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NATIONAL BUREAU OF STANDARDS REPORT

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THE COEFFICIENTS OF CERTAIN INFINITE PRODUCTS

by

M. Newman

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U. S. DEPARTMENT OF COMMERCE

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● Office of Basic Instrumentation ● Office of Weights and Measures.

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The coefficients of certain infinite products

Introduction. The identities

(a) $$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=-\infty}^{\infty}(-1)^n x^{(3n^2+n)/2}$$

(b) $$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1) x^{(n^2+n)/2}$$

are classical, (a) being the so-called 'pentagonal number theorem' of Euler, and (b) being an identity of Jacobi's well-known in the theory of elliptic modular functions. It is remarkable that for no other values of r are the coefficients of $\prod_{n=1}^{\infty}(1-x^n)^r$ known explicitly. It will be the purpose of this paper to give various recursion formulas for these coefficients, which are consequences of identities between functions on certain modular subgroups. In particular, the coefficients will be investigated thoroughly for $r = 2,4,6$. A typical theorem is that every integer occurs as coefficient for $r = 2$.

1. Preliminary identities. In what follows all sums will be extended from 0 to ∞ and all products from 1 to ∞, unless otherwise indicated. We denote the coefficient of x^n in $\prod(1-x^n)^r$ by $P_r(n)$; i.e.,

(1.1) $$\prod(1-x^n)^r = \sum P_r(n)x^n .$$

Noting that $\prod(1-x^n)^r = \prod(1-x^n)^a . \prod(1-x^n)^{r-a}$, we obtain the identity

(1.2) $$P_r(n) = \sum_{j=0}^{n} P_a(j)\, P_{r-a}(n-j)$$

Differentiating (1.1) logarithmically, we also obtain

(1.3) $$n\, P_r(n) = -r \sum_{j=1}^{n} \sigma(j)\, P_r(n-j), \quad \sigma(j) = \sum_{d|j} d .$$

(1.3) is especially suited to numerical calculation.

Using (a), (b), and (1.2), we can obtain the following explicit formulas for $P_2(n)$, $P_4(n)$, $P_6(n)$:

(1.4) $$P_2(n) = \sum_{n = (3x^2 \pm x)/2 + (3y^2 \pm y)/2} (-1)^{x+y}$$

(1.5) $$P_4(n) = \sum_{n = (x^2+x)/2 + (3y^2 \pm y)/2} (-1)^{x+y}(2x+1)$$

(1.6) $$P_6(n) = \sum_{n = (x^2+x)/2 + (y^2+4)/2} (-1)^{x+y}(2x+1)(2y+1)$$

It is understood in formulas (1. 4)-(1-6) that (x,y) runs over all positive integral solutions of the defining equations. These formulas, though complicated, are arithmetically enlightening and will be studied in detail. We set $\langle x \rangle = [x + \frac{1}{2}]$, so that $\langle x \rangle$ is the integer nearest to x. A little consideration shows that we may rewrite formulas (1.4)-(1.6) as follows:

$$(1.7) \qquad P_2(n) = \sum_{u^2+v^2 = 24n+2} (-1)^{\langle (u+v)/6 \rangle}$$

$$(1.8) \qquad P_4(n) = \sum_{u^2+3v^2 = 24n+4,\ u,v \text{ odd}} (-1)^{[(u+3v)/6]} \cdot v$$

$$(1.9) \qquad P_6(n) = -\sum_{u^2+v^2 = 8n+2} (-1)^{(u+v)/2} \cdot uv$$

It is understood in formulas (1.7)-(1.9) that (u,v) runs over all positive integral solutions of the defining equations.

Let $a = \prod_{p|a} p^{e_p}$ be the canonical decomposition of a. If e_p is odd for some $p \equiv -1(4)$, then the equation $x^2+y^2 = a$ has no solutions in integers. If e_p is odd for some $p \equiv -1(3)$, then the equation $x^2+3y^2 = a$ has no solutions in integers. We can say therefore:

(1.10) Let $r = 2,4,6$. Let $\frac{24}{r}n+1 = \prod_{p \mid \frac{24}{r}n+1} p^{e_p}$ be the canonical decomposition of $\frac{24}{r}n+1$. If e_p is odd for some $p \equiv -1(m_r)$, where $m_r = 4,3,4$ for $r = 2,4,6$, then $P_r(n) = 0$.

Hence we need only consider n such that $\frac{24}{r}\ n+1 = q_r Q_r^2$, where q_r is a product of primes $\equiv 1\ (m_r)$ and Q_r is a product of primes $\equiv -1\ (m_r)$.

We note that if $p \equiv -1(4)$, then $x^2+y^2 \equiv 0(p)$ implies $x \equiv y \equiv 0(p)$, and if $p \equiv -1(3)$, then $x^2+3y^2 \equiv 0(p)$ implies $x \equiv y \equiv 0(p)$.

Making use of the above, we obtain the following reduction formulas:

(1.11) $P_2(\ (q_2 Q_2^2-1)/12) = \sum_{u^2+v^2=2q_2} (-1)^{\langle Q_2(u+v)/6 \rangle}$

(1.12) $P_4(\ (q_4 Q_4^2-1)/6) = Q_4 \sum_{u^2+3v^2=4q_4,\ u,v\ \text{odd}} (-1)^{[Q_4(u+3v)/6]}$

(1.13) $P_6(\ (\ Q_6^2-1)/4) = Q_6^2\ P_6(\ q_6\ -1)/4)$.

In particular,

(1.14) $P_2((Q_2^2-1)/12) = (-1)^{\langle Q_2/3 \rangle}$

(1.15) $P_4((Q_4^2-1)/6) = (-1)^{[2Q_4/3]} Q_4$

(1.16) $P_6((Q_6^2-1)/4) = Q_6^2$

A similar set of explicit formulas can be written down when

q_r is a prime. We note only that if q_2 is a prime, then

$$(1.17) \qquad P_2((q_2 Q_2^2-1)/12) = 2(-1)^{\langle Q_2(\alpha+\beta)/6 \rangle}$$

where α, β are the uniquely detèrmined positive integers such that $2q_2 = \alpha^2 + \beta^2$.

2. <u>The principal identities.</u> We are going to extend the scope of the previous identities by recursion formulas derived from identities between modular functions. The following theorem has been proved by the author in his paper [1](1):

Theorem 1. Suppose that r is even, $0 < r \leq 24$. Let ρ be a prime such that $r(\rho-1) \equiv 0(24)$. Set $\delta = r(\rho-1)/24$. Then

$$(2.1) \qquad \sum P_r(n\rho+\delta)x^n = P_r(\delta)\prod(1-x^n)^r - \rho^{\frac{r}{2}-1}x^{\delta}\prod(1-x^{n\rho})^r.$$

By comparing coefficients of corresponding powers of x, we find

$$(2.2) \qquad P_r(n\rho+\delta) = P_r(\delta)\,P_r(n) - \rho^{\frac{r}{2}-1}\,P_r((n-\delta)/\rho),$$ (2)

with the convention that $P_r(x)$ is zero if x is not an integer.

(1) Numbers in square brackets refer to bibliography at end of paper.

(2) The case r=24 reduces to Mordell's identity for Ramanujan's function $\tau(n)$.

We propose to derive some more identities of this type for $r = 2,4,6$ and for various associated p's. For this purpose we will require a number-theoretic lemma, the proof of which is straightforward and will be omitted.

Lemma 1. Suppose that r is 2,4, or 6. Let p be a prime such that $r(p+1) \equiv 0(24)$. Set $\Delta = r(p^2-1)/24$, $\lambda_n = \Delta - p[\Delta/p] + np$, $n = 0,1,\ldots$. Then $[\Delta/p]$ is the least value of n for which λ_n is the sum of two pentagons ($r=2$); the sum of a triangle and a pentagon ($r=4$); the sum of two triangles ($r=6$).

Lemma 1 leads easily to a proof that the functions S_r defined in [1] are constant, under the conditions imposed above (3). We are thus lead to the following theorem:

Theorem 2. With the notation of lemma 1, we have

$$(2.3) \qquad \sum P_r(np+\Delta)x^n = (-p)^{r/2-1} \prod (1-x^{np})^r$$

By comparing coefficients of corresponding powers of x, we find

$$(2.4) \qquad P_r(np+\Delta) = (-p)^{r/2-1} P_r(n/p) .$$

(3) The coefficients of the pole terms are essentially $P_r(\lambda_n)$, $n=0,1,\ldots,[\Delta/p]-1$. These coefficients are of necessity zero, in view of the formulas (1-4)-(1.5).

3. <u>Deductions from the principal formulas.</u> We will consider recursion formulas (2.2) and (2.4) more closely. It will be understood that a formula derived from a theorem is subject to the hypotheses of that theorem.

We see from (2.4) that

$$(3.1) \qquad P_r(n\rho + \Delta) = 0, \text{ if } (n,\rho) = 1.$$

Setting $n = \rho\mu_t$, $\mu_{t+1} = \rho^2\mu_t + \Delta$ $(t = 0,1,\ldots)$, and $A_t = P_r(\mu_t)$, formula (2.4) reads $A_{t+1} = (-p)^{r/2-1} A_t$. This implies that $P_r(\mu_0\rho^{2t} + r(\rho^{2t}-1)/24) = (-\rho)^{(r/2-1)t} P_r(\mu_0)$. In particular, choosing $\mu_0 = 0,1,2$ we obtain

$$(3.2) \qquad P_r(r(\rho^{2t}-1)/24) = (-p)^{(r/2-1)t}$$

$$(3.3) \qquad P_r(\rho^{2t} + r(\rho^{2t}-1)/24) = -r(-\rho)^{(r/2-1)t}$$

$$(3.4) \qquad P_r(2\rho^{2t} + r(\rho^{2t}-1)/24) = r(r-3)/2 \,.\, (-\rho)^{(r/2-1)t}$$

The recursion formula (2.2) requires a slightly more elaborate discussion. Setting $a = P_r(\delta)$, $b = \rho^{r/2-1}$, $n = \mu_t$, $\mu_{t+1} = \rho\mu_t + \delta$ $(t = 0,1,\ldots)$, and $A_t = P_r(\mu_t)$, (2.2) reads $A_{t+2} - a A_{t+1} + b A_t = 0$. The solution of this difference equation depends on the nature of the roots of the characteristic equation $x^2 - ax + b = 0$. This in turn depends on the dis-

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Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

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