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NATIONAL BUREAU OF STANDARDS REPORT

1996

CHANGES OF SIGN OF SUMS OF RANDOM VARIABLES

by

P. Erdös and G. A. Hunt

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U. S. DEPARTMENT OF COMMERCE

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CHANGES OF SIGN OF SUMS OF RANDOM VARIABLES *

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CHANGES OF SIGN OF SUMS OF RANDOM VARIABLES *

by

P. Erdos ** and G. A. Hunt ***

1. Let $x_1, x_2, \ldots$ be independent random variables all having the same continuous symmetric distribution, let $s_k = x_1 + \cdots + x_k$, and let N_n be the number of changes of sign in the sequence $s_1, \ldots, s_{n+1}$.

Th 1.
$$\sum_{k=1}^{n} \frac{1}{2(k+1)} \leq E\{N_n\} \leq \frac{1}{2} \sum_{k=1}^{n} \phi(k)$$

Here $E\{N_n\}$ denotes the expectation of N_n and $\phi(k)$ is

$$\frac{2([k/2]+1)}{k+1} \binom{k}{[k/2]} 2^{-k} \approx (2\pi k)^{-1/2} \quad .$$

Th 2. With probability one $\underline{\lim}\, N_n / \log n \geq \frac{1}{2}$. We conjecture, but cannot prove, that also $\overline{\lim}\, N_n / (n \log \log n)^{1/2} \leq 1$.

By considering certain subsequences we obtain an exact limit theorem which is still independent of the distribution of the x_i. Let α be positive and a the first integer such that $(1 + \alpha)^a \geq 2$; let $1', 2', \ldots$ be any sequence of natural numbers satisfying $(k + 1)' \geq (1 + \alpha)k'$; and N_k' be the number of changes of sign in $s_1', \ldots, s_{k+1}'$, where $s_j' = s_{j'}$.

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Th 3. $$E\{N'_k\} \geq \frac{1}{8}[k/a] \quad \text{and} \quad N'_k/E\{N'_k\} \to 1$$

with probability one.

It is likely that $\frac{1}{8}[k/a]$ can be replaced by $\frac{1}{2}\alpha n/(1+\alpha)$. For $k' = 2^k$ it is easy to see that $E\{N'_k\} = k/4$; so with probability one the number of changes of sign in the first k terms of $s_1, s_2, \ldots, s_{2^k}, \ldots$ is asymptotic to $k/4$.

Our proofs are elementary and hardly use more than Lemma 3 of the next section, which gives Theorem 1 immediately. We prove Theorem 3 in §3 and then demonstrate Th 2 by considering particular sequences $1', 2', \ldots$. A sequence $x_1, x_2, \ldots$ for which $N_n/\log n \to 1/2$ is exhibited in §4. And finally we sketch the proof of the following theorem, which was discovered by Paul Levy [1] when the x_i are the Rademacher functions.

Th 4. With probability one

$$\sum_{k=1}^{n} \frac{\operatorname{sgn} s_k}{k} = o(\log n)$$

Our results are stated only for random variables with continuous distributions. Lemma 3, slightly altered to take into account cases of equality, remains true however for discontinuous distributions; the altered version is strong enough to prove the last three theorems as they stand and the first theorem with the extreme members slightly changed. The symmetry of the x_i is of course essential in all our theorems.

2. Let $a_1, \ldots, a_n$ be positive numbers which are free in the sense that no two of the sums $\pm a_1 \pm \ldots \pm a_n$ have the same value. These sums, arranged in decreasing order, we denote by $S_1, S_2, \ldots, S_{2^n}$; q_i is the excess of +'s over -'s occurring in S_i; and $Q_i = q_1 + \ldots + q_i$. It is clear that $Q_i = Q_{2^n - i}$ for $1 \leqq i < 2^n$.

Lemma 1 $\quad 0 \leqq Q_i - i \leqq ([n/2]+1)\binom{n}{[n/2]} - 2^{n-1}$ for $i \leqq 2^{n-1}$

The proof of the first inequality, which is evident for $n = 1$, goes by induction. Let $n > 1$ and $i \leqq 2^{n-1}$. Define S'_j and Q'_j for $1 \leqq j \leqq 2^{n-1}$ just as S_j and Q_j above, but using only $a_1, \ldots, a_{n-1}$; and let k and ℓ be the largest integers such that $S'_k - a_n \geqq S_i$ and $S'_\ell + a_n \geqq S_i$. It may happen that no such k exists; then $i = \ell$ and

$$Q_i = Q'_\ell + \ell \geqq \ell + \ell > i$$

if $\ell \leqq 2^{n-2}$,

$$Q_i = Q'_{2^{n-1}-\ell} + \ell \geqq 2^{n-1} - \ell + \ell > i$$

if $2^{n-2} < \ell < 2^{n-1}$, and

$$Q_i = Q'_{2^{n-1}} + 2^{n-1} = 2^{n-1} \geqq i$$

if $\ell = 2^{n-1}$. The remaining cases are dealt with in the same way.

In order to prove the second inequality we note that for each i the maximum of Q_i is attained if the a_i are given such values that $S_j > S_k$ implies $q_j \geqq q_k$, — this happens if the a_j are nearly equal. If n is odd q_i is positive for $i \leqq i_o = 2^{n-1}$ and $Q_i - i$ is maximum for $i = i_o$. We have

$$Q_{i_o} - i_o = \sum_{k=0}^{[n/2]} (n - 2k)\binom{n}{k} - 2^{n-1} = ([n/2] + 1)\binom{n}{[n/2]} - 2^{n-1}$$

A similar computation for n even gives $i_o = 2^{n-1} - \binom{n}{n/2}$ and the same expression for $Q_{i_o} - i_o$. This completes the proof.

If $c_1, \dots, c_{n+1}$ are real numbers let $m(c_1, \dots, c_{n+1})$ be the number of indices j for which $|c_j| > |\Sigma_{i \neq j} c_i|$. We now consider n + 1 positive numbers $a_1, \dots, a_{n+1}$ which are 'free' and define

$$M = M(a_1, \dots, a_{n+1}) = \Sigma m(\pm a_1, \dots, \pm a_{n+1}) ,$$

the summation being taken over all combinations of + and - signs.

Lemma 2. $$2^{n+1} \leq M \leq 4([n/2] + 1)\binom{n}{[n/2]} .$$

It is clear that $M = 2^{n+1}$ if $a_{n+1} > a_1 + \dots + a_n$, and we reduce the other cases to this one by computing the change in M as a_{n+1} is raised to $a_1 + \dots + a_n + 1$. Let i be the integer between 1 and 2^{n-1} for which $S_{i+1} < a_{n+1} < S_i$ (we use the notation of Lemma 1) and let a'_{n+1} be slightly greater than S_i. Then $a_{n+1} < S_i$ becomes $a'_{n+1} > S_i$ if a_{n+1} is replaced by a'_{n+1} and we see that there is a contribution +4 to M coming from the terms $\pm a_{n+1}$ in the four sums $\pm S_i \pm a_{n+1}$. In like manner, each $+a_j$ occurring in S_i gives -4 to M and each $-a_j$ in S_i gives +4, so that

$$M(a_1, \dots, a_n, a_{n+1}) - M(a_1, \dots, a_n, a'_{n+1}) = 4(q_i - 1)$$

Thus raising a_{n+1} to $1 + a_1 + \cdots + a_n$ lowers M by $4(Q_i - i) = 4\Sigma_{j \leq i}(q_j - 1)$ and Lemma 2 follows from Lemma 1.

There is another more direct way of establishing the first inequality of Lemma 2. Considering the n numbers $(a_1 + a_2)$, $a_3, \ldots, a_{n+1}$, we assume that there are at least 2^{n-2} inequalities of the form

(1) $$a_j > U \qquad j > 2$$

or

(2) $$(a_1 + a_2) > V$$

where the right members are positive, and U is a sum over $(a_1 + a_2)$, $a_3, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n+1}$ with appropriate signs, and V is a sum over $a_3, \ldots, a_{n+1}$. From (1) we obtain an inequality (1') by dropping the parentheses from $(a_1 + a_2)$ in U; from (2) we obtain (2'): $a_1 > a_2 - V$ or $a_1 > V - a_2$ according as a_2 is greater or less than V (we assume without loss of generality that $a_1 > a_2$). We consider also the n numbers $(a_1 - a_2)$, $a_3, \ldots, a_{n+1}$ and the inequalities

(3) $$a_j > U' \qquad j > 2$$

(4) $$(a_1 - a_2) > V'$$

of which there are at least 2^{n-2}. From (3) we derive (3') by dropping the parentheses from $(a_1 - a_2)$ in U'; from (4) we obtain (4') $a_1 > a_2 + V'$. It is easy to see that no two of the primed inequalities are the same. Hence there must be at least $2 \cdot 2^{n-2} = 2^{n-1}$ inequalities

$$a_i > \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \pm a_j$$

in which the right member is positive. Taking into account the four possibilities of attributing signs to the two members of each inequality we get the first statement of the lemma.

Lemma 3. $\frac{1}{n+1} \leq \Pr\left\{|x_{n+1}| > |x_1 + \cdots + x_n|\right\} \leq \phi(n)$

Here of course the x_i satisfy the conditions imposed at the beginning of §1 and $\phi(n)$ is the function defined there. Since the joint distribution of the x_i is unchanged by permutations of the x_i and multiplication of x_i by -1, we have

$$\Pr\left\{|x_{n+1}| > \left|\sum_{i=1}^{n} x_i\right|\right\} = \frac{1}{n+1} \sum_{i=1}^{n+1} \Pr\left\{|x_i| > \left|\sum_{j \neq i} x_j\right|\right\}$$

$$= \frac{1}{n+1} E\left\{m(x_1, \cdots, x_{n+1})\right\}$$

$$= \frac{1}{n+1} E\left\{\frac{1}{2^{n+1}} \sum_{+,-} m(\pm|x_1|, \cdots, \pm|x_{n+1}|)\right\}$$

$$= \frac{2^{-n-1}}{n+1} E\left\{M(|x_1|, \cdots, |x_{n+1}|)\right\}$$

where m and M are the functions defined above. So Lemma 3 follows at once from Lemma 2.

We cannot prove the inequality

$$\frac{m}{m+n} \leq P_{m,n} = \Pr\left\{ \left| \sum_{1}^{n} x_i \right| < \left| \sum_{n+1}^{n+m} x \right| \right\} \leq \phi([n/m])$$

for $m \leq n$, which would make our later proofs somewhat easier. But we shall use

(5) $$P_{m,n} \leq 6\phi([n/m]) < 3[n/m]^{-1/2} ,$$

and establish it in the following manner: Let $a = [n/m]$ and write $z = x_{n+1} + \cdots + x_{n+m}$, $u = x_1 + \cdots + x_{am}$, $v = x_{am+1} + \cdots + x_n$, $w = x_{n+1} + \cdots + x_{am+m}$. It follows from Lemma 3 that the set E on which the four inequalities $|z| < |u \pm v \pm w|$ hold has probability at least $1 - 4\phi(a + 1)$ and that the set F on which the two inequalities $|v \pm w| < |u|$ hold has probability at least $1 - 2\phi(a)$. So

$$\Pr\{EF\} \geq 1 - 2\phi(a) - 4\phi(a + 1) > 1 - 6\phi(a) .$$

And clearly $|u + v| > |z|$ on EF.

3. It is easy to see that

$$E\{N_n\} = \sum_{k=1}^{n} Pr\{s_k s_{k+1} < 0\} = \frac{1}{2}\sum_{k=1}^{n} Pr\{|x_{k+1}| > |s_k|\}$$

so that Lemma 3 implies Theorem 1.

Let us turn to Theorem 3. Clearly the probability of s'_k and $s_{2k'} = \Sigma_1^{2k'} x_i$ differing in sign is 1/4. Also $s'_{k+a} - s_{2k'}$ is independent of s'_k and $s_{2k'}$, for $(k+a)' \geq (1+\alpha)^a k' \geq 2k'$. Since $s'_{k+a} - s_{2k'}$ has an even chance of taking on the same sign as $s_{2k'}$ we must have $Pr\{s'_k s'_{k+a} < 0\} \geq \frac{1}{2} Pr\{s'_k s_{2k'} < 0\} = 1/8$. Now, if $s'_k s'_{k+a} < 0$ there must be at least one change of sign in the sequence $s'_k, s'_{k+1}, \ldots, s'_{k+a}$. Hence $p_k + p_{k+1} + \cdots + p_{k+a-1} \geq 1/8$, where $p_k = Pr\{s'_k s'_{k+1} < 0\}$. Consequently

$$E\{N'_k\} = \sum_1^k p_\ell \geq \frac{1}{8}[k/a] \quad , \tag{6}$$

and the first part of the theorem is proved.

We next show that the variance of N'_k is $O(k)$ by estimating $p_{i,j} = Pr\{s'_i s'_{i+1} < 0 \text{ and } s'_j s'_{j+1} < 0\}$ for $i < j$. Let $u = s'_i$, $v = s'_{i+1} - s'_i$, $w = s'_j - s'_{i+1}$, $z = s'_{j+1} - s'_j$ and define the events

A: $uv < 0$

B: $|u| < |v|$

C: $(u + v + w)z < 0$

D: $|u + v + w| < |z|$

D': $|w| < |z|$

E: $|z - w| > |u + v|$,

so that $p_i = \Pr\{AB\}$, $p_j = \Pr\{CD\}$, $p_{i,j} = \Pr\{ABCD\}$. One sees immediately that A, B, C, D' are independent and that ED = ED'. Writing $\tilde{E}$ for the complement of E, we have

$$ABCD = \tilde{E}ABCD + EABCD' \\ \subset \tilde{E} + ABCD'$$

and

$$D' \subset \tilde{E} + D \quad .$$

Hence

$$\begin{aligned}\Pr\{ABCD\} &\leq \Pr\{\tilde{E}\} + \Pr\{ABC\}\Pr\{D'\} \\ &\leq \Pr\{\tilde{E}\} + \Pr\{ABC\}(\Pr\{\tilde{E}\} + \Pr\{D\}) \\ &\leq \Pr\{AC\}\Pr\{C\}\Pr\{D\} + 2\Pr\{\tilde{E}\} \\ &= p_i\, p_j + 2\Pr\{\tilde{E}\}\end{aligned}$$

Now z - w is the sum of $(j + 1)' - (i + 1)'$ of the x's and u + v is the sum of $(i + 1)'$ of the x's; moreover

$$(j + 1)' - (i + 1)' \geq \{(1 + \alpha)^{j-i} - 1\}\,(i+1)' \quad .$$

We may thus apply the inequality (5) following Lemma 3 to obtain

$$\Pr\{\tilde{E}\} < 3\{(1 + \alpha)^{j-i} - 2\}^{-1/2}$$

provided $j - i \geq a$. A similar argument gives a lower bound for $p_{i,j}$. We have finally

$$p_{i,j} = p_i\, p_j + O\left\{(1 + \alpha)^{-\frac{1}{2}|i-j|}\right\}$$

for all i,j. This estimate yeilds

$$E\{N_k'^2\} = \sum_{1 \le i,j \le k} p_{i,j}$$

(7)
$$= \sum p_i\, p_j + \sum O\left\{(1+\alpha)^{-\frac{1}{2}|i-j|}\right\}$$

$$= E\{N_k'\}^2 + O(k) \quad .$$

Let us denote $E\{N_k'\}$ by b_k. It follows from (6), (7) and Tchebycheff's inequality that

$$Pr\{|N_k'/b_k - 1| > \epsilon| < \frac{c}{\epsilon^2 k}$$

for an appropriate c and all positive ϵ. Hence

$$Pr\left\{\left|\frac{N_{k(j)}'}{b_{k(j)}} - 1\right| > \epsilon\right\}$$

is the j-th term of a convergent series if $k(j) = j^2$, so that $N_{k(j)}'/b_{k(j)} \to 1$ with probability one. Since also $b_{k(j)}/b_{k(j+1)} \to 1$ and

$$\frac{N_{k(j)}'}{b_{k(j+1)}} \le \frac{N_k'}{b_k} \le \frac{N_{k(j+1)}}{b_{k(j)}}$$

for $k(j) \le k < k(j+1)$, the proof of Theorem 3 is complete.

Theorem 2 is obtained in the following way. Let r be a large integer and let 1', 2', ... be the sequence

$$r, \ (r+1),$$
$$r^2, \ r(r+1), \ (r+1)^2,$$
$$\cdots$$
$$r^{\ell}, \ r^{\ell-1}(r+1), \ \cdots, \ (r+1)^{\ell},$$
$$r^m, \ r^{m-1}(r+1), \ \cdots, \ (r+1)^m,$$
$$\cdots$$

with m defined by $r^{m+1} \geq (r+1)^{\ell+1} > r^m$. Let us call j 'favorable' if $(j+1)' = (1 + 1/r)j'$. Then

a) $(1 + 1/r)j' \leq (j+1)' \leq (1+r)j'$ for all j

b) There are $k + o(k)$ favorable j less than k (as $k \to \infty$).

We see at once that

$$\log k' = k \log (1 + 1/r) + o(k) \quad .$$

Furthermore, if j is favorable then $j' = r\left\{(j+1)' - j'\right\}$; so, according to Lemma 3,

$$\Pr\left\{s'_j \ s'_{j+1} < 0\right\} = \frac{1}{2} \Pr\left\{\left|s'_{j+1} - s'_j\right| > \left|s'_j\right|\right\}$$
$$\geq \frac{1}{2(1+r)} \quad .$$

Hence

$$E\left\{N'_k\right\} = \sum_{j=1}^{k} \Pr\left\{s'_j \ s'_{j+1} < 0\right\}$$
$$\geq \sum_{j \text{ favorable}} \Pr\left\{s'_j \ s'_{j+1} < 0\right\}$$
$$\geq \frac{k}{2(r+1)} + o(k) \quad ,$$

and consequently

$$\underline{\lim}_{n \to \infty} \frac{2N_n}{\log n} \geq \underline{\lim}_{k \to \infty} \frac{2N_k'}{\log(k + 1)'}$$

$$\geq \underline{\lim} \frac{2N_k'}{(k + 1) \log(1 + 1/r)}$$

$$\geq \underline{\lim} \frac{N_k'}{E\{N_k'\}(r + 1) \log (1 + 1/r)}$$

$$= \frac{1}{(r + 1) \log 1 + 1/r} .$$

Letting $r \to \infty$ we have Theorem 2.

4. Our construction of a sequence $x_1, x_2, \ldots$ for which $N_n / \log n \to 1/2$ depends on the following observations. For a given k define the random index $i = i(k)$

$$|x_i| = \max_{1 \leq j \leq n+1} |x_j|$$

and let A_k be the event $|x_i| > \Sigma\, |x_j|$, where the summation is over $j \neq i$, $1 \leq j \leq k+1$. Let f_k and g_k be the characteristic functions of the events '$s_k\, s_{k+1} < 0$' and '$i(k) = k+1$ and $x_{k+1}(x_1 + \cdots + x_k) < 0$. It is clear $g_1, g_2, \ldots$ are independent random variables, that $2\Pr\{g_k = 1\} = 1/(k+1)$, and that $f_k = g_k$ on A_k. If moreover $\Sigma \Pr\{\tilde{A}_k\} < \infty$ (here $\tilde{A}_k$ is the complement of A_k) then with probability one $f_k = g_k$ for all but a finite number of indices k. In this case we have

$$N_n = \sum_{k=1}^{n} f_k$$

$$= \sum_{k=1}^{n} g_k + O(1)$$

$$= \sum_{k=1}^{n} \frac{1}{2(k+1)} + o(\log n) \quad ,$$

the last step being the strong law of large numbers applied to $g_1, g_2, \ldots$ Thus we have only to show that the x_j may so be chosen that $\Pr\{\tilde{A}_k\} = O\{k^{-2}\}$, say.

To do this we take $x_j = \pm\exp(\exp 1/u_j)$ where $u_1, u_2, \ldots$ is a sequence of independent random variables each of which is uniformly distributed on the interval $(0, 1)$. For a given k let y and z be

the least and next to least of $u_1, u_2, \ldots, u_{k+1}$; the joint density function of y and z is

$$(k + 1)\, k(1 - z)^{k-1} \qquad 0 < y < z < 1 \quad .$$

Consequently the event D_k: $1/y > 1/z + 1/k^2$ has probability

$$\int_0^{k^2/(k^2+1)} dy \int_{\frac{y}{1-y/k^2}}^{1} (1 - z)^{k-1} dz = 1 + O(k^{-2})$$

and the event E_k: $1/z > 3 \log k$ also has probability $1 + O(k^{-2})$. It is easy to verify that A_k as defined above contains $D_k E_k$; so $Pr\left\{\tilde{A}_k\right\} = O\left\{k^{-2}\right\}$ and our example is complete.

5. We prove Theorem 4 in the form

$$T_n = \sum_{\substack{1 \le k \le n \\ s_k > 0}} 1/k = \frac{1}{2} \log n + o(\log n) \quad .$$

First $E\{T_n\} = \sum_1^n 1/k = \frac{1}{2} \log n + o(1)$. Next, the inequality following Lemma 3 yields

$$\Pr\{|s_\ell - s_k| < |s_k|\} = O(\tfrac{k}{\ell})^{1/2} \qquad \ell > k \quad ,$$

so that

$$\Pr\{s_k > 0 \text{ and } s_\ell > 0\} = \frac{1}{4} + O(\tfrac{k}{\ell})^{1/2} \qquad \ell > k \quad .$$

This implies

$$E\{T_n^2\} = \frac{1}{2} \sum_1^n 1/k + 2 \sum_{k<\ell} \{1/4 + O(\tfrac{k}{\ell})^{1/2}\} \frac{1}{k\ell}$$

$$= \frac{1}{4} (\log n)^2 + O(\log n) \quad .$$

Tchebycheff's inequality and the Borel-Cantelli lemma then yield

$$\frac{T_{n_k}}{\log n_k} \rightarrow \frac{1}{2} \quad , \tag{8}$$

where $n_k = 2^{k^2}$; and the sequence n_k is dense enough for (8) to imply the theorem.

[1] P. Levy Théorie de l'addition des variables aléatoires Paris 1937.

October 13, 1952

THE NATIONAL BUREAU OF STANDARDS

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