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ON NEARLY TRIANGULAR MATRICES

By

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Introduction

Consider a system of linear equations

\[ \sum_{\nu=1}^{n} a_{\mu\nu} x_{\nu} = y_{\mu} \quad (\mu = 1, \ldots, n) \]

with the matrix A, in which all diagonal elements \( a_{\mu\mu} \) (\( \mu = 1, \ldots, n \)) are \( \neq 0 \) and the elements off the diagonal satisfy for two positive numbers \( m, M \) the inequalities

\[ |a_{\mu\nu}| \leq m |a_{\mu\mu}| \quad (\nu < \mu ; \mu = 1, \ldots, n) \]
\[ |a_{\mu\nu}| \leq M |a_{\mu\mu}| \quad (\nu > \mu ; \mu = 1, \ldots, n) . \]

If \( n \) is very small, the system does not essentially differ from the corresponding "triangular" system in which all \( a_{\mu\nu} \) with \( \nu < \mu \) are replaced by zeros and the matrix of which will be denoted by \( A^{(0)} \). It appears then plausible that the solutions of this "triangular" system does not differ very much from that of the system (1).

However the value of the determinant of the order n
shows that if \( M \) is for instance \( \geq 10 \), the determinant of our system will not be even necessarily different from zero unless \( m < 10^{-(n-1)} \). A detailed study of the problems connected with the matrices characterized by (2) appears therefore to be of importance and interest.

As the first problem in this connection we give a necessary and sufficient condition for the determinants of all matrices \( A \) satisfying the conditions (2) being non-singular. This condition is, if \( m < M \), given by

\[
\frac{m}{(1+m)^n} < \frac{M}{(1+M)^n}
\]

and, if \( m = M \), by

\[
m < \left( \frac{1}{n-1} \right)^{1/(n-1)}
\]

In order to obtain a precise measure of the influence of the change from \( A \) to \( A^{(0)} \), we have to discuss the estimates for convenient norms of the matrix \( A^{-1} - A^{(0)-1} \).

We consider in particular two such norms defined in the section IV and denoted by \( \| A^{-1} - A^{(0)-1} \|_p \) (\( p = 1, \infty \)), which are particularly suitable for the problems of Numerical Analysis. We show then, in assuming without loss of generality that \( a_{\mu \mu} = 1 \) (\( \mu = 1, \ldots, n \)),
that for given values of \( m, M \) we have, if \( M \geq \frac{1.5}{n} \), \( n \geq 4 
\)

\[
|A^{-1} - A^{(m)}^{-1}|_p = \left(1 + M\right)^{n-1} \frac{\delta}{1 - \delta} \quad (M \geq \frac{1.5}{n}, \, n \geq 4)
\]

where \( 1 - \delta \) is the smallest modulus of the determinant attainable for the matrices \( A \) and is connected with \( m \) by the relation

\[
\delta = \frac{M_n m}{1 - \varepsilon m} , \quad 0 < \varepsilon < 1
\]

where

\[
M_n = \frac{(1+M)^{n} - n - M - 1}{M}
\]

If \( M < \frac{1.5}{n} \), the formula (4) need not be valid any longer, but we can prove in this case the relation

\[
|A^{-1} - A^{(m)}^{-1}|_p \leq \frac{6n m}{1 - 2nm} \quad (M \leq 1.5/n)
\]

valid as long as \( m \) remains less than \( \frac{1}{2n} \).

The estimate (7) is not a "best" estimate for all values of \( M \leq \frac{1.5}{n} \) but still it is not very far from the best, since for \( m = M \leq \frac{1}{n-1} \) we have

\[
|A^{-1} - A^{(m)}^{-1}|_p \leq \frac{(n-1)m}{1 - (n-1)m} \quad (m = M < \frac{1}{n-1})
\]
The condition (3) is derived in the section III of this paper, theorem B. However, we derive it as a special case of a more general theorem concerning the case where in the inequalities (2) the expressions \( m, M \) depend on \( \mu \), that is to say, change from one row to another. The necessary and sufficient condition of all matrices \( A \) being regular (theorem A, section III) is in this case rather unwieldy, but may be still very useful in some cases, since it contains \( 2n-2 \) instead of two essential parameters. The direct derivation of theorem B is of course much simpler since, as the reader will immediately see, the computations of the determinant \( \Omega_n \) in the section I can be considerable shortened in this case. The connection between the formal algebra of the section I and II and the theorems A and B is provided by a result concerning the so-called H-determinants and M-determinants which I published 15 years ago\(^{[4]} \). The results about the norms \[ |A^{-1} - A^{(o)}^{-1}|_p \] are obtained in using the explicit representation of the inverse matrix of a certain matrix \( \Delta_n \) which provides a majorant for all matrices \( A^{-1} \). The formulae giving \( \Delta_n^{-1} \) are derived in the second part of the section II, and in the section V the norms \[ |\Delta_n^{-1} - \Delta_n^{(o)}^{-1}|_p \] are derived and discussed. The corresponding inequalities for \[ |A^{-1} - A^{(o)}^{-1}|_p \] are then obtained in the section VII, in using a new theorem (lemma III) concerning the connection between the H-determinants and the M-determinants.
We give in this section another application of this theorem in estimating the variation in the inverse matrix of a triangular matrix satisfying the conditions (2) with \( m = 0 \). We obtain an unexpected simple and elegant formula (VI,33).

In the section VII we apply our results explicitly to the problems concerning the linear system (1). It may be finally remarked that our results remain with obvious changes valid if in the matrix
of (1) the rows and the columns are interchanged, although we did not care to mention it explicitly at every step 2).

I. The Value of the Determinant \( \Omega_n \)

Let \( K_n \) be defined by

\[
K_n = \begin{vmatrix}
K_n & S_n & 0 & 0 & \cdots & 0 & 0 \\
K_{n-1} & K_{n-1} & S_{n-1} & 0 & \cdots & 0 & 0 \\
K_{n-2} & K_{n-2} & K_{n-2} & S_{n-2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
K_2 & K_2 & K_2 & K_2 & \cdots & K_2 & S_a \\
K_1 & K_1 & K_1 & K_1 & \cdots & K_1 & K_1 \\
\end{vmatrix}
\]

\[ (I,1) \]

where in the \( \mu \)-th row all elements to the left of the main diagonal and on this diagonal are equal to \( K_{n-\mu+1} \), the next element to the right is \( S_{n-\mu+1} \) and all other elements are 0. \( K_1, S_a \) are here independent variables. In subtracting the second column from the first we obtain \( K_n = (K_n - S_n)K_{n-1} \) and therefore the following formula, valid also for \( n = 1, 2 \):

\[
K_n = K_1 \prod_{\nu=2}^{n} (K_\nu - S_\nu).
\]

\[ (I,2) \]
Consider now for $n \geq 3$ the determinant

$$T_n = \begin{vmatrix}
\delta_n & 0 & 0 & \ldots & 0 & \theta_n \\
\eta_{n-1} & \delta_{n-1} & 0 & \ldots & 0 & \theta_{n-1} \\
\eta_{n-2} & \eta_{n-2} & \delta_{n-2} & \ldots & 0 & \theta_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\eta_2 & \eta_2 & \eta_2 & \eta_2 & \delta_2 & \theta_2 \\
1 & 1 & 1 & 1 & \ldots & 1 & 0
\end{vmatrix} \quad (n \geq 3).$$
We have in particular

\[(I,4) \quad T_3 \begin{bmatrix} \delta_3 & 0 & \beta_3 \\ \gamma_2 & s_2 & \beta_2 \\ 1 & 1 & 0 \end{bmatrix} = - (s_3 \beta_2 + \beta_3 (s_2 - \gamma_2)).\]

Developing $T_n$ in the elements of the first line and using the value \((I,2)\) of $K_n$ we obtain for $n \geq 4$

\[
T_n = s_n T_{n-1} - \beta_n \prod_{\nu=2}^{n-1} (s_\nu - \gamma_\nu),
\]

\[
\frac{T_n}{s_2 \cdots s_n} = \frac{T_{n-1}}{s_2 \cdots s_{n-1}} - \frac{\beta_n}{s_n} \prod_{\nu=2}^{n-1} \left(1 - \frac{\gamma_\nu}{s_\nu}\right),
\]

and therefore generally for $n \geq 4$

\[(I,5) \quad \frac{T_n}{s_2 \cdots s_n} = - \left[ \frac{\beta_n}{s_n} \prod_{\nu=2}^{n-1} \left(1 - \frac{\gamma_\nu}{s_\nu}\right) + \frac{\beta_{n-1}}{s_{n-1}} \prod_{\nu=2}^{n-2} \left(1 - \frac{\gamma_\nu}{s_\nu}\right) + \cdots + \frac{\beta_2}{s_2} \prod_{\nu=2}^{3} \left(1 - \frac{\gamma_\nu}{s_\nu}\right) \right] + \frac{T_3}{s_3}.
\]

Since by \((I,4)\)

\[
\frac{T_3}{s_3 s_2} = - \frac{\beta_3}{s_3} \left(1 - \frac{\gamma_2}{s_2}\right) - \frac{\beta_2}{s_2},
\]

we obtain

\[
\frac{T_n}{s_2 \cdots s_n} = \sum_{K=2}^{n} \frac{\beta_K}{s_K} \prod_{\nu=2}^{K-1} \left(1 - \frac{\gamma_\nu}{s_\nu}\right),
\]
where $\mathbf{A}_{\nu=2}^{\nu}$ is identically $1$, and therefore finally

\[
T_n = -\sum_{\mu=2}^{n} \beta_{\mu} \prod_{\nu=\mu+1}^{n} S_{\nu} \prod_{\nu=2}^{\nu} (s_{\nu} - \chi_{\nu}) .
\]

If we now put

\[
T_n^* = \begin{bmatrix}
S_1 & 0 & 0 & \ldots & 0 & \beta_1 \\
\gamma_2 & S_2 & 0 & \ldots & 0 & \beta_2 \\
& & \ddots & \ddots & \ddots & \ddots \\
\gamma_{n-1} & \gamma_{n-1} & \gamma_{n-1} & \ldots & \gamma_{n-1} & \beta_{n-1} \\
1 & 1 & 1 & \ldots & 1 & 0
\end{bmatrix}_{(n \neq 3)}
\]

\[
T_{3}^* = \begin{bmatrix}
S_1 & 0 & \beta_1 \\
\gamma_2 & S_2 & \beta_2 \\
1 & 1 & 0
\end{bmatrix}
\]

this goes over into $T_n$ if the indices of $\beta_\nu$, $\chi_\nu$, $S_\nu$ are replaced by their complements with respect to $n + 1$. We obtain then from (I,6)

\[
T_n^* = -\sum_{\mu=2}^{n} \beta_{n+1-\mu} \prod_{\nu=\mu+1}^{n} S_{n+1-\nu} \prod_{\nu=2}^{\nu} (s_{n+1-\nu} - \chi_{n+1-\nu}) ;
\]

or in replacing the summation index $\mu$ by $n + 1 - \lambda$

\[
T_n^* = \sum_{\lambda=2}^{n-1} \beta_{\lambda} \prod_{\nu=\lambda+1}^{n-1} S_{\lambda+1-\nu} \prod_{\nu=2}^{\nu} (s_{\lambda+1-\nu} - \chi_{\lambda+1-\nu}) ;
\]

and finally, if in both products $\nu$ is replaced by $n + 1 - \lambda$

\[
T_n^* = -\sum_{\lambda=1}^{n-2} \beta_{\lambda} \prod_{\nu=\lambda+1}^{k-1} S_{\lambda+1} \prod_{\lambda=2}^{\lambda+1} (s_{\lambda} - \chi_{\lambda}) .
\]
Consider now for $n \geq 3$ the determinant

$$
\begin{vmatrix}
\alpha_1 & -M_1 & -M_2 & \cdots & -M_2 \\
-m_1 & \alpha_1 & -M_2 & \cdots & -M_2 \\
-m_2 & -m_1 & \alpha_2 & \cdots & -M_3 \\
& & \ddots & \ddots & \ddots \\
-m_{n-1} & -m_{n-1} & -m_n & \cdots & \alpha_n \\
\end{vmatrix}
$$

(I,9) $\Omega_n = \begin{vmatrix}
\end{vmatrix}$

If we subtract here the last column from each of the proceeding ones we obtain

$$
\begin{vmatrix}
M_1 + \alpha_1 & 0 & 0 & \cdots & 0 - M_1 \\
M_2 - m_2 & M_1 + \alpha_2 & 0 & \cdots & 0 - M_2 \\
M_3 - m_3 & M_2 - m_2 & M_2 + \alpha_3 & \cdots & 0 - M_3 \\
& & \ddots & \ddots & \ddots \\
M_{n-1} - m_{n-1} & M_{n-2} - m_{n-2} & \cdots & M_{n-1} - m_{n-1} & 0 - M_{n-1} \\
-(\alpha_n + m_n) & -(\alpha_n + m_n) & -(\alpha_n + m_n) & \cdots & \alpha_n \\
\end{vmatrix}
$$

Here the subdeterminant corresponding to the last element of the last row is obviously $\prod_{j=1}^{n-1} (M_j + \alpha_j)$, so that $\Omega_n$ is the sum of $\alpha_n \prod_{j=1}^{n-1} (M_j + \alpha_j)$ and
\[
\begin{bmatrix}
M_1 + \alpha_2 & 0 & 0 & \cdots & 0 & -M_4 \\
M_2 - m_2 & M_2 + \alpha_2 & 0 & \cdots & 0 & -M_2 \\
M_3 - m_3 & M_3 - m_3 & M_3 + \alpha_3 & \cdots & 0 & -M_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
M_{n-1} - m_{n-1} & M_{n-1} - m_{n-1} & M_{n-1} - m_{n-1} & \cdots & M_{n-1} + \alpha_{n-1} & -M_{n-1} \\
1 & 1 & 1 & \cdots & 1 & 0
\end{bmatrix}
\]
This last determinant becomes $T_n^*$ if we put

$$S_\nu = M_\nu + \alpha_\nu, \quad \gamma_\nu = M_\nu - m_\nu, \quad \beta_\nu = M_\nu \quad (\nu = 1, \ldots, n-1)$$

and has therefore by (1,8) the value

$$-\sum_{K=1}^{n-1} \left( -M_K \right)^{K-1} (M_2 + \alpha_2) \prod_{2=K+1}^{n-1} (m_2 + \alpha_2).$$

We obtain therefore for $\Omega_n$ the expression

$$(I,10) \quad \Omega_n = \alpha_n \prod_{\nu=1}^{n-1} (M_\nu + \alpha_\nu) - \sum_{K=1}^{n-1} M_k \prod_{2=K+1}^{n-1} (M_2 + \alpha_2) \prod_{2=K+1}^{n-1} (m_2 + \alpha_2).$$

The determinant $\Omega_n$ can be also written in the form

$$(I,11) \quad \Omega_n = \begin{vmatrix}
\alpha_n - m_n & -m_n & \cdots & -m_n & -m_n \\
-M_n-1 & \alpha_n-1 & -m_n-1 & \cdots & -m_n-1 & -m_n-1 \\
-M_n-2 & -M_n-2 & \alpha_n-2 & \cdots & -m_n-2 & -m_n-2 \\
& & & \ddots & & \ddots \\
-M_2 & -M_2 & \cdots & \alpha_2 & -m_2 \\
-M_1 & -M_1 & \cdots & -M_1 & \alpha_1
\end{vmatrix}
and we obtain therefore from (I,10)

\[
(I,12) \quad \Omega_n = \sum_{\nu=2}^{n} (\mu_{\nu} + \alpha_{\nu}) - \sum_{\kappa=2}^{n} m_{\kappa} \prod_{\mu=\kappa+1}^{n} (\mu_{\mu} + \alpha_{\mu}) \prod_{\mu=1}^{k-1} (M_{\mu} + \alpha_{\mu}).
\]

II. The Matrix $\Delta_n$ and its Inverse

We consider now the matrix

\[
(II,1) \quad \Delta_n = \begin{pmatrix}
1 & -M & -M & \cdots & -M \\
-M & 1 & -M & \cdots & -M \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-M & -M & -M & \cdots & 1
\end{pmatrix}
\]

Its determinant is obtained from $\Omega_n$ in putting in (I,12)

\[
\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1, \quad m_2 = m_3 = \cdots = m_n = m, \quad M = N_2 = \cdots = N_{n-1} = M.
\]

We obtain

\[
| \Delta_n | = (m+1)^{n-1} - m \sum_{\kappa=2}^{n} (m+1)^{n-\kappa} (M+1)^{\kappa-1}.
\]

and this becomes, if $M \neq m$, $= (m+1)^{n-1} - m(M+1) \frac{(M+1)^{n-1} - (m+1)^{n-1}}{M - M}$

\[
= \frac{1}{M - m} \left[ (m+1)^{n-1} (M-m+m(M+1)) - m(M+1)^n \right].
\]

\[
(II,2) \quad | \Delta_n | = \frac{1}{M - m} \left[ M(m+1)^n - m(M+1)^n \right] \quad (m \neq M).
\]
while for \( m = M \) we obtain from (II,2) in letting \( M \to m \):

\[
\text{(II,3)} \quad |\Delta_n| = (1 - (n-1)m)(1 + m)^{n-1} \quad (m = M).
\]

In particular we see that under hypothesis \( m < M \) necessary and sufficient for \( |\Delta_n| > 0 \) is

\[
\text{(II,4)} \quad \frac{m}{(m+1)^n} < \frac{M}{(M+1)^n}
\]

We assume now in particular

\[
\text{(II,5)} \quad m < M.
\]

If we introduce the abbreviations

\[
\text{(II,6)} \quad M_n = \frac{(1+m)^n - nM - 1}{M}, \quad m_n = \frac{(1+m)^n - nM - 1}{m}
\]

we can write (II,2) in the form

\[
(M-m)|\Delta_n| = M(l+nM+m_m) - m(l+nM+M_m) = M - m - mM(M_n - m_n)
\]

and therefore, it we put
\[(II,7)\]
\[S = 1 - |\Delta_n|,\]

\[(M-m) S = m M (M_n - m_n)\]

\[(II,8)\]
\[\frac{S}{m M} = \frac{M_n - m_n}{M - m}\]

It follows from \((II,5)\) for \(n > 2\)

\[\frac{M_n - m_n}{M - m} = \sum_{\nu = 2}^{n} \binom{n}{\nu} \frac{M^{\nu-1} m^{\nu-1} - \nu}{M - m} > \sum_{\nu = 2}^{n} \binom{n}{\nu} M^{\nu-2} \frac{(1+M)^{\nu-1} - n M}{M^2} = \frac{M_n}{M},\]
so that from (II,8) we have

\[(II,9)\quad \frac{M_n}{1-\frac{m}{M}} > \frac{S}{3} > M_n.\]

It follows in particular if (II,4) holds $S > 0$, $0 < |\Delta_n| < 1$.

In solving (II,9) with respect to $m$ we obtain

\[(II,10)\quad \frac{S}{M_n + \frac{S}{M}} < m < \frac{S}{M_n} \quad \Rightarrow \quad m = \frac{S}{M_n + \Theta \frac{S}{M}} \quad , \quad 0 < \Theta < 1.\]

\[(II,10^o)\quad M_n m < S < \frac{M_n m}{1-\frac{m}{M}} \quad \Rightarrow \quad S = \frac{M_n m}{1-\Theta \frac{m}{M}} \quad , \quad 0 < \Theta < 4.\]

In order to discuss the meaning of the condition (II,4), consider the curve

\[(II,11)\quad y = f(x) = \frac{x}{(1+x)^n};\]
we have

\[ f'(x) = \frac{1-(n-1)x}{(1+x)^{n+1}} \]

\[ f''(x) = n \frac{(n-1)x-2}{(1+x)^{n+2}} \]

For \( x > 0 \) the curve (II,11) has one maximum at \( x = \frac{1}{n-1} \) and an inflection point at \( x = \frac{2}{n-1} \).

In the diagram 1 the curve (II,11) is drawn (computed for \( n = 5 \)). The portion of this curve from the point 0 to the highest point T will be denoted as the ascending branch and the portion between T and \( x = \infty \) as the descending branch.
If we assume that in \((II, 4)\) \(m\) is less than \(M\), we see that either

\[(II, 12) \quad 0 < m < \frac{1}{n-1}, \quad M > \frac{1}{n-1}\]

or

\[(II, 12') \quad m < M \frac{1}{n-1} .\]

In order to find for a given \(M > \frac{1}{n-1}\) the range of the values of \(m\) satisfying \((II, 4)\) we find on the curve the second point \(P_M\) with the same ordinate as the point \(x = M\), then if to \(P_M\) corresponds the abcissa \(m(M)\) we have \(0 < m < m(M)\).

We prove finally that from \((II, 4)\) and \(m < M\) always follows

\[(II, 13) \quad m M^{n-1} < 1 .\]

Indeed in proving \((II, 13)\) we can obviously assume that \(M > 1\) and therefore on the descending branch in the diagram but then we have

\[
\frac{1}{1 + \frac{1}{M^{n-1}}} = \frac{M^{n-1}}{1 + M^{n-1}} > \frac{M}{1 + M}
\]

and therefore in raising it to the \(n\)th power

\[
\frac{1}{(M^{n-1})} > \frac{M}{(1 + M)^n} ;
\]

we see that \(\frac{1}{M^{n-1}}\) must lie in the fig. 1 between \(M\) and \(m(M)\) and it follows \(\frac{1}{M^{n-1}} > m\), that is \((II, 13)\).

We are going now to determine the inverse matrix

\[(II, 14) \quad \Delta_n^{-1} = (\alpha_{\mu, \nu}) \quad (\mu, \nu = 1, \ldots, n)\]
of $\Delta_n$, if its determinant $(\text{II}, 2)$ resp. $(\text{II}, 3)$ is $\neq 0$. It is sufficient for this purpose to solve the corresponding linear system

$$
\text{(II, 15)} \quad x_\mu - m \sum_{\nu=1}^{\mu-1} x_\nu - M \sum_{\nu=\mu+1}^{n} x_\nu = z_\mu \quad (\mu = 1, \ldots, n)
$$

for indeterminants $z_\mu$. Put

$$
\frac{x_1 + \ldots + x_n = s}{M - M = \frac{m - M}{M + 1} = \sigma}, \quad \frac{M}{1 + M} = \lambda, \quad 1 + \sigma = \frac{1 + m}{1 + M} = \phi)
$$

Then the system $(\text{II}, 15)$ is equivalent with

$$
(1 + M) x_\mu = (m - M)(x_1 + \ldots + x_{\mu - 1}) + Ms + z_\mu \quad (\mu = 1, \ldots, n)
$$

or in putting

$$
S_\nu = x_1 + \ldots + x_\nu \quad (\nu = 1, \ldots, n), \quad S_0 = 0
$$

$$
S_\mu - S_{\mu - 1} = \sigma S_{\mu - 1} + y_\mu \quad S_{\mu} = (1 + \sigma) S_{\mu - 1} + y_\mu = \phi S_{\mu - 1} + y_\mu
$$

$$
S_\mu q^{-\mu} = S_{\mu - 1} q^{-\mu + 1} + y_\mu q^{-\mu} \quad (\mu = 1, \ldots, n)
$$

$$
S_\mu q^{-\mu} = \sum_{\nu=2}^{\mu} y_\nu q^{-\nu}
$$
(II,17) \[ S_\mu = \sum_{v=1}^{\mu} y_v q^{\mu-v} \quad (\mu = 1, \ldots, n) \]

(II,18) \[ S_\mu = \sum_{v=1}^{\mu-1} y_v (q-1) q^{\mu-v-1} + y_\mu \quad (\mu = 1, \ldots, n). \]

From (II,17) for \( \mu = n \) we have by (II,16)

\[ S_n = s = \sum_{v=1}^{n} y_v q^{n-v} = 2 \sum_{v=1}^{n} q^{n-v} + \sum_{v=1}^{n} \alpha_v q^{n-v}, \]

(II,19) \[ s = 2s \frac{q^n-1}{q-1} + \sum_{v=1}^{n} \alpha_v q^{n-v}, \]

where for \( m = M, q = 1 \) the coefficient of \( \alpha s \) is to be replaced by \( n \).

Since for \( m \neq M \)

\[ 1 - 2 \frac{q^n-1}{q-1} = \frac{m-Mq^n}{m-M} = \frac{Mq^n-m}{M-m} \]

we have from (II,19) and (II,2)

(II,20) \[ s = Q \sum_{v=1}^{n} \alpha_v q^{n-v} \quad (m \neq M), \]

(II,21) \[ Q = \frac{M-m}{Mq^n-m} = \frac{(m+1)^n}{|A_n|} \quad (m \neq M) \]

while for \( m = M \) we obtain again (II,20) in defining \( Q \) by

(II,21°) \[ Q = \frac{m+1}{1-(m+1)m} = \frac{(m+1)^n}{|A_n|} \quad (m = M). \]
If we now replace in (II,18) $\gamma_\nu$ by $\lambda s + \alpha_\nu$ and use for $s$ the expressions from (II,20)

$$x_\mu = (q-1) \sum_{v=1}^{\mu} \alpha_\nu q^{\mu-v-1} + \alpha_\nu + \lambda s \left[ (q-1) \sum_{v=1}^{\mu-1} q^{\mu-v-1} + 1 \right].$$

But here the expression within the brackets is $q^{\mu-1}$ and we obtain therefore

$$x_\mu = \sum_{v=1}^{\mu-1} \alpha_\nu q^{\mu-v-1}(q-1) + \alpha_\mu + \lambda Q q^{\mu-1} + \sum_{v=1}^{n} \alpha_\nu q^{n-v} \quad (\mu = 1, \ldots, n),$$

and finally in introducing $\alpha_\nu$ from (II,16)

$$(II,22) \quad x_\mu = \sum_{v=1}^{\mu-1} \alpha_\nu q^{\mu-v-1}(q-1) + \frac{\alpha_\mu}{M+1} + \frac{\lambda Q q^{\mu-1}}{M+1} \sum_{v=1}^{n} \alpha_\nu q^{n-v} \quad (\mu = 1, \ldots, n).$$

Introduce the two following triangular matrices

$$U_n(q) = (u_{\mu,\nu}) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \vdots \\ q & 1 & 0 & \vdots \\ q^2 & q & 1 & 0 \end{pmatrix}$$

$$(II,23)$$

$$V_n(q) = (v_{\mu,\nu}) = \begin{pmatrix} q^{\mu-v-1} & (\mu > \nu) \\ 0 & (\mu \leq \nu) \\ \vdots & \vdots \end{pmatrix}$$

$$(II,24)$$

$$(II,25)$$
We see then that in (II,22) the matrix corresponding to the first right hand side is

\[ \frac{q-1}{M+1} U_n = \frac{m-M}{(M+1)^2} U_n \]

while to the second right hand sum in (II,22) corresponds the matrix

\[ \frac{\lambda q}{M+1} q^{n-1} (q U_n + E_n + V_n) . \]

We obtain

\[ \Delta_n^1 = \left( \frac{m-M}{(M+1)^2} + \frac{Mq^n}{(M+1)^2} \right) U_n + \left( \frac{1}{M+1} + \frac{Mq^n}{(M+1)^2} \right) E_n + \frac{MQq^{n-1}}{(M+1)^2} V_n. \]

The coefficient of \( U_n \) is here equal to \( \frac{m q}{(M+1)^2} \) both in the case (II,21) and (II,21°) and we have finally

\[ \Delta_n^1 = \frac{m q}{(M+1)^2} U_n + \left( \frac{1}{M+1} + \frac{Mq^n}{(M+1)^2} \right) E_n + \frac{MQq^{n-1}}{(M+1)^2} V_n. \]

We now denote by \( s^\mu \) the sum of the elements in the \( \mu \)-th row of the matrix (II,27) and by \( t^\mu \) the corresponding sum in the \( \mu \)-th column. If we assume first \( m = M \) for the matrices \( U_n \) and \( V_n \) the \( \mu \)-th row terms are respectively equal to

\[ \frac{q^{\mu-1}}{q-1} \cdot \frac{1-q^{\mu-n}}{q-1} \]

while the \( \psi \)-th column sums are

\[ \frac{q^{\psi-1}}{q-1} \cdot \frac{1-q^{\psi-n}}{q-1} . \]
We obtain now from (II, 27) by (II, 16)
\[(M+1)^2 s = m Q \frac{q^{n-1}}{q - 1} + M + 1 + M Q q^n + M Q \frac{q^{n-1}}{q - 1} = \]
\[= \frac{Q}{\Delta_n} \left[ m q^{n-1} + M q^n - M q^{n-1} + M q^{n-1} M q^{n-1} \right] + M + 1 = \]
\[= \frac{M+1}{m-M} Q \left[ M q^n - m + (m-M) q^{n-1} \right] + M + 1 \]
and this is by (II, 21) = (M+1) q^{\mu-1}. We have finally

\[(II, 28) \quad s = \frac{Q}{M+1} q^{\mu-1} = \frac{(M+1)^{n-1} q^{\mu-1}}{|\Delta_n|} \]

and this remains true also for \( m = M \), as is immediately seen.

Similarly, we have
\[(M+1)^2 t = m Q \frac{q^{n-1}}{q - 1} + M + 1 + M Q q^n + M Q \frac{q^{n-1}}{q - 1} = \]
\[= \frac{Q}{\Delta_n} \left[ m q^{n-1} - m + M q^n - M q^{n-1} + M q^{n-1} M q^{n-1} \right] + M + 1 = \]
\[= \frac{M+1}{m-M} Q \left[ M q^n - m + (m-M) q^{n-1} \right] + M + 1 = (M+1) q^{n-\nu} \]
and we have

\[(II, 29) \quad t = \frac{Q}{M+1} q^{n-\nu} = \frac{(M+1)^{n-1} q^{n-\nu}}{|\Delta_n|} \]
a relation which is also immediately verified for the case \( m = M \).
III. Bounds for the Determinants, depending on $\Omega_n$ and $\Delta_n$

A determinant

\[
M = \begin{vmatrix}
\alpha_1 & -m_{h2} & \ldots & -m_{hn} \\
-m_{21} & \alpha_2 & \ldots & -m_{2n} \\
& & \ddots & \ddots \\
-m_{n1} & -m_{n2} & \ldots & \alpha_n 
\end{vmatrix}
\]
will be called an \textit{M-determinant} if all diagonal elements \(\alpha_i\) are positive, all elements off the main diagonal, \(-m_{\mu \nu}(\mu, \nu)\) are not positive and the determinant \(M\) as well as of all principal (coaxial) minors of \(M\) of all degrees are positive.

In what follows we will have to use a theorem given by the author in 1937 \cite{1} and which will be formulated as Lemma I. If we have for the \(M\)-determinant (III,1) and a determinant \(H = \begin{vmatrix} h_{\mu \nu} \end{vmatrix} (\mu, \nu = 1, \ldots, n)\) of the order \(n\) the inequalities

\[
(III,2) \quad |h_{\nu \nu}| \geq \alpha_{\nu}, \quad |h_{\mu \nu}| \leq m_{\mu \nu} (\mu, \nu = 1, \ldots, n)
\]

then \(H \neq 0\) and we have

\[
(III,3) \quad |H| \leq M.
\]

If \((H), (M)\) respectively denote the matrices of the determinants \(H\) and \(M\) the inverse matrix of \((H)\) is majorated by the inverse matrix of \((M)\)

\[
(III,3') \quad (H)^{-1} \ll (M)^{-1}
\]

Suppose now that in the determinant \(\bigwedge_{n}\) given by (I,9) all \(\alpha_i\) are positive and \(m_{\nu}, M_{\nu}\) non negative: Then it follows from (I,10) that \(\bigwedge_{n}\) is a monotonically decreasing function of all \(m_{\nu}\) and from (I,12) that \(M\) is also monotonically decreasing in all \(M_{\nu}\). Suppose now that for a certain set of values of \(\alpha_{\nu}, m_{\nu}, M_{\nu}\) the determinant \(\bigwedge_{n}\neq 0\).
Then replace the small $m_v$ and $M_v$ corresponding to certain rows $(v_1, \ldots, v_r)$ by zeros; the determinant $\Omega_n$ cannot decrease and remains therefore positive; but then $\Omega_n$ becomes equal to the product of $\alpha_{v_1} \alpha_{v_2} \cdots \alpha_{v_r}$ with the principal minor complementary to the set of indices $v_1, \ldots, v_r$. Therefore all principal minors of $\Omega_n$ are positive. We see that $\Omega_n$ is an M-determinant, if the conditions (III, 4) are satisfied and $\Omega_n \neq 0$. But now we can easily deduce the following theorem.
A. Consider the set of all determinants \( A = \left| a_{\mu \nu} \right| \) satisfying

\[
| a_{\mu \nu} | \leq m_{\mu} \quad (\nu < \mu) \quad | a_{\mu \nu} | \leq M_{\mu} \quad (\nu > \mu),
\]

(III,5)

where \( \alpha_{\mu} \) are \( n \) given positive constants and \( m_{\mu}, M_{\mu} \) \( 2n-2 \) given non-negative constants. Then necessary and sufficient in order that all determinants \( A \) of this set do not vanish is the inequality \( \sum_{\mu} > 0 \). If this condition is satisfied, we have

\[
| A | \geq \sum_{\mu}.
\]

(III,6)

Proof: Since in the case \( \sum_{\mu} > 0 \), \( \sum_{\mu} \) is an M-determinant, the sufficiency of our condition follows immediately from the inequality (III,3) mentioned above.

If \( \sum_{\mu} = 0 \) we can take obviously \( a_{\mu \mu} = \alpha_{\mu} \) and \( a_{\mu \nu} = m_{\mu} \) or \( a_{\mu \nu} = M_{\mu} \) according as \( \nu < \mu \) or \( \nu > \mu \) and obtain a vanishing determinant \( A \) satisfying the condition (III,5).

Suppose now \( \sum_{\mu} < 0 \), then it follows from the form (I,9) of \( \sum_{\mu} \) that \( \sum_{\mu} \) becomes positive if all \( m_{\mu} \) are replaced by zeros. There exists therefore such a positive \( t < 1 \) that \( \sum_{\mu} \) vanishes if all \( m_{\mu} \) in (I,9) are replaced by \( t \cdot m_{\mu} \), but then we obtain a vanishing determinant \( A \) of our set in taking \( a_{\mu \mu} = \alpha_{\mu} \) and \( a_{\mu \nu} = t \cdot m_{\mu} \) or \( a_{\mu \nu} = M_{\mu} \) according as \( \nu > \mu \) or \( \nu < \mu \). The theorem A is proved.

In specializing the matrix of \( \sum_{\mu} \) to \( \Delta \) and in assuming that in particular \( 0 < m < M \) we obtain immediately from (II,2) and the theorem A:

B. Consider for two positive constants \( m, M, m < M \) the set of all determinants \( A = \left| a_{\mu \nu} \right| \) of the \( n \)th order for which

\[
(III,7) \quad | a_{\mu \mu} | \leq 1 \quad (\mu = 1, \ldots, n) \quad | a_{\mu \nu} | \leq m \quad (\nu < \mu) \quad | a_{\mu \nu} | \leq M \quad (\nu > \mu).
\]
then necessary and sufficient in order that all determinants $A$ of this set do not vanish is for $m < M$, the inequality

$$(III,3) \quad \frac{m}{(1+m)^n} < \frac{M}{(1+M)^n} \quad (m < M)$$

and if this inequality is satisfied we have for each determinant $A$ of the set

$$(III,9) \quad |A| \geq |\Delta_n| = \frac{(1+m)^n(1+M)^n}{M-m} \left[ \frac{M}{(1+M)^n} - \frac{m}{(1+m)^n} \right].$$

If $m = M$, necessary and sufficient for all determinants $A$ being $\neq 0$ is

$$(III,80) \quad m < \frac{1}{n-1} \quad (M = m)$$

and we have, if $(III,80)$ is satisfied,

$$(III,90) \quad |A| \geq |\Delta_n| = (1-(n-1)m)(1+m)^{n-1} \quad (m = M).$$

From the theorem B we can deduce the following theorem.

C. Let $A = (a_{\mu,\nu})$ ($\mu, \nu = 1, \ldots, n$) be a matrix satisfying the conditions

$$(III,10) \quad |a_{\mu,\nu}| \leq m \quad (\nu < \mu), \quad |a_{\mu,\nu}| \leq M \quad (\nu > \mu)$$

where $m, M$ are two constants with $0 < m < M$.

Put

$$(III,11) \quad S(m, M) = \frac{Mm^\frac{1}{n} - mM^\frac{1}{n}}{M^\frac{1}{n} - m^\frac{1}{n}}.$$
then all fundamental roots of the matrix $A$ are contained in the set of the $n$ closed circles described around the elements $a_{\mu\nu}$ with the radius $S(m, M)$. The value (III,11) of $S(m, M)$ is indeed assumed if $A = \Delta_n^3$.

Proof:

Let $\lambda$ be a fundamental root of $A$ so that the matrix $\lambda E - A$ is singular. Put $\min_{\mu} |\lambda - \frac{a_{\mu\nu}}{\lambda}| = \alpha$, we have to prove that $\alpha \leq S(m, M)$. If $\alpha = 0$ there is nothing to prove. Suppose $\alpha > 0$ and consider the matrix

$$\frac{\lambda E - A}{\alpha} = (b_{\mu\nu}).$$

For this matrix we have

$$|b_{\mu\nu}| \leq 1, \quad |b_{\mu\nu}| \leq \frac{m}{\alpha} (\alpha < \mu), \quad |b_{\mu\nu}| \leq \frac{M}{\alpha} (\alpha > \mu);$$

therefore we have by the theorem B

$$\frac{m}{\alpha (1 + \frac{m}{\alpha})^n} \leq \frac{M}{\alpha (1 + \frac{M}{\alpha})^n}$$

and therefore

$$\frac{1 + \frac{m}{\alpha}}{m^{\mu\nu}} \leq \frac{1 + \frac{M}{\alpha}}{M^{\mu\nu}}$$

$$\alpha \left( \frac{1}{m^{\mu\nu}} - \frac{1}{M^{\mu\nu}} \right) \leq \frac{M}{M^{\mu\nu}} - \frac{m}{m^{\mu\nu}}$$

$$\alpha \leq \frac{M - m}{m^{\mu\nu} - M^{\mu\nu}} = S(m, M).$$

IV. The Bounds of the Matrix $\Delta_n^3$

For an $n$ dimensional vector $\xi = (x_1, \ldots, x_n)$ the Hölder norms corresponding to the exponent $p \geq 1$ is given by

$$|\xi|_p = \sqrt[p]{|x_1|^p + \ldots + |x_n|^p} \quad (p \geq 1).$$
We will only use the three cases corresponding to \( p = 1, 2, \infty \):

\[
(IV, 2) \quad \|\mathbf{x}\|_2 = |x_1| + \ldots + |x_n|, \quad \|\mathbf{x}\|_\infty = M_x, \quad \|\mathbf{x}\|_1 = \sum |x_i|^p.
\]

We have between these three norms the following inequalities

\[
(IV, 3a) \quad \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n \|\mathbf{x}\|_\infty,
\]

\[
(IV, 3b) \quad \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n} \|\mathbf{x}\|_\infty,
\]

\[
(IV, 4c) \quad \|\mathbf{x}\|_1 \leq \sqrt{n} \|\mathbf{x}\|_2 \leq \sqrt{n} \|\mathbf{x}\|_\infty,
\]

which are immediately verified. The left hand inequality (IV, 3c) implies the well known inequality between the arithmetical mean and the arithmetical mean of the squares. If \( A = (a_{ij}) \) is an \( n \times n \) matrix we define its norm corresponding to the exponent \( p(1 \leq p \leq \infty) \) by

\[
(IV, 4) \quad \|A\|_p = \max_{\mathbf{x} \neq 0} \frac{|A\mathbf{x}|_p}{\|\mathbf{x}\|_p},
\]

and denote it by \( \|A\|_p \). We will use here too only the cases \( p = 1, 2, \infty \).

In applying to the definition (IV, 4) the formulae (IV, 3a), (IV, 3b) and (IV, 3c) we obtain immediately

\[
(IV, 5a) \quad \frac{1}{\sqrt{n}} \|A\|_4 \leq \|A\|_2 \leq \sqrt{n} \|A\|_1,
\]

\[
(IV, 5b) \quad \frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{n} \|A\|_\infty.
\]
For $p = 1, \infty$ the expressions of $|A|_1, |A|_\infty$ are easy to write down; we have as is well known and very easy to prove

(IV, 6a) 

$$|A|_1 = \max \sum_{\mu} |a_{\mu \nu}|$$

(IV, 6b) 

$$|A|_\infty = \max \sum_{\mu} |a_{\mu \nu}|.$$ 

As to $|A|_2$, its expression is irrational, $|A|_2$ is the square root of the maximum fundamental root of the symmetric and non-negative matrix $AA^*$. Since the direct computation of $|A|_2$ is in most cases difficult we prove in what follows the following estimate for $|A|_2$:

**Lemma II.** We have for any matrix $A$

(IV, 7) 

$$\frac{1}{n} \max(|A|_1, |A|_\infty) \leq |A|_2 \leq \sqrt{|A|_1 |A|_\infty},$$

The first part of (IV, 7) follows from (IV, 5a) and (IV, 5b). To prove the second part we introduce the notations

$$s_\mu = \sum_{\nu} |a_{\mu \nu}|, \quad t_\nu = \sum_{\mu} |a_{\mu \nu}| \quad (\mu, \nu = 1, \ldots, n).$$

The sum of the moduli of all elements in the $\mu$-th row of $AA^*$ can be estimated as follows

$$|\sum_{\gamma} a_{\mu \gamma} a_{\gamma \nu}| \leq \sum_{\mu} \sum_{\nu} |a_{\mu \nu} a_{\gamma \nu}| \leq \sum_{\mu} |a_{\mu \nu}| t_\nu \leq s_\mu |A|_1 \leq |A|_\infty |A|_1.$$
(IV, 7) follows now from the theorem of Frobenius that the modulus of each fundamental root of a square matrix does not exceed the greatest sum of the moduli of the elements of this matrix in different rows.

If we apply now these results to the matrix $\Delta_n^i$ discussed in the section II, we obtain from (II,18, (II,19), IV,6a) and (IV,6b)

$$ (IV,8) \quad |\Delta_n^i|_1 = |\Delta_n^i|_\infty = \frac{\varphi}{m+1} = \frac{(m+1)^{n-1}}{|\Delta_n|} $$
In combining these inequalities with the result given in the section III, we obtain the following theorem.

D. Let \( A = (a_{ij}) \) be a square matrix of order \( n \) satisfying to the conditions (III,7) and let (III,8) be satisfied, then we have

\[
|A^{-1}|_p \leq \frac{(M+1)^{n-1}}{|\Delta_n|} \quad (p = 1, 2, \infty)
\]

and therefore for any vector \( \xi \)

\[
|A\xi|_p \leq \frac{|\Delta_n|}{(M+1)^{n-1}} |\xi|_p \quad (p = 1, 2, \infty).
\]

Proof. It follows from (III,30) of the lemma I at once in virtue of (IV 6a) and (IV,6b) that

\[
|A^{-1}|_p \leq |\Delta_n^{-1}|_p = \frac{Q}{M+1} \quad (p = 1, \infty);
\]

further, the matrix \( A^{-1}(A^{-1})^* \) is majorated by \( \Delta_n^{-1}(\Delta_n^{-1})^* \). Since therefore by a well known theorem of Frobenius, the maximum modulus of a fundamental root of \( A^{-1}(A^{-1})^* \) is majorated by that of \( \Delta_n^{-1}(\Delta_n^{-1})^* \), we have

\[
|A^{-1}|_2 \leq |\Delta_n^{-1}|_2 \leq \frac{Q}{M+1}
\]

and therefore by definition (IV,4) in putting \( A\xi = \gamma \)

\[
|A^{-1}\gamma|_2 \leq \frac{Q}{M+1} |\gamma|_2
\]

and this is equivalent with (IV,11).
V. The bounds of the matrix $\Delta_n^{-1} - \Delta_n^{(0)-1}$

We denote by $\Delta_n^{(0)}$ the matrix obtained from $(II,1)$ in replacing there everywhere by zero.

\[
(V,1) \quad \Delta_n^{(0)} = \begin{pmatrix}
1 & -M & -M & \cdots & -M \\
0 & 1 & -M & \cdots & -M \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]

We will prove the following theorem:

E. If we have $0 < m < M$ and $(III,8)$ is satisfied, then the matrix $\Delta_n^{-1} - \Delta_n^{(0)-1}$ is a non-negative matrix and we have

\[
(V,2) \quad |\Delta_n^{-1} - \Delta_n^{(0)-1}|_p = \max_{1 \leq \nu \leq n} (1+m)^{\nu-1} \frac{(1+m)^{n-\nu} - |\Delta_n|}{|\Delta_n|} \quad (p = 1, \infty).
\]

Proof: Since $\Delta_n^{(0)}$ satisfies the inequalities (III,2) it follows from (III,30) that $\Delta_n^{-1} - \Delta_n^{(0)-1}$ is non-negative. Denote by $s_k(0)$ and $t_k(0)$ the sums of the elements in the $\mu$th and in the $\nu$th column of $\Delta_n^{(0)}$ and by $s_\mu$ and $t_\nu$ the corresponding expressions for the matrix $\Delta_n^{-1} - \Delta_n^{(0)-1}$. We have

\[
(V,3) \quad \overline{s}_\mu = s_\mu - s(0), \quad \overline{t}_\nu = t_\nu - t(0) \quad (\forall \mu, \nu = 1, \ldots, n).
\]

It follows then from the formulae (II,22) and (II,23) that $\overline{s}_\mu$ run for $\mu = n, n-1, \ldots, 1$ through the same set of values as $\overline{t}_\nu$ for $\nu = 1, \ldots, n$.

The formula (II,29) can be written in the form

\[
(V,4) \quad t_\nu = \frac{q^{n-\nu}(M+1)^{n-1}}{|\Delta_n|} = \frac{(1+m)^{n-\nu}(1+M)^{n-1}}{|\Delta_n|}.
\]
Since $\Delta_n$ becomes 1 for $m = 0$, we have $(V, 2)$ and $E$ is proved.

Discussion of $|\Delta_n^{(m)} - \Delta_n^{(0)}|$. We prove first that $t_\nu - t_\nu^{(0)}$ goes increasing with $\nu$ as long as the condition

$$(V, 5) \quad 1 - \frac{m}{M} \triangleq |\Delta_n|$$

is satisfied. Indeed we put

$$(V, 6) \quad k_\nu = \frac{t_{\nu+1} - t_{\nu}^{(0)}}{t_\nu - t_\nu^{(0)}} = (M+1) \frac{(1+m)^{n-\nu-1} |\Delta_n|}{(1+m)^{n-\nu} |\Delta_n|}.$$

In solving the 3 inequalities

$$(V, 7) \quad k_\nu \triangleq 1$$

with respect to $|\Delta_n|$, we obtain correspondingly

$$(V, 8) \quad |\Delta_n| \leq \sum_\nu \triangleq (1 - \frac{m}{M})(1+m)^{n-\nu-1}.$$

The inequalities $k_\nu \triangleq 1 (\forall = 1, \ldots, n)$ are obviously satisfied in virtue of $(V, 5)$ as long as $\forall \leq n-1$. We have therefore in this case, in using (II,7)

$$\max_\nu (t_\nu - t_\nu^{(0)}) = (1+M)^{n-1} \frac{1-|\Delta_n|}{|\Delta_n|} = (1-M)^{n-1} \frac{\xi}{1 - \xi},$$
(V,9) \[ |\Delta_n^1 - \Delta_n^{0-1}| = (1 + M)^{n-1} \frac{1 - |\Delta_n^1|}{1 - |\Delta_n|} = (1 + M)^{n-1} \frac{\delta}{\xi - \delta} \quad (\xi = 1, \delta \leq 1 - \frac{m}{M}). \]

The condition (V,5) can be written using the notation (II,7)

\[ \delta \geq \frac{m}{M} \]

and this is in virtue of (II,10) certainly satisfied, if we have \( M M_n \geq 1, \)

(V,10) \[ (M + 1)^n - n M - 2 \geq 0. \]
The condition \((V,10)\) becomes for \(n = 3\): \(M \geq 0.5321\). For \(n \geq 4(V,10)\) is in any case satisfied if we have

\[
(V,11) \quad M \geq \frac{1.5}{n} \quad (n \geq 4).
\]

Indeed if the relation \((V,10)\) is satisfied for a positive \(M\) it is satisfied for any greater value since the coefficients of all positive powers of \(M\) in the left side expression are greater or equal 0. To prove the sufficiency of \((V,11)\) it is sufficient to prove that

\[
(1 + \frac{1.5}{n})^n \geq 3.5 \quad (n \geq 4).
\]

But here the left side expression is monotonically growing with \(n\) and this inequality follows therefore from

\[
\left(\frac{5.5}{4}\right)^4 = 3.75... > 3.5
\]

Suppose now that we have

\[
(V,12) \quad 1 - \frac{m}{M} \leq |\Delta_n| \leq 1 \quad 5 \leq \frac{M}{m}.
\]

Then we have, since \(0 < m < \frac{1}{n-1}\),

\[
\frac{(1+m)^{n-1}}{m} \leq \frac{(1 + \frac{1}{n-1})^{n-1} - 4}{\frac{1}{n-1}} \leq (n-1) \left((1 + \frac{1}{n-1})^{n-1} - 1\right) \leq (e-1)(n-1)
\]

\[
(1+m)^{n-1} = 1 + 1.72 \Theta n m, \quad 0 < \Theta < 1
\]

\[
(1+m)^{n-1} |\Delta_n| = 1.72 \Theta n m + (1 - |\Delta_n|)
\]

and therefore by \((V,12)\)

\[
(1+m)^{n-1} |\Delta_n| \leq 1.72 \Theta n m + \frac{m}{M}.
\]

For \(n \geq 4\) we have in the case \((V,12)\) since the inequality \((V,11)\) is not verified,

\[
\frac{1}{M} \leq \frac{n}{1.5}
\]
and therefore finally

\[(V,13) \quad (1 + m)^n - |\Delta_n| \leq 2.39 \, n \cdot m.\]

On the other hand, since \( M < \frac{1}{n} \), \((1 + M)^{\frac{1}{n}} \leq (1 + \frac{1}{n})^n \leq e^{1.5} = 4.481689\)
and therefore from \((V,2)\) and \((V,3)\)

\[t_v - t_v^{(0)} \leq 10.72 \, n \frac{m}{|\Delta_n|} \quad (v = 1, \ldots, n; |\Delta_n| \geq 1 - \frac{m}{M}; j \geq 3),\]

\[(V,14) \quad |\Delta_n - \Delta_n^{(0)}|_p \leq 10.72 \, n \frac{m}{|\Delta_n|} \quad (p = 1, \infty; |\Delta_n| \geq 1 - \frac{m}{M}; j \geq 3).\]

To obtain a lower bound, we take in \((V,4)\) \( v = 1. \) We obtain, since

\[0 < |\Delta_n| \leq 1, \quad t_1 - t_1^{(0)} = (1 + m)^n - 1 \leq \frac{(n - 1)^n}{|\Delta_n|} \geq \frac{75 \, n \, m}{|\Delta_n|},\]

and therefore

\[(V,15) \quad |\Delta_n - \Delta_n^{(0)}|_p \geq \frac{75 \, n \, m}{|\Delta_n|} \quad (p = 1, \infty; j \geq 4).\]

To obtain the exact value of \(|\Delta_n - \Delta_n^{(0)}|_p\) if \((V,5)\) is not satisfied, we return to the inequalities \((V,8)\) equivalent to \((V,7)\).

Denote by \( n_0 \) the smallest integer between 1 and \( n \) such that we have

\[(V,16) \quad |\Delta_{n_0 - 1} - \Delta_{n_0}^{(0)}| > \frac{1}{n_0} \quad (1 \leq n_0 \leq n).\]

The parts of this inequality implying \( \delta_{n_0} \) or \( \bar{s}_n \) must be disregarded, that is to say, this inequality reduces to \(|\Delta_n| > \delta_{n_0}^{(0)}\) for \( n_0 = 1 \) and to \(|\Delta_{n - 1} | \geq |\Delta_n| \) for \( n_0 = n \). Then we see at once that

\[\text{Max}(t_v - t_v^{(0)}) = t_{n_0} - t_0^{(0)} \quad \text{and therefore}\]

\[(V,17) \quad |\Delta_n - \Delta_n^{(0)}|_p = t_{n_0} - t_0^{(0)} = \frac{(1 + m)^{n_0 - 1}(1 + m)^n - |\Delta_n|}{|\Delta_n|} \quad (p = 1, \infty),\]

For \( m = M \) we have \( n_0 = 1 \) and therefore
In order to obtain the theorem corresponding to E for a determinant \( A \) satisfying the conditions (III,7), (III,8) of the theorem B, we prove first the following important lemma which generalizes considerably the relation (III,3') of the lemma I.

Lemma III. Consider \( n \) positive numbers \( \alpha_1 \) to \( \alpha_n \) and \( 2n^2 - 2n \) non-negative numbers such that the matrices

\[
M = \begin{pmatrix}
\alpha_1 & -m_{12} & \cdots & -m_{1n} & -\epsilon_{12} & \cdots & -\epsilon_{1n} \\
-m_{21} & \alpha_2 & \cdots & -m_{2n} & -\epsilon_{21} & \cdots & -\epsilon_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-m_{n1} & \cdots & -m_{nn} & \alpha_n & & & \\
\end{pmatrix}
\]

\[
M(0) = \begin{pmatrix}
\alpha_1 & -m_{12} & \cdots & -m_{1n} \\
-m_{21} & \alpha_2 & \cdots & -m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-m_{n1} & \cdots & -m_{nn} & \alpha_n \\
\end{pmatrix}
\]

are \( M \)-matrices.

Consider \( n \) constants \( A_\mu \) (\( \mu = 1, \ldots, n \)) such that

\[
|A_\mu| \geq \alpha_\mu \quad (\mu = 1, \ldots, n)
\]

and \( 2n^2 - 2n \) constants \( a_{\mu\nu} \) \( b_{\mu\nu} \) (\( \mu = \nu; \mu, \nu = 1, \ldots, n \)) such that
and form the two matrices

\[(VI, 5) \quad A = \begin{pmatrix}
A_1 & b_{12} & \cdots & b_{1n} \\
b_{21} & A_2 & \cdots & b_{2n} \\
& & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & A_n
\end{pmatrix}
\]

\[(VI, 6) \quad A^{(0)} = \begin{pmatrix}
A_1 & a_{12} & \cdots & a_{1n} \\
a_{21} & A_2 & \cdots & a_{2n} \\
& & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & A_n
\end{pmatrix}
\]

then we have

\[(VI, 7) \quad A^{-1} - A^{(0)}^{-1} \ll M^{-1} - M^{(0)}^{-1}
\]

**Proof:** We can write \(M = P - T, M^{(0)} = P - T^{(0)}\) where the matrices \(T, T^{(0)}\) have in the main diagonal zeros and off the main diagonal respectively the non-negative elements \(m_{\mu \nu} + \xi_{\mu \nu}, m_{\mu \nu}\), while \(P\) is the diagonal matrix

\[(VI, 8) \quad P = \begin{pmatrix}
\mu_1 & \xi_{\mu_1} & \cdots & 0 \\
0 & \mu_2 & \cdots & \xi_{\mu_2} \\
& & \ddots & \vdots \\
0 & 0 & \cdots & \mu_n
\end{pmatrix}
\]
We can develop the inverse of \( P - T = P(E - P^{-1}T) \) (\( E \) is the unit matrix) in the following way.

\[
(VI, 9) \quad (P - T)^{-1} = \sum_{k=0}^{\infty} (P^{-1}T)^k P^{-1}.
\]

The convergence of this development and the validity of \((VI, 9)\) follows.
easily from the fact that the determinant of the matrix $P-tT$ does not vanish for $|t| \leq 1$ as follows immediately from the lemma I. The corresponding development holds also for $(P-T(0))^{-1}$ and we obtain therefore

$$(VI,10) \quad M^{-1}M(0)-1 = \sum_{k=0}^{\infty} [ (P^{-1}T)^k - (P^{-1}T(0))^k ] P-1$$

The elements of $P^{-1}T$ are here \( \frac{m_{\mu \nu} + \epsilon_{\mu \nu}}{x_\mu} \) or zeros and those of $P^{-1}T^0$ are \( \frac{a_{\mu \nu}}{x_\mu} \) or zeros, therefore all elements of the matrices

$$(VI,11) \quad (P^{-1}T)^k - (P^{-1}T(0))^k \quad (k = 1, 2, \ldots)$$

are polynomials in \( \frac{m_{\mu \nu}}{x_\mu} \) and \( \frac{\epsilon_{\mu \nu}}{x_\mu} \) with nonnegative coefficients. Denote by $Q$ the diagonal matrix

$$Q = \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \\ & & & A_n \end{pmatrix}$$

and write

$$A = Q - S, \quad A(0) = Q - S(0),$$

where the elements of $S$ are \( \frac{b_{\mu \nu}}{x_\mu} \) or zeros and those of $S(0)$ \( \frac{a_{\mu \nu}}{x_\mu} \) or zeros. Since therefore we have

$$Q^{-1}S \ll P^{-1}T, \quad Q^{-1}S(0) \ll P^{-1}T(0),$$

we have

$$A^{-1} = \sum_{k=0}^{\infty} (Q^{-1}S)^k Q^{-1}, \quad A(0)^{-1} = \sum_{k=0}^{\infty} (Q^{-1}S(0))^k Q^{-1},$$

$$(VI,12) \quad A^{-1} - A(0)^{-1} = \sum_{k=0}^{\infty} \left[ (Q^{-1}S)^k - (Q^{-1}S(0))^k \right] Q^{-1}$$

But now the elements of

$$(Q^{-1}S)^k - (Q^{-1}S(0))^k$$

are obtained from those of $(VI,11)$ in substituting there \( \frac{m_{\mu \nu}}{x_\mu} \) instead of
\[
\frac{m_{\mu \nu}}{\alpha_{\mu}} \quad \text{and} \quad \frac{b_{\mu \nu} - a_{\mu \nu}}{A_{\mu}} \quad \text{instead of} \quad \frac{\varepsilon_{\mu \nu}}{\alpha_{\mu}}.
\]
Here we have by (VI,3) and (VI,4)

\[
\left| \frac{a_{\mu \nu}}{\alpha_{\mu}} \right| \leq \frac{m_{\mu \nu}}{\alpha_{\mu}} \quad \text{and} \quad \left| \frac{b_{\mu \nu} - a_{\mu \nu}}{A_{\mu}} \right| \leq \frac{\varepsilon_{\mu \nu}}{\alpha_{\mu}}
\]

and therefore since the coefficients in (VI,11) are, as already mentioned, not negative

\[
(Q^{-1}S)^k \cdot (Q^{-1}S(0))^k \ll (P^{-1}T)^k - (P^{-1}T(0))^k
\]

and

\[
(\text{VI,13}) \quad \sum_{k=1}^{\infty} [(Q^{-1}S)^k \cdot (Q^{-1}S(0))^k] \ll \sum_{k=1}^{\infty} [(P^{-1}T)^k - (P^{-1}T(0))^k]
\]

But from (VI,13) in virtue of (VI,3) it follows that the development (VI,12) is majorated by (VI,10) and our lemma is proved.

Under the conditions of the theorem B in the section III, if the inequalities (III,7) and (III,8) are satisfied, we can apply the lemma III in replacing there \(M\) by \(\mathcal{M}(0)\) by \(\mathcal{M}^{(0)}\). We obtain

F. Under the conditions of the theorem B, if the inequalities (III,7) and (III,8) are satisfied, we have

\[
(\text{VI,14}) \quad \left| A^{-1} - A^{(0)-1} \right|_p \leq \left| \Delta_n^{-1} - A^{(0)-1} \right|_p = \max_{1 \leq l, \leq n} \left( \frac{(1 + M)^{n-1}}{\mathcal{M}^{(0)} - \mathcal{M}} \right) \frac{\Delta_n}{l \Delta_n} \quad (p = 1, \infty)
\]

where the values and the estimates for the right side expression in (VI,14) are obtained from the formulae (V,9) to (V,17). The lemma III can be applied in many problems similar to that solved by the theorem F. For instance, in the theory of the solutions of linear equations the following problem has to be dealt with, although its complete discussion is usually avoided:

Consider a "triangular" system of linear equations

\[
(\text{VI,15}) \quad \sum_{\mu} a_{\mu \nu} x_{\nu} = y_{\mu} \quad (\mu = 1, \ldots, n)
\]
where the coefficients $a_{\mu \nu}$ are only approximate values to the "true" values $b_{\mu \nu}$. Suppose that we have generally

$$|b_{\mu \nu} - a_{\mu \nu}| \leq \varepsilon (\nu \geq \mu),$$

how far the solution of the system

$$\sum_{\nu=\mu}^{n} b_{\mu \nu} x_{\nu} = y_{\mu}$$

is influenced if the $b_{\mu \nu}$ are replaced by $a_{\mu \nu}$?

If we denote the matrix of (VI,15) by $A^{(0)}$ and that of (VI,17) by $A$, the question can be in particular answered by giving estimates of

$$|A^{-1} - A^{(0)}^{-1}|_{p} (p = 1, \infty).$$

Suppose that we have generally

$$a_{\mu \mu} = 1, \quad |a_{\mu \nu}| \leq M_{\mu} \quad (\mu > \nu; \mu = 1, \ldots, n-1)$$

and consider the matrix

$$\Delta(M_{1}, \ldots, M_{n-1}) = 
\begin{pmatrix}
1 & -M_{1} & \cdots & -M_{1} \\
0 & 1 & \cdots & -M_{2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix},$$

where all elements to the left of the main diagonal are zeros, while all elements to the right of the main diagonal in the $\mu$th row are equal to $M_{\mu}$.

We obtain from the lemma III at once the majoration

$$A^{-1} A^{(0)^{-1}} \ll \Delta(M_{1}, \varepsilon, \ldots, M_{n-1}, \varepsilon^{-1})^{-1} - \Delta(M_{1}, \ldots, M_{n-1})^{-1}.$$

To obtain the inverse of the matrix $\Delta(M_{1}, \ldots, M_{n-1})$, consider the system of linear equations

$$x_{\mu} - M_{\mu} (x_{\mu+1} + \cdots + x_{n}) = u_{\mu} \quad (\mu = 1, \ldots, n).$$
To solve it explicitly we put

\[(VI,21)\]
\[s_\mu = \sum_{k=0}^{n} x_k \quad (\mu = 1, \ldots, n)\]

\[(VI,22)\]
\[1 + M_\mu = N_\mu, \quad N_\mu N_{\mu+1} \cdots N_n = P_\mu \quad (\mu = 1, \ldots, n-1)\]

\[N_n = 1, \quad P_n = 1. \]

Then \((VI,20)\) becomes

\[(VI,23)\]
\[x_\mu = M_\mu s_{\mu+1} + u_\mu \quad (\mu = 1, \ldots, n)\]

\[(VI,24)\]
\[s_\mu = N_\mu s_{\mu+1} + u_\mu \quad (\mu = 1, \ldots, n)\]

where \(S_{n+1} = 0.\) Dividing \((VI,24)\) by \(P_\mu\) we obtain

\[
\frac{s_\mu}{P_\mu} = \frac{s_{\mu+1}}{P_{\mu+1}} + \frac{u_\mu}{P_\mu}
\]

and therefore

\[
s_\mu = P_\mu \sum_{k=0}^{n} \frac{u_k}{P_k}
\]

introducing this in \((VI,23)\) we have

\[(VI,25)\]
\[x_\mu = u_\mu + M_\mu P_{\mu+1} \sum_{k=0}^{n} \frac{u_k}{P_k} \quad (\mu = 1, \ldots, n).\]

We obtain therefore for the inverse of our \(\Delta(M_1, \ldots, M_{n-1}),\) in denoting by \(E\) the unit matrix,

\[(VI,26)\]
\[\Delta(M_1, \ldots, M_{n-1})^{-1} = E + D \cdot T,
\]

where \(D\) is the diagonal matrix

\[(VI,27)\]
\[D = \begin{pmatrix}
M_1 & P_2 & 0 \\
M_2 & P_3 & 0 \\
& & \ddots \\
0 & & & M_{n-1} \end{pmatrix}
\]

and \(T\) the triangular matrix
The expressions

\[(VI, 29)\]

\[\Delta(M_1, \ldots, M_{n-1})^{-1} \Delta(M_1, \ldots, M_{n-1})^{-1}\]

obtained from (VI, 26) to (VI, 28) is of course rather unwieldy; however we shall obtain for its norm corresponding to \( p = 1 \) (cf IV, 6a) a very simple and elegant expression.

Indeed, if we take the sum of the elements in the \( v \)th column of \( \Delta(M_1, \ldots, M_{n-1})^{-1} \) and denote it by \( t' \), we have

\[t'_v = 1 + \frac{1}{p_v} \sum_{k=1}^{n-1} M_k p_{k+1} = 1 + \sum_{k=1}^{n-1} M_k \frac{1}{1+M_k} (1+M_k) .\]

If we write now \( t'_{v+1} \) out, we obtain

\[t'_{v+1} = 1 + \left[ \sum_{k=1}^{v+1} M_k \frac{1}{1+M_k} (1+M_k) \right] (1+M_v) + M_v ;\]

and in comparing this with the expression of \( t' \) we see that we have

\[t'_{v+1} = (1+M_v) t'_v \]

and therefore

\[(VI, 30)\]

\[t'_v = \prod_{k=1}^{v-1} (1+M_k) .\]

We obtain now for the sum of the elements in the \( v \)th column of (VI, 29)

\[t_1 = 0, \quad t_2 = \epsilon \]

\[t_v = \prod_{k=1}^{v-1} (1+M_k+\epsilon) - \prod_{k=1}^{v-1} (1+M_k) \quad (v=1, \ldots, n).\]

In multiplying this by \( 1+M_v+\epsilon \) we obtain

\[\prod_{k=1}^{v-1} (1+M_k+\epsilon) - \prod_{k=1}^{v-1} (1+M_k) - \epsilon \frac{1}{1+M_k} (1+M_k),\]

and comparing this with \( t'_{v+1} \)
\( (VI, 31) \quad t_{v+1} = (1 + M \nu + \epsilon) t_{v} + \epsilon \prod_{k=1}^{\nu-1} (1 + M_{k}) \).

Therefore, \( t_{\nu} \) is monotonically increasing with \( \nu \) and we see that the norm of \((VI, 29)\) corresponding to \( p = 1 \) has the value

\( (VI, 32) \quad \prod_{\nu=1}^{n-1} (1 + M \nu + \epsilon) - \prod_{\nu=1}^{n-1} (1 + M \nu) \)

and obtain therefore

\( (VI, 33) \quad \| A^{-1} - A^{(\alpha)-1} \|_1 \leq \prod_{\nu=1}^{n-1} (1 + M \nu + \epsilon) - \prod_{\nu=1}^{n-1} (1 + M \nu) \)

as the solution of our problem. In applications it may be better to use the recurrent formula \((VI, 31)\). If all \( M_{\mu} \) have the same value \( M \), the expression \((VI, 26)\) coincides with that obtained in IV for \( \Delta_{n}^{-1} \) under the hypothesis \( m = 0 \). But in this case we see from \((II, 28)\) and \((II, 29)\) that the row sum run through the same values as the sums of the columns. We obtain therefore in this case the expression

\( (M+1+\epsilon)^{n-1} - (M+1)^{n-1} = (n-1) \epsilon (M+1+\epsilon) \), \( 0 < \epsilon < 1 \),

as the norm of \((VI, 29)\); both for \( p = 1 \) and \( p = \infty \).
VII. Linear systems with a Nearly Triangular Matrix

The results of the preceding sections give the means to discuss the following problem concerning the system (1) in the introduction under the conditions (2) and the "triangular" system.

\[(\text{VII,1}) \sum_{v=\mu}^{n} a_{\mu v} x_v = x_\mu \quad (\mu = 1, \ldots, n),\]

with the matrix \(A^{(0)}\). In discussing this problem we can obviously assume that

\[a_{\mu \mu} = 1 \quad (\mu = 1, \ldots, n).\]
Then the difference between the solutions (1) and of (VII,1) is given by the vector

\((A^{-1} - A(0)^{-1}) \gamma_0, \quad \gamma_0 = (\gamma_1, \ldots, \gamma_n)\)

and the norm of this vector corresponding to one of the indices \(p = 1, \infty\) does not exceed

\[ |A^{-1} - A(0)^{-1}|_p \| \gamma_0 \|_p \quad (p = 1, \infty), \]

and can indeed for suitable choice of the vector \(\gamma_0\) attain this limit. Therefore the norms \(|A^{-1} - A(0)^{-1}|_p\) measure the error committed in replacing the system (7) by (VII,1).

For an \(\epsilon > 0, M > 0\) being given, how small must \(m > 0\) be taken in order that we have

\[ (VII,2) \quad |A^{-1} - A(0)^{-1}|_p \leq \epsilon \quad (p = 1, \infty). \]

If we introduce the quantities \(|\Delta_n|\) and \(\delta\) corresponding by (II,2), (II,3) and (II,7) to \(m\) and \(M\), we obtain from (V,9) and (VI,7) the condition

\[ (1 + M)^{n-1} \frac{\delta}{1 - \delta} \leq \epsilon \]

\[ (VII,3) \quad \delta \leq \frac{\epsilon}{(1 + M)^{n-1} \epsilon} \quad (|\Delta_n| \leq 1 - \frac{m}{M}) \]

as long as the condition (V,5) is satisfied and therefore certainly as long as \(M > \frac{1.5}{n} (n \geq 4)\).

On the other hand we have by (2,10)

\[ (VII,4) \quad m = \frac{\delta}{Mn + \epsilon \frac{\delta}{M}} \]

and from (VII,3) and (VII,4)

\[ (VII,5) \quad m \leq \frac{\epsilon}{Mn (1 + M)^{n-1} (M + \frac{\theta}{M})^2} \quad (|\Delta_n| \leq 1 - \frac{m}{M}). \]
It will be therefore sufficient for (VII, 2) to take

\[(\text{VII, 6}) \quad m \leq m_0 \equiv \frac{\varepsilon}{M_n(1+M)^{n-1}(M_n+\frac{1}{n})\varepsilon} \quad (|A_n| \leq 1 - \frac{M}{n}).\]

solve our problem, for instance, if we have \(M \geq \frac{1.5}{n} \quad (n \geq 4)\) or \(M \geq .5321 \quad (n = 3)\). For small values of \(\varepsilon\) obviously, only the first term in the denominator is essential and we have

\[(\text{VII, 7}) \quad m_0 = K(n, M)\varepsilon \quad , \quad K(n, M) = \frac{1}{M_n(1+M)^{n-1}}\]

The tables I and III give the values of \(K(n, M)\) for a set of integer \(M\) from \(1\) to \(10\) and some values of \(M > \frac{1.5}{n}\). We have obviously

\[(\text{VII, 8}) \quad K(n, M) < \frac{1}{M^{2n-2}}\]

The bound in (VII, 6) is obviously the "best" under the condition (V, 5), save that the factor \(\frac{1}{M}\) of \(\varepsilon\) in the denominator could be replaced by an (unknown) fraction of it.

If \(M < \frac{1.5}{n}\), the use of the general formula (V, 17) is very cumbersome; we can however obtain a good working limit for \(m\) in the following way. If the inequalities (2) are valid for an \(M = \frac{1.5}{n} = M^{(n)}\), then are also satisfied if \(M\) is replaced by \(M^{(n)}\), but then the limit for \(m\) obtained for \(M^{(n)}\) is also sufficient for our \(M\). We obtain therefore in this case for \(m\) the sufficient condition for (VII, 2).
\begin{equation}
\begin{aligned}
(\text{VII},9) \quad m \leq m'_0 \equiv \frac{\epsilon}{M_n^{(n)}(1 + M^{(n)})^{n-1} + (M_n^{(n)} + \frac{1}{M_n}) \epsilon}
\end{aligned}
\end{equation}

The table II gives the values of $K(n, M^{(n)})$ for $m = 1, 2, \ldots, 50$.

The expression for $m'_0$ can be written in introducing the value of $M_n^{(n)}$ as

\begin{equation}
\frac{3}{2n} \cdot \frac{\epsilon}{(1 + \frac{1.5}{n})^{n-1}((1 + \frac{1.5}{n})^{n-2} - 2.5) + ((1 + \frac{1.5}{n})^{n} - 1.5) \epsilon}
\end{equation}
We observe now that for any positive \( \alpha \) and positive \( x \) the expression \((1 + \frac{x}{n})^n \cdot x\) monotonically increases and tends to \( e^x \) if the positive \( n \) increases monotonically to \( \infty \). Indeed, if we put \( u = \frac{1}{n} \), take the logarithm of this expression, differentiate it with respect to \( u \) and multiply by \( u^2 \), we obtain \( \frac{x(u - \alpha u^2)}{1 + u x} - \log(1+ux) \); but this expression vanishes for \( u = 0 \) and decreases for positive \( u \) since its derivative is
\[
-\frac{xu}{(1 + ux)^2} \quad (2x + x + \alpha x u)
\]
We see therefore that \((1 + \frac{1.5}{n})^{n-1} \uparrow e^{1.5}, (1 + \frac{1.5}{n})^n \uparrow e^{1.5}\)
and our bound for \( m \) can therefore be replaced by
\[
\frac{3}{2n} \cdot \frac{\epsilon}{e^{1.5}(e^{1.5} - 2.5) + (e^{1.5} - 1.5)\epsilon} = \frac{\epsilon}{(5.93 + 1.99)\epsilon} = \frac{\epsilon}{(5.93 + 1.99)\epsilon} = \frac{\epsilon}{5.93 + 1.99}
\]
we can replace therefore the condition \( m \leq m' _0 \) by the simpler condition
\[
(VII,10) \quad m \leq m_0 = \frac{\epsilon}{n} \cdot \frac{1}{6 + 2\epsilon}
\]
On the other hand we obtain at once the solution of our problem in the case \( m = M < \frac{1}{n-1} \) from (V,17).
\[
(VII,11) \quad m \leq m_0 \leq \frac{1}{n-1} \cdot \frac{\epsilon}{1+\epsilon} \quad (m = M < \frac{1}{n-1}) \quad .
\]
This is an "exact" condition, while the condition (VII,10) is only an estimate. We see however, in comparing (VII,10) with (VII,11) that the bound in (VII,10) cannot be improved by a factor \( > 6 \).
From the condition (VII,10) we can finally derive the inequality (7) of the introduction. Indeed, if positive \( m_0 < \frac{1}{2n} \) is given, we can solve (VII,10) with respect to \( \varepsilon \),

\[
\varepsilon = \frac{6 \cdot n \cdot m}{1 - 2n \cdot m_0}
\]

and obtain therefore in applying (VII,10) for \( m = m_0 \) the inequality

\[
| A^{-1} - A(0)^{-1} |_{\infty} \leq \frac{6 \cdot n \cdot m_0}{1 - 2n \cdot m_0}
\]

which holds for all positive \( m_0 < \frac{1}{2n} \) and gives the inequality (7) if we replace here \( m_0 \) by \( m \).
Bibliography


Foot Notes

1) This condition \((\delta^0)\) is already contained in some results of my paper [2], and can be also deduced from a well known theorem of the theory of determinants discussed in [4].

2) Some of the results contained in the section I - IV have been published without proof in my note [3].

3) For \(M = m\), \(8(m,M)\) becomes \((n-1)m\), but in this case our result is obtained in Gerschgorin's theorem cf. [4].

From the theorem c the result given by Stein and Rosenberg in [3a] as theorem III follows immediately.

2a) This formula can be also obtained from the formulae given in the proof of theorem III in [3a], p. 113.
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Table II

\[ M = M(n) = \frac{1.5}{n} \]

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THE NATIONAL BUREAU OF STANDARDS

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