NATIONAL BUREAU OF STANDARDS REPORT

1859

PROBABILITY TABLES FOR THE WILCOXON TEST WHEN THERE ARE TIES

Statistical Engineering Laboratory

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U. S. DEPARTMENT OF COMMERCE
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U. S. DEPARTMENT OF COMMERCE
Charles Sawyer, *Secretary*

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1103-11-1107 AUG 13 1952 1859

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Statistical Engineering Laboratory

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FOREWORD

This report was prepared as part of a continuing program of research on mathematical statistics and its applications carried out at the National Bureau of Standards under the general supervision of Dr. Churchill Eisenhart, Chief of the Statistical Engineering Laboratory. The Statistical Engineering Laboratory is Section 11.3 of the National Applied Mathematics Laboratories (Division 11, National Bureau of Standards, and is concerned with the development and application of modern statistical methods in the physical sciences and engineering.

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PROBABILITY TABLES FOR THE WILCOXON TEST WHEN THERE ARE TIES

Statistical Engineering Laboratory

Summary: Tables are given for the exact distribution of the Wilcoxon statistic in five selected cases where both of the samples contain five observations, some of which are tied.

Introduction: William H. Kruskal and W. Allen Wallis in "Use of Ranks in One-Criterion Variance Analysis" [SRC-20609KW] suggest "the average-rank method" for handling the Wilcoxon statistic when some of the observations are tied. They point out that this method of treating ties does not affect the mean value of the Wilcoxon statistic. Their formula (3.6) gives the variance of the Wilcoxon statistic when ties are treated according to the average-rank method.

Unfortunately, it would be expensive to make complete tables of the exact distribution of the Wilcoxon statistic when there are ties, since these ties may occur in a good many ways. However, when specific problems arise, if the samples are small, one can make the necessary tables as needed.

The Statistical Engineering Laboratory (National Bureau of Standards, Division 11.3), in using the Wilcoxon test in a specific problem, treated the ties as suggested by Kruskal and Wallis. The tables of this report give the exact distribution of the Wilcoxon statistic for the cases that arose.

It is realized that these probability tables as such will not be found useful, for it is unlikely that others will have samples which have the same ties occurring in them. But it was felt that these tables might be helpful in indicating the nature of the exact distributions of the Wilcoxon statistic that do arise when ties occur.

TABLE I.

Observed Ranks[1]: 1, 2, 3, 4, 6, 6, 6, 8, 9, 10

Score[2]	Frequency[3]	Cumulative Probability[4]	Score	Frequency	Cumulative Probability
16	3	.012	28	21	.583
16.5	0	.012	28.5	0	.583
17	0	.012	29	18	.655
17.5	0	.012	29.5	0	.655
18	4	.028	30	19	.730
18.5	0	.028	30.5	0	.730
19	4	.044	31	15	.790
19.5	0	.044	31.5	0	.790
20	7	.071	32	15	.849
20.5	0	.071	32.5	0	.849
21	10	.111	33	10	.889
21.5	0	.111	33.5	0	.889
22	10	.151	34	10	.929
22.5	0	.151	34.5	0	.929
23	15	.210	35	7	.956
23.5	0	.210	35.5	0	.956
24	15	.270	36	4	.973
24.5	0	.270	36.5	0	[illegible]
25	19	.345	37	4	.988
25.5	0	.345	37.5	0	.988
26	18	.417	38	0	[illegible]
26.5	0	.417	38.5	0	.988
27	21	.500	39	3	1.000
27.5	0	.500			

[5]$M = 27.5$; [6]$V = 22.36$; [7]$\sigma = 4.73$

$M \pm 1.96\sigma = 36.77$ or 18.23

TABLE I.

Observed Ranks[1]: 1, 2, 3, 4, 6, 6, 6, 8, 9, 10

Score[2]	Frequency[3]	Cumulative[4] Probability	Score	Frequency	Cumulative Probability
16	3	.012	28	21	.544
16.5	0	.012	28.5	0	.544
17	0	.012	29	18	.615
17.5	0	.012	29.5	0	.615
18	4	.028	30	19	.690
18.5	0	.028	30.5	0	.690
19	4	.044	31	15	.790
19.5	0	.044	31.5	0	.790
20	7	.071	32	15	.849
20.5	0	.071	32.5	0	.849
21	10	.111	33	10	.889
21.5	0	.111	33.5	0	.889
22	10	.111	34	10	.929
22.5	0	.111	34.5	0	.929
23	15	.171	35	7	.956
23.5	0	.171	35.5	0	.956
24	15	.230	36	4	.972
24.5	0	.230	36.5	0	.972
25	19	.306	37	4	.988
25.5	0	.306	37.5	0	.988
26	18	.377	38	0	.988
26.5	0	.377	38.5	0	.988
27	21	.460	39	3	1.000
27.5	0	.460			

[5] $M = 27.5$; [6] $V = 22.36$; [7] $\sigma = 4.73$

$M \pm 1.96\sigma = 36.77$ or 18.23

TABLE II.

Observed Ranks[1]: 1½, 1½, 3½, 3½, 6½, 6½, 6½, 6½, 9, 10

Score[2]	Frequency[3]	Cumulative[4] Probability	Score	Frequency	Cumulative Probability
16.5	4	.016	28	24	.603
17	0	.016	28.5	4	.619
17.5	0	.016	29	6	.643
18	0	.016	29.5	2	.651
18.5	0	.016	30	14	.706
19	1	.020	30.5	16	.770
19.5	12	.067	31	8	.802
20	1	.071	31.5	0	.802
20.5	0	.071	32	8	.833
21	0	.071	32.5	4	.849
21.5	12	.119	33	8	.881
22	8	.151	33.5	12	.929
22.5	4	.167	34	0	.929
23	8	.198	34.5	0	.929
23.5	0	.198	35	1	.933
24	8	.230	35.5	12	.980
24.5	16	.294	36	1	.984
25	14	.349	36.5	0	.984
25.5	2	.357	37	0	.984
26	6	.381	37.5	0	.984
26.5	4	.397	38	0	.984
27	24	.492	38.5	4	1.000
27.5	4	.508			

[5] $M = 27.5$; [6] $V = 21.25$; [7] $\sigma = \sqrt{V} = 4.61$

$M \pm 1.96\sigma = 36.54$ or 18.46

TABLE III.

Observed Ranks[1]: 1, 3, 3, 3, 6, 6, 6, 8½, 8½, 10

Score[2]	Frequency[3]	Cumulative Probability[4]	Score	Frequency	Cumulative Probability
15.5	0	.000	27.5	4	.508
16	3	.012	28	9	.544
16.5	0	.012	28.5	18	.615
17	0	.012	29	10	.655
17.5	0	.012	29.5	6	.679
18	0	.012	30	3	.690
18.5	2	.020	30.5	18	.762
19	9	.056	31	6	.786
19.5	0	.056	31.5	6	.810
20	1	.060	32	9	.845
20.5	0	.060	32.5	0	.845
21	3	.071	33	3	.857
21.5	18	.143	33.5	18	.929
22	3	.155	34	3	.940
22.5	0	.155	34.5	0	.940
23	9	.190	35	1	.944
23.5	6	.214	35.5	0	.944
24	6	.238	36	9	.980
24.5	18	.310	36.5	2	.988
25	3	.321	37	0	.988
25.5	6	.345	37.5	0	.988
26	10	.385	38	0	.988
26.5	18	.456	38.5	0	.988
27	9	.492	39	3	1.000

[5] $M = 27.5$; [6] $V = 21.67$; [7] $\sigma =$ ~~4.76~~ 4.65

$M \pm 1.96\sigma = 36.83$ or 18.17

TABLE IV.

Observed Ranks[1]: 1, 2½, 2½, 4½, 4½, 6½, 6½, 8, 9, 10

Score[2]	Frequency[3]	Cumulative Probability[4]	Score	Frequency	Cumulative Probability
15	1	.004	28	14	.563
15.5	0	.004	28.5	8	.593
16	0	.004	29	13	.647
16.5	0	.004	29.5	4	.663
17	4	.020	30	9	.698
17.5	0	.020	30.5	14	.754
18	0	.020	31	6	.778
18.5	2	.028	31.5	8	.810
19	5	.048	32	3	.821
19.5	2	.056	32.5	14	.877
20	0	.056	33	1	.881
20.5	8	.087	33.5	4	.897
21	4	.103	34	4	.912
21.5	4	.119	34.5	8	.944
22	1	.123	35	0	.944
22.5	14	.179	35.5	2	.952
23	3	.190	36	5	.972
23.5	8	.222	36.5	2	.980
24	6	.246	37	0	.980
24.5	14	.302	37.5	0	.980
25	9	.337	38	4	.996
25.5	4	.353	38.5	0	.996
26	13	.405	39	0	.996
26.5	8	.437	39.5	0	.996
27	14	.492	40	1	1.000
27.5	4	.508			

[5] $M = 27.5$; [6] $V = 22.5$; [7] $\sigma = 4.74$

$M \pm 1.96\sigma = 36.79$ or 18.21

TABLE V.

Observed Ranks[1]: 1, 2½, 2½, 4, 5, 6, 7, 8, 9, 10

Score[2]	Frequency[3]	Cumulative[4] Probability	Score	Frequency	Cumulative Probability
15	1	.004	28	8	.556
15.5	0	.004	28.5	12	.603
16	1	.008	29	8	.635
16.5	0	.008	29.5	10	.675
17	2	.016	30	7	.702
17.5	0	.016	30.5	12	.750
18	2	.024	31	6	.774
18.5	2	.032	31.5	8	.805
19	3	.044	32	6	.829
19.5	2	.052	32.5	8	.861
20	4	.067	33	4	.877
20.5	4	.083	33.5	6	.901
21	4	.099	34	4	.917
21.5	6	.123	34.5	4	.933
22	4	.139	35	4	.948
22.5	8	.171	35.5	2	.956
23	6	.194	36	3	.968
23.5	8	.226	36.5	2	.976
24	6	.250	37	2	.984
24.5	12	.298	37.5	0	.984
25	7	.325	38	2	.992
25.5	10	.365	38.5	0	.992
26	8	.397	39	1	.996
26.5	12	.444	39.5	0	.996
27	8	.476	40	1	1.000
27.5	12	.524			

[5] $M = 27.5$; [6] $V = 22.67$; [7] $\sigma = 4.76$

$M \pm 1.96\sigma = 36.83$ or 18.17

FOOTNOTES ON TABLES.

[1] The "Observed Ranks" are the ranks given to the observations from two samples of five each, using the average-rank method.

[2] The "Score" column contains all of the possible sums of five of the ranks from the observed ranks.

[3] The "Frequency" column gives the number of ways that a particular sum can be formed from the observed ranks.

[4] The "Cumulative Probability" column gives the partial sums of the "Frequency" column divided by 252 (252 is the number of possible sums that can be formed from the "observed ranks" when repetitions are permitted — 252 being the number of possible ways of selecting five things from ten things).

[5] "M" is the average of the tabulated distribution. For this distribution the average is exactly the theoretical value and hence serves as a check that the distribution was tabulated correctly.

[6] "V" is the variance of the tabulated distribution. For this distribution the variance agrees exactly with the theoretical value given by equation (3.6) in Kruskal and Wallis. The fact that the variances are correct serves as another check on the tabulation of the distributions.

[7] "σ" is the standard deviation of the tabulated distribution.

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

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