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THE CONVERGENCE OF NUMERICAL ITERATION

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Iteration arises frequently in the numerical solution of applied problems. Textbook statements on its convergence are often cursory. We hope this note will put students on their guard.

We shall deal with solving

(1) $x = f(x)$

by means of the iteration

(2) $x_n = f(x_{n-1})$

starting with a trial value x_0. For simplicity we shall suppose throughout that $f(x)$ is a real function of a real variable x, that (1) possesses a unique solution, and that this solution is $x = 0$. From the point of view of theory, there is no real loss of generality involved in this last assumption; for if $x = a \neq 0$ were the solution of $x = f(x)$, then $x = 0$ would be the solution of $x = g(x) = f(x+a) - a$. To avoid triviality, we shall also suppose throughout that the initial value x_0 is not zero.

We shall confine our attention to equations in a single unknown x; the difficulties in the case of several unknowns are more severe.

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Under certain conditions upon the function $f(x)$ the sequence $x_0, x_1, x_2, \ldots$ will converge to the desired root $x = 0$. We now invite the reader, who cares to test his appreciation of such conditions, to answer the following questions (with the aid of a textbook if he so desires).

Question 1: Is it sufficient for convergence that, for some given $k<1$, $|f'(\xi)| \leq k$ for every ξ in a (sufficiently small) neighborhood of the root $x = 0$, and that x_0 shall belong to this neighborhood? (In this question and subsequently, $f'(x)$ denotes the derivative of $f(x)$.)

Question 2: Is it sufficient for divergence that, for some given $k \geq 1$, $|f'(\xi)| \geq k$ for every ξ in a (sufficiently small) neighborhood of the root $x = 0$, and that x_0 shall belong to this neighborhood?

Question 3: Consider two functions $f_1(x)$ and $f_2(x)$ which both satisfy the conditions of Question 1 in a common (sufficiently small) neighborhood of the root $x = 0$; and suppose that k_1 and k_2 are respectively the smallest possible values of k for which these conditions hold [i.e. there exist no constants $k_1' < k_1$, $k_2' < k_2$ such that $|f'_1(\xi)| \leq k'_1$, $|f'_2(\xi)| \leq k'_2$ for all ξ in this neighborhood]. If $k_1 < k_2$ and if both iterative processes converge, will the convergence for $f_1(x)$ be more rapid than that for $f_2(x)$?

Question 4: Is it necessary for convergence that all conditions stated in Question 1 shall hold (a) if we restrict ourselves to the class of functions $f(x)$ which are everywhere differentiable or (b) if we make no such restriction?

Question 5: Can a condition bearing on the derivative $f'(0)$ or on a Lipschitz condition solely at the root $x = 0$ be sufficient for convergence?

Question 6: Are there any functions $f(x)$, satisfying $f(0) = 0$ and being discontinuous both to the left and to the right of $x = 0$, for which the iteration (2) converges whenever x_0 lies within some (sufficiently small) neighborhood of $x = 0$?

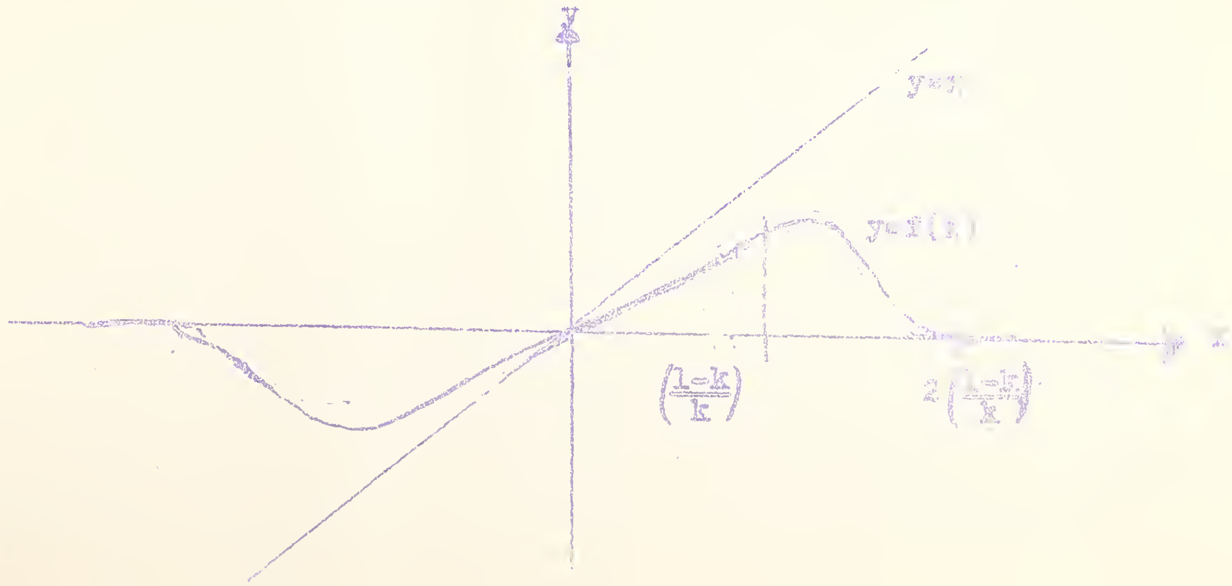
The answer to Question 1 is "Yes." This is a corollary to our answer to Question 5 (see below).

The answer to Question 2 is "No" as shown by the example

$$f(x) = \begin{cases} 2x & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \tag{3}$$

for which the process (2) converges.

The answer to Question 3 is "No." The example



$$(4)\quad f(x) = \begin{cases} \left(\frac{3}{4}\right)^2 kx & |x| \leq (1-k)/k \\ \left(\frac{3}{4}\right)^2 (kx+2k-2)^2(3kx+2k-2)/(1-k)^2 & (1-k)/k < |x| < 2(1-k)/k \\ 0 & |x| \geq 2(1-k)/k \end{cases}$$

satisfies the conditions of Question 1; and yet just one step of the iteration will yield the desired root $x = 0$ if

$$k \geq 1/(1+\tfrac{1}{2}|x_0|).$$

whereas an infinite number of steps is needed if

$$k < 1/(1+\tfrac{1}{2}|x_0|).$$

The convergence is slowest in the case $k = 1/(1+\frac{9}{10}|x_0|)$. The smallness of k, therefore, is not always a guarantee for rapid convergence as stated in textbooks. In practice, however, it will be a useful (though not wholly reliable) guide for the rapidity of convergence.

The answer to Question 4(a) is "No." The process (2) converges (very rapidly) for the function

$$(5)\qquad f(x) = \begin{cases} |x|^{3/2} e^{-x^2} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

whatever the initial value x_0; although $f'(x)$ is unbounded in every neighborhood of the root $x = 0$. <u>A fortiori,</u> the answer to Question 4(b) is "No." In one of the standard textbooks the conditions of Question 1 are falsely stated as both <u>necessary</u> and sufficient for convergence.

The answer to Question 5 is "Yes." It is sufficient for convergence that, solely at the root $x = 0$, $f(x)$ shall satisfy a Lipschitz condition (of order unity) with an implied constant k less than unity: that is to say

(6) $$\limsup_{x \to 0} \left|\frac{f(x)}{x}\right| \leq k < 1.$$

For, if (6) holds, there exists a number $\delta > 0$ such that

$$|x_n| = |f(x_{n-1})| \leq K|x_{n-1}|, \quad k < K < 1$$

whenever $|x_{n-1}| \leq \delta$. Therefore, if x_0 belongs to the neighborhood $|x| \leq \delta$, $|x_n| \leq K^n |x_0| \to 0$ as $n \to \infty$. The reader will see that f(x) can satisfy (6) even though it may not be differentiable; if, however, f(x) is differentiable, we may replace (6) by the condition

(7) $$|f'(0)| < 1.$$

Condition (7) is a weaker condition than the condition of Question 1. The reader will also notice that the foregoing proof of convergence is very much shorter than the corresponding proofs of weaker versions found in some textbooks. Finally, a condition sufficient for convergence need hold only at a single point, as we have just shown; but an analoguous condition sufficient for divergence would, it seems, have to hold for all points; see for instance example (3) above.

The answer to Question 6 is "Yes." A simple example is

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Here at most two steps yield the root $x = 0$, although the function is everywhere discontinuous. A more elaborate example shows