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NATIONAL BUREAU OF STANDARDS REPORT

1856

ON THE SPECTRUM OF A ONE PARAMETRIC FAMILY OF MATRICES

by

A. M. Ostrowski

American University
and
University of Basle, Switzerland

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NBS PROJECT
1101-10-5100

July 30, 1952

NBS REPORT
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PREPRINT

*This work was performed on a National Bureau of Standards contract with the American University.

INTRODUCTION

By the transformation

$$y_\mu = \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu \qquad (\mu = 1, \cdots, n)$$

with the matrix $A(\alpha_{\mu\nu})$ the vector $\xi(x_1, \cdots, x_n)$ is transformed into the vector $\eta(y_1, \cdots, y_n)$. The expressions

$$\Lambda(A) = \operatorname*{Max}_{|\xi| \neq 0} \frac{|\eta|}{|\xi|}, \qquad \lambda(A) = \operatorname*{Min}_{|\xi| \neq 0} \frac{|\eta|}{|\xi|}$$

are then respectively called the <u>upper</u> and the <u>lower bounds of the matrix</u> A. They are given by the square roots of the greatest and smallest eigenvalues of the non-negative matrix AA^*.

$\Lambda(A)$ and $\lambda(A)$ are very important in the discussion of metric properties of the matrix A, at least when the euclidean length is used. On the other hand, in practice, only estimates of Λ and λ can be used which contain the corresponding quantities for some special matrices. However, only very few types of matrices are known for which Λ and λ can be given explicitly, and in all of these cases these values are immediately obtained.

In what follows we discuss Λ and λ for a not trivial case of a one parametric class of matrices given by

$$D_\alpha^{(n)} = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 1 & \alpha & 0 & \cdots & 0 \\ 1 & 1 & \alpha & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdots & \alpha \end{pmatrix}$$

where all elements of the principal diagonal are equal to the real parameter α, all elements to the right of this diagonal are 0 and all elements to the left are 1.

As a matter of fact, our discussion covers the whole spectrum of the matrix $D_\alpha^{(n)} D_\alpha^{(n)*}$. If the eigenvalues of this matrix are denoted in the order of their magnitude

$$\lambda_1(\alpha) \leqq \lambda_2(\alpha) \leqq \dots \leqq \lambda_n(\alpha)$$

we write

$$\lambda_\nu(\alpha) = \alpha^2 - \alpha + v_\nu(\alpha)$$

and find that the $v_\nu(\alpha)$ are roots of a polynomial $\Delta_n(\alpha, \nu)$, which is a linear combination of two Chebyshev polynomials and contains α as a linear parameter. We have

$$\Delta_n(\alpha, 4\cos^2\theta) \equiv \frac{\alpha \sin(2n+1)\theta + (\alpha - 1)\sin(2n-1)\theta}{\sin\theta} .$$

It follows then that for each ν, $v_\nu(\alpha)$ is a monotonically increasing function of α, $-\infty < \alpha < \infty$. In particular, for three values of α we obtain the values of

$$\Lambda(\alpha) \equiv \Lambda(D_\alpha^{(n)}), \quad \lambda(\alpha) \equiv \lambda(D_\alpha^{(n)})$$

explicitly. We have

$$\Lambda(0) = \frac{1}{2\sin\frac{\pi}{4n-2}}, \quad \Lambda(\tfrac{1}{2}) = \tfrac{1}{2}\cot\frac{\pi}{4n}, \quad \Lambda(1) = \frac{1}{2\sin\frac{\pi}{4n+2}},$$

$$\lambda(0) = 0, \quad \lambda(\tfrac{1}{2}) = \tfrac{1}{2}\tan\frac{\pi}{4n}, \quad \lambda(1) = \frac{1}{2\cos\frac{\pi}{2n+1}} .$$

For other values of α, $\lambda(\alpha)$ and $\Lambda(\alpha)$ are continuous functions of α. We have in particular

$$\lambda(\alpha) \sim \alpha^{2n} \qquad (\alpha \to 0) ,$$

$$\Lambda(\alpha) = \alpha + \frac{n-1}{2} + \frac{n^2-1}{24\alpha} + O\left(\frac{1}{\alpha^2}\right) \qquad (\alpha \to \infty) ,$$

$$\Lambda(\alpha) = \alpha - \frac{1}{2} + \frac{1}{8\alpha} \cot^2\frac{\pi}{2n} + O\left(\frac{1}{\alpha^2}\right) \qquad (\alpha \to -\infty),$$

$$\lambda(\alpha) = \alpha - \frac{1}{2} + \frac{1}{8\alpha} \tan^2\frac{\pi}{2n} + O\left(\frac{1}{\alpha^2}\right) \qquad (\alpha \to \infty) ,$$

$$\lambda(\alpha) = \alpha + \frac{n-1}{2} + \frac{n^2-1}{24\alpha} + O\left(\frac{1}{\alpha^2}\right) \qquad (\alpha \to -\infty) .$$

These results are deduced in the section XI of the paper. Some "finite" estimates for $\lambda(\alpha)$ can be deduced from the results of the section X.

The equation satisfied by the $v_\nu(\alpha)$ is deduced in the sections IV and V while in the preceding sections I - III the properties of some determinants and of some special polynomials are developed as the preparation for the discussion of this equation. The discussion is pursued further in the sections VI and VII. In the section VIII we discuss in particular $v_1(\alpha)$ and $v_n(\alpha)$ and deduce from all these results the corresponding results for $\lambda_\nu(\alpha)$ in the section XI.

We may finally mention that the matrices $D_\alpha^{(n)}$ play a certain role in the discussion of the rounding off errors in the operations with the matrices.

I. SOME SPECIAL DETERMINANTS

If u, v are independent variables we will denote by $\psi_n(u,v)$ for n = 0, 1, 2, ... the expressions

$$(I,1) \qquad \psi_n(u,v) = \begin{vmatrix} u & 1 & 1 & \cdots & 1 & 1 \\ v & u & 1 & \cdots & 1 & 1 \\ 0 & v & u & \cdots & 1 & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & v & u \end{vmatrix} \qquad (n > 2)$$

$$(I,1^o) \qquad \psi_o \equiv 1 , \quad \psi_1 = u , \quad \psi_2 \equiv \begin{vmatrix} u & 1 \\ v & u \end{vmatrix} \equiv u^2 - v ,$$

where the determinant is a determinant of the order n in which all elements of the principal diagonal are equal to u, all elements to the right of the principle diagonal are equal to one, all elements of the first diagonal to the left of and parallel to the principal diagonal are equal to v and all other elements are equal to 0.

If we subtract in the determinant for ψ_n the second row from the first and then develop in the elements of the first row, the coefficient of (u - v) is ψ_{n-1} while (1 - u) is multiplied by

$$- \begin{vmatrix} v & 1 & 1 & \cdots & 1 & 1 \\ 0 & u & 1 & \cdots & 1 & 1 \\ 0 & v & u & \cdots & 1 & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & v & u \end{vmatrix} = - v \, \psi_{n-2} ,$$

and this is at once verified also for n = 2. We obtain therefore

$$(I,2) \quad \psi_n(u,v) = (u-v)\psi_{n-1}(u,v) + v(u-1)\psi_{n-2}(u,v) \qquad (n \geq 2) .$$

We put now

$$(I,3)\qquad f_n(x,u,v) = \begin{vmatrix} u & 1 & 1 & \cdots & 1 & 1 \\ v & u & 1 & \cdots & 1 & 1 \\ 0 & v & u & \cdots & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & u & 1 \\ 0 & 0 & 0 & \cdots & v & x \end{vmatrix} \qquad (n > 2)$$

$$(I,3^\circ)\qquad f_1 = x\,, \quad f_2 = \begin{vmatrix} u & 1 \\ v & x \end{vmatrix} = ux - v\,,$$

where the determinant is obtained from that in (I,1) in replacing the last element in the last row by x. The last line of the determinant in (I,3) is the sum of the two lines

$$\begin{matrix} 0 & 0 & 0 & \cdots & 0 & 0 & x-u \\ 0 & 0 & 0 & \cdots & 0 & v & u \end{matrix}\,,$$

and it follows

$$(I,4)\qquad f_n(x,u,v) = \psi_n(u,v) + (x-u)\,\psi_{n-1}(u,v) \qquad (n > 2)\,.$$

This is immediately verified also for $n = 2$.

We put finally

$$(I,5)\qquad F_n(\alpha,x,u,v) = \begin{vmatrix} u & 1 & 1 & \cdots & 1 & \alpha \\ v & u & 1 & \cdots & 1 & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & & u & 1 & \alpha \\ 0 & \cdots & & v & u & \alpha \\ 0 & \cdots & & 0 & v & x \end{vmatrix} \qquad (n > 2)\,,$$

$$(I,5^\circ)\qquad F_1 = x\,, \quad F_2 = ux - \alpha v$$

where the determinant is obtained from that in (I,3) in replacing all elements of the last column by α with the exception of the last element of this column which remains equal to x.

If $\alpha \neq 0$, it follows from (I,5), (I,3)

$$F_n(\alpha, x, u, v) = \alpha f_n(\frac{x}{\alpha}, u, v)$$

and therefore by (I,4)

(I,6) $$F_n(\alpha, x, u, v) = \alpha \psi_n(u, v) + (x - \alpha u)\psi_{n-1}(u, v) \quad .$$

This formula remains by continuity also true for $\alpha = 0$ and is also immediately verified for n = 2.

II. $\phi_n(v)$

We define now the polynomials $\phi_n(v)$ in putting u = 1 - v in ψ_n:

(II,1) $$\phi_n(v) = \psi_n(1-v, v) \quad .$$

From (I,1), (I,1°) we obtain

(II,2) $$\phi_o = 1 \ , \quad \phi_1 = 1 - v \ , \quad \phi_2 = v^2 - 3v + 1$$

and from (I,2)

(II,3) $$\phi_n = (1 - 2v)\ \phi_{n-1} - v^2\ \phi_{n-2} \quad .$$

We put now in $\phi_n(v)$

(II,4) $$v = \frac{1}{4 \cos^2 \theta}$$

and multiply the resulting expression by $4^n \cos^{2n}\theta$; we obtain then an expression

(II,5) $$U_n(\theta) = 4^n \cos^{2n}\theta\, \phi_n\left(\frac{1}{4\cos^2\theta}\right) .$$

From (II,5), (II,2) and (II,3) it follows

(II,6) $$U_o = 1 , \quad U_1 = 4\cos^2\theta - 1 = \frac{\sin 3\theta}{\sin\theta} ,$$

(II,7) $$U_n(\theta) = 2\cos 2\theta\, U_{n-1}(\theta) - U_{n-2}(\theta) \qquad (n = 2, 3, \cdots) .$$

From (II,6) and (II,7) follows now

(II,8) $$U_n(\theta) = \frac{\sin (2n+1)\theta}{\sin\theta} .$$

Indeed, (II,8) is true for n = 0, 1 and the right hand expression in (II,8) verifies obviously the recurrent relation (II,7). From (II,5) and (II,8) we have

(II,9) $$4^n \cos^{2n}\theta\, \phi_n\left(\frac{1}{4\cos^2\theta}\right) = \frac{\sin (2n+1)\theta}{\sin\theta} .$$

On the other hand, we have by a well known trigonometric formula, in putting $\gamma = 2\cos\theta$,

$$\frac{\sin (2n+1)\theta}{\sin\theta} = \sum_{\mu=o}^{n} (-1)^\mu \binom{2n-\mu}{\mu} \gamma^{2n-2\mu}$$

and therefore in introducing $n - \nu = \mu$ as the new variable of summation

$$\frac{\sin (2n+1)\theta}{\sin\theta} = (-1)^n \sum_{\nu=o}^{n} (-1)^\nu \binom{n+\nu}{n-\nu} \gamma^{2\nu} = \gamma^{2n}(-1)^n \sum_{\nu=o}^{n} (-1)^\nu \binom{n+\nu}{2\nu} \left(\frac{1}{\gamma^2}\right)^{n-\nu} .$$

In comparing this with (II,9) we obtain finally

$$\text{(II,10)}\quad (-1)^n\phi_n(v) = v^n - \binom{n+1}{2} v^{n-1} + \cdots + (-1)^\nu \binom{n+\nu}{2\nu} v^{n-\nu} + \cdots + (-1)^n \ .$$

III. THE ROOTS OF ϕ_n, $\phi_n \pm v\phi_{n-1}$

The right hand side expression in (II,8) vanishes for $\theta = \frac{\nu}{2n+1}\pi$ (v = 1, ⋯ , n). On the other hand we have by (II,8)

$$\text{(III,1)}\qquad U_n(\theta) + U_{n-1}(\theta) = 2 \sin 2n\,\theta \cot \theta \ ,$$

$$\text{(III,2)}\qquad U_n(\theta) - U_{n-1}(\theta) = 2 \cos 2n\,\theta \ ;$$

therefore $U_n(\theta) + U_{n-1}(\theta)$ vanishes for the n values

$$\text{(III,3)}\qquad \frac{\nu\pi}{2n} \qquad (\nu = 1, \cdots, n) ,$$

while $U_n(\theta) - U_{n-1}(\theta)$ has the n roots

$$\text{(III,4)}\qquad \frac{2\nu-1}{4n}\pi \qquad (\nu = 1, \cdots, n) .$$

The roots of $\phi_n(v)$ are obtained from those of $U_n(\theta)$ by the transformation (II,4) and give

$$\text{(III,5)}\qquad \frac{1}{4\cos^2 \frac{\nu\pi}{2n+1}} \qquad (\nu = 1, \cdots, n) .$$

We obtain from (II,10)

$$\text{(III,6)}\qquad \phi_n(v) = (-1)^n \prod_{\nu=1}^{n} \left(v - \frac{1}{4\cos^2 \frac{\nu\pi}{2n+1}} \right) .$$

On the other hand by the transformation (II,4) and by (II,5)

$$U_n(\theta) + U_{n-1}(\theta), \quad U_n(\theta) - U_{n-1}(\theta)$$

become after multiplication by v^n

$$\phi_n(v) + v\phi_{n-1}(v)$$

and

$$\phi_n(v) - v\phi_{n-1}(v) .$$

To the roots (III,1) and (III,2) correspond respectively the (n-1) roots

$$\text{(III,7)} \qquad \frac{1}{4\cos^2 \frac{\nu\pi}{2n}} \qquad (\nu = 1, \ldots, n-1)$$

of $\phi_n + v\phi_{n-1}$ (the n-th root becomes infinite since $\phi_n + v\phi_{n-1}$ is only of degree $n - 1$) and the roots

$$\text{(III,8)} \qquad \frac{1}{4\cos^2 \frac{2\nu-1}{4n}\pi} \qquad (\nu = 1, \ldots, n)$$

of $\phi_n - v\phi_{n-1}$.

By (II,10) $\phi_n + v\phi_{n-1}$ begins with the term

$$(-1)^n v^n + (-1)^{n-1} \frac{n(n+1)}{2} v^{n-1} + (-1)^{n-1} v^n + (-1)^{n-2} \frac{(n-1)n}{2} v^{n-1}$$

$$= (-1)^{n-1} n \, v^{n-1}$$

and $\phi_n - v\phi_{n-1}$ with the term $2(-1)^n v^n$; we obtain finally

$$(\text{III},9) \qquad \phi_n + v\,\phi_{n-1} = (-1)^{n-1}\, n \prod_{\nu=1}^{n-1} \left(v - \frac{1}{4\cos^2 \frac{\nu\pi}{2n}} \right)$$

$$(\text{III},10) \qquad \phi_n - v\,\phi_{n-1} = (-1)^{n}\, 2 \prod_{\nu=1}^{n} \left(v - \frac{1}{4\cos^2 \frac{2\nu-1}{4n}\pi} \right) .$$

IV. GENERAL REMARKS ON THE SPECTRUM OF AA^*

If $A(a_{\mu\nu})$ is a quadratic matrix of the order n corresponding to the transformation

$$(\text{IV},1) \qquad y_\mu = \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu \qquad (\mu = 1, \cdots, n) ,$$

the spectrum of the symmetric matrix

$$(\text{IV},2) \qquad S = AA^* = (s_{\mu\nu})$$

is given by the set of n roots of the corresponding secular equation

$$(\text{IV},3) \qquad |\lambda E - S| = 0 .$$

This spectrum can be interpreted in a different way. If we form the quadratic form

$$(\text{IV},4) \qquad S(x_\nu) = \sum_{\mu=1}^{n} y_\mu^{\,2} = \sum_{\mu,\nu} s_{\mu\nu} x_\mu x_\nu ,$$

the extremal problem on the unit sphere

$$(\text{IV},5) \qquad S(x_\nu) = \text{Extremum} \qquad (x_1^{\,2} + \cdots + x_n^{\,2} = 1)$$

gives for the Lagrangian multiplier λ in

$$(IV,6) \qquad S(x_\nu) - \lambda\left(\sum_{\nu=1}^{n} x_\nu - 1\right) ,$$

after elimination of the variables x_ν, again the equation (IV,3) as condition, although usually only two roots of (IV,3) correspond to the solutions of (IV,5).

In differentiating (IV,6) we obtain the n conditions

$$(IV,7) \qquad \sum_{\mu=1}^{n} a_{\mu\nu}\, y_\mu - \lambda\, x_\nu = 0 \qquad (\nu = 1, \ldots, n)$$

in which we would have to introduce the values of y_μ from (IV,1).

If however we introduce instead of x_ν and y_ν another set of n variables z_ν connected with the x_ν by a non-singular linear transformation, the elimination of the z_ν from the transformed equation (IV,7) gives the same equation for λ since the equation (IV,3) is then only multiplied by a non zero constant.

V. THE SECULAR EQUATION FOR $D_\alpha^{(n)}\, D_\alpha^{(n)*}$ $(n \geq 2)$

In order to obtain the secular equation for the $D_\alpha^{(n)}\, D_\alpha^{(n)*}$ we have after what has been said in the section IV to apply the Lagrangian procedure to the expression

$$(V,1) \quad (\alpha x_1)^2 + (x_1 + \alpha x_2)^2 + \cdots + (x_1 + \cdots + x_{n-1} + \alpha x_n)^2 - \lambda \sum_{\nu=1}^{n} x_\nu^2 .$$

If we put

$$(V,2) \qquad z_\nu = x_1 + \cdots + x_\nu \qquad (\nu = 1, \ldots, n)$$

and use the convention $z_o = 0$, (V,1) becomes

$$(V,3)\qquad \sum_{\nu=1}^{n} (\alpha z_\nu - \alpha z_{\nu-1} + z_{\nu-1})^2 - \lambda \sum_{\nu=1}^{n} x_\nu^2 \;;$$

in differentiating this with respect to x_ν we obtain the set of equations

$$(V,4)\qquad \alpha(\alpha z_\nu - \alpha z_{\nu-1} + z_{\nu-1}) + \sum_{k=\nu+1}^{n} (\alpha z_k - \alpha z_{k-1} + z_{k-1}) = \lambda x_\nu$$

$$(\nu = 1, \ldots, n)\,.$$

Here we write on the right hand side $z_\nu - z_{\nu-1}$ for x_ν and bring everything on the left

$$\lambda z_{\nu-1} - \lambda z_\nu + \alpha(\alpha z_\nu - \alpha z_{\nu-1} + z_{\nu-1}) + \sum_{k=\nu+1}^{n} (\alpha z_k - \alpha z_{k-1} + z_{k-1}) = 0$$

$$(\nu = 1, \ldots, n)\,.$$

In reordering this and in using the notations

$$(V,5)\qquad v = \lambda - \alpha^2 + \alpha\,, \quad u = \alpha^2 - \alpha + 1 - \lambda$$

we obtain finally

$$(V,6)\qquad \begin{cases} v\, z_{\nu-1} + u z_\nu + \sum_{k=\nu+1}^{n-1} z_k + \alpha z_n = 0 & (\nu = 1, \ldots, n-1) \\ v\, z_{n-1} + (u + \alpha - 1)\, z_n = 0\,. \end{cases}$$

The determinant of this system is now just $F_n(\alpha, x, u, v)$ given by (I,5), for $x = u + \alpha - 1$. (I,6) gives therefore for this determinant

the expression

$$\alpha \psi_n(u, v) + (u + \alpha - 1 - \alpha u)\, \psi_{n-1}(u, v) \quad .$$

Further we have from (V, 5) $u = 1 - v$ and our determinant becomes therefore by (II, 1)

$$\text{(V, 7)} \qquad \Delta_n(v, \alpha) \equiv \alpha \phi_n(v) + (\alpha - 1) v\, \phi_{n-1}(v) \quad .$$

The spectrum of $D_\alpha^{(n)} D_\alpha^{(n)*}$ is therefore obtained in solving with respect to λ the equation

$$\text{(V, 8)} \quad \Delta_n(v, \alpha) \equiv \alpha \phi_n(v) + (\alpha - 1) v\, \phi_{n-1}(v) = 0 \qquad (v = \lambda - \alpha^2 + \alpha) \text{ .}$$

VI. THE DISCUSSION OF Δ_n AND Q_n

In the following discussion we will keep n fixed and denote the roots (III, 5) of $\phi_n(v)$ and the roots of $\phi_{n-1}(v)$ respectively by x_ν, x'_ν:

$$\text{(VI, 1)} \quad x_\nu = \frac{1}{4\cos^2 \frac{\nu\pi}{2n+1}} \; (\nu = 1, \ldots, n) \; ; \quad x'_\nu = \frac{1}{4\cos^2 \frac{\nu\pi}{2n-1}} \; (\nu = 1, \ldots, n-1) \; .$$

We verify immediately the inequalities

$$\frac{\nu}{2n+1} < \frac{\nu}{2n} < \frac{\nu}{2n-1} < \frac{2\nu+1}{4n} < \frac{\nu+1}{2n+1} \qquad (\nu = 1, \ldots, n-1)$$

and obtain therefore

$$\text{(VI, 2)} \quad x_\nu < \frac{1}{4\cos^2 \frac{\nu\pi}{2n}} < x'_\nu < \frac{1}{4\cos^2 \frac{2\nu+1}{4n}\pi} < x_{\nu+1} \quad (\nu = 1, \ldots, n-1) \; .$$

In particular, the roots of ϕ_{n-1} separate the roots of ϕ_n.

If we develope now

$$- v \frac{\phi_{n-1}(v)}{\phi_n(v)}$$

into partial fractions

$$(VI,3) \qquad - v \frac{\phi_{n-1}(v)}{\phi_n(v)} = \sum_{\nu=1}^{n} \frac{p_\nu}{v - x_\nu} + 1$$

where the "polynomial term" is 1 as follows immediately from (II,10), we have

$$(VI,4) \qquad p_\nu = - \frac{x_\nu \phi_{n-1}(x_\nu)}{\phi_n'(x_\nu)}$$

and in putting in (VI,3) $v = 0$,

$$(VI,5) \qquad \sum_{\nu=1}^{n} \frac{p_\nu}{x_\nu} = 1 \quad .$$

It follows from (VI,4), since both the roots of ϕ_{n-1} and of ϕ_n' separate the positive roots of ϕ_n, that <u>all</u> p_ν have the same sign and therefore by (VI,5) are <u>positive</u>.

Consider now the quotient

$$(VI,6) \qquad Q_n(v, \alpha) \equiv \frac{\Delta_n(\alpha, v)}{(1 - \alpha)\phi_n(v)} \quad ;$$

we have from (V,8)

$$Q_n(v, \alpha) = \frac{- v \phi_{n-1}(v)}{\phi_n(v)} - 1 - \frac{1}{\alpha - 1} \quad ,$$

and in using (VI,3)

(VI,7) $$Q_n(v, \alpha) = \sum_{\nu=1}^{n} \frac{p_\nu}{v - x_\nu} - \frac{1}{\alpha - 1}, \qquad p_\nu > 0 .$$

It follows from (VI,7) that $Q_n(v, \alpha)$ is steadily <u>decreasing</u> if v grows through any interval containing none of the x_ν. Indeed we have

$$\frac{d\, Q_n(v, \alpha)}{dv} = -\sum_{\nu=1}^{n} \frac{p_\nu}{(v - x_\nu)^2} < 0 .$$

In particular, if v grows from x_ν to $x_{\nu+1}$, $Q_n(v, \alpha)$, decreases monotonically from $+\infty$ to $-\infty$ and assumes therefore in $(x_\nu, x_{\nu+1})$ any real value exactly once. We see that for any value of $\alpha \neq 1$ the equation (V,8) has exactly one root in any of the intervals $\langle x_\nu, x_{\nu+1})$.

VII. THE ROOTS OF $\Delta_n(\alpha, v)$

In what follows we denote the roots of $\Delta_n(\alpha, v)$ with respect to v, ordered increasingly by

(VII,1) $$v_1(\alpha) \leq v_2(\alpha) \leq \cdots .$$

Since $\frac{1}{\alpha - 1}$ decreases with increasing $\alpha \neq 1$, the corresponding root of Q_n laying in $(x_\nu, x_{\nu+1})$ is increasing with increasing α. If we take now into account the roots of (III,9) and (III,10) corresponding to $\alpha = \infty$ and $\alpha = \frac{1}{2}$, we obtain the subdivision of the interval from x_ν to $x_{\nu+1}$ corresponding to the subdivision

of the α-axis from 1 to ∞, from $-\infty$ to 0, from 0 to $\frac{1}{2}$ and from $\frac{1}{2}$ to 1. In the diagram 1 below, the roots of Δ_n from the interval $\langle x_\nu, x_{\nu+1})$ for α from one of the partial intervals of the lower line lie in the corresponding interval on the upper line.

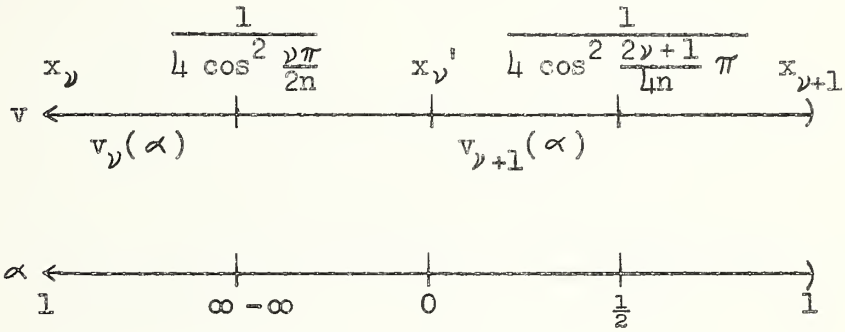

DIAGRAM 1

On the other hand if v goes from $-\infty$ to x_1, we can take for α the values between $-\infty$ and 1 and obtain therefore the following diagram:

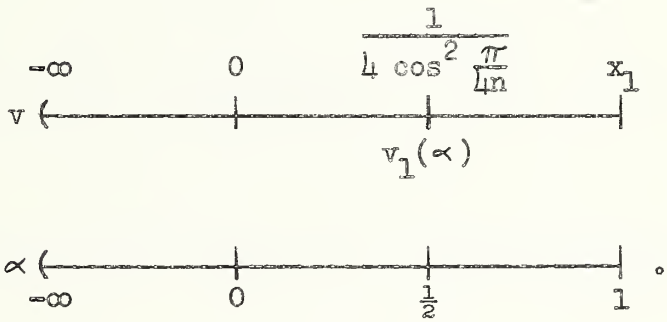

DIAGRAM 2

Finally, if v grows from x_n to ∞, we can take for α all values from 1 to ∞ and obtain the following diagram:

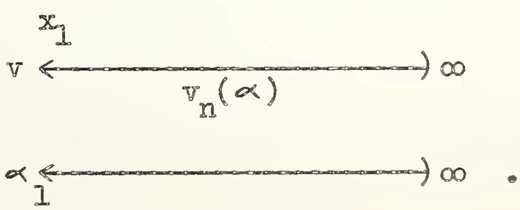

DIAGRAM 3

The resulting distribution of the roots from x_ν to $x_{\nu+1}$ between $v_\nu(\alpha)$ and $v_{\nu+1}(\alpha)$ has been already indicated in the Diagram 1. We see in particular that $\Delta_n(v, \alpha)$ has exactly n real and different roots for any α, except the case $\alpha = \infty$ where, as follows from (III,9) and from the diagram 3, the n-th root becomes infinite, but the n roots remain different.

In combining the information contained in the Diagrams 1, 2 and 3 we obtain the result, that each $v_\nu(\alpha)$, $\nu = 1, \ldots, n$ is a monotonically increasing function of α if α runs from $-\infty$ to ∞. We have in particular

$$\text{(VII,2)} \begin{cases} v_\nu(-\infty) = \dfrac{1}{4\cos^2 \frac{\nu-1}{2n}\pi}, \quad v_\nu(0) = x'_{\nu-1} = \dfrac{1}{4\cos^2 \frac{\nu-1}{2n-1}\pi}, \\ v_\nu(\tfrac{1}{2}) = \dfrac{1}{4\cos^2 \frac{2\nu-1}{4n}\pi}, \quad v_\nu(1) = x_\nu = \dfrac{1}{4\cos^2 \frac{\nu\pi}{2n+1}}, \\ v_\nu(\infty) = \dfrac{1}{4\cos^2 \frac{\nu\pi}{2n}} \qquad (1 < \nu < n); \end{cases}$$

$$\text{(VII,3)} \begin{cases} v_1(-\infty) = -\infty, \quad v_1(0) = 0, \quad v_1(\tfrac{1}{2}) = \dfrac{1}{4\cos^2 \frac{\pi}{4n}}, \\ v_1(1) = x_1 = \dfrac{1}{4\cos^2 \frac{\pi}{2n+1}}, \quad v_1(\infty) = \dfrac{1}{4\cos^2 \frac{\pi}{2n}}; \end{cases}$$

$$\text{(VII,4)} \begin{cases} v_n(-\infty) = \dfrac{1}{4\cos^2 \frac{n-1}{2n}\pi}, \quad v_n(0) = x'_{n-1} = \dfrac{1}{4\cos^2 \frac{n-1}{2n-1}\pi}, \\ v_n(\tfrac{1}{2}) = \dfrac{1}{4\cos^2 \frac{2n-1}{4n}\pi}, \quad v_n(1) = x_n = \dfrac{1}{4\cos^2 \frac{n\pi}{2n+1}}, \quad v_n(\infty) = \infty. \end{cases}$$

On the other hand it follows from (II,10) and (V,7)

(VII,5) $$\phi_n(0) = 1 , \quad \Delta_n(0, \alpha) = \alpha ,$$

(VII,6) $$\Delta_n(v, \alpha) = (-1)^n \left[v^n - n \left(\alpha + \frac{n-1}{2} \right) v^{n-1} + \cdots \right] .$$

Therefore we obtain

(VII,7) $$v_1(\alpha) \cdots v_n(\alpha) = \alpha$$

(VII,8) $$v_1(\alpha) + \cdots + v_n(\alpha) = n \left(\alpha + \frac{n-1}{2} \right) .$$

For $\alpha = 1$ we obtain from (VII,7)

(VII,9) $$x_1 \cdots x_n = 1$$

and, if we apply this to $\phi_{n-1}(v)$,

(VII,10) $$x_1' \cdots x_{n-1}' = 1 .$$

Since $\Delta_n(\alpha)$ has for any α, including ∞, n <u>different</u> roots, we see further, that the development of each of $v_\nu(\alpha)$ in powers of $\alpha - \alpha_0$ or in powers of $\frac{1}{\alpha}$ has <u>integer exponents</u> only.

VIII. THE DISCUSSION OF $v_1(\alpha)$ AND $v_n(\alpha)$

It follows from (II,10), that

$$(-1)^n (\phi_n + v\, \phi_{n-1}) = \sum_{\nu=1}^{n} (-1)^\nu \binom{n+\nu-1}{2\nu-1} v^{n-\nu} =$$

(VIII,1)
$$= -n v^{n-1} + \binom{n+1}{3} v^{n-2} - \binom{n+2}{5} v^{n-3} + \cdots + (-1)^{n-1}(2n-2)v + (-1)^n ,$$

and we have therefore for the sum and the product of the roots of this polynomial as they are given in (III, 9),

(VIII, 2) $$\sum_{\nu=1}^{n-1} \frac{1}{4\cos^2 \frac{\nu\pi}{2n}} = \frac{1}{n}\binom{n+1}{3} = \frac{n^2-1}{6},$$

(VIII, 3) $$\prod_{\nu=1}^{n-1} \frac{1}{4\cos^2 \frac{\nu\pi}{2n}} = \frac{1}{n}.$$

If α runs between 1 and ∞, it follows from (VII, 2) and (VII, 3), that

$$x_\nu < v_\nu(\alpha) < \frac{1}{4\cos^2 \frac{\nu\pi}{2n}} = v_\nu(\infty) \quad (\alpha > 1, \ \nu = 1, \ldots, n-1),$$

$$v_\nu(\alpha) = v_\nu(\infty) + O(\tfrac{1}{\alpha}) \quad (\alpha \to \infty, \ \nu = 1, \ldots, n-1),$$

and therefore in summing, in virtue of (VIII, 2),

(VIII, 4)
$$\sum_{\nu=1}^{n-1} x_\nu < \sum_{\nu=1}^{n-1} v_\nu(\alpha) < \frac{n^2-1}{6},$$
$$\sum_{\nu=1}^{n-1} v_\nu(\alpha) = \frac{n^2-1}{6} + O(\tfrac{1}{\alpha}) \qquad (\alpha \to \infty).$$

On the other hand it follows from (II, 10), (III, 6) and (VI, 1)

(VIII, 5) $$\sum_{\nu=1}^{n} x_\nu = \binom{n+1}{2}$$

and therefore

(VIII, 6) $$\binom{n+1}{2} - x_n < \sum_{\nu=1}^{n-1} v_\nu(\alpha) < \frac{n^2-1}{6}.$$

Here we have

$$x_n = \frac{1}{4\cos^2 \frac{n\pi}{2n+1}} = \frac{1}{4\sin^2 \frac{\pi}{2(2n+1)}} \quad ;$$

if we introduce this in (VIII,6) and use (VII,8) we obtain, since

$$v_n(\alpha) - n\alpha = \binom{n}{2} - \sum_{\nu=1}^{n-1} v_\nu(\alpha) \quad ,$$

$$\text{(VIII,7)} \begin{cases} \frac{(2n-1)(n-1)}{6} < v_n(\alpha) - n\alpha < \frac{1}{4\sin^2 \frac{\pi}{4n+2}} - n & (\alpha > 1) , \\ v_n(\alpha) = n\alpha + \frac{(2n-1)(n-1)}{6} + O(\frac{1}{\alpha}) & (\alpha \to \infty) . \end{cases}$$

On the other hand it follows from (VII,3) for $\alpha > 1$

$$\text{(VIII,8)} \begin{cases} \frac{1}{4\cos^2 \frac{\pi}{2n+1}} < v_1(\alpha) < \frac{1}{4\cos^2 \frac{\pi}{2n}} , & \\ \frac{1}{4}\tan^2 \frac{\pi}{2n+1} < v_1(\alpha) - \frac{1}{4} < \frac{1}{4}\tan^2 \frac{\pi}{2n} , & (\alpha > 1) \\ v_1 = \frac{1}{4} + \frac{1}{4}\tan^2 \frac{\pi}{2n} + O(\frac{1}{\alpha}) & (\alpha \to \infty) . \end{cases}$$

We assume now $\alpha < 1$. Then we obtain from (VII,4) the inequalities

$$\text{(VIII,9)} \begin{cases} \frac{1}{4\cos^2 \frac{(n-1)\pi}{2n}} < v_n(\alpha) < \frac{1}{4\cos^2 \frac{(n-1)\pi}{2n-1}} & (\alpha < 0) , \\ \frac{1}{4\cos^2 \frac{(n-1)\pi}{2n-1}} < v_n(\alpha) < \frac{1}{4\cos^2 \frac{2n-1}{4n}\pi} & (0 < \alpha < \frac{1}{2}) , \\ \frac{1}{4\cos^2 \frac{2n-1}{4n}\pi} < v_n(\alpha) < \frac{1}{4\cos^2 \frac{n}{2n+1}\pi} & (\frac{1}{2} < \alpha < 1) . \end{cases}$$

For $v_1(\alpha)$ we have from (VII,3) at once

$$\text{(VIII,10)} \quad \begin{cases} \dfrac{1}{4\cos^2\frac{\pi}{4n}} < v_1(\alpha) < \dfrac{1}{4\cos^2\frac{\pi}{2n+1}} & (\frac{1}{2} < \alpha < 1), \\ 0 < v_1(\alpha) < \dfrac{1}{4\cos^2\frac{\pi}{4n}} & (0 < \alpha < \frac{1}{2}), \end{cases}$$

and it remains to consider $v_1(\alpha)$ for negative α. Here we have by (VII,2) and (VII,4)

$$\frac{1}{4\cos^2\frac{\nu-1}{2n}\pi} < v_\nu(\alpha) < x'_{\nu-1} \quad (\nu = 2, \ldots, n),$$

$$v_\nu(\alpha) = \frac{1}{4\cos^2\frac{\nu-1}{2n}\pi} + O(\tfrac{1}{\alpha}) \qquad (\alpha \to -\infty),$$

$$\sum_{\nu=2}^{n} v_\nu(\alpha) = \sum_{\nu=2}^{n} \frac{1}{4\cos^2\frac{\nu-1}{2n}\pi} + O(\tfrac{1}{\alpha}) \quad (\alpha \to -\infty),$$

$$\sum_{\nu=2}^{n} \frac{1}{4\cos^2\frac{\nu-1}{2n}\pi} < \sum_{\nu=2}^{n} v_\nu(\alpha) < \sum_{\nu=2}^{n} x'_{\nu-1} .$$

The sum to the right is obtained from (VII,8) for $\alpha = 0$ as $\binom{n}{2}$, while the sum to the left is given by (VIII,2); we obtain

$$\frac{n^2-1}{6} < \sum_{\nu=2}^{n} v_\nu(\alpha) < \binom{n}{2}, \quad \sum_{\nu=2}^{n} v_\nu(\alpha) = \frac{n^2-1}{6} + O(\tfrac{1}{\alpha}) \quad (\alpha \to \infty),$$

and it follows now from (VII,8)

$$(VIII,11)\begin{cases} 0 < v_1(\alpha) - n\alpha < \dfrac{(2n-1)(n-1)}{6} & (\alpha < 0), \\ v_1(\alpha) = n\alpha + \dfrac{(2n-1)(n-1)}{6} + O(\frac{1}{\alpha}) & (\alpha \to -\infty). \end{cases}$$

We have now from (VII,7)

$$(VIII,12) \qquad v_1(\alpha) = \frac{\alpha}{v_2(\alpha) \cdots v_n(\alpha)} ;$$

here we have by (VII,2) and (VII,4) for $\alpha > 0$:

$$v_\nu(\alpha) > x'_{\nu-1} \qquad (\alpha > 0,\ \nu = 2, \cdots, n),$$

and it follows from (VII,10)

$$(VIII,13) \qquad v_1(\alpha) < \alpha \qquad (\alpha > 0).$$

On the other hand, if we assume now $\frac{1}{2} > \alpha > 0$, we have

$$v_\nu(\alpha) < v_\nu(\tfrac{1}{2}) \qquad (\nu = 2, \cdots, n)$$

and therefore by (VIII,12) and (VII,7) applied to $\alpha = \frac{1}{2}$:

$$(VIII,14) \quad v_1(\alpha) > \frac{\alpha}{v_2(\frac{1}{2})\cdots v_n(\frac{1}{2})} = \frac{\alpha\, v_1(\frac{1}{2})}{v_1(\frac{1}{2})\cdots v_n(\frac{1}{2})} = \frac{\alpha}{2\cos^2\frac{\pi}{4n}}$$

$$(\tfrac{1}{2} > \alpha > 0).$$

IX. $\lambda_\nu(\alpha)$

Consider now the fundamental roots $\lambda_\nu(\alpha)$ of $D_\alpha^{(n)} D_\alpha^{(n)*}$, ordered according to their magnitude

$$(IX,1) \qquad \lambda_1(\alpha) \leqq \lambda_2(\alpha) = \cdots \leqq \lambda_n(\alpha) \quad .$$

We have in virtue of (V,5)

$$(IX,2) \qquad \lambda_\nu(\alpha) = \alpha^2 - \alpha + v_\nu(\alpha) \qquad (\nu = 1, \cdots , n) \quad ,$$

so that the equality in (IX,1) is impossible. The product of the λ_ν is equal to the square of the determinant of $D_\alpha^{(n)}$:

$$(IX,3) \qquad \lambda_1(\alpha) \cdots \lambda_n(\alpha) = \alpha^{2n} \quad .$$

On the other hand it follows by (VII,8) and (IX,2)

$$(IX,4) \qquad \lambda_1(\alpha) + \cdots + \lambda_n(\alpha) = n\alpha^2 + \binom{n}{2} \quad ,$$

and further from (IX,3)

$$(IX,5) \qquad \lambda_1(\alpha) = \frac{\alpha^{2n}}{\prod_{\nu=2}^{n} \lambda_\nu(\alpha)} \quad .$$

We have from (IX,2), (VII,2) and (VII,4)

$$\lambda_\nu(\alpha) \rightarrow v_\nu(0) = x'_{\nu-1} \qquad (\nu > 1, \ \alpha \rightarrow 0) \quad ,$$

and therefore in virtue of (VII,10)

$$(IX,6) \qquad \prod_{\nu=2}^{n} \lambda_\nu(\alpha) \rightarrow \prod_{\nu=1}^{n-1} x'_\nu = 1 \qquad (\alpha \rightarrow 0) \quad .$$

We obtain now from (IX,5)

(IX,7) $$\lambda_1(\alpha) \sim \alpha^{2n} \qquad (\alpha \to 0) .$$

For $\lambda_n(\alpha)$ it follows from (IX,4)

(IX,8) $$\lambda_n(\alpha) = n\alpha^2 + \binom{n}{2} - \Big(\lambda_1(\alpha) + \cdots + \lambda_{n-1}(\lambda)\Big) ,$$

and further from (IX,2), (VII,2) and (VII,3)

(IX,9) $$\lambda_\nu(\alpha) = \alpha^2 - \alpha + \frac{1}{4\cos^2 \frac{\nu\pi}{2n}} + 0\left(\frac{1}{\alpha}\right)$$
$$(\nu = 1, \cdots, n-1 \ ; \quad \alpha \to \infty) ,$$

and therefore by (VIII,2)

$$\sum_{\nu=1}^{n-1} \lambda_\nu(\alpha) = (n-1)\alpha^2 - (n-1)\alpha + \frac{n^2-1}{6} + 0\left(\frac{1}{\alpha}\right) \qquad (\alpha \to \infty) .$$

Introducing this in (IX,8) we obtain finally

(IX,10) $$\lambda_n(\alpha) = \alpha^2 + (n-1)\alpha + \frac{(2n-1)(n-1)}{6} + 0\left(\frac{1}{\alpha}\right) \qquad (\alpha \to \infty) .$$

If $\alpha \to -\infty$ we have

(IX,11) $$v_n(\alpha) \to \frac{1}{4\cos^2 \frac{n-1}{2n}\pi} = \frac{1}{4\sin^2 \frac{\pi}{2n}} ,$$

and therefore

(IX,12) $$\lambda_n(\alpha) = \alpha^2 - \alpha + \frac{1}{4} + \frac{1}{4}\cot^2 \frac{\pi}{2n} + 0\left(\frac{1}{\alpha}\right) \qquad (\alpha \to -\infty) .$$

Further, in the case of $\lambda_n(\alpha)$, we can obtain from the bounds for $v_n(\alpha)$ derived in the section VIII, immediately further estimates for great $|\alpha|$.

X. BOUNDS FOR $\lambda_1(\alpha)$ FOR SMALL $|\alpha|$.

We have from the definition of $v_\nu(\alpha)$, since $\Delta_n(v, \alpha)$ begins with $(-1)^n v^n$,

$$(X,1) \qquad \alpha \phi_n(v) + (\alpha - 1) v \phi_{n-1}(v) = \prod_{\nu=1}^{n} (v_\nu(\alpha) - v) .$$

In replacing here v by $\alpha - \alpha^2$ and in using (IX,2) and (IX,3) we obtain then the identity

$$\alpha \phi_n(\alpha - \alpha^2) - (\alpha - 1)^2 \alpha \phi_{n-1}(\alpha - \alpha^2) = \alpha^{2n} ,$$

and in dividing this by α,

$$(X,2) \qquad \phi_n(\alpha - \alpha^2) = (\alpha - 1)^2 \phi_{n-1}(\alpha - \alpha^2) + \alpha^{2n-1} ,$$

and therefore

$$\phi_n(\alpha - \alpha^2) = \sum_{\nu=0}^{n-1} (\alpha - 1)^{2\nu} \alpha^{2n-2\nu-1} .$$

On the other hand we have from (X,1) for $\alpha = 1/2$ and $\alpha = 0$

$$\phi_n(v) - v\phi_{n-1}(v) = 2 \prod_{\nu=1}^{n} (v_\nu(\tfrac{1}{2}) - v) ,$$

$$v\phi_{n-1}(v) = - \prod_{\nu=1}^{n} (v_\nu(0) - v) = v \prod_{\nu=2}^{n} (v_\nu(0) - v) ,$$

and in replacing in both formulae v by $\alpha - \alpha^2$:

$$(X,3) \qquad \phi_n(\alpha - \alpha^2) - \alpha(1 - \alpha)\phi_{n-1}(\alpha - \alpha^2) = 2\prod_{\nu=1}^{n}\left(v_\nu(\tfrac{1}{2}) + \alpha^2 - \alpha\right) ,$$

$$(X,4) \qquad \phi_{n-1}(\alpha - \alpha^2) = \prod_{\nu=2}^{n}\left(v_\nu(0) + \alpha^2 - \alpha\right) .$$

To obtain a representation of $\phi_n(\alpha - \alpha^2)$ in the form of a product we replace in (III,6) v by $(\alpha - \alpha^2)$ and use (VI,1) and (VII,9). Then we have

$$\phi_n(\alpha - \alpha^2) = \prod_{\nu=1}^{n}(x_\nu + \alpha^2 - \alpha) = \prod_{\nu=1}^{n}\left(1 + \frac{1}{x_\nu}(\alpha^2 - \alpha)\right) ,$$

$$\frac{1}{x_\nu} = 4\cos^2\frac{\nu\pi}{2n+1} = 4 - 4\sin^2\frac{\nu\pi}{2n+1} ,$$

and therefore finally

$$(X,5) \quad \phi_n(\alpha - \alpha^2) = \prod_{\nu=1}^{n}\left((1 - 2\alpha)^2 + 4\alpha(1 - \alpha)\sin^2\frac{\nu\pi}{2n+1}\right) .$$

If we have now

$$(X,6) \qquad 0 < \alpha < \frac{1}{2} ,$$

we get from (X,5)

$$0 < \phi_n(\alpha) < 1$$

and by (IX,2), since $v_\nu(\alpha)$ monotonically increases with α,

$$(X,7) \quad v_\nu(0) + \alpha^2 - \alpha < \lambda_\nu(\alpha) < v_\nu(\tfrac{1}{2}) + \alpha^2 - \alpha \qquad (\nu > 1) .$$

But here $0 < \alpha - \alpha^2 = \alpha(1-\alpha) < 1/4$, while by (VII,2) and (VI,1) $v_\nu(0) > 1/4$ $(\nu > 1)$. Therefore all terms in (X,7) are positive and

$$(X,8)\quad \prod_{\nu=2}^{n} (v_\nu(0) + \alpha^2 - \alpha) < \prod_{\nu=2}^{n} \lambda_\nu(\alpha) < \prod_{\nu=2}^{n}\left(v_\nu(\tfrac{1}{2}) + \alpha^2 - \alpha\right) .$$

We have now from (X,8)

$$\frac{\lambda_1(\alpha)}{v_1(\frac{1}{2}) + \alpha^2 - \alpha} > \frac{\lambda_1(\alpha)\cdots\lambda_n(\alpha)}{\prod_{\nu=1}^{n}(v_\nu(\frac{1}{2}) + \alpha^2 - \alpha)} ,$$

and therefore in virtue of (IX,3), (X,2) and (X,3)

$$\frac{\lambda_1(\alpha)}{v_1(\frac{1}{2}) + \alpha^2 - \alpha} > \frac{2\alpha^{2n}}{\phi_n(\alpha - \alpha^2) - (\alpha - \alpha^2)\phi_{n-1}(\alpha - \alpha^2)} =$$

$$= \frac{2\alpha^{2n}}{\alpha^{2n-1} + (1-\alpha)(1-2\alpha)\phi_{n-1}(\alpha - \alpha^2)} .$$

We obtain therefore finally, as $\phi_{n-1}(\alpha - \alpha^2)$ is positive and < 1,

$$(X,9)\quad \lambda_1(\alpha) > \frac{2\alpha^n\left(\dfrac{1}{4\cos^2\frac{\pi}{4n}} + \alpha^2 - \alpha\right)}{(1-\alpha)(1-2\alpha) + \alpha^{2n-1}} \qquad (0 < \alpha < \tfrac{1}{2}) .$$

On the other hand we have again from (X,8) in using (IX,3) and (X,4):

$$\lambda_1(\alpha) < \frac{\prod_{\nu=1}^{n}\lambda_\nu(\alpha)}{\prod_{\nu=2}^{n}(v_\nu(0) + \alpha^2 - \alpha)} = \frac{\alpha^{2n}}{\phi_{n-1}(\alpha - \alpha^2)} ,$$

and therefore by (X,5) applied to n - 1,

$$(X,10) \qquad \lambda_1(\alpha) < \frac{\alpha^{2n}}{\prod_{\nu=1}^{n-1}\left((1-2\alpha)^2+4\alpha(1-\alpha)\sin^2\frac{\nu\pi}{2n-1}\right)} \qquad (0<\alpha<\tfrac{1}{2}) \quad .$$

If we make now the hypothesis

$$(X,11) \qquad \alpha < 0 \quad ,$$

we have then

$$v_\nu(0) + \alpha^2 - \alpha > \lambda_\nu(\alpha) > v_\nu(-\infty) + \alpha^2 - \alpha \qquad (\nu > 1) \quad ,$$

where

$$v_\nu(-\infty) = \frac{1}{4\cos^2\frac{(\nu-1)}{2n}\pi} \qquad (\nu = 2, \cdots, n)$$

and all terms are positive; therefore, in multiplying and in using (X,4):

$$(X,12) \quad \phi_{n-1}(\alpha-\alpha^2) > \prod_{\nu=2}^{n}\lambda_\nu(\alpha) > \prod_{\nu=1}^{n-1}\left(\alpha^2 - \alpha + \frac{1}{4\cos^2\frac{\nu\pi}{2n}}\right) \quad .$$

It follows now from (IX,3)

$$(X,13) \quad \frac{\alpha^{2n}}{\phi_{n-1}(\alpha-\alpha^2)} \leq \lambda_1(\alpha) \leq \frac{\alpha^{2n}}{\prod_{\nu=1}^{n-1}\left(\alpha^2-\alpha+\frac{1}{4\cos^2\frac{\nu\pi}{2n}}\right)} \quad .$$

On the other hand, if we put in (III,9) $v = \alpha - \alpha^2$, we obtain

$$\frac{1}{n}\left(\phi_n(\alpha - \alpha^2) + (\alpha - \alpha^2)\,\phi_{n-1}(\alpha - \alpha^2)\right) = \prod_{\nu=1}^{n-1}\left(\alpha^2 - \alpha + \frac{1}{4\cos^2\frac{\nu\pi}{2n}}\right),$$

and therefore by (X,2)

$$\text{(X,14)} \quad \prod_{\nu=1}^{n-1}\left(\alpha^2 - \alpha + \frac{1}{4\cos^2\frac{\nu\pi}{2n}}\right) = \frac{1}{n}\left[\alpha^{2n-1} + (1-\alpha)\,\phi_{n-1}(\alpha - \alpha^2)\right].$$

We have, since the roots of ϕ_{n-1} are x'_ν,

$$\text{(X,15)} \qquad \phi_{n-1}(\alpha - \alpha^2) = \prod_{\nu=1}^{n-1}(x'_\nu + \alpha^2 - \alpha),$$

and this is in virtue of (VII,10) > 1 and tends to 1 if $\alpha \uparrow 0$. We obtain finally

$$\text{(X,16)} \quad \frac{\alpha^{2n}}{\phi_{n-1}(\alpha - \alpha^2)} \leq \lambda_1(\alpha) \leq \frac{n\,\alpha^{2n}}{(1-\alpha)\phi_{n-1}(\alpha-\alpha^2) + \alpha^{2n-1}} \quad (\alpha < 0),$$

where both denominators are > 1 and tend to 1 with $\alpha \to 0$.

XI. $\Lambda(D_\alpha^{(n)})$ AND $\lambda(D_\alpha^{(n)})$

We have by definition

$$\text{(XI,1)} \qquad \Lambda(\alpha) \equiv \Lambda(D_\alpha^{(n)}) = \sqrt{\lambda_n(\alpha)},$$

(XI,2) $$\lambda(\alpha) \equiv \lambda(D_{\alpha}^{(n)}) = \sqrt{\lambda_1(\alpha)} \quad .$$

For $\Lambda(\alpha)$ we obtain at once from (IX,2) and (VII,4) the values

(XI,3)
$$\Lambda(-\infty) = \infty \quad , \Lambda(0) = \frac{1}{2\sin\frac{\pi}{4n-2}} \quad , \quad \Lambda(\tfrac{1}{2}) = \tfrac{1}{2}\cot\frac{\pi}{4n} \quad ,$$
$$\Lambda(1) = \frac{1}{2\sin\frac{\pi}{4n+2}} \quad , \quad \Lambda(\infty) = \infty \quad .$$

The asymptotic behavior for $\alpha \to \infty$ is easily obtained from (IX,10) and (IX,12). From (IX,10) we obtain,in developing the square root of the right hand expression,

$$\alpha\sqrt{1 + \frac{n-1}{\alpha} + \frac{(n-1)(2n-1)}{6\alpha^2} + 0\left(\frac{1}{\alpha^3}\right)} = \alpha\left[1 + \frac{n-1}{2\alpha} + \frac{n^2-1}{24\alpha^2} + 0\left(\frac{1}{\alpha^3}\right)\right] ,$$

(XI,4) $$\Lambda(\alpha) = \alpha + \frac{n-1}{2} + \frac{n^2-1}{24\alpha} + 0\left(\frac{1}{\alpha^2}\right) \qquad (\alpha \to \infty) \quad .$$

From (IX,12) we obtain in the same way

$$\alpha\sqrt{1 - \frac{1}{\alpha} + \frac{1}{4\alpha^2} + \frac{1}{4\alpha^2}\cot^2\frac{\pi}{2n} + 0\left(\frac{1}{\alpha^3}\right)} =$$

$$= \alpha\left[1 - \frac{1}{2\alpha} + \frac{1}{8\alpha^2} + \frac{1}{8\alpha^2}\cot^2\frac{\pi}{2n} - \frac{1}{8\alpha^2} + 0\left(\frac{1}{\alpha^3}\right)\right] ,$$

(XI,5) $\Lambda(\alpha) = \alpha - \frac{1}{2} + \frac{1}{8\alpha}\cot^2\frac{\pi}{2n} + O\left(\frac{1}{\alpha^2}\right) \qquad (\alpha \to -\infty)$.

On the other hand it is easy to see that $\Lambda(\alpha)$ is <u>monotonically increasing</u> with α from $\alpha = 0$ to $\alpha \to \infty$. Indeed $\Lambda^2(\alpha) = \lambda_n(\alpha)$ is, as we have used in the section V, the maximum of the quadratic form

$$(\alpha x_1)^2 + (\alpha x_2 + x_1)^2 + (\alpha x_3 + x_1 + x_2)^2 + \dots + (\alpha x_n + x_1 + \dots + x_{n-1})^2$$

on the unit sphere $x_1^2 + \dots + x_n^2 = 1$. This maximum is obviously assumed for non negative values of the variables and is therefore not decreasing with $\alpha \geqq 0$.

We consider now $\lambda(\alpha)$. Here we obtain immediately from (VII,3) and (IX, 2) the three special values:

(XI,6) $\lambda(0) = 0 \quad , \quad \lambda(\tfrac{1}{2}) = \frac{1}{2}\tan\frac{\pi}{4n} \quad , \quad \lambda(1) = \frac{1}{2}\,\frac{1}{\cos\frac{\pi}{2n+1}}$.

Further, for $\alpha \to \infty$ we have from (VIII,8) by (IX,2)

$$\lambda_1(\alpha) = \alpha^2 - \alpha + \frac{1}{4} + \frac{1}{4}\tan^2\frac{\pi}{2n} + O\left(\frac{1}{\alpha}\right) \qquad (\alpha \to \infty) ,$$

$$\lambda(\alpha) = \alpha\left[1 - \frac{1}{\alpha} + \frac{1}{4\alpha^2} + \frac{1}{4\alpha^2}\tan^2\frac{\pi}{2n} + O\left(\frac{1}{\alpha^3}\right)\right]^{\frac{1}{2}} =$$

$$= \alpha\left\{1 - \frac{1}{2\alpha} + \frac{1}{8\alpha^2} + \frac{1}{8\alpha^2}\tan^2\frac{\pi}{2n} - \frac{1}{8\alpha^2} + O\left(\frac{1}{\alpha^3}\right)\right\} ,$$

(XI,7) $\lambda(\alpha) = \alpha - \frac{1}{2} + \frac{1}{8\alpha}\tan^2\frac{\pi}{2n} + O\left(\frac{1}{\alpha^2}\right) \qquad (\alpha \to \infty)$.

Finally, for $\alpha \to -\infty$ we have from (VIII,11) and (IX,2)

$$\lambda(\alpha) = \left[1 + \frac{n-1}{\alpha} + \frac{(2n-1)(n-1)}{6\alpha^2} + 0\left(\frac{1}{\alpha^3}\right)\right]^{\frac{1}{2}} =$$

$$= \alpha\left\{1 + \frac{n-1}{2\alpha} + \frac{n^2-1}{24\alpha^2} + 0\left(\frac{1}{\alpha^3}\right)\right\} ,$$

(XI,8) $\lambda(\alpha) = \alpha + \frac{n-1}{2} + \frac{n^2-1}{24\alpha} + 0\left(\frac{1}{\alpha^2}\right) \qquad (\alpha \to -\infty)$.

The discussion of the monotony properties of $\Lambda(\alpha)$ for $\alpha < 0$ and $\lambda(\alpha)$ for $\alpha > 0$ appears to present considerable difficulties. On the other hand, we can prove that $\lambda(\alpha)$ is <u>monotonically decreasing</u> for $\alpha < 0$. Indeed, if we put

$$\alpha = -\beta , \quad \beta > 0 , \qquad \gamma = \frac{1}{\beta} ,$$

the transformation with the matrix $D_\alpha^{(n)}$ is

$$\begin{aligned} y_1 &= -\beta x_1 \\ y_2 &= \quad x_1 - \beta x_2 \\ &\cdots\cdots\cdots \\ y_n &= \quad x_1 + x_2 + \cdots + x_{n-1} - \beta x_n . \end{aligned}$$

If we write here $y_\nu = -z_\nu$ $(\nu = 1, \cdots, n)$, we obtain the inverse of the above transformation in eliminating successively $x_1, x_2, \cdots$ from the right side expression in

$$x_1 = \gamma z_1$$
$$x_2 = \gamma z_2 + \gamma x_1$$
$$\cdots\cdots$$
$$x_n = \gamma z_n + \gamma (x_1 + x_2 + \cdots + x_{n-1}) \ .$$

The result will be a transformation S:

$$\text{(S)} \quad x_\nu = \sum_{\mu=1}^{\nu} A_\mu(\gamma)\, z_\mu \qquad (\nu = 1, \cdots, n) \quad ,$$

where $A_\mu(\gamma)$ are polynomials in γ with <u>non negative</u> <u>coefficients</u>, and we have

$$\text{(XI,9)} \qquad \lambda(D_\alpha^{(n)}) = \frac{1}{\Lambda(S)} \ .$$

On the other hand, the maximum of the quadratic form

$$\sum_{\nu=1}^{n} \left(\sum_{\mu=1}^{\nu} A_\mu(\gamma)\, z_\mu \right)^2$$

on the unit sphere $\Sigma_{\mu=1}^{n} z_\mu^{\ 2} = 1$ is obviously attained for non negative z and is therefore monotonically increasing with γ. By (XI,9) $\lambda(\alpha)$ is therefore monotonically decreasing with γ and monotonically increasing with β. We see that if α goes from 0 to $-\infty$, $\lambda(\alpha)$ monotonically increases from 0 to ∞.

August 7, 1952

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