NATIONAL BUREAU OF STANDARDS REPORT

1977

On Gauss' Speeding Up Device in the Theory of

Single Step Iteration

by

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U. S. DEPARTMENT OF COMMERCE

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NBS PROJECT | **NBS REPORT**

1101-10-5100 | 1977

October 8, 1952

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1. In solving the linear system

(1) $$\sum_{\nu=1}^{n} a_{\mu\nu} x_{\nu} = y_{\mu} \qquad (\mu = 1,\ldots,n)$$

with the matrix A, $a_{\mu\mu} \neq 0$ $(\mu = 1,\ldots,n)$ and non vanishing determinant by the single step iteration, we form,

starting from an arbitrary vector ξ_0, a sequence of vectors

(2) $$\xi_k = (x_1^{(k)},\ldots,x_n^{(k)}) \qquad (k = 1,2,\ldots)$$

obtained in the following way: For any integer k $(k \geqq 0)$ choose a value N_k of the "leading index" from the indices $1,\ldots,n$; then if

(3) $$\rho_k = (r_1^{(k)},\ldots,r_n^{(k)}) \qquad (k = 0,1,2,\ldots)$$

is the k-th residual vector defined by

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*This work was performed under a contract of the NBS with the Am. Un. I gratefully acknowledge discussions with T. S. Motzkin and O. Taussky-Tod

(4) $$r_\mu^{(k)} = \sum_{\nu=1}^{n} a_{\mu\nu}\, x_\nu^{(k)} - y_\mu \qquad (\mu = 1,\ldots,n; k=0,1,\ldots)$$

we put

(5) $$x_\mu^{(k+1)} = x_\mu^{(k)} \; (\mu \neq N_k), \quad x_{N_k}^{(k+1)} = x_{N_k}^{(k)} - \frac{r_{N_k}^{(k)}}{a_{N_k N_k}} .$$

In studying this iteration, we will only consider the case where all y_μ vanish, since we can always by a convenient change of the origin make all $y_\mu = 0$, without changing the ς_k.

2. We consider in what follows only the case where the matrix of the system (1) is <u>symmetric</u> and the quadratic form

(6) $$k(\xi) = \sum_{\mu,\nu=1}^{n} a_{\mu\nu} x_\mu x_\nu \quad ,$$

defined for an arbitrary vector $\xi = (x_1,\ldots,x_n)$, is <u>positive definite</u>. In this case it is well known and immediately verified that if the vector ξ_{k+1} is obtained from the vector ξ_k by the transformation (5), we have

(7) $$K(\xi_{k+1}) = K(\xi_k) - \frac{r_{N_k}^{(k)2}}{a_{N_k N_k}} .$$

In using (7) it was proved by Seidel, 1874 [9], that the single step iteration is always convergent if N_k is chosen at each step so that

(8) $$\frac{\left| r_{N_k}^{(k)} \right|^2}{\left| a_{N_k N_k} \right|} = \operatorname*{Max}_{\mu=1,\ldots,n} \frac{\left| r_\mu^{(k)} \right|^2}{\left| a_{\mu\mu} \right|} .$$

This is the "relaxation" procedure [1]. On the other hand Schmeidler (1949) proved in using [8] [2]), (7) that the single step procedure is convergent in the cyclic case when N_k runs periodically through all indices 1,2,...,n.

3. Gauss [2], [1], [4] proposed the following modification of the above procedure in order to speed up the convergence. Put

$$(9) \qquad x_\nu = z_\nu - z_0;\ a_{0\nu} = a_{\nu 0} = -\sum_{\mu=1}^{n} a_{\nu\mu} \qquad (\nu = 1,\dots,n)$$

$$a_{00} = -\sum_{\nu=1}^{n} a_{0\nu} = \sum_{\mu,\nu=1}^{n} a_{\mu\nu} ,$$

where z_0 can be arbitrarily chosen. Then the system (1) can be written in the form (assuming $y_\mu = 0$)

$$(10) \qquad \begin{aligned} \sum_{\nu=0}^{n} a_{0\nu} z_\nu &= 0 \\ \sum_{\nu=0}^{n} a_{\mu\nu} z_\nu &= 0 \qquad (\mu = 1,\dots,n) , \end{aligned}$$

where the first equation is, of course, not independent of the last n equations but is useful for the sake of uniformity and for checking purposes.

In particular a_{00} is positive since by (9) a_{00} is the value of the quadratic form (6) for $x_\nu = 1 (\nu = 1,\dots,n)$.

4. From a solution $(z_0, z_1,\dots,z_n)$ of the system (10) we obtain at once by (9) the solution $(x_1,\dots,x_n)$ of the system (1). The idea of Gauss is now to apply the procedure, described in (4) and (5), to the system (10). If we obtain then, starting from a vector

$\zeta_o = (z_o^{(0)}, z_1^{(0)}, \dots, z_n^{(0)})$ a sequence of the vectors

$$\zeta_k = (z_o^{(k)}, z_1^{(k)}, \dots, z_n^{(k)}) ,$$

we consider at the same time the corresponding vectors

$$\xi_k = (z_1^{(k)} - z_o^{(k)}, z_2^{(k)} - z_o^{(k)}, \dots, z_n^{(k)} - z_o^{(k)}) .$$

If then in the passage from ζ_k to ζ_{k+1} the leading index N_k is $\neq 0$ we have

$$\text{(11)} \qquad \sum_{\nu=0}^{n} a_{\mu\nu} z_\nu^{(k)} = \sum_{\nu=1}^{n} a_{\mu\nu} (z_\nu^{(k)} - z_o^{(k)}) = r_\mu^{(k)} \quad (\mu=1,\dots,n)$$

$$z_{N_k}^{(k+1)} = z_{N_k}^{(k)} - \frac{\sum_{\nu=0}^{n} a_{N_k\nu} z_\nu^{k}}{a_{N_k N_k}} = z_{N_k}^{(k)} - \frac{r_{N_k}^{(k)}}{a_{N_k N_k}}$$

$$z_\mu^{(k+1)} = z_\mu^{(k)} \qquad (k \neq N_k) .$$

Since here $z_o^{(k+1)} = z_o^{(k)}$, we see that the corresponding n-dimensional vectors ξ_k, ξ_{k+1} are connected exactly by the formulae (5), so that in this case there is no essential change compared with the original method.

5. If however $N_k = 0$, then only $z_o^{(k)}$ is changed and therefore <u>all components</u> $x_1^{(k)}, \dots, x_n^{(k)}$ are changed by the same amount. In this case we have obviously a new possibility and the question arises

whether in this case the convergence is indeed speeded up. Of course under the convergence in this case is not meant the convergence of the vectors ζ_k but the convergence of the corresponding vectors ξ_k. This question is apparently not as yet settled as widely contradictory opinions are to be found in the literature.

6. In what follows we will prove that in the case of the relaxation rule (8) the procedure remains convergent, and for $n > 2$ the convergence is speeded up by Gauss' transformation "in the statistical sense", if

$$\sqrt{\sum_{\mu,\nu=1}^{n} a_{\mu\nu}} < \sum_{\mu=1}^{n} \sqrt{a_{\mu\mu}} - \operatorname*{Min}_{\mu} \sqrt{a_{\mu\mu}}$$

and is not speeded up in the same sense if

$$\sqrt{\sum_{\mu,\nu=1}^{n} a_{\mu\nu}} \geqq \sum_{\mu=1}^{n} \sqrt{a_{\mu\mu}} - \operatorname*{Min}_{\mu} \sqrt{a_{\mu\mu}}$$

The probability that the decrease of $K(\xi_k)$ is greater for $N_k = 0$ than for $N_k > 0$ is always positive for some values λ of N_{k-1} in the first case and vanishes in the second case.

For $n = 2$ Gauss' device has no effect in the second case and speeds up (this time in the "absolute" sense)[3].

It is quite different in the case of the cyclic one step iteration. In this case we will prove that the procedure remains convergent, but for any $n \geqslant 2$ there exists matrices for which the modified procedure is slower and others for which the modified procedure

is indeed faster than the original one.[3])

7. In what follows we will say the two (n+1)-dimensional vectors

$\zeta = (z_0, z_1, \ldots, z_n)$ and $\zeta' = (z_0', z_1', \ldots, z_n')$ are <u>equivalent</u>

if we have $z_\nu - z_0 = z_\nu' - z_0'$. In the class of vectors equivalent to ζ there exists a <u>reduced</u> one $\hat{\zeta} = (0, x_1, \ldots, x_n)$ and the corresponding n dimensional vector $\xi = (x_1, \ldots, x_n)$ is uniquely determined.

We have for the component of the residual vector corresponding to the index 0

$$(11^0) \qquad r_0^{(k)} = \sum_{\nu=0}^{n} a_{0\nu} z_\nu^{(k)} = \sum_{\nu=1}^{n} a_{0\nu}\left(z_\nu^{(k)} - z_0^{(k)}\right) = -\sum_{\nu=1}^{n} r_\nu^{(k)} ,$$

and we see from (11) and (11°) that the residual vector for the system (10) does not depend on the component z_0 but only on the corresponding vector ξ. It follows from (9) and (6)

$$\sum_{\mu,\nu=0}^{n} a_{\mu\nu} z_\mu z_\nu = \sum_{\mu=0}^{n} z_\mu \sum_{\nu=0}^{n} a_{\mu\nu} z_\nu$$

$$= \sum_{\mu=0}^{n} z_\mu \left(\sum_{\nu=1}^{n} a_{\mu\nu} z_\nu + a_{\mu 0} z_0\right)$$

$$= \sum_{\mu=0}^{n} z_\mu \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = \sum_{\nu=1}^{n} x_\nu \sum_{\mu=0}^{n} a_{\mu\nu} z_\mu$$

$$= \sum_{\nu=1}^{n} x_\nu \left(\sum_{\mu=1}^{n} a_{\mu\nu} z_\mu + a_{0\nu} z_0\right)$$

$$= \sum_{\nu=1}^{n} x_\nu \sum_{\mu=1}^{n} a_{\mu\nu} x_\mu ,$$

(12) $$\sum_{\mu,\nu=0}^{n} a_{\mu\nu} z_{\mu} z_{\nu} = \sum_{\mu,\nu=1}^{n} a_{\mu\nu} x_{\mu} x_{\nu}.$$

8. It is obvious that the algebraic identity corresponding to (7) remains also true for the system (10), although the corresponding quadratic form is only <u>semi-definite</u>. Therefore and from (12) it follows that the relation (7) is also true for $N_k = 0$ where $r_0^{(k)}$ is given by (11), and the quadratic form K is the positive definite quadratic form (6). But then it follows

(13) $$\frac{r_{N_k}^{(k)2}}{a_{N_k N_k}} \to 0 \qquad (k \to \infty) \quad ,$$

9. If now the relaxation rule (8) is used, it follows obviously

$$\frac{r_{\mu}^{(k)2}}{a_{\mu\mu}} \to 0 \quad (k \to \infty;\ \mu = 0,1,\ldots,n)$$

$$r_{\mu}^{(k)} \to 0 \ (k \to \infty;\ \mu = 0,1,\ldots,n)$$

and therefore since $y_{\mu} = 0$ and the determinant in (4) does not vanish $x_{\mu}^{(k)} \to 0 \quad (k \to \infty;\ \mu = 1,\ldots,n)$, and we see that the modified procedure in the case of the relaxation rule (8) is always convergent.

10. The rate of the convergence in this case can be measured by the decreases of the quadratic form $K(\xi_k)$ at each step. But then obviously the convergence is each time speeded up in choosing $N_k = 0$ if we have

$$(14) \qquad \frac{r_o^{(k)2}}{a_{oo}} > \operatorname*{Max}_{\mu=1,\ldots,n} \frac{r_\mu^{(k)2}}{a_{\mu\mu}} \quad .$$

In estimating the probability of (14) we must of course assume that $r_o^{(k)} \neq 0$, that is to say, that N_{k-1} is not = 0. But then one of the $r_\mu^{(k)}$ (μ = 1,2,...n) must vanish. For n = 2 there remains only one $r_\mu^{(k)} \neq 0$. If for instance $N_{k-1} = 1$, $r_1^{(k)} = 0$, we have $r_2^{(k)} = - r_o^{(k)}$ and (14) is true if and only if $a_{oo} < a_{22}$. If $r_2^{(k)} = 0$, (14) is true if and only if $a_{oo} < a_{11}$. We see that in the case $a_{oo} > \text{Max}\ (a_{11}, a_{22})$, (14) never occurs and Gauss' device has no effect at all.[3]

If however $a_{oo} < \text{Max}\ (a_{11}, a_{22})$, the procedure is indeed speeded up.[3] (and becomes cyclic if an additional convention is used for $a_{11} = a_{22}$).

Suppose now $n > 2$. In order to estimate the probability of (14) put

$$\text{Max}\ \frac{r_\mu^2}{a_{\mu\mu}} = m^2$$

and ask for the probability of the inequality

$$\frac{\left(\sum_{\mu=1}^{n} r_\mu\right)^2}{a_{oo}} > m^2$$

if one of the $r_\mu (\mu = 1,\ldots,n)$ is = 0. If we put

(15) $$\alpha_\mu = \sqrt{a_{\mu\mu}}\, m, \alpha_o = \sqrt{a_{oo}}\, m ,$$

we have to determine the probability of the inequality

(16) $$\sum_{\mu=1}^{n} r_\mu > \alpha_o ,$$

in assuming that the r_μ are independent variables, uniformly distributed conformally to the condition

(17) $$\operatorname*{Max}_{\mu=1,..,n} \frac{|r_\mu|}{\alpha_\mu} = 1; \quad \alpha_\mu > 0 \ (\mu = 1,\ldots,n;\ \mu \neq \lambda); \alpha_\lambda = 0,$$

where $\lambda = N_{k-1}$.

11. In a previous paper [6] we have proved that the probability $F^*(\sigma)$ of the inequality $r_1 +,,,+r_N < \sigma$ under the condition

$\operatorname*{Max}_{\mu=1,..,N} \frac{r_\mu}{\alpha_\mu} = 1$, α_μ $0(\mu = 1,\ldots,N)$ for uniformly distributed r_μ is given by the formula

$$F^*(\sigma) = \frac{1}{(N-1)!\,2^N} \frac{1}{\alpha_1 \cdots \alpha_N \left(\frac{1}{\alpha_1}+\ldots+\frac{1}{\alpha_N}\right)} \sum_{\mu=1}^{N} \frac{1+S^{2\alpha_\mu}}{1-S^{2\alpha_\mu}} \prod_{\nu=1}^{N} \left(1-S^{2\alpha_\nu}\right)(\alpha+\sigma)_+^{N-1}$$

$$\alpha = \alpha_1 + \ldots + \alpha_N$$

and is a strictly monotonically increasing function of σ as long as we have

(18) $$-\sum_{\nu=1}^{N} \alpha_\nu \leqq \sigma \leqq \sum_{\nu=1}^{N} \alpha_\nu \quad .$$

The symbolism used in the formula (18) is the following; we denote by k_+

(19) $$k_+ = \begin{cases} k & k \geqq 0 \\ 0 & k < 0 \end{cases}$$

and by S^{η} the displacement operator defined by

(20) $$S^{\eta} f(\sigma) = f(\sigma - \eta)$$

The probability of (16) under the conditions (17) is therefore

(21) $$2(1 - F(\alpha_0)) ,$$

where

(22) $$F(\sigma) = \frac{2^{-n+1}}{(n-2)!} \frac{1}{\alpha_1 \cdots \alpha_n} \frac{1}{\sum_{\nu=1}^{n} \frac{1}{\alpha_\nu \alpha_\lambda} - \frac{1}{\alpha_\lambda^2}} \sum_{\substack{\mu=1 \\ \mu \neq \lambda}}^{n} \frac{1 + S^{2\alpha_\mu}}{(1 - S^{2\alpha_\mu})(1 - S^{2\alpha_\lambda})} \prod_{\nu=1}^{n} (1 - S^{2\alpha_\nu}) (\alpha - \alpha_\lambda + \sigma)_+^{n-2}$$

for $\sigma = \alpha$. In the case n = 3, (22) becomes for $\lambda = 3$, for instance,

$$\frac{1}{4} \frac{1}{\alpha - \alpha_3} \left[(1 + S^{2\alpha_1})(1 - S^{2\alpha_2}) + (1 + S^{2\alpha_2})(1 - S^{2\alpha_1}) \right] (\alpha - \alpha_3 + \sigma)_+ = \frac{1}{2} \frac{1}{\alpha - \alpha_3} (1 - S^{2(\alpha - \alpha_3)}) (\alpha - \alpha_3 + \sigma)_+$$

and more generally for any λ

(23) $$\frac{1}{2} \frac{1}{\alpha - \alpha_\lambda} \left[(\sigma + \alpha - \alpha_\lambda)_+ - (\sigma - \alpha + \alpha_\lambda)_+ \right].$$

12. (21) is positive if and only if $\alpha_0 < \alpha - \alpha_\lambda$. It is easy to prove that

$$\alpha_0 < \sum_{\mu=1}^{n} \alpha_\mu = \alpha \quad ,$$

that is

(24)
$$\sqrt{a_{00}} < \sum_{\mu=1}^{n} \sqrt{a_{\mu\mu}} \quad .$$

Indeed (24) follows immediately from

$$|a_{00}| \leq \sum_{\mu=1}^{n} |a_{\mu\mu}| + \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{n} |a_{\mu\nu}| < \sum_{\mu=1}^{n} a_{\mu\mu} + \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{n} \sqrt{a_{\mu\mu}}\sqrt{a_{\nu\nu}} = \left(\sum_{\mu=1}^{n} \sqrt{a_{\mu\mu}}\right)^2 ,$$

since we have for $\mu \neq \nu$

$$a_{\mu\mu}\, a_{\nu\nu} - a_{\mu\nu}^{\ 2} > 0 \qquad (\mu \neq \nu).$$

On the other hand the inequality $\alpha_0 < \alpha - \alpha_\lambda$ is not necessarily satisfied even for one λ only. Consider indeed the symmetric determinant

$$\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} = (x-1)^2\,(2x+1) \quad ,$$

the condition that the corresponding quadratic form is positive definite is obviously $-\frac{1}{2} < x < 1$ and we have here

$$\alpha_1 = \alpha_2 = \alpha_3 = 1 \, , \quad a_0 = 3 + 6x \, , \quad \alpha_0 = \sqrt{3+6x} \quad ,$$

and we have here $\alpha_0 < \alpha - \alpha_\lambda$ if $x < \frac{1}{6}$ and $\alpha_0 > \alpha - \alpha_\lambda$ if $x > \frac{1}{6}$. In the general case (21) is positive for all λ if we have

(25,a) $$\alpha_0 < \sum_{\nu=1}^{n} \alpha_\nu - \operatorname*{Max}_\nu \alpha_\nu$$

and (21) vanishes for all λ if we have

(25,b) $$\alpha_0 \geqq \sum_{\nu=1}^{n} \alpha_\nu - \operatorname*{Min}_\nu \alpha_\nu \quad .$$

14. In the example considered by Gauss the matrix of the equations (1) is

$$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

and we have with m = 1

$$\alpha_0 = 2, \alpha_1 = \sqrt{3}, \alpha_2 = 2, \alpha_3 = \sqrt{3}, \alpha = 2 + 2\sqrt{3},$$

and we have for every λ: $\alpha_0 < \alpha - \alpha_\lambda$ and the probability (21) becomes

$$1 - \frac{\alpha_0}{\alpha - \alpha_\lambda} = 1 - \frac{2}{2 + 2\sqrt{3} - \alpha_\lambda} \quad .$$

This is .46410, .42265, .46410 according as $\lambda = 1,2,3$. The probability for $N_k = 0$ is therefore in this case $> .42265$, that is fairly great, due to the $a_{\mu\nu}$ ($\mu \neq \nu$) being negative.

15. We consider now the cyclic case. Here it follows from (13)

$$r_{N_k}^{(k)} \to 0 \qquad (k \to \infty)$$

and therefore from (5) and the corresponding formula for $\mathfrak{Z}_k$ and $N_k = 0$.

$$x_\mu^{(k+1)} - x_\mu^{(k)} \to 0 \qquad (k \to \infty;\ \mu = 1,\ldots,n) ,$$

and therefore by (4)

$$r_\mu^{(k+1)} - r_\mu^{(k)} \to 0. \qquad (k \to \infty;\ \mu = 1,\ldots,n).$$

Or more generally for each constant integer γ

$$r_\mu^{(k+\gamma)} - r_\mu^{(k)} \to 0 \qquad (k \to \infty;\ \mu = 1,\ldots,n).$$

But for any fixed μ, among n+1 consecutive values of k there is one for which $N_k = \mu$, therefore it follows that

$$r_\mu^{(k)} \to 0 \qquad (k \to \infty;\ \mu = 1,\ldots,n),$$

and, since the determinant in (4) does not vanish,

$$x_\mu^{(k)} \to 0 \qquad (k \to \infty;\ \mu = 1,\ldots,n) .$$

We see that the modified procedure is indeed convergent.

16. In comparing the rate of convergence of the original and the modified cyclic single step iteration it is better to change our notations in the following way. If we start with a vector ξ_0 and apply the complete n-cycle of single steps corresponding to $N_k = 1,\ldots,n$, the obtained vector will be denoted by ξ_1 and the vectors obtained in repeating each time the complete n cycle will be denoted by $\xi_2, \xi_3, \ldots$.

In the same way, in the modified cyclic procedure we obtain, starting from a vector ζ_0 and applying each time the whole (n+1) cycle corresponding to $N_k = 0,1,\ldots,n$ the sequence of vectors $\zeta_1, \zeta_2, \ldots$.

17. The rate of convergence of the usual cyclic single step iteration depends on the maximum modulus λ_N of the roots of the equation

$$(26) \qquad N(\lambda) \equiv \begin{vmatrix} \lambda a_{11} & a_{12} & \cdots & a_{1\,n-1} & a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & a_{2\,n-1} & a_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \lambda a_{n-1\,1} & \lambda a_{n-1\,2} & \cdots & \lambda a_{n-1\,n-1} & a_{n-1\,n} \\ \lambda a_{n1} & \lambda a_{n2} & \cdots & \lambda a_{n\,n-1} & \lambda a_{nn} \end{vmatrix} = 0$$

We have then, if $\lambda_N > 0$

$$(27) \qquad \xi_k = O(\lambda_N^k \, k^{n-1}) \qquad (k \to \infty)$$

while the starting vector ξ_0 can be chosen so that $\xi_k \lambda_N^{-k}$ does not tend to 0 with $k \to \infty$.[4]

18. We will now characterize in a similar way the rate of convergence of the modified cyclic single step iteration.

We decompose A in the following way

$$A = L + D + L^* , \tag{28}$$

where D is the diagonal matrix with the elements $a_{11}, \ldots, a_{nn}$, while in L all elements to the right and on the main diagonal and in L^* all elements to the left and on the main diagonal vanish. We have then for the matrix $\hat{A}$ of the system (10) the corresponding decomposition

$$\hat{A} \equiv \begin{pmatrix} a_{oo} & a_{o\nu} \\ a_{\nu o} & A \end{pmatrix} = \hat{L} + \hat{D} + \hat{L}^* . \tag{29}$$

Then we have between ζ_o and ζ_1, as in the theory of the usual cyclic one step iteration, the relations

$$(\hat{L} + \hat{D})\, \zeta_1 + \hat{L}^* \zeta_o = 0 ,$$

$$\zeta_1 = - (\hat{L} + \hat{D})^{-1} \hat{L}^* \zeta_o . \tag{30}$$

19. As has been mentioned above the result of this operation is not changed if ζ_o is replaced by the corresponding reduced vector

$\hat{\mathfrak{Z}}_o = (0, x_1^{(0)}, \ldots, x_n^{(0)})$. Before we go on from $\mathfrak{Z}_1$, we replace therefore $\mathfrak{Z}_1$ again by the corresponding reduced vector $\hat{\mathfrak{Z}}_1 = (0, x_1^{(1)}, \ldots, x_n^{(1)})$. For this purpose we apply the transformation $x_\mu = z_\mu - z_o$ $(\mu = 1, \ldots, n)$ which is equivalent to the multiplication by the matrix

$$(31) \qquad N_o = \begin{pmatrix} 0 & 0. & . & . & . & .0 & 0 \\ -1 & 1. & . & . & . & .0 & 0 \\ & . & . & . & . & . & \\ -1 & 0. & . & . & . & .1 & 0 \\ -1 & 0. & . & . & . & .0 & 1 \end{pmatrix} .$$

We have then finally in putting

$$(32) \qquad Q_o = -N_o \, (\hat{L} + \hat{D})^{-1} \, \hat{L}^* ,$$

$$(33) \qquad \hat{\mathfrak{Z}}_k = Q_o^{\,k} \, \hat{\mathfrak{Z}}_o \qquad (k = 1, 2, \ldots) .$$

20. We use now the following result due to Werner Gautschi [3]. If for any matrix $c = (c_{\mu\nu})$ we define as its "norm"

$$N(c) = \sqrt{\sum_{\mu,\nu} |c_{\mu\nu}|^2}$$

and if B is a square matrix of the order n for which the greatest modulus of the fundamental root is Λ then we have

(34) $$N(B^k) = O(\Lambda^k k^{p-1}) \qquad (k \to \infty),$$

where p is the greatest multiplicity of a fundamental root of B with the modulus Λ.

21. If we apply this to the singular matrix (32) and denote the maximal modulus of a fundamental root of Q_o by λ_g, the maximal multiplicity of a root with modulus λ_g is $\leqq n-1$ if $\lambda_g > 0$, as will follow later from (42). We have therefore

(35) $$N(Q_o^k) = O(\lambda_g^k k^{n-2}) (k \to \infty).$$

On the other hand it follows from (33) in applying Cauchy-Schwartz inequality

$$|\hat{\zeta}_k| \leqq N(Q_o^k) |\hat{\zeta}_o| \qquad ,$$

and we obtain therefore

(36) $$\hat{\zeta}_k = O(\lambda_g^k k^{n-1}) \qquad (k \to \infty).$$

22. On the other hand it is easy to show that for a conveniently chosen starting vector $\hat{\zeta}_o$ the expression $\hat{\zeta}_k \lambda_g^{-k}$ does not tend to zero. Indeed if η is an eigenvector of Q_o corresponding to a fundamental root λ with $|\lambda| = \lambda_g$, we have

$$\lambda \eta = Q_o \eta$$

and iterating

$$\lambda^k \eta = Q_o{}^k \eta \qquad .$$

But, since the first row in Q_o consists of zeros, the vector η is a reduced one and can be taken as $\hat{\zeta}_o$. Then we have

$$\hat{\zeta}_k = \lambda^k \hat{\zeta}_o, \quad \hat{\zeta}_k \lambda_g{}^k = \left(\frac{\lambda}{\lambda_g}\right)^k \hat{\zeta}_o$$

and this does not tend to zero with $k \to \infty$.)[5]

23. We are going now to transform the fundamental equation of Q_o and introduce for this purpose the matrix

$$(37) \qquad N_\epsilon = \begin{pmatrix} \epsilon & 0 & 0 . \ . \ . 0 & 0 \\ -1 & 1 & 0 . \ . \ . 0 & 0 \\ -1 & 0 & 1 . \ . \ . 0 & 0 \\ & . & . \ . \ . \ . \ . & \\ -1 & 0 & 0 \quad\ \ 1 & 0 \\ -1 & 0 & 0 . \ . \ . 0 & 1 \end{pmatrix},$$

which corresponds to the transformation

$$(38) \qquad y_o = \epsilon z_o, \quad y_\nu = z_\nu - z_o \qquad (\nu > 0)$$

and goes for $\epsilon \to 0$ into N_o. Since the inverse of (38) is for $\epsilon \neq 0$:

$$z_o = \frac{1}{\epsilon} y_o, \quad z_\nu = y_\nu + \frac{1}{\epsilon} y_o \qquad (\nu > 0)$$

we have

$$(39)\quad N_\epsilon^{-1} = \begin{pmatrix} \frac{1}{\epsilon} & 0. & . & .0 & 0 \\ \frac{1}{\epsilon} & 1. & . & .0 & 0 \\ & . & . & . & . \\ \frac{1}{\epsilon} & 0. & . & .1 & 0 \\ \frac{1}{\epsilon} & 0. & . & .0 & 1 \end{pmatrix} .$$

The fundamental equation of Q_0 can be written in the form

$$(40)\qquad \lim_{\epsilon \to 0} \left| \lambda E + N_\epsilon (\hat{L} + \hat{D})^{-1} \hat{L}^* \right| = 0 .$$

24. On the other hand we have identically since $|N_\epsilon| = \epsilon$

$$(41)\quad \left(|\hat{L} + \hat{D}| \right)\left(|\lambda E + N_\epsilon (\hat{L} + \hat{D})^{-1} \hat{L}^*| \right) = \epsilon \left| \lambda (\hat{L}+\hat{D}) N_\epsilon^{-1} + \hat{L}^* \right|$$

and obtain therefore the fundamental equation of Q in taking the limit for $\epsilon \to 0$ on the right in (41).

Now we have

$$(\hat{L}+\hat{D})N_\epsilon^{-1} = \begin{pmatrix} \frac{1}{\epsilon} a_{00} & 0 & 0 & . & . & 0 & 0 \\ \frac{1}{\epsilon}(a_{10}+a_{11}) & a_{11} & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ \frac{1}{\epsilon}(a_{n-1\,0}+..+a_{n-1\,n-1}) & a_{n-1\,1} & a_{n-1\,2} & . & . & a_{n-1\,n-1} & 0 \\ \frac{1}{\epsilon}(a_{n0}+..+a_{nn}) & a_{n1} & a_{n2} & . & . & a_{n\,n-1} & a_{nn} \end{pmatrix} ,$$

where $a_{n0} + a_{n1} + \dots + a_{nn} = 0$ by (9) and we have therefore identically

$$\epsilon\left|\lambda(\hat{L}+\hat{D})N_\epsilon^{-1}+\hat{L}^*\right| = \lambda^2 \begin{vmatrix} a_{00} & a_{01} & a_{02} & \cdot & \cdot & a_{0n-1} & a_{0n} \\ a_{10}+a_{11} & \lambda a_{11} & a_{12} & \cdot & \cdot & a_{1n-1} & a_{1n} \\ a_{20}+a_{21}+a_{22} & \lambda a_{21} & \lambda a_{22} & \cdot & \cdot & a_{2n-1} & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-10}+\dots+a_{n-1n-1} & \lambda a_{n-11} & \lambda a_{n-12} & \cdot & \cdot & \lambda a_{n-1n-1} & a_{n-1n} \\ 0 & a_{n1} & a_{n2} & \cdot & \cdot & a_{nn-1} & a_{nn} \end{vmatrix}$$

λ_G is therefore the maximum modulus of the fundamental roots of the equation

(42) $$G(\lambda) \equiv \begin{vmatrix} a_{00} & a_{01} & \cdot & \cdot & a_{0n-1} & a_{0n} \\ a_{10}+a_{11} & \lambda a_{11} & \cdot & \cdot & a_{1n-1} & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-10}+\dots+a_{n-1n-1} & \lambda a_{n-11} & \cdot & \cdot & \lambda a_{n-1n-1} & a_{n-1n} \\ 0 & a_{n1} & \cdot & \cdot & a_{nn-1} & a_{nn} \end{vmatrix} = 0$$

25. In specializing for $n = 2$ we obtain in particular, if we put

$a_{11} = a_1$, $a_{22} = a_2$, $a_{12} = a_{21} = \sigma$ and assume $\sigma \neq 0$ for

$G_2(\lambda)$:

$$G_2(\lambda) = \begin{vmatrix} a_1 + a_2 + 2\sigma & -a_1 - \sigma & -a_2 - \sigma \\ -\sigma & \lambda a_1 & \sigma \\ 0 & \sigma & a_2 \end{vmatrix} =$$

$$= (a_1 + a_2 + 2\sigma) a_2 a_1 \lambda - \sigma (a_1 + \sigma)(a_2 + \sigma),$$

(43) $$\lambda_G = \frac{|\sigma (a_1 + \sigma)(a_2 + \sigma)|}{a_1 a_2 (a_1 + a_2 + 2\sigma)},$$

while the equation for λ_N reduces to

$$N_2(\lambda) = \begin{vmatrix} \lambda a_1 & \sigma \\ \lambda \sigma & \lambda a_2 \end{vmatrix} = 0$$

and gives

(44) $$\lambda_N = \frac{\sigma^2}{a_1 a_2}.$$

From (43) and (44) we have

(45) $$\frac{\lambda_G}{\lambda_N} = \frac{|a_1 + \sigma| \, |a_2 + \sigma|}{|\sigma| (a_1 + a_2 + 2\sigma)}.$$

26. If we square this, subtract 1 and multiply by the square of the denominator we obtain

$$[(a_1+\sigma)(a_2+\sigma)-\sigma(a_1+a_2)-2\sigma^2][(a_1+\sigma)(a_2+\sigma)+\sigma(a_1+a_2)+2\sigma^2] =$$

$$= (a_1a_2-\sigma^2)(a_1a_2-2(a_1+a_2)\sigma+3\sigma^2).$$

Since the first factor is positive, we see that $\lambda_G \gtreqless \lambda_N$ according as the second factor is $\gtreqless 0$, but this factor is $=(a_1+2\sigma)(a_2+2\sigma)-\sigma^2$ and we see that $\lambda_G \gtreqless \lambda_N$ according as

$$(46)\qquad (a_1 + 2\sigma)\ (a_2 + 2\sigma) \gtreqless \sigma^2 .$$

Here $\sigma \neq 0$ is subject only to the condition $\sigma^2 < a_1 a_2$

In particular for $\sigma > 0$ we have always $\lambda_G > \lambda_N$. We see that for $n = 2$, λ_G can be as well $> \lambda_N$ as $< \lambda_N$.[3])

27. To prove the corresponding result for $n > 2$ consider the matrix A corresponding to the quadratic form

$$(47)\qquad K(\xi) = a_1 x_1^2 + 2\sigma x_1x_2 + a_2x_2^2 + \sum_{\mu=3}^{n} x_\mu^2 .$$

In the corresponding determinant (42) for $G(\lambda)$ the elements in the first column are

$$a_{\mu 0} + a_{\mu 1} + \ldots + a_{\mu\mu} = -(a_{\mu\,\mu+1} + \ldots + a_{\mu n})$$

and vanish therefore for $\mu \geqq 2$. The same is true for the elements

to the left of the main diagonal $\lambda a_{\mu\nu}$ with $\mu > 2$ and $\nu < \mu$ and the elements $a_{n\nu}$ $(\nu < n)$, while the elements on the diagonal $\lambda a_{\mu\mu}$ $(\mu > 2)$ and a_{nn} become respectively λ and 1. We obtain therefore

$$G(\lambda) = \lambda^{n-2} G_2(\lambda)$$

so that λ_G is in this case given also by (43).

28. In the same way it follows from (26) that N_λ in our case is equal to $\lambda^{n-1} N_2(\lambda)$ and therefore λ_N is given by (44). We can have therefore in this case according to the chosen values of σ as well $\lambda_G > \lambda_N$ as well $\lambda_G < \lambda_N$.

It may be finally remarked that the value of λ_G is not changed if the (n+1) equation in (10) and the corresponding new variable z_o are not put at the beginning but interpolated between two indices μ, $\mu + 1$ or even put at the end. Indeed this amounts to the old process applied to a transformation of $\hat{\zeta}_o$ by a finite sequence of single step iterations, but then $\hat{\zeta}_o$ is carried over in the <u>general</u> reduced vector and the invariency of λ_G follows then from the characterization of λ_G contained in the development of numbers 21 and 22.

Footnotes

1. This special rule goes back to F. R. Helmert (1872) [5]. The relaxation rule indicated previously by Gauss [2] and Gerling [4] is different as well as that proposed by Southwell [10], but the rule (8) is apparently the most advantageous one.

2. The same result was proved, 1949, by E. Reich [7] independently and with a completely different method. 4. The proof of it is quite similar as in what follows the proof of the corresponding results for the modified single step iterations, (see (35)).

   If $\lambda_N = 0$, then already ζ_n vanishes identically and the solution is obtained at the most (in) n steps.

5. If $\lambda_G = 0$, then already ζ_{n+1} vanishes identically. 3. This agrees with the results mentioned in the paper [1A] of Forsythe and Motzkin, footnote 24.

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