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ON NEARLY TRIANGULAR MATRICES

By

A. M. Ostrowski

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Introduction

Consider a system of linear equations

$$(1)\qquad \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = y_\mu \qquad (\mu = 1,\ldots,n)$$

with the matrix A, in which all diagonal elements $a_{\mu\mu}$ $(\mu = 1,\ldots,n)$ are $\neq 0$ and the elements off the diagonal satisfy for two positive numbers m, M the inequalities

$$(2)\qquad \begin{aligned} |a_{\mu\nu}| &\leqq m|a_{\mu\mu}| \quad (\nu<\mu;\ \mu = 1,\ldots,n)\ , \\ |a_{\mu\nu}| &\leqq M|a_{\mu\mu}| \quad (\nu>\mu;\ \mu = 1,\ldots,n)\ . \end{aligned}$$

If n is very small, the system does not essentially differ from the corresponding "triangular" system in which all $a_{\mu\nu}$ with $\nu<\mu$ are replaced by zeros and the matrix of which will be denoted by $A^{(0)}$. It appears then plausible that the solutions of this "triangular" system does not differ very much from that of the system (1).

However the value of the determinant of the order n

$$\begin{vmatrix} 1 & -M & 0 & \cdots & 0 & 0 \\ 0 & 1 & -M & \ddots & & 0 \\ & & \ddots & \ddots & & \\ 0 & & & \ddots & 1 & -M \\ -m & 0 & \cdots & & 0 & 1 \end{vmatrix} = 1 - mM^{n-1},$$

shows that if M is for instance ≥ 10, the determinant of our system will not be even necessarily different from zero unless $m < 10^{-(n-1)}$. A detailed study of the problems connected with the matrices characterized by (2) appears therefore to be of importance and interest.

As the first problem in this connection we give a necessary and sufficient condition for the determinants of all matrices A satisfying the conditions (2) being non singular. This condition is, if $m < M$, given by

(3) $$\frac{m}{(1+m)^n} < \frac{M}{(1+M)^n},$$

and, if $m = M$, by

(3°) $$m < \frac{1}{n-1}$$ [1)].

In order to obtain a precise measure of the influence of the change from A to $A^{(0)}$, we have to discuss the estimates for convenient norms of the matrix $A^{-1} - A^{(0)-1}$.

We consider in particular two such norms defined in the section IV and denoted by $|A^{-1} - A^{(0)-1}|_p$ $(p = 1, \infty)$, which are particularly suitable for the problems of Numerical Analysis. We show then, in assuming without loss of generality that $a_{\mu\mu} = 1$ $(\mu = 1, \ldots, n)$, that for given values of m, M we have, if $M \geqslant \frac{1.5}{n}$, $n \geqq 4$:

(4) $$|A^{-1} - A^{(0)-1}|_p = (1+M)^{n-1}\frac{\delta}{1-\delta} \qquad \left(M \geqq \frac{1.5}{n}, n \geqq 4\right)$$

where $1-\delta$ is the smallest modulus of the determinant attainable for the matrices A and is connected with m by the relation

(5) $$\delta = \frac{M_n\, m}{1-\theta\frac{m}{M}}, \quad 0<\theta<1$$

where

(6) $$M_n = \frac{(1+M)^n - n\,M - 1}{M}.$$

If $M < \frac{1.5}{n}$, the formula (4) need not be valid any longer, but we can prove in this case the relation

(7) $$|A^{-1} - A^{(0)-1}|_p \leqslant \frac{6nm}{1-2nm} \qquad (M \leqq 1.5/n)$$

valid as long as m remains less than $\frac{1}{2n}$.

The estimate (7) is not a "best" estimate for all values of $M \leqq \frac{1.5}{n}$ but still it is not very far from the best, since for $m = M \leqq \frac{1}{n-1}$ we have

(8) $$|A^{-1} - A^{(0)-1}|_p \leqq \frac{(n-1)m}{1-(n-1)m} \qquad \left(m = M < \frac{1}{n-1}\right),$$

which cannot be improved for any value of $m < \frac{1}{n-1}$.

The condition (3) is derived in the section III of this paper, theorem B. However, we derive it as a special case of a more general theorem concerning the case where in the inequalities (2) the expressions m, M depend on μ , that is to say, change from one row to another. The necessary and sufficient condition of all matrices A being regular(theorem A, section III) is in this case rather unwieldly, but may be still very useful in some cases, since it contains 2n-2 instead of two essential parameters. The direct derivation of theorem B is of course much simpler since, as the reader will immediately see, the computations of the determinant Ω_n in the section I can be considerable shortened in this case. The connection between the formal algebra of the section I and II and the theorems A and B is provided by a result concerning the so-called H-determinants and M-determinants which I published 15 years ago[1]. The results about the norms $|A^{-1} - A^{(0)-1}|_p$ are obtained in using the explicit representation of the inverse matrix of a certain matrix Δ_n which provides a majorant for all matrices A^{-1}. The formulae giving Δ_n^{-1} are derived in the second part of the section II, and in the section V the norms $|\Delta_n^{-1} - \Delta_n^{(0)-1}|_p$ are derived and discussed. The corresponding inequalities for $|A^{-1} - A^{(0)-1}|_p$ are then obtained in the section VII, in using a new theorem (lemma III) concerning the connection between the H-determinants and the M-determinants.

We give in this section another application of this theorem in estimating the variation in the inverse matrix of a triangular matrix satisfying the conditions (2) with $m = 0$. We obtain an unexpected simple and elegant formula (VI,33).

In the section VII we apply our results explicitly to the problems concerning the linear system (1). It may be finally remarked that our results remain with obvious changes valid if in the matrix of (1) the rows and the columns are interchanged, although we did not care to mention it explicitly at every step [2].

I. The Value of the Determinant Ω_n

Let K_n be defined by

$$(\mathrm{I},1) \qquad K_n = \begin{vmatrix} \kappa_n & \delta_n & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \kappa_{n-1} & \kappa_{n-1} & \delta_{n-1} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \kappa_{n-2} & \kappa_{n-2} & \kappa_{n-2} & \delta_{n-2} & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \kappa_2 & \kappa_2 & \kappa_2 & \kappa_2 & \cdot & \cdot & \cdot & \kappa_2 & \delta_2 \\ \kappa_1 & \kappa_1 & \kappa_1 & \kappa_1 & \cdot & \cdot & \cdot & \kappa_1 & \kappa_1 \end{vmatrix} \quad (n \geqq 3),$$

$$K_1 = \kappa_1 \quad , \quad K_2 = \begin{vmatrix} \kappa_2 & \delta_2 \\ \kappa_1 & \kappa_1 \end{vmatrix} = \kappa_1 (\kappa_2 - \delta_2) ,$$

where in the μ-th row all elements to the left of the main diagonal and on this diagonal are equal to $\kappa_{n-\mu+1}$, the next element to the right is $\delta_{n-\mu+1}$ and all other elements are 0. κ_μ, δ_μ are here independent variables. In subtracting the second column from the first we obtain $K_n = (\kappa_n - \delta_n) K_{n-1}$ and therefore the following formula, valid also for n = 1, 2:

$$(\mathrm{I},2) \qquad K_n = \kappa_1 \prod_{\nu=2}^{n} (\kappa_\nu - \delta_\nu) .$$

Consider now for $n \geq 3$ the determinant

$$(I,3) \qquad T_n = \begin{vmatrix} \delta_n & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \beta_n \\ \gamma_{n-1} & \delta_{n-1} & 0 & 0 & \cdot & \cdot & \cdot & 0 & \beta_{n-1} \\ \gamma_{n-2} & \gamma_{n-2} & \delta_{n-2} & 0 & \cdot & \cdot & \cdot & 0 & \beta_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma_2 & \gamma_2 & \gamma_2 & \gamma_2 & \cdot & \cdot & \cdot & \delta_2 & \beta_2 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 0 \end{vmatrix} \qquad (n \geqq 3) \text{ .}$$

We have in particular

$$(\mathrm{I},4)\qquad T_3 \begin{vmatrix} \delta_3 & 0 & \beta_3 \\ \gamma_2 & \delta_2 & \beta_2 \\ 1 & 1 & 0 \end{vmatrix} = -\left(\delta_3\beta_2 + \beta_3(\delta_2 - \gamma_2)\right).$$

Developing T_n in the elements of the first line and using the value (I,2) of K_n we obtain for $n \geqq 4$

$$T_n = \delta_n T_{n-1} - \beta_n \prod_{\nu=2}^{n-1} (\delta_\nu - \gamma_\nu) ,$$

$$\frac{T_n}{\delta_2 \cdots \delta_n} = \frac{T_{n-1}}{\delta_2 \cdots \delta_{n-1}} - \frac{\beta_n}{\delta_n} \prod_{\nu=2}^{n-1} \left(1 - \frac{\gamma_\nu}{\delta_\nu}\right) ,$$

and therefore generally for $n \geqq 4$

$$(\mathrm{I},5)\quad \frac{T_n}{\delta_2 \cdots \delta_n} = -\left[\frac{\beta_n}{\delta_n} \prod_{\nu=2}^{n-1}\left(1 - \frac{\gamma_\nu}{\delta_\nu}\right) + \frac{\beta_{n-1}}{\delta_{n-1}} \prod_{\nu=2}^{n-2}\left(1 - \frac{\gamma_\nu}{\delta_\nu}\right) + \cdots + \frac{\beta_4}{\delta_4} \prod_{\nu=2}^{3}\left(1 - \frac{\gamma_\nu}{\delta_\nu}\right)\right] + \frac{T_3}{\delta_2 \delta_3} .$$

Since by (I,4)

$$\frac{T_3}{\delta_2 \delta_3} = -\frac{\beta_3}{\delta_3}\left(1 - \frac{\gamma_2}{\delta_2}\right) - \frac{\beta_2}{\delta_2}$$

we obtain

$$\frac{-T_n}{\delta_2 \cdots \delta_n} = \sum_{\mu=2}^{n} \frac{\beta_\mu}{\delta_\mu} \prod_{\nu=2}^{\mu-1} \left(1 - \frac{\gamma_\nu}{\delta_\nu}\right) ,$$

where $\prod_{\nu=2}^{1}$ is identically 1, and therefore finally

$$(\mathrm{I},6) \qquad T_n = -\sum_{\mu=2}^{n} \beta_\mu \prod_{\nu=\mu+1}^{n} \delta_\nu \prod_{\nu=2}^{\mu-1} (\delta_\nu - \gamma_\nu) .$$

If we now put

$$(\mathrm{I},7) \qquad T_n^* = \begin{vmatrix} \delta_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \beta_1 \\ \gamma_2 & \delta_2 & 0 & \cdot & \cdot & \cdot & 0 & \beta_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \gamma_{n-1} & \gamma_{n-1} & \gamma_{n-1} & \cdot & \cdot & \cdot & \gamma_{n-1} & \beta_{n-1} \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 0 \end{vmatrix} \quad (n \geqq 3),$$

$$T_3^* = \begin{vmatrix} \delta_1 & 0 & \beta_1 \\ \gamma_2 & \delta_2 & \beta_2 \\ 1 & 1 & 0 \end{vmatrix} ,$$

this goes over into T_n if the indices of β_ν, γ_ν, δ_ν are replaced by their complements with respect to $n + 1$. We obtain then from (I,6)

$$T_n^* = -\sum_{\mu=2}^{n} \beta_{n+1-\mu} \prod_{\nu=\mu+1}^{n} \delta_{n+1-\nu} \prod_{\nu=2}^{\mu-1} (\delta_{n+1-\nu} - \gamma_{n+1-\nu}) ;$$

or in replacing the summation index μ by $n + 1 - \kappa$

$$T_n^* = -\sum_{\kappa=1}^{n-1} \beta_\kappa \prod_{\nu=n+2-\kappa}^{n} \delta_{n+1-\nu} \prod_{\nu=2}^{n-\kappa} (\delta_{n+1-\nu} - \gamma_{n+1-\nu}) ,$$

and finally, if in both products ν is replaced by $n + 1 - \lambda$,

$$(\mathrm{I},8) \qquad T_n^* = -\sum_{\kappa=1}^{n-1} \beta_\kappa \prod_{\lambda=1}^{\kappa-1} \delta_\lambda \prod_{\lambda=\kappa+1}^{n-1} (\delta_\lambda - \gamma_\lambda) .$$

Consider now for $n \geqq 3$ the determinant

$$(I,9) \qquad \Omega_n = \begin{vmatrix} \alpha_1 & -M_1 & -M_1 & \cdot & \cdot & \cdot & -M_1 \\ -m_2 & \alpha_2 & -M_2 & \cdot & \cdot & \cdot & -M_2 \\ -m_3 & -m_3 & \alpha_3 & \cdot & \cdot & \cdot & -M_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -m_n & -m_n & -m_n & \cdot & \cdot & \cdot & \alpha_n \end{vmatrix} .$$

If we subtract here the last column from each of the proceeding ones we obtain

$$\begin{vmatrix} M_1+\alpha_1 & 0 & 0 & \cdot & \cdot & 0 & -M_1 \\ M_2-m_2 & M_2+\alpha_2 & 0 & \cdot & \cdot & 0 & -M_2 \\ M_3-m_3 & M_3-m_3 & M_3+\alpha_3 & \cdot & \cdot & 0 & -M_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ M_{n-1}-m_{n-1} & M_{n-1}-m_{n-1} & M_{n-1}-m_{n-1} & \cdot & \cdot & M_{n-1}-m_{n-1} & -M_{n-1} \\ -(\alpha_n+m_n) & -(\alpha_n+m_n) & -(\alpha_n+m_n) & \cdot & & -(\alpha_n+m_n) & \alpha_n \end{vmatrix}$$

Here the subdeterminant corresponding to the last element of the last row is obviously $\prod_{\nu=1}^{n-1}(M_\nu+\alpha_\nu)$, so that Ω_n is the sum of $\alpha_n\prod_{\nu=1}^{n-1}(M_\nu+\alpha_\nu)$ and

$$-(\alpha_n+m_n)\begin{vmatrix} M_1+\alpha_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -M_1 \\ M_2-m_2 & M_2+\alpha_2 & 0 & \cdot & \cdot & \cdot & 0 & -M_2 \\ M_3-m_3 & M_3-m_3 & M_3+\alpha_3 & \cdot & \cdot & \cdot & 0 & -M_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ M_{n-1}-m_{n-1} & M_{n-1}-m_{n-1} & M_{n-1}-m_{n-1} & \cdot & \cdot & \cdot & M_{n-1}+\alpha_{n-1} & -M_{n-1} \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 0 \end{vmatrix}$$

This last determinant becomes $T_n{}^*$ if we put

$$\delta_\nu = M_\nu + \alpha_\nu, \quad \gamma_\nu = M_\nu - m_\nu, \quad \beta_\nu = M_\nu \quad (\nu = 1, \ldots, n-1),$$

and has therefore by (I,8) the value

$$-\sum_{\kappa=1}^{n-1} (-M_\kappa) \prod_{\lambda=1}^{\kappa-1} (M_\lambda + \alpha_\lambda) \prod_{\lambda=\kappa+1}^{n-1} (m_\lambda + \alpha_\lambda).$$

We obtain therefore for Ω_n the expression

$$\text{(I,10)} \quad \Omega_n = \alpha_n \prod_{\nu=1}^{n-1} (M_\nu + \alpha_\nu) - \sum_{\kappa=1}^{n-1} M_\kappa \prod_{\lambda=1}^{\kappa-1} (M_\lambda + \alpha_\lambda) \prod_{\lambda=\kappa+1}^{n} (m_\lambda + \alpha_\lambda).$$

The determinant Ω_n can be also written in the form

$$\text{(I,11)} \qquad \Omega_n = \begin{vmatrix} \alpha_n & -m_n & -m_n & \cdot & \cdot & \cdot & -m_n & -m_n \\ -M_{n-1} & \alpha_{n-1} & -m_{n-1} & \cdot & \cdot & \cdot & -m_{n-1} & -m_{n-1} \\ -M_{n-2} & -M_{n-2} & \alpha_{n-2} & \cdot & \cdot & \cdot & -m_{n-2} & -m_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -M_2 & -M_2 & -M_2 & \cdot & \cdot & \cdot & \alpha_2 & -m_2 \\ -M_1 & -M_1 & -M_1 & \cdot & \cdot & \cdot & -M_1 & \alpha_1 \end{vmatrix}$$

and we obtain therefore from (I,10)

$$(\mathrm{I},12)\qquad \Omega_n = \alpha_1 \prod_{\nu=2}^{n} (m_\nu + \alpha_\nu) - \sum_{\kappa=2}^{n} m_\kappa \prod_{\mu=\kappa+1}^{n} (m_\mu + \alpha_\mu) \prod_{\mu=1}^{\kappa-1} (M_\mu + \alpha_\mu).$$

II. The Matrix Δ_n and its Inverse

We consider now the matrix

$$(\mathrm{II},1)\quad \Delta_n = \begin{pmatrix} 1 & -M & -M & . & . & -M \\ -m & 1 & -M & . & . & -M \\ . & . & . & . & . & . \\ -m & -m & -m & . & . & 1 \end{pmatrix}$$

Its determinant is obtained from Ω_n in putting in (I,12)

$$\alpha_1 = \alpha_2 = \ldots \alpha_n = 1;\quad m_2 = m_3 = \ldots = m_n = m;\quad M_1 = M_2 = \ldots = M_{n-1} = M.$$

We obtain

$$|\Delta_n| = (m+1)^{n-1} - m \sum_{\kappa=2}^{n} (m+1)^{n-\kappa} (M+1)^{\kappa-1}$$

and this becomes, if $M \neq m$, $= (m+1)^{n-1} - m(M+1)\ \dfrac{(M+1)^{n-1} - (m+1)^{n-1}}{M-M} =$

$$= \frac{1}{M-m} \left[(m+1)^{n-1}(M-m+m(M+1)) - m(M+1)^n\right],$$

$$(\mathrm{II},2)\quad |\Delta_n| = \frac{1}{M-m}\left[M(m+1)^n - m(M+1)^n\right] \qquad (m \neq M),$$ [2a)]

while for m = M we obtain from (II,2) in letting $M \to m$:

(II,3) $$|\Delta_n| = (1-(n-1)m)(1+m)^{n-1} \quad (m=M).$$

In particular we see that under hypothesis $m < M$ necessary and sufficient for $|\Delta_n| > 0$ is

(II,4) $$\frac{m}{(m+1)^n} < \frac{M}{(M+1)^n}$$

We assume now in particular

(II,5) $$m < M \quad .$$

If we introduce the abbreviations

(II,6) $$M_n = \frac{(1+M)^n - nM - 1}{M}, \quad m_n = \frac{(1+m)^n - nM - 1}{m},$$

we can write (II,2) in the form

$$(M-m)|\Delta_n| = M(1+nm+mm_n) - m(1+nM+MM_n) = M-m - mM(M_n - m_n)$$

and therefore, it we put

$$\delta = 1 - |\Delta_n|, \tag{II,7}$$

$$(M-m)\,\delta = m\,M(M_n - m_n)$$

$$\frac{\delta}{mM} = \frac{M_n - m_n}{M - m} \tag{II,8}$$

It follows from (II,5) for $n > 2$

$$\frac{M_n - m_n}{M - m} = \sum_{\nu=2}^{n} \binom{n}{\nu} \frac{M^{\nu-1} - m^{\nu-1}}{M - m} > \sum_{\nu=2}^{n} \binom{n}{\nu} M^{\nu-2} = \frac{(1+M)^n - 1 - nM}{M^2} = \frac{M_n}{M} \; ;$$

so that from (II,8) we have

(II,9) $$\frac{M_n}{1-\frac{m}{M}} > \frac{\delta}{m} > M_n \ .$$

It follows in particular if (II,4) holds $\delta > 0$, $0 < |\Delta_n| < 1$.

In solving (II,9) with respect to m we obtain

(II,10) $$\frac{\delta}{M_n + \frac{\delta}{M}} < m < \frac{\delta}{M_n} \ , \quad m = \frac{\delta}{M_n + \theta\frac{\delta}{M}} \ , \quad 0 < \theta < 1 \ .$$

(II,10°) $$M_n m < \delta < \frac{M_n m}{1-\frac{m}{M}} \ , \qquad \delta = \frac{M_n m}{1-\theta\frac{m}{M}} \ , \quad 0 < \theta < 1 \ .$$

In order to discuss the meaning of the condition (II,4), consider the curve

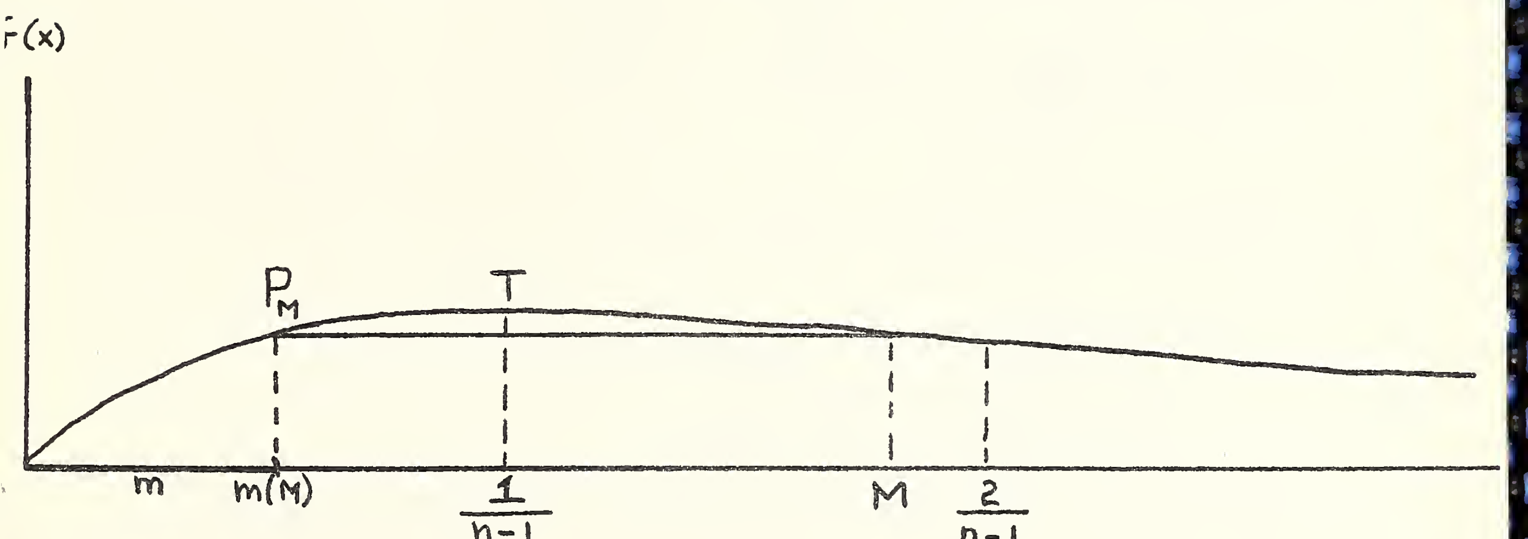
(II,11) $$y = f(x) = \frac{x}{(1+x)^n} \ ;$$

we have

$$f'(x) = \frac{1-(n-1)x}{(1+x)^{n+1}}, \quad f''(x) = n\,\frac{(n-1)x-2}{(1+x)^{n+2}}$$

For $x > 0$ the curve (II,11) has one maximum at $x = \frac{1}{n-1}$ and an inflection point at $x = \frac{2}{n-1}$.

In the diagram 1 the curve (II,11) is drawn (computed for $n = 5$). The portion of this curve from the point 0 to the highest point T will be denoted as the <u>ascending</u> branch and the portion between T and $x = \infty$ as the <u>descending branch</u>.



$$f(X) = \frac{X}{(1+X)^5}$$

fig. 1

If we assume that in (II,4) m is less than M, we see that either

(II,12) $$0 < m < \frac{1}{n-1}, \quad M > \frac{1}{n-1}$$

or

(II,12°) $$m \quad M \quad \frac{1}{n-1}.$$

In order to find for a given $M > \frac{1}{n-1}$ the range of the values of m satisfying (II,4) we find on the curve the second point P_M with the same ordinate as the point $x = M$, then if to P_M corresponds the abcissa $m(M)$ we have $0 < m < m(M)$.

We prove finally that from (II,4) and $m < M$ always follows

(II,13) $$m\,M^{n-1} < 1 .$$

Indeed in proving (II,13) we can obviously assume that M is > 1 and therefore on the descending branch in the diagram but then we have

$$\frac{1}{1+\frac{1}{M^{n-1}}} = \frac{M^{n-1}}{1+M^{n-1}} > \frac{M}{1+M}$$

and therefore in raising it to the nth power

$$\frac{\frac{1}{M^{n-1}}}{\left(1+\frac{1}{M^{n-1}}\right)^n} > \frac{M}{(1+M)^n} ;$$

we see that $\frac{1}{M^{n-1}}$ must lie in the fig. 1 between M and m(M) and it follows $\frac{1}{M^{n-1}} > m$, that is (II,13).

We are going now to determine the inverse matrix

(II,14) $$\Delta_n^{-1} = (\alpha_{\mu\nu}) \qquad (\mu, \nu = 1, \dots, n)$$

of Δ_n, if its determinant (II,2) resp. (II,3) is $\neq 0$. It is sufficient for this purpose to solve the corresponding linear system

$$\text{(II,15)}\quad x_\mu - m\sum_{\nu=1}^{\mu-1} x_\nu - M\sum_{\nu=\mu+1}^{n} x_\nu = z_\mu \qquad (\mu = 1,\dots,n)$$

for indeterminants z_μ . Put

$$\text{(II,16)}\quad \begin{aligned} &x_1+\dots\dots+x_n = s, \quad \frac{m-M}{M+1} = \sigma, \\ &\frac{M}{1+M} = \lambda, \quad 1+\sigma = \frac{1+m}{1+M} = q, \\ &\frac{z_\nu}{1+M} = \alpha_\nu, \quad \alpha_\nu + \lambda s = y_\nu . \end{aligned}$$

Then the system (II,15) is equivalent with

$$(1+M)x_\mu = (m-M)(x_1+\dots+x_{\mu-1}) + Ms + z_\mu \quad (\mu=1,\dots,n)$$

$$x_\mu = \sigma(x_1+\dots+x_{\mu-1}) + y_\mu \qquad (\mu=1,\dots,n)$$

or in putting

$$S_\nu = x_1+\dots\dots+x_\nu \ (\nu=1,\dots,n), \quad S_0 = 0,$$

$$S_\mu - S_{\mu-1} = \sigma S_{\mu-1} + y_\mu, \quad S_\mu = (1+\sigma)S_{\mu-1} + y_\mu = q\,S_{\mu-1} + y_\mu,$$

$$S_\mu q^{-\mu} = S_{\mu-1} q^{-\mu+1} + y_\mu q^{-\mu} \quad (\mu = 1,\dots,n)$$

$$S_\mu q^{-\mu} = \sum_{\nu=1}^{\mu} y_\nu q^{-\nu},$$

(II,17) $$S_\mu = \sum_{\nu=1}^{\mu} y_\nu q^{\mu-\nu} \qquad (\mu = 1, \ldots, n)$$

(II,18) $$x_\mu = \sum_{\nu=1}^{\mu-1} y_\nu (q-1) q^{\mu-\nu-1} + y_\mu \qquad (\mu = 1, \ldots, n).$$

From (II,17) for $\mu = n$ we have by (II,16)

$$S_n = s = \sum_{\nu=1}^{n} y_\nu q^{n-\nu} = \lambda s \sum_{\nu=1}^{n} q^{n-\nu} + \sum_{\nu=1}^{n} \alpha_\nu q^{n-\nu},$$

(II,19) $$s = \lambda s \frac{q^n - 1}{q-1} + \sum_{\nu=1}^{n} \alpha_\nu q^{n-\nu},$$

where for m = M, q = 1 the coefficient of λs is to be replaced by n. Since for $m \neq M$

$$1 - \lambda \frac{q^n - 1}{q - 1} = \frac{m - Mq^n}{m - M} = \frac{Mq^n - m}{M - m}$$

we have from (II,19) and (II,2)

(II,20) $$s = Q \sum_{\nu=1}^{n} \alpha_\nu q^{n-\nu} \qquad (m \neq M),$$

(II,21) $$Q = \frac{M - m}{Mq^n - m} = \frac{(M+1)^n}{|\Delta_n|} \qquad (m \neq M)$$

while for m = M we obtain again (II,20) in defining Q by

(II,21°) $$Q = \frac{m+1}{1-(n-1)m} = \frac{(m+1)^n}{|\Delta_n|} \qquad (m = M).$$

If we now replace in (II,18) Y_ν by $\lambda s + \alpha_\nu$ and use for s the expressions from (II,20)

$$x_\mu = (q-1)\sum_{\nu=1}^{\mu-1}\alpha_\nu q^{\mu-\nu-1} + \alpha_\nu + \lambda s\left[(q-1)\sum_{\nu=1}^{\mu-1} q^{\mu-\nu-1} + 1\right].$$

But here the expression within the brackets is $q^{\mu-1}$ and we obtain therefore

$$x_\mu = \sum_{\nu=1}^{\mu-1}\alpha_\nu q^{\mu-\nu-1}(q-1) + \alpha_\mu + \lambda Q q^{\mu-1} + \sum_{\nu=1}^{n}\alpha_\nu q^{n-\nu} \qquad (\mu=1,\dots,n),$$

and finally in introducing α_ν from (II,16)

$$\text{(II,22)} \qquad x_\mu = \sum_{\nu=1}^{\mu-1} z_\nu q^{\mu-\nu-1}\frac{q-1}{M+1} + \frac{z_\mu}{M+1} + \frac{\lambda Q q^{\mu-1}}{M+1}\sum_{\nu=1}^{n} z_\nu q^{n-\nu} \qquad (\mu=1,\dots,n).$$

Introduce the two following triangular matrices

$$\text{(II,23)} \qquad U_n(q) = (u_{\mu\nu}) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & \cdots & \vdots \\ q & 1 & 0 & & & \vdots \\ q^2 & q & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & q^2 & q & 1 & 0 \end{pmatrix},$$

$$\text{(II,24)} \qquad u_{\mu\nu} = \begin{cases} q^{\mu-\nu-1} & (\mu > \nu) \\ 0 & (\mu \leqq \nu) \end{cases}$$

$$\text{(II,25)} \qquad V_n(q) = (v_{\mu\nu}) = \begin{pmatrix} 0 & q^{-1} & q^{-2} & q^{-3} \cdots & q^{-n+1} \\ 0 & 0 & q^{-1} & q^{-2} \cdots & q^{-n+2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 \cdots & 0 \end{pmatrix}$$

(II,26) $$v_{\mu\nu} = \begin{cases} 0 & (\mu \geqq \nu) \\ q^{\mu-\nu} & (\mu < \nu) \end{cases}$$

We see then that in (II,22) the matrix corresponding to the first right hand side is

$$\frac{q-1}{M+1} U_n = \frac{m-M}{(M+1)^2} U_n$$

while to the second right hand sum in (II,22) corresponds the matrix

$$\frac{\lambda Q}{M+1} q^{n-1} (q\, U_n + E_n + V_n) .$$

We obtain

$$\Delta_n^{-1} = \left(\frac{m-M}{(M+1)^2} + \frac{MQq^n}{(M+1)^2}\right) U_n + \left(\frac{1}{M+1} + \frac{MQq^{n-1}}{(M+1)^2}\right) E_n + \frac{MQq^{n-1}}{(M+1)^2} V .$$

The coefficient of U_n is here equal to $\frac{m\,Q}{(M+1)^2}$ both in the case (II,21) and (II,21°) and we have finally

(II,27) $$\Delta_n^{-1} = \frac{mQ}{(M+1)^2} U_n + \left(\frac{1}{M+1} + \frac{MQq^{n-1}}{(M+1)^2}\right) E_n + \frac{MQq^{n-1}}{(M+1)^2} V_n .$$

We now denote by s_μ the sum of the elements in the μth row of the matrix (II,27) and by t_μ the corresponding sum in the μ th column. If we assume first $m = M$, for the matrices U_n and V_n the μ th row terms are respectively equal to

$$\frac{q^{\mu-1}-1}{q-1} \; ; \; \frac{1-q^{\mu-n}}{q-1} ,$$

while the ν-th column sums are

$$\frac{q^{\mu-\nu}-1}{q-1} \; ; \; \frac{1-q^{1-\nu}}{q-1} .$$

We obtain now from (II,27) by (II,16)

$$(M+1)^2 s_\mu = mQ\frac{q^{\mu-1}-1}{q-1} + M+1 + MQq^{n-1} + MQ\frac{q^{n-1}-q^{\mu-1}}{q-1} =$$

$$= \frac{Q}{\sigma}\left[mq^{\mu-1}-m + Mq^n - Mq^{n-1} + Mq^{n-1} - Mq^{\mu-1}\right] + M + 1 =$$

$$= \frac{M+1}{m-M}Q\left[Mq^n - m + (m-M)q^{\mu-1}\right] + M + 1$$

and this is by (II,21) $= (M+1)\, Q\, q^{\mu-1}$. We have finally

$$\text{(II,28)} \qquad s_\mu = \frac{Q}{M+1}q^{\mu-1} = \frac{(M+1)^{n-1}}{|\Delta_n|}q^{\mu-1}$$

and this remains true also for $m = M$, as is immediately seen.

Similarly, we have

$$(M+1)^2 t_\nu = mQ\frac{q^{n-\nu}-1}{q-1} + M+1 + MQq^{n-1} + MQ\frac{q^{n-1}-q^{n-\nu}}{q} =$$

$$= \frac{Q}{\sigma}\left[mq^{n-\nu}-m + Mq^n - Mq^{n-1} + Mq^{n-1} - Mq^{n-\nu}\right] + M+1 =$$

$$= \frac{M+1}{m-M}Q\left[Mq^n - m + (m-M)q^{n-\nu}\right] + M+1 = (M+1)Qq^{n-\nu},$$

and we have

$$\text{(II,29)} \qquad t_\nu = \frac{Q}{M+1}q^{n-\nu} = \frac{(M+1)^{n-1}}{|\Delta_n|}q^{n-\nu},$$

a relation which is also immediately verified for the case $m = M$.

III. Bounds for the Determinants, depending on Ω_n and Δ_n

A determinant

$$(\mathrm{III},1) \qquad M = \begin{vmatrix} \alpha_1 & -m_{12} & \cdots & -m_{1n} \\ -m_{21} & \alpha_2 & \cdots & -m_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ -m_{n1} & -m_{n2} & \cdots & \alpha_n \end{vmatrix}$$

will be called an <u>M-determinant</u> if all diagonal elements α_ν are positive, all elements off the main diagonal, $-m_{\mu\nu}$ $(\mu \neq \nu)$ are not positive and the determinant M as well as of all principal (coaxial) minors of M of all degrees are positive.

In what follows we will have to use a theorem given by the author in 1937 [1] and which will be formulated as Lemma I. <u>If we have for the M-determinant (III,1) and a determinant</u> $H = |h_{\mu\nu}|$ $(\mu, \nu = 1, \ldots, n)$ <u>of the order n the inequalities</u>

$$(III,2) \qquad |h_{\nu\nu}| \geqq \alpha_\nu \,, \quad |h_{\mu\nu}| \leqq m_{\mu\nu} \quad (\mu \neq \nu; \mu, \nu = 1, \ldots, n),$$

then $H \neq 0$ and we have

$$(III,3) \qquad |H| \geqq M \,.$$

<u>f (H), (M) respectively denote the matrices of the determinants H and M, the inverse matrix of (H) is majorated by the inverse matrix of (M)</u>

$$(III,3^o) \qquad (H)^{-1} \ll (M)^{-1} \,.$$

Suppose now that in the determinant Ω_n given by (I,9) all α_ν are posi ive and m_ν , M_ν non negative: Then it follows from (I,10) that Ω_n is a monotonically decreasing function of all m_ν and from (I,12) that M is also monotonically decreasing in all M_ν . Suppose now that for a certain set of values of α_ν, m_ν , M_ν the determinant $\Omega_n \neq 0$.

Then replace the small m_ν and M_ν corresponding to certain rows $(\nu_1,\ldots,\nu_k)$ by zeros; the determinant Ω_n cannot decrease and remains therefore positive; but then Ω_n becomes equal to the product of $\alpha_{\nu_1}\alpha_{\nu_2}\ldots\alpha_{\nu_k}$ with the principal minor complementary to the set of indices $\nu_1,\ldots,\nu_k$. Therefore all principal minors of Ω_n are positive. We see that Ω_n <u>is an M-determinant, if the conditions (III,4) are satisfied and</u> $\Omega_n \neq 0$. But now we can easily deduce the following theorem.

A. Consider the set of all determinants $A = |a_{\mu\nu}|$ satisfying the conditions

(III,5) $$|a_{\mu\nu}| \leqq m_\mu \ (\nu < \mu), \quad |a_{\mu\nu}| \leqq M_\mu \ (\nu > \mu),$$
$$|a_{\mu\mu}| \geqq \alpha_\mu \qquad (\mu, \nu = 1, ---, n)$$

where α_μ are n given positive constants and m_μ, M_μ 2n-2 given non-negative constants. Then necessary and sufficient in order that all determinants A of this set do not vanish is the inequality $\Omega_n > 0$. If this condition is satisfied, we have

(III,6) $$|A| \geqq \Omega_n \quad .$$

Proof: Since in the case $\Omega_n > 0$, Ω_n is an M-determinant, the sufficiency of our condition follows immediately from the inequality (III,3) mentioned above.

If $\Omega_n = 0$ we can take obviously $a_{\mu\mu} = \alpha_\mu$ and $a_{\mu\nu} = m_\mu$ or $a_{\mu\nu} = M_\mu$ according as $\nu < \mu$ or $\nu > \mu$ and obtain a vanishing determinant A satisfying the condition (III,5).

Suppose now $\Omega_n < 0$, then it follows from the form (I,9) of Ω_n that Ω_n becomes positive if all m_μ are replaced by zeros. There exists therefore such a positive $t < 1$ that Ω_n vanishes if all m_μ in (I,9) are replaced by $t\,m_\mu$, but then we obtain a vanishing determinant A of our set in taking $a_{\mu\mu} = \alpha_\mu$ and $a_{\mu\nu} = t\,m_\mu$ or $a_{\mu\nu} = M_\mu$ according as $\nu > \mu$ or $\nu < \mu$. The theorem A is proved.

In specializing the matrix of Ω_n to Δ_n and in assuming that in particular $0 < m < M$ we obtain immediately from (II,2) and the theorem A:

B. Consider for two positive constants m, M, $m < M$ the set of all determinants A = [redacted] $|a_{\mu\nu}|$ of the n^{th} order for which

(III,7) $|a_{\mu\mu}| \geqq 1$ $(\mu = 1, ---, n)$, $|a_{\mu\nu}| \leqq m$ $(\nu < \mu)$, $|a_{\mu\nu}| \leqq M$ $(\nu > \mu)$, then necessary and sufficient in order that all determinants A of this set do not vanish is for m < M, the inequality

(III,8) $$\frac{m}{(1+m)^n} < \frac{M}{(1+M)^n} \quad (m < M),$$

and if this inequality is satisfied we have for each determinant A of the set

(III,9) $$|A| \geqq |\Delta_n| = \frac{(1+m)^n(1+M)^n}{M-m}\left[\frac{M}{(1+M)^n} - \frac{m}{(1+m)^n}\right].$$

If m = M, necessary and sufficient for all determinants A being ≠ 0 is

(III,8°) $$m < \frac{1}{n-1} \qquad (M = m)$$

and we have, if (III,8°) is satisfied,

(III,9°) $$|A| \geqq |\Delta_n| = (1-(n-1)m)(1+m)^{n-1} \qquad (m = M).$$

From the theorem B we can deduce the following theorem.

C. Let $A = (a_{\mu\nu})$ $(\mu, \nu = 1, \ldots, n)$ be a matrix satisfying the conditions

(III,10) $$|a_{\mu\nu}| \leqq m \quad (\nu < \mu), \quad |a_{\mu\nu}| \leqq M \quad (\nu > \mu),$$

where m, M are two constants with 0 < m < M.

Put

(III,11) $$\delta(m, M) = \frac{M m^{1/n} - m M^{1/n}}{M^{1/n} - m^{1/n}},$$

then all fundamental roots of the matrix A are contained in the set of the n closed circles described around the elements $a_{\mu\mu}$ with the radius $\delta(m,M)$. The value (III,11) of $\delta(m, M)$ is indeed assumed if $A = \Delta_n$ [3]).

Proof:

Let λ be a fundamental root of A so that the matrix λE-A is singular. Put $\operatorname{Min}_{\mu} |\lambda - a_{\mu\mu}| = \alpha$, we have to prove that $\alpha \leqq \delta(m,M)$. If $\alpha = 0$ there is nothing to prove. Suppose $\alpha > 0$ and consider the matrix

$$\frac{\lambda E - A}{\alpha} = (b_{\mu\nu}),$$

For this matrix we have

$$|b_{\mu\mu}| \geqq 1, \quad |b_{\mu\nu}| \leqq \frac{m}{\alpha} \quad (\nu < \mu), \quad |b_{\mu\nu}| \leqq \frac{M}{\alpha} \quad (\nu > \mu),$$

therefore we have by the theorem B

$$\frac{\frac{m}{\alpha}}{(1+\frac{m}{\alpha})^n} \geqq \frac{\frac{M}{\alpha}}{(1+\frac{M}{\alpha})^n}$$

and therefore

$$\frac{1+\frac{m}{\alpha}}{m^{1/n}} \leqq \frac{1+\frac{M}{\alpha}}{M^{1/n}},$$

$$\alpha\left(\frac{1}{m^{1/n}} - \frac{1}{M^{1/n}}\right) \leqq \frac{M}{M^{1/n}} - \frac{m}{m^{1/n}}$$

$$\alpha \leqq \frac{M - m}{m^{-1/n} - M^{-1/n}} = \delta(m, M).$$

IV. The Bounds of the Matrix Δ_n^{-1}

For an n dimensional vector $\xi = (x_1, \ldots, x_n)$ the Hölder norms corresponding to the exponent $p \geqq 1$ is given by

(IV,1) $$|\xi|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p} \quad (p \geqq 1).$$

We will only use the three cases corresponding to $p = 1,2,\infty$:

(IV,2) $$|\xi|_1 = |x_1| + \dots + |x_n|, \quad |\xi|_\infty = \operatorname*{Max}_{\nu} |x_\nu|, \quad |\xi|_2 = \sqrt{|x_1|^2 + \dots + |x_n|^2}.$$

We have between these three norms the following inequalities

(IV,3a) $$|\xi|_\infty \leqq |\xi|_1 \leqq n|\xi|_\infty,$$

(IV,3b) $$|\xi|_\infty \leqq |\xi|_2 \leqq \sqrt{n}\,|\xi|_\infty,$$

(IV,4) $$|\xi|_1 \leqq \sqrt{n}\,|\xi|_2 \leqq \sqrt{n}\,|\xi|_1\,,$$

which are immediately verified. The left hand inequality (IV, 3c) implies the well known inequality between the arithmetical mean and the arith - metical mean of the squares. If $A = (a_{\mu\nu})$ is an n x n matrix we define its norm corresponding to the exponent $p(1 \leqq p \leqq \infty)$ by

(IV,4) $$|A|_p = \operatorname*{Max}_{\xi \neq 0} \frac{|A\xi|_p}{|\xi|_p}$$

and denote it by $|A|_p$. We will use here too only the cases $p = 1,2,\infty$.

In applying to the definition (IV 4) the formulae (IV,3a), (IV,3b) and (IV,3c) we obtain immediately

(IV,5a) $$\frac{1}{\sqrt{n}}|A|_1 \leqq |A|_2 \leqq \sqrt{n}\,|A|_1,$$

(IV,5b) $$\frac{1}{\sqrt{n}}|A|_\infty \leqq |A|_2 \leqq \sqrt{n}\,|A|_\infty$$

For $p = 1, \infty$ the expressions of $|A|_1$, $|A|_\infty$ are easy to write down; we have as is well known and very easy to prove

(IV,6a) $$|A|_1 = \operatorname{Max}_\nu \sum_\mu |a_{\mu\nu}|$$

(IV,6b) $$|A|_\infty = \operatorname{Max}_\mu \sum_\nu |a_{\mu\nu}|.$$

As to $|A|_2$, its expression is irrational, $|A|_2$ is the square root of the maximum fundamental root of the symmetric and non negative matrix $A\,A^*$. Since the direct computation of $|A|_2$ is in most cases difficult we prove in what follows the following estimate for $|A|_2$:

Lemma II. We have for any matrix A

(IV,7) $$\frac{1}{n} \operatorname{Max}(|A|_1, \ |A|_\infty) \leq |A|_2 \leq \sqrt{|A|_1\,|A|_\infty}\,,$$

The first part of (IV,7) follows from (IV,5a) and (IV, 5b). To prove the second part we introduce the notations

$$s_\mu = \sum_\nu |a_{\mu\nu}|, \quad t_\nu = \sum_\mu |a_{\mu\nu}| \qquad (\mu, \nu = 1, \dots, n).$$

The sum of the moduli of all elements in the μ th row of AA^* can be estimated as follows

$$\Big|\sum_{\nu,\kappa} a_{\mu\kappa} a_{\nu\kappa}\Big| \leq \sum_\kappa \sum_\nu |a_{\mu\kappa} a_{\nu\kappa}| \leq \sum_\kappa |a_{\mu\kappa}|\, t_\kappa \leq s_\mu |A|_1 \leq |A|_\infty |A|_1 .$$

(IV,7) follows now from the theorem of Frobenius that the modulus of each fundamental root of a square matrix does not exceed the greatest sum of the moduli of the elements of this matrix in different rows.

If we apply now these results to the matrix Δ_n^{-1} discussed in the section II, we obtain from (II,18, (II,19), IV,6a) and (IV,6b)

(IV,8) $$|\Delta_n^{-1}|_1 = |\Delta_n^{-1}|_\infty = \frac{Q}{M+1} = \frac{(M+1)^{n-1}}{|\Delta_n|},$$

and from (IV,7) and (IV, 5a)

$$\text{(IV,9)} \qquad \frac{Q}{\sqrt{n}(M+1)} \leqq |\Delta_n^{-1}|_2 \leqq \frac{Q}{M+1} = \frac{(M+1)^{n-1}}{|\Delta_n|} .$$

In combining these inequalities with the result given in the section III, we obtain the following theorem.

D. <u>Let $A = (a_{\mu\nu})$ be a square matrix of order n satisfying to the conditions (III,7) and let (III,8) be satisfied, then we have</u>

$$|A^{-1}|_p \leqq \frac{(M+1)^{n-1}}{|\Delta_n|} \quad (p = 1,2,\infty)$$

and therefore for any vector ξ

$$\text{(IV,10)} \qquad |A\xi|_p \geqq \frac{|\Delta_n|}{(M+1)^{n-1}} |\xi|_p \quad (p = 1, 2, \infty).$$

<u>Proof</u>. It follows from (III,3°) of the lemma I at once in virtue of (IV 6a) and (IV,6b) that

$$\text{(IV,13)} \qquad |A^{-1}|_p \leqq |\Delta_n^{-1}|_p = \frac{Q}{M+1} \quad (p = 1, \infty);$$

further, the matrix $A^{-1}(A^{-1})^*$ is majorated by $\Delta_n^{-1}(\Delta_n^{-1})^*$. Since therefore by a well known theorem of Frobenius, the maximum modulus of a fundamental root of $A^{-1}(A^{-1})^*$ is majorated by that of $\Delta_n^{-1}(\Delta_n^{-1})^*$, we have

$$\text{(IV,14)} \qquad |A^{-1}|_2 \leqq |\Delta_n^{-1}|_2 \leqq \frac{Q}{M+1},$$

and therefore by definition (IV,4) in putting $A\xi = \eta$

$$|A^{-1}\eta|_2 \leqq \frac{Q}{M+1} |\eta|_2$$

and this is equivalent with (IV,11).

V. The bounds of the matrix $\Delta_n^{-1} - \Delta_n^{(0)-1}$

We denote by $\Delta_n^{(0)}$ the matrix obtained from (II,1) in replacing there m everywhere by zero.

$$\Delta_n^{(0)} = \begin{pmatrix} 1 & -M & -M & \dots & -M \\ 0 & 1 & -M & \dots & -M \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{V,1}$$

We will prove the following theorem:

E. <u>If we have $o < m < M$ and (III,8) is satisfied, then the matrix</u> $\Delta_n^{-1} - \Delta_n^{(0)-1}$ is a non negative matrix and we have

$$|\Delta_n^{-1} - \Delta_n^{(0)-1}|_p = \operatorname*{Max}_{1 \leq \nu \leq n} (1+M)^{\nu-1} \frac{(1+m)^{n-\nu} - |\Delta_n|}{|\Delta_n|} \qquad (p = 1, \infty). \tag{V,2}$$

<u>Proof:</u> Since $\Delta_n^{(0)}$ satisfies the inequalities (III,2) it follows from (III,3°) that $\Delta_n^{-1} - \Delta_n^{(0)-1}$ is non negative. Denote by $s_\mu^{(0)}$ and $t_\nu^{(0)}$ the sums of the elements in the μ th and in the ν th column of $\Delta_n^{(0)-1}$ and by $\bar{s}_\mu$ and $\bar{t}_\nu$ the corresponding expressions for the matrix $\Delta_n^{-1} - \Delta_n^{(0)-1}$. We have

$$\bar{s}_\mu = s_\mu - s_\mu^{(0)}, \quad \bar{t}_\nu = t_\nu - t_\nu^{(0)} \qquad (\mu, \nu = 1, \dots, n). \tag{V,3}$$

It follows then from the formulae (II,22) and (II,23) that $\bar{s}_\mu$ run for $\mu = n, n-1, \dots, 1$ through the same set of values as $\bar{t}_\nu$ for $\nu = 1, \dots, n$.

The formula (II,29) can be written in the form

$$t_\nu = \frac{q^{n-\nu}(M+1)^{n-1}}{|\Delta_n|} = \frac{(1+m)^{n-\nu}(1+M)^{\nu-1}}{|\Delta_n|}. \tag{V,4}$$

Since Δ_n becomes 1 for m = 0, we have (V,2) and E is proved.

Discussion of $|\Delta_n^{-1} - \Delta_n^{(0)-1}|_p$. We prove first that $t_\nu - t_\nu^{(0)}$ goes increasing with ν as long as the condition

$$1 - \frac{m}{M} \geqq |\Delta_n| \tag{V,5}$$

is satisfied. Indeed we put

$$K_\nu \equiv \frac{t_{\nu+1} - t_{\nu+1}^{(0)}}{t_\nu - t_\nu^{(0)}} = (M+1)\frac{(1+m)^{n-\nu-1} - |\Delta_n|}{(1+m)^{n-\nu} - |\Delta_n|} . \tag{V,6}$$

In solving the 3 inequalities

$$k_\nu \gtreqless 1 \tag{V,7}$$

with respect to $|\Delta_n|$, we obtain correspondingly

$$|\Delta_n| \lesseqgtr \sigma_\nu \equiv \left(1 - \frac{m}{M}\right)(1+m)^{n-\nu-1} . \tag{V,8}$$

The inequalities $k_\nu \geqq 1 (\nu = 1, \ldots, n)$ are obviously satisfied in virtue of (V,5) as long as $\nu \leqq n-1$. We have therefore in this case, in using (II,7)

$$\operatorname*{Max}_\nu (t_\nu - t_\nu^{(0)}) = (1+M)^{n-1}\frac{1-|\Delta_n|}{|\Delta_n|} = (1-M)^{n-1}\frac{\delta}{1-\delta} ,$$

$$(V,9)\qquad |\Delta_n^{-1} - \Delta_n^{(0)-1}|_p = (1+M)^{n-1}\frac{1-|\Delta_n|}{|\Delta_n|} = (1+M)^{n-1}\frac{\delta}{1-\delta} \quad (p=1,\infty;\ |\Delta_n| \leqq 1-\frac{m}{M}).$$

The condition (V,5) can be written using the notation (II,7)

$$\delta \geqq \frac{m}{M}$$

and this is in virtue of (II,10) certainly satisfied, if we have $M\, M_n \geqq 1$,

$$(V,10)\qquad (M+1)^n - n\,M - 2 \geqq 0.$$

The condition (V,10) becomes for $n = 3$: $M \geqq .5321$. For $n \geqq 4$ (V,10) is in any case satisfied if we have

$$M \geqq \frac{1.5}{n} \qquad (n \geqq 4). \tag{V,11}$$

Indeed if the relation (V,10) is satisfied for a positive M it is satisfied for any greater value since the coefficients of all positive powers of M in the left side expression are greater or equal 0. To prove the sufficiency of (V,11) it is sufficient to prove that $(1 + \frac{1.5}{n})^n \geqq 3.5 \quad (n \geqq 4)$. But here the left side expression is monotonically growing with n and this inequality follows therefore from

$$\left(\frac{5.5}{4}\right)^4 = 3.75\ldots > 3.5 \quad .$$

Suppose now that we have

$$1 - \frac{m}{M} \leqq |\Delta_n| \leqq 1 \qquad \delta \leqq \frac{m}{M} \ . \tag{V,12}$$

Then we have, since $0 < m < \frac{1}{n-1}$,

$$\frac{(1+m)^{n-\nu} - 1}{m} \leqq \frac{\left(1 + \frac{1}{n-1}\right)^{n-\nu} - 1}{\frac{1}{n-1}} \leqq (n-1)\left(\left(1 + \frac{1}{n-1}\right)^{n-1} - 1\right) \leqq (e-1)(n-1),$$

$$(1+m)^{n-\nu} = 1 + 1.72\,\theta n m \ , \qquad 0 < \theta < 1$$

$$(1+m)^{n-\nu} - |\Delta_n| = 1.72\,\theta n m + 1 - |\Delta_n|$$

and therefore by (V,12) $(1+m)^{n-\nu} - |\Delta_n| \leqq 1.72\,\theta n m + \frac{m}{M}$.

For $n \geqq 4$ we have in the case (V,12) since the inequality (V,11) is not verified,

$$\frac{1}{M} \leqq \frac{n}{1.5}$$

and therefore finally

$$(V,13) \qquad (1+m)^{n-\nu} - |\Delta_n| \leqq 2.39\, n\, m\,.$$

On the other hand, since $M < \frac{1.5}{n}$, $(1+M)^{\nu-1} \leqq (1+\frac{1.5}{n})^n < e^{1.5} = 4.481689$

and therefore from (V,2) and (V,3)

$$t_\nu - t_\nu^{(0)} \leqq 10.72\, n \frac{m}{|\Delta_n|} \quad (\nu = 1, \ldots, n;\ |\Delta_n| \geqq 1 - \frac{m}{M};\ n > 3),$$

$$(V,14) \qquad |\Delta_n^{-1} - \Delta_n^{(0)-1}|_p \leqq 10.72\, n \frac{m}{|\Delta_n|} \quad (p = 1, \infty;\ |\Delta_n| \geqq 1 - \frac{m}{M};\ n > 3).$$

To obtain a lower bound, we take in (V,4) $\nu = 1$. We obtain, since $0 < |\Delta_n| \leqq 1$, $t_1 - t_1^{(0)} = \frac{(1+m)^{n}-1}{|\Delta_n|} \geqq \frac{(n-1)m}{|\Delta_n|} \geqq .75\, n \frac{m}{|\Delta_n|}$

and therefore

$$(V,15) \qquad |\Delta_n^{-1} - \Delta_n^{(0)-1}|_p \geqq .75\, n \frac{m}{|\Delta_n|} \quad (p = 1, \infty;\ n \geqq 4).$$

To obtain the exact value of $|\Delta_n^{-1} - \Delta_n^{(0)-1}|_p$ if (V,5) is not satisfied, we return to the inequalities (V,8) equivalent to (V,7).

Denote by n_o the smallest integer between 1 and n such that we have

$$(V,16) \qquad \sigma_{n_o-1} \geqq |\Delta_n| > \sigma_{n_o} \quad (1 \leqq n_o \leqq n).$$

The parts of this inequality implying σ_0 or σ_n must be disregarded, that is to say, this inequality reduces to $|\Delta_n| > \sigma_1$ for $n_o = 1$ and to $\sigma_{n-1} \geqq |\Delta_n|$ for $n_o = n$. Then we see at once that $\mathrm{Max}(t_\nu - t_\nu^{(0)}) = t_{n_o} - t_o^{(0)}$ and therefore

$$(V,17) \qquad |\Delta_n^{-1} - \Delta_n^{(0)-1}|_p = t_{n_o} - t_o^{(0)} = \frac{(1+M)^{n_o-1}(1+m)^{n-n_o} - |\Delta_n|}{|\Delta_n|} \quad (p = 1, \infty),$$

For $m = M$ we have $n_o = 1$ and therefore

(V,17°) $$|\Delta_n^{-1} - \Delta_n^{(0)-1}|_p = \frac{(n-1)m}{1-(n-1)m} \quad \left(m = M < \frac{1}{n-1}\right).$$

VI. The bounds of $A^{-1} - A^{(0)-1}$

In order to obtain the theorem corresponding to E for a determinant A satisfying the conditions (III,7), (III,8) of the theorem B, we prove first the following important lemma which generalizes considerably the relation (III,3°) of the lemma I.

Lemma III. Consider n positive numbers α_1 to α_n and $2n^2 - 2n$ non negative numbers

$$m_{\mu\nu}, \; \epsilon_{\mu\nu} \quad (\mu \neq \nu; \mu, \nu = 1, \ldots, n),$$

such that the matrices

(VI,1) $$M = \begin{pmatrix} \alpha_1 & -m_{12}-\epsilon_{12} & \cdots & -m_{1n}-\epsilon_{1n} \\ -m_{21}-\epsilon_{21} & \alpha_2 & \cdots & -m_{2n}-\epsilon_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ -m_{n1}-\epsilon_{n1} & -m_{n2}-\epsilon_{n2} & \cdots & \alpha_n \end{pmatrix},$$

(VI,2) $$M^{(0)} = \begin{pmatrix} \alpha_1 & -m_{12} & \cdots & -m_{1n} \\ -m_{21} & \alpha_2 & \cdots & -m_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ -m_{n1} & -m_{n2} & \cdots & \alpha_n \end{pmatrix}$$

are M-matrices.

Consider n constants A_μ $(\mu = 1, \ldots, n)$ such that

(VI,3) $$|A_\mu| \geqq \alpha_\mu \quad (\mu = 1, \ldots, n)$$

and $2n^2 - 2n$ constants $a_{\mu\nu}$, $b_{\mu\nu}$ $(\mu \neq \nu; \mu, \nu = 1, \ldots, n)$ such that

(VI,4) $$|a_{\mu\nu}| \leq m_{\mu\nu}\,,\quad |b_{\mu\nu} - a_{\mu\nu}| \leq \epsilon_{\mu\nu} \quad (\mu \neq \nu;\ \mu, \nu = 1, \dots, n),$$

and form the two matrices

(VI,5) $$A = \begin{pmatrix} A_1 & b_{12} & \cdots & b_{1n} \\ b_{21} & A_2 & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & A_n \end{pmatrix},$$

(VI,6) $$A^{(0)} = \begin{pmatrix} A_1 & a_{12} & \cdots & a_{1n} \\ a_{21} & A_2 & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & A_n \end{pmatrix};$$

then we have

(VI,7) $$A^{-1} - A^{(0)-1} \ll M^{-1} - M^{(0)-1} .$$

Proof: We can write $M = P - T$, $M^{(0)} = P - T^{(0)}$ where the matrices T, $T^{(0)}$ have in the main diagonal zeros and off the main diagonal respectively the non negative elements $m_{\mu\nu} + \epsilon_{\mu\nu}$, $m_{\mu\nu}$, while P is the diagonal matrix

(VI,8) $$P = \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{pmatrix}.$$

We can develop the inverse of $P-T = P(E-P^{-1}T)$ (E is the unit matrix) in the following way

$$(VI,9) \qquad (P-T)^{-1} = \sum_{k=0}^{\infty} (P^{-1}T)^k P^{-1} .$$

The convergence of this development and the validity of (VI,9) follows easily from the fact that the determinant of the matrix $P-tT$ does not vanish for $|t| \leqq 1$ as follows immediately from the lemma I. The corresponding development holds also for $(P-T^{(0)})^{-1}$ and we obtain therefore

$$\text{(VI,10)} \qquad M^{-1}-M^{(0)-1} = \sum_{k=1}^{\infty} \left[(P^{-1}T)^k - (P^{-1}T^{(0)})^k\right] P^{-1}$$

The elements of $P^{-1}T$ are here $\dfrac{m_{\mu\nu}+\epsilon_{\mu\nu}}{\alpha_\mu}$ or zeros and those of $P^{-1}T^{0}$ are $\dfrac{m_{\mu\nu}}{\alpha_\mu}$ or zeros, therefore all elements of the matrices

$$\text{(VI,11)} \qquad (P^{-1}T)^k - (P^{-1}T^{(0)})^k \qquad\qquad (k = 1,2,\ldots)$$

are polynomials in $\dfrac{m_{\mu\nu}}{\alpha_\mu}$ and $\dfrac{\epsilon_{\mu\nu}}{\alpha_\mu}$ with nonnegative coefficients. Denote by Q the diagonal matrix

$$Q = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix}$$

and write

$$A = Q - S \;, \quad A^{(0)} = Q - S^{(0)} \;,$$

where the elements of S are $b_{\mu\nu}$ or zeros and those of $S^{(0)}$ $a_{\mu\nu}$ or zeros. Since therefore we have

$$Q^{-1}S \ll P^{-1}T, \quad Q^{-1}S^{(0)} \ll P^{-1}T^{(0)} \;,$$

we have

$$A^{-1} = \sum_{k=0}^{\infty} (Q^{-1}S)^k\, Q^{-1}, \quad A^{(0)-1} = \sum_{k=0}^{\infty} (Q^{-1}S^{(0)})^k\, Q^{-1} \;,$$

$$\text{(VI,12)} \quad A^{-1} - A^{(0)-1} = \sum_{k=0}^{\infty} \left[(Q^{-1}S)^k - (Q^{-1}S^{(0)})^k\right] Q^{-1} \;.$$

But now the elements of

$$(Q^{-1}S)^k - (Q^{-1}S^{(0)})^k$$

are obtained from those of (VI,11) in substituting there $\dfrac{a_{\mu\nu}}{A_\mu}$ instead of

$\frac{m_{\mu\nu}}{\alpha_\mu}$ and $\frac{b_{\mu\nu}-a_{\mu\nu}}{A_\mu}$ instead of $\frac{\epsilon_{\mu\nu}}{\alpha_\mu}$. Here we have by (VI,3) and (VI,4)

$$\left|\frac{a_{\mu\nu}}{A_\mu}\right| \leqq \frac{m_{\mu\nu}}{\alpha_\mu}, \quad \left|\frac{b_{\mu\nu}-a_{\mu\nu}}{A_\mu}\right| \leqq \frac{\epsilon_{\mu\nu}}{\alpha_\mu}$$

and therefore since the coefficients in (VI,11) are, as already mentioned, not negative

$$(Q^{-1}S)^k - (Q^{-1}S^{(0)})^k \ll (P^{-1}T)^k - (P^{-1}T^{(0)})^k$$

and

$$\text{(VI,13)} \qquad \sum_{k=1}^{\infty}\left[(Q^{-1}S)^k-(Q^{-1}S^{(0)})^k\right] \ll \sum_{k=1}^{\infty}\left[(P^{-1}T)^k-(P^{-1}T^{(0)})^k\right].$$

But from (VI,13) in virtue of (VI,3) it follows that the development (VI,12) is majorated by (VI,10) and our lemma is proved.

Under the conditions of the theorem B in the section III, if the inequalities (III,7) and (III,8) are satisfied, we can apply the lemma III in replacing there M by Δ_n, $M^{(0)}$ by $\Delta_n^{(0)}$. We obtain

F. <u>Under the conditions of the theorem B, if the inequalities (III,7) and (III,8) are satisfied, we have</u>

$$\text{(VI,14)} \quad \left|A^{-1}-A^{(0)-1}\right|_p \leqq \left|\Delta_n^{-1}-\Delta_n^{(0)-1}\right|_p = \underset{1\leqq\nu\leqq n}{\mathrm{Max}}\,(1+M)^{\nu-1}\frac{(1+m)^{n-\nu}\,|\Delta_n|}{|\Delta_n|} \quad (p=1,\infty),$$

<u>where the values and the estimates for the right side expression in (VI,14) are obtained from the formulae (V,9) to (V,17°)</u>.

The lemma III can be applied in many problems similar to that solved by the theorem F. For instance, in the theory of the solutions of linear equations the following problem has to be dealt with, although its complete discussion is usually avoided:

Consider a "triangular" system of linear equations

$$\text{(VI,15)} \qquad \sum^{n} a_{\mu\nu}x_\nu = y_\mu \quad (\mu=1,\ldots,n),$$

where the coefficients $a_{\mu\nu}$ are only approximate values to the "true" values $b_{\mu\nu}$. Suppose that we have generally

(VI,16) $$|b_{\mu\nu} - a_{\mu\nu}| \leqq \epsilon \quad (\nu \geqq \mu);$$

how far the solution of the system

(VI,17) $$\sum_{\nu=\mu}^{n} b_{\mu\nu} x_\nu = y_\mu$$

is influenced if the $b_{\mu\nu}$ are replaced by $a_{\mu\nu}$?

If we denote the matrix of (VI,15) by $A^{(0)}$ and that of (VI,17) by A, the question can be in particular answered by giving estimates of

$$|A^{-1} - A^{(0)-1}|_p \; (p = 1, \infty) .$$

Suppose that we have generally

(VI,18) $$a_{\mu\mu} = 1, \; |a_{\mu\nu}| \leqq M_\mu \quad (\nu > \mu \; ; \; \mu = 1, \ldots, n-1)$$

and consider the matrix

(VI,19) $$\Delta(M_1, \ldots, M_{n-1}) = \begin{pmatrix} 1 & -M_1 & \ldots & -M_1 \\ 0 & 1 & \ldots & -M_2 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

where all elements to the left of the main diagonal are zeros, while all elements to the right of the main diagonal in the μ th row are equal to M_μ.

We obtain from the lemma III at once the majoration

$$A^{-1} - A^{(0)-1} << \Delta(M_1+\epsilon, \ldots, M_{n-1}+\epsilon)^{-1} - \Delta(M_1, \ldots, M_{n-1})^{-1}.$$

To obtain the inverse of the matrix $\Delta(M_1, \ldots, M_{n-1})$, consider the system of linear equations

(VI,20) $$x_\mu - M_\mu(x_{\mu+1} + \ldots + x_n) = u_\mu \quad (\mu = 1, \ldots, n).$$

To solve it explicitly we put

(VI,21) $$s_\mu = \sum_{k=\mu}^{n} x_k \qquad (\mu = 1,\dots,n)$$

(VI,22) $$1+M_\mu = N_\mu,\quad N_\mu N_{\mu+1}\cdots N_n = P_\mu \qquad (\mu=1,\dots,n-1)$$
$$N_n = 1,\quad P_n = 1\,.$$

Then (VI,20) becomes

(VI,23) $$x_\mu = M_\mu\, s_{\mu+1} + u_\mu \qquad (\mu = 1,\dots,n)$$

(VI,24) $$s_\mu = N_\mu\, s_{\mu+1} + u_\mu \qquad (\mu = 1,\dots,n)$$

where $S_{n+1} = 0$. Dividing (VI,24) by P_μ we obtain

$$\frac{s_\mu}{P_\mu} = \frac{s_{\mu+1}}{P_{\mu+1}} + \frac{u_\mu}{P_\mu}$$

and therefore

$$s_\mu = P_\mu \sum_{k=\mu}^{n} \frac{u_k}{P_k}\;;$$

introducing this in (VI,23) we have

(VI,25) $$x_\mu = u_\mu + M_\mu P_{\mu+1} \sum_{k=\mu+1}^{n} \frac{u_k}{P_k} \qquad (\mu=1,\dots,n).$$

We obtain therefore for the inverse of our $\Delta(M_1,\dots,M_{n-1})$, in denoting by E the unit matrix,

(VI,26) $$\Delta(M_1,\dots,M_{n-1})^{-1} = E + D\,T,$$

where D is the diagonal matrix

(VI,27) $$D = \begin{pmatrix} M_1P_2 & & & 0 \\ & M_2P_3 & & \\ & & \ddots & \\ 0 & & & M_{n-1}P_n \\ & & & & 0 \end{pmatrix}$$

and T the triangular matrix

(VI,28)
$$T = \begin{pmatrix} 0 & \frac{1}{P_2} & \frac{1}{P_3} & \cdots & \frac{1}{P_n} \\ & 0 & \frac{1}{P_3} & \cdots & \frac{1}{P_n} \\ & & 0 & \cdots & \frac{1}{P_n} \\ & \text{\Large 0} & & \ddots & \vdots \\ & & & & 0 \end{pmatrix} .$$

The expressions

(VI,29)
$$\Delta(M_1+\epsilon, \ldots, M_{n-1}+\epsilon)^{-1} - \Delta(M_1, \ldots, M_{n-1})^{-1}$$

obtained from (VI,26) to (VI,28) is of course rather unwieldy; however we shall obtain for its norm corresponding to p = 1 (cf IV,6a) a very simple and elegant expression.

Indeed, if we take the sum of the elements in the ν th column of $\Delta(M_1, \ldots, M_{n-1})^{-1}$ and denote it by t'_ν, we have

$$t'_\nu = 1 + \frac{1}{P_\nu} \sum_{\mu=1}^{\nu-1} M_\mu P_{\mu+1} = 1 + \sum_{\mu=1}^{\nu-1} M_\mu \prod_{\kappa=\mu+1}^{\nu-1} (1+M_\kappa) .$$

If we write now $t'_{\nu+1}$ out, we obtain

$$t'_{\nu+1} = 1 + \left[\sum_{\mu=1}^{\nu-1} M_\mu \prod_{\kappa=\mu+1}^{\nu-1} (1+M_\kappa)\right](1+M_\nu) + M_\nu ;$$

and in comparing this with the expression of t'_ν we see that we have $t'_{\nu+1} = (1 + M_\nu)\, t'_\nu$ and therefore

(VI,30)
$$t'_\nu = \prod_{\kappa=1}^{\nu-1} (1 + M_\kappa) .$$

We obtain now for the sum of the elements in the ν th column of (VI,29)

$$t_1 = 0, \quad t_2 = \epsilon$$

$$t_\nu = \prod_{\kappa=1}^{\nu-1} (1+M_\kappa+\epsilon) - \prod_{\kappa=1}^{\nu-1} (1+M_\kappa) \qquad (\nu = 1, \ldots, n).$$

In multiplying this by $1 + M_\nu + \epsilon$ we obtain

$$\prod_{\kappa=1}^{\nu} (1+M_\kappa+\epsilon) - \prod_{\kappa=1}^{\nu} (1+M_\kappa) - \epsilon \prod_{\kappa=1}^{\nu-1} (1+M_\kappa) ,$$

and comparing this with $t_{\nu+1}$

$$(VI,31)\qquad t_{\nu+1} = (1+M_\nu+\epsilon)t_\nu + \epsilon\prod_{\kappa=1}^{\nu-1}(1+M_\kappa).$$

Therefore, t_ν is monotonically increasing with ν and we see that the norm of (VI,29) corresponding to p = 1 has the value

$$(VI,32)\qquad \prod_{\nu=1}^{n-1}(1+M_\nu+\epsilon) - \prod_{\nu=1}^{n-1}(1+M_\nu)$$

and obtain therefore

$$(VI,33)\qquad |A^{-1}-A^{(0)-1}|_1 \leqq \prod_{\nu=1}^{n-1}(1+M_\nu+\epsilon) - \prod_{\nu=1}^{n-1}(1+M_\nu)$$

as the solution of our problem. In applications it may be better to use the recurrent formula (VI,31). If all M_μ have the same value M, the expression (VI,26) coincides with that obtained in IV for Δ_n^{-1} under the hypothesis m = 0. But in this case we see from (II,28) and (II,29) that the row sum run through the same values as the sums of the columns. We obtain therefore in this case the expression

$$(M+1+\epsilon)^{n-1}-(M+1)^{n-1} = (n-1)\,\epsilon\,(M+1+\theta\epsilon), \quad 0<\theta<1,$$

as the norm of (VI,29); both for p = 1 and p = ∞.

VII. Linear systems with a Nearly Triangular Matrix

The results of the preceeding sections give the means to discuss the following problem concerning the system (1) in the introduction under the conditions (2) and the "triangular" system.

(VII,1) $$\sum_{\nu=\mu}^{n} a_{\mu\nu} x_{\nu} = y_{\mu} \qquad (\mu = 1, \ldots, n),$$

with the matrix $A^{(0)}$. In discussing this problem we can obviously assume that

$$a_{\mu\mu} = 1 \qquad (\mu = 1,\ldots,n).$$

Then the difference between the solutions (1) and of (VII,1) is given by the vector

$$(A^{-1}-A^{(0)-1})\eta\ , \qquad \eta = (y_1\ , \ldots , y_n)\ ,$$

and the norm of this vector corresponding to one of the indices $p = 1, \infty$ does not exceed

$$|A^{-1} - A^{(0)-1}|_p |\eta|_p \qquad (p = 1, \infty),$$

and can indeed for suitable choice of the vector η attain this limit. Therefore the norms $|A^{-1}-A^{(0)-1}|_p$ measure the error committed in replaci the system (7) by (VII,1).

For an $\epsilon > 0$, $M > 0$ being given, how small must $m > 0$ be taken in order that we have

$$\text{(VII,2)} \qquad |A^{-1} - A^{(0)-1}|_p \leqq \epsilon \qquad (p = 1, \infty).$$

If we introduce the quantities $|\Delta_n|$ and δ corresponding by (II,2), (II,3) and (II,7) to m and M, we obtain from (V,9) and (VI,7) the condition

$$(1+M)^{n-1} \frac{\delta}{1-\delta} \leqq \epsilon$$

$$\text{(VII,3)} \qquad \delta \leqq \frac{\epsilon}{(1+M)^{n-1}+\epsilon} \qquad \left(|\Delta_n| \leqq 1 - \frac{m}{M}\right)$$

as long as the condition (V,5) is satisfied and therefore certainly as long as $M \gtreqless \frac{1.5}{n}$ $(n \geqq 4)$.

On the other hand we have by (2,10)

$$\text{(VII,4)} \qquad m = \frac{\delta}{M_n + \theta \frac{\delta}{M}}$$

and from (VII,3) and (VII,4)

$$\text{(VII,5)} \qquad m \leqq \frac{\epsilon}{M_n (1+M)^{n-1} + (M_n + \frac{\theta}{M})\epsilon} \qquad \left(|\Delta_n| \leqq 1 - \frac{m}{M}\right).$$

It will be therefore sufficient for (VII,2) to take

$$m \leqq m_0 \equiv \frac{\epsilon}{M_n(1+M)^{n-1}+(M_n+\frac{1}{M})\epsilon} \quad (|\Delta_n| \leqq 1-\frac{m}{M}). \tag{VII,6}$$

solve our problem, for instance, if we have $M \geqq \frac{1.5}{n}$ $(n \geqq 4)$ or $M \geqq .5321$ $(n = 3)$. For small values of ϵ obviously, only the first term in the denominator is essential and we have

$$m_0 \approx K(n,M)\epsilon \quad , \quad K(n,M) = \frac{1}{M_n(1+M)^{n-1}} \tag{VII,7}$$

The tables I and III give the values of $K(n,M)$ for a set of integer M from 1 to 10and some values of $M > \frac{1.5}{n}$. We have obviously

$$K(n,M) < \frac{1}{M^{2n-2}} \tag{VII,8}$$

The bound in (VII,6) is obviously the "best" under the condition (V,5), save that the factor $\frac{1}{M}$ of ϵ in the denominator could be replaced by an (unknown) fraction of it.

If $M < \frac{1.5}{n}$, the use of the general formula (V,17) is very cumbersome; we can however obtain a good working limit for m in the following way. If the inequalities (2) are valid for an M $\frac{1.5}{n} = M^{(n)}$, then are also satisfied if M is replaced by $M^{(n)}$, but then the limit for m obtained for $M^{(n)}$ is also sufficient for our M. We obtain therefore in this case for m the sufficient condition for (VII,2).

$$(VII,9) \qquad m \leq m_0' \equiv \frac{\epsilon}{M_n^{(n)}(1+M^{(n)})^{n-1} + \left(M_n^{(n)} + \frac{1}{M_n}\right)\epsilon} .$$

The table IIgives the values of $K(n,M^{(n)})$ for $n = 1,2,\ldots,50$.

The expression for m'_o can be written in introducing the value of $M_n^{(n)}$ as

$$\frac{3}{2n} \cdot \frac{\epsilon}{(1+\frac{1.5}{n})^{n-1}((1+\frac{1.5}{n})^n - 2.5) + \left((1+\frac{1.5}{n})^n - 1.5\right)\epsilon} .$$

We observe now that for any positive α and positive x the expression $(1 + \frac{x}{n})^{n-\alpha}$ monotonically increase and tends to e^{x} if the positive n increases monotonically to ∞. Indeed, if we put $u = \frac{1}{n}$, take the logarithm of this expression, differentiate it with respect to u and multiply by u^2, we obtain $\frac{x(u - \alpha u^2)}{1 + u x} - \log(1+ux)$; but this expression vanishes for $u = 0$ and decreases for positive u since its derivative is

$$\frac{-xu}{(1 + u x)^2} \qquad (2a + x + a x u) \quad .$$

We see therefore that $(1 + \frac{1.5}{n})^{n-1} \uparrow e^{1.5}$, $(1 + \frac{1.5}{n})^{n} \uparrow e^{1.5}$

and our bound for m can therefore be replaced by

$$\frac{3}{2n} \cdot \frac{\epsilon}{e^{1.5}(e^{1.5} - 2.5) + (e^{1.5} - 1.5)\epsilon} =$$

$$= \frac{\epsilon}{(5.93 + 1.99 \quad)n} \quad ;$$

we can replace therefore the condition [redacted] $m \leq m'_0$ by the simpler condition

(VII,10) $$m \leq m_0 = \frac{\epsilon}{n} \cdot \frac{1}{6 + 2\epsilon} \quad .$$

On the other hand we obtain at once the solution of our problem in the case $m = M < \frac{1}{n-1}$ from (V,17).

(VII,11) $$m \leq m_0 \leq \frac{1}{n-1} \frac{\epsilon}{1+\epsilon} \qquad (m = M < \frac{1}{n-1}) \quad .$$

This is an "exact" condition, while the condition (VII,10) is nly an estimate. We see however, in comparing (VII,10) with (VII,11) that the bound in (VII,10) cannot be improved by a factor > 6.

From the condition (VII,10) we can finally derive the inequality (7) of the introduction. Indeed, if positive $m_0 < \frac{1}{2n}$ is given, we can solve (VII,10) with respect to ϵ,

$$\epsilon = \frac{6\ n\ m}{1-2\ n\ m_0}$$

and obtain therefore in applying (VII,10) for $m = m_0$ the inequality

$$\left| A^{-1} - A^{(0)-1} \right|_p \leqq \frac{6\ n\ m_0}{1-2n\ m_0}$$

which holds for all positive $m_0 < \frac{1}{2m}$ and gives the inequality (7) if we replace here m_0 by m.

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Foot Notes

1) This condition (3^0) is already contained in some results of my paper [2] , and can be also deduced from a well known theorem of the theory of determinants discussed in [4] .

2) Some of the results contained in the section I - IV have been published without proof in my note [3] .

3) For $M = m$, $\delta(m,M)$ becomes $(n-1)m$, but in this case our result is obtained in Gerschgorin's theorem cf. [4] .

From the theorem c the result given by Stein and Rosenberg in [3a] as theorem III follows immediately.

2a) This formula can be also obtained from the formulae given in the proof of theorem III in [3a] , p. 113.

Table I

K(n,M)

M	n=3	n=4	n=5	n=6	n=7	n=8
1	.0625	.0114	.(2)240	.(3)548	.(3)130	.(4)316
2	.0111	.(2)103	.(3)106	.(4)115	.(5)126	.(6)140
3	.(2)347	.(3)193	.(4)116	.(6)719	.(7)448	.(8)280
4	.(2)143	.(4)526	.(5)206	.(7)821	.(8)328	.(9)131
5	.(3)694	.(4)182	.(6)498	.(7)138	.(9)383	.(10)106
6	.(3)378	.(5)736	.(6)149	.(8)304	.(10)619	.(11)126
7	.(3)223	.(5)336	.(7)522	.(9)815	.(10)127	.(12)199
8	.(3)140	.(5)168	.(7)207	.(9)255	.(11)315	.(13)389
9	.(4)926	.(6)903	.(8)900	.(10)900	.(12)900	.(14)900
10	.(4)636	.(6)515	.(8)424	.(10)351	.(12)290	.(14)239

M	n=9	n=10
1	.(5)778	.(5)193
2	.(7)155	.(8)172
3	.(9)175	.(10)109
4	.(11)524	.(12)210
5	.(12)295	.(14)821
6	.(13)258	.(15)526
7	.(14)311	.(16)486
8	.(15)480	.(17)592
9	.(16)900	.(18)900
10	.(16)198	.(18)164

Table II

$$M = M^{(n)} = \frac{1.5}{n}$$

n	$(1+M)^n - n\,M - 1$	K(n,M)	n	$(1+M)^n - n\,M - 1$	K(n,M)
3	.871	.255	41	1.863	.00466
4	1.074	.134	42	1.866	.00454
5	1.213	.0866	43	1.869	.00442
6	1.315	.0623	44	1.871	.00431
7	1.393	.0480	45	1.873	.00420
8	1.454	.0387	46	1.876	.00410
9	1.504	.0323	47	1.878	.00401
10	1.546	.0276	48	1.880	.00391
11	1.580	.0240	49	1.882	.00383
12	1.610	.0213	50	1.884	.00374
13	1.635	.0190			
14	1.658	.0172			
15	1.677	.0157			
16	1.695	.0144			
17	1.710	.0133			
18	1.724	.0124			
19	1.736	.0116			
20	1.748	.0109			
21	1.758	.0102			
22	1.768	.00965			
23	1.776	.00914			
24	1.784	.00869			
25	1.792	.00827			
26	1.799	.00789			
27	1.805	.00755			
28	1.811	.00723			
29	1.817	.00694			
30	1.822	.00667			
31	1.827	.00642			
32	1.831	.00619			
33	1.836	.00597			
34	1.840	.00577			
35	1.844	.00558			
36	1.848	.00540			
37	1.851	.00524			
38	1.854	.00508			
39	1.857	.00493			
40	1.860	.00480			

Table III

K(n,M)

M	n=3	n=4	n=5	n=6	n=7	n=8	n=9
.1							
.2						.0328	.0197
.3			.0866	.0399	.0196	.0101	.(2)533
.4		.117	.0438	.0180	.(2)788	.(2)359	.(2)169
.5	.254	.0718	.0241	.(2)891	.(2)349	.(2)142	.(3)592
.6	.181	.0464	.0141	.(2)470	.(2)165	.(3)602	.(3)224
.7	.134	.0313	.00864	.(2)260	.(3)825	.(3)270	.(4)902
.8	.102	.0218	.00548	.(2)150	.(3)431	.(3)127	.(4)382
.9	.0789	.0156	.00359	.(3)894	.(3)233	.(4)623	.(4)169
.0	.0625	.0114	.00240	.(3)548	.(3)130	.(4)316	.(5)778

M	n=10	n=15	n=20	n=25	n=30	n=35
.1		.0157	.(2)439	.(2)138	.(3)469	.(3)166
.2	.0121	.(2)137	.(3)188	.(4)281	.(5)439	.(6)697
.3	.(2)289	.(3)167	.(4)112	.(6)793	.(7)570	.(8)413
.4	.(3)809	.(4)242	.(6)809	.(7)277	.(9)957	.(10)331
.5	.(3)252	.(5)399	.(7)681	.(8)118	.(10)204	.(12)354
.6	.(4)848	.(6)729	.(8)658	.(10)597	.(12)543	.(14)494
.7	.(4)305	.(6)146	.(9)721	.(11)357	.(13)177	.(16)879
.8	.(4)116	.(7)317	.(10)886	.(12)248	.(15)695	.(17)195
.9	.(5)462	.(8)743	.(10)121	.(13)197	.(16)322	.(19)525
.0	.(5)193	.(8)186	.(11)182	.(14)178	.(17)173	.(20)169

M	n=40	n=45	n=50
.1	.(4)604	.(4)224	.(5)841
.2	.(6)112	.(7)180	.(8)290
.3	.(9)299	.(10)217	.(11)157
.4	.(11)114	.(13)395	.(14)137
.5	.(14)613	.(15)106	.(17)184
.6	.(16)449	.(18)409	.(20)372
.7	.(18)436	.(20)216	.(22)107
.8	.(20)545	.(22)153	.(25)428
.9	.(22)856	.(24)140	.(27)228
.0	.(23)165	.(26)162	.(29)158

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