rho(z)dz - \int_0^{\gamma}\mu(z)L'(z,\sigma,k_2)dz &= F_1'(\sigma)\ ;\\ \text{V.p.}\int_0^{\gamma}\frac{N(z,\sigma)}{z-\sigma}\mu(z)dz + \int_0^{\gamma}\rho(z)L'(z,\sigma,k_1)dz &= F_2'(\sigma)\ .\end{aligned}\right\} \tag{55}$$

Let us remark once again that the functions $M(z,\sigma)$, $N(z,\sigma)$, $L'(z,\sigma,k_2) = \frac{d}{d\sigma}L(z,\sigma,k_2)$ and $L'(z,\sigma,k_1) = \frac{d}{d\sigma}L(z,\sigma,k_1)$ are regular functions of z and σ. Besides this, not one of these functions vanishes for $z = \sigma$. We shall not adduce here proof of this statement, which reduces to a simple calculation of the values at the point $z = \sigma$ of the functions mentioned.

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[1]Hobson, Functions of a Real Variable, p. 599.

Multiplying all of the elements of equation (55) by $\frac{d\sigma}{\sigma - s}$ and integrating with respect to σ across the interval from 0 to χ, we obtain, on the basis of formula (I), § 24, which we utilize repeatedly,

$$- \pi^2 M(s,s)\rho(s) + \text{V.p.} \int_0^\chi \rho(z)dz \int_0^\chi \frac{M(z,\sigma)}{(\sigma - s)(z-\sigma)} d\sigma - \text{V.p.}$$
$$\times \int_0^\chi \mu(z)dz \int_0^\chi \frac{L'(z,\sigma,k_2)}{\sigma - s} d\sigma = \text{V.p.} \int_0^\chi \frac{F'(\sigma)}{\sigma - s} d\sigma ,$$

$$- \pi^2 N(s,s)\mu(s) + \text{V.p.} \int_0^\chi \mu(z)dz \cdot \int_0^\chi \frac{N(z,\sigma)}{(\sigma - s)(z-\sigma)} d\sigma + \text{V.p.}$$
$$\times \int_0^\chi \rho(z)dz \int_0^\chi \frac{L'(z,\sigma,k_1)}{\sigma - s} d\sigma = \text{V.p.} \int_0^\chi \frac{F'(\sigma)}{\sigma - s} d\sigma .$$

Reintroduce the designations

$$\text{V.p.} \int_0^\chi - \frac{1}{\pi^2(\sigma - s)(z-\sigma)} \frac{M(z,\sigma)}{M(s,s)} d\sigma = a(s,z) ;$$
$$\text{V.p.} \int_0^\chi \frac{L'(z,\sigma,k_2)}{\sigma - s} d\sigma = c(s,z) ,$$
$$\text{V.p.} \int_0^\chi - \frac{1}{\pi^2(\sigma - s)(z-\sigma)} \frac{N(z,\sigma)}{N(s,s)} d\sigma = b(s,z) ;$$
$$\text{V.p.} \int_0^\chi \frac{L'(z,\sigma,k_1)}{\sigma - s} d\sigma = d(s,z) ,$$
$$\text{V.p.} \int_0^\chi \frac{F'(\sigma)}{\sigma - s} d\sigma = e(s) ; \qquad \text{V.p.} \int_0^\chi \frac{F'(\sigma)}{\sigma - s} d\sigma = g(s) .$$

Then at last we have

$$\left. \begin{aligned} \rho(s) + \int_0^\chi \rho(z)a(s,z)dz - \int_0^\chi \mu(z)c(s,z)dz &= e(s) ; \\ \mu(s) + \int_0^\chi \mu(z)b(s,z)dz + \int_0^\chi \rho(z)d(s,z)dz &= g(s) . \end{aligned} \right\} \quad (56)$$

And system (56) is in fact a regular system of Fredholm equations, the obtaining of which constituted our final task.

EDITOR'S NOTES

Note 1. In equations (18), (19), and (20), page 38 of translation, Kupradze failed to state the following relations, without which the coefficients are not completely defined:

(a) Equation (19) is incorrect for $s = n - 1$.

Write instead

If $s = n - 1$, $n \geq 1$, replace $\alpha_{s+1}^{(2n)}$ by $2\,\alpha_{s+1}^{(2n)}$.

If $n = 0$, then $ce_0(\eta, q_1) = \sum_{s=0}^{\infty} \beta_s^{o} \cos 2s\eta$;

$$\beta_0^o = 1 \; ; \; \beta_1^o = -a\beta_0^o/8q_1 \; ; \; \beta_2^o = \left\{(-a + 4)/8q_1\right\}\beta_1^o - 2\beta_0^o$$

$$s^2\beta_s^o = 2p\,\beta_s^o + 2q_1(\beta_{s-1}^o + \beta_{s+1}^o) \quad , \; s \geq 3 \quad .$$

Note 2. In equations (22), (22*) and (23), pages 45 - 48 of the translation, the author gives bounds for $\beta_s^{(2n)}$. It has not been shown that $\Delta_o \neq 0$. On page 46, immediately following (22*) it is stated that Δ_o is different from zero for small values of q; this is true. No corresponding statement can be made for general values of q. It has been pointed out in the Editor's Preface that Kupradze used Whittaker's normalization, and in that normalization β_s can in fact become infinite for a certain set of discrete values of q. This was shown by S. Goldstein in 1927.

Note 3. It is not clear what the author meant, in the statement after equation (50), page 65 of the translation, about replacing η by $-\eta$; such replacement is unnecessary. Moreover, the Russian text has

the wrong sign in the expansion for $H_{2k}^{(2)}(K_1 r) \sin(2k\theta + y\frac{\pi}{2})$, $y = 0, 1$, the first two equations of (50*). This was corrected in the translation. Similarly the sign was wrong in the equations for $Se_{2n}(\xi, q_1)$ and $Ze_{2n}(\xi, q_1)$ in equations (51) on page 67. While it is true that the change in sign merely affects the normalization of the solution, it is best to retain the sign required by the fundamental definitions I, II, III, IV, on page 55 of the translation.

Note 4. A slight liberty was taken in translating the second sentence on page 68 of the text. A literal translation reads

> "utilizing the well-known asymptotic expressions of the Bessel and Hankel functions for large values of the parameter."

The author obviously did not mean that.

Note 5. Page 60 of translation, immediately following (40): The phrase "dropping the factor $(-1)^n$ exp in η" does not appear in the Russian text. A literal translation reads: "Let us study the expansion in a Fourier series of the following solution of the vibrational equation:

$$\left[H_n^{(2)}(\sqrt{R^2 + \rho^2 - 2R\cos\infty}) \cdot \right] e^{in\theta} .$$

The above function by itself is not a solution of the vibrational equation, and it is believed the author really meant what the altered translation implies.

ERRORS IN RUSSIAN TEXT

Page	
15	Footnote. Debye (spelling)
17-18	Summation index and independent variable both denoted by r in the same equations. Correct as in translation.
17	Line 14 - Replace kr in the denominator by $\sqrt{k}\,r$
27	Footnote 1. E. L. Ince.
28	Eq. 19. Superfluous factor 2; replace $\alpha_s^{(2n)}$ by $\alpha_{s+1}^{(2n)}$. Correct as in translation.
	Eq. 20. Superfluous factor 2.
	Also see Editor's Note 1.
30	Third line from bottom. Replace $b_{2s+1}^{(2n)}$ by $b_{2s+1}^{(2n+1)}$.
35	Eq. 25. Replace $k_1 \cos \eta \cos \theta$ by $i\, k_1 \cos \eta \cos \theta$ in exponent.
36	Equation immediately following eq.(27) and eq. (28); signs incorrect, superfluous factors, and missing factors. Correct as in translation.
	Last equation on page, add factor $(-1)^r$.
	See Whittaker and Watson, Modern Analysis p. 411 for comparison.
39	Equation immediately preceding (I): Read $\theta = \text{arc tg}\left\{\text{tg}\,\eta\ \text{th}\,\xi\right\}$.
	Eq. IV. Read $ce_{2n+1}(\eta, q_1)$ under the integral, in place of $se_{2n+1}(\eta, q_1)$.
40	First equation. Read $+ O(\frac{1}{r})$.
	Line 10. Replace Se_{2n} by $Se_{2n}(\xi, q_1)$.
	Last line and 5 lines from bottom. Replace $O(1/\cosh \xi)$ by $O(1/\cosh^2 \xi)$.

Page

41 Line 3. Sign incorrect. See translation.

43 Top of page. Add "Dropping the factor $(-1)^n$ exp i n η".
See Editor's Note. 5.

44 Nine lines from bottom. Right-hand member of equation for $C_{s-n}(R)\, J_s(\rho)$: Exponential factor wrong. See translation.

46 4 lines from bottom. Replace $(-1)^{s-k}$ by $(-1)^{s-k+1}$, as in translation.

47 Signs wrong in first line of page, and in first terms of $Se_{2n}(\xi_1 q_1)$ and $Ze_{2n}(\xi_1 q_1)$. Also notation errors in coefficients. Correct as in translation.

48 First line of second paragraph. Replace Se_{1m} by Se_{2m}.
9 lines from bottom. Replace N - n by N - m.

50 Equation immediately preceding (57). Symbols wrong. See translation.

52 Replace ce'_{1s} by ce'_{2s} in first eq., as in translation.

66 Equation preceding (92). Symbol should be χ_1.

68 Eq. (96). Replace $a_{s,r}$ by $\lambda_{s,r}$.

77 Second equation. Add = 0.

78 Equation for Φ. Replace F(cty) by F(ct - y).

108 Last equation. Replace Ξ by Ξ_1.

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