NATIONAL BUREAU OF STANDARDS REPORT

1846

SOME PROPERTIES OF STRONGLY CONTINUOUS STOCHASTIC PROCESSES

by

Eugene Lukacs

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

U. S. DEPARTMENT OF COMMERCE
Charles Sawyer, *Secretary*

NATIONAL BUREAU OF STANDARDS
A. V. Astin, [redacted] *Director*

THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.

1. **Electricity.** Resistance Measurements. Inductance and Capacitance. Electrical Instruments. Magnetic Measurements. Electrochemistry.

2. **Optics and Metrology.** Photometry and Colorimetry. Optical Instruments. Photographic Technology. Length. Gage.

3. **Heat and Power.** Temperature Measurements. Thermodynamics. Cryogenics. Engines and Lubrication. Engine Fuels.

4. **Atomic and Radiation Physics.** Spectroscopy. Radiometry. Mass Spectrometry. Physical Electronics. Electron Physics. Atomic Physics. Neutron Measurements. Nuclear Physics. Radioactivity. X-Rays. Betatron. Nucleonic Instrumentation. Radiological Equipment. Atomic Energy Commission Instruments Branch.

5. **Chemistry.** Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Gas Chemistry. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

6. **Mechanics.** Sound. Mechanical Instruments. Aerodynamics. Engineering Mechanics. Hydraulics. Mass. Capacity, Density, and Fluid Meters.

7. **Organic and Fibrous Materials.** Rubber. Textiles. Paper. Leather. Testing and Specifications. Organic Plastics. Dental Research.

8. **Metallurgy.** Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion.

9. **Mineral Products.** Porcelain and Pottery. Glass. Refractories. Enameled Metals. Building Stone. Concreting Materials. Constitution and Microstructure. Chemistry of Mineral Products.

10. **Building Technology.** Structural Engineering. Fire Protection. Heating and Air Conditioning. Exterior and Interior Coverings. Codes and Specifications.

11. **Applied Mathematics.** Numerical Analysis. Computation. Statistical Engineering. Machine Development.

12. **Electronics.** Engineering Electronics. Electron Tubes. Electronic Computers. Electronic Instrumentation.

13. **Ordnance Development.** Mechanical Research and Development. Electromechanical Fuzes. Technical Services. Missile Fuzing Research. Missile Fuzing Development. Projectile Fuzes. Ordnance Components. Ordnance Tests. Ordnance Research.

14. **Radio Propagation.** Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Frequency Utilization Research. Tropospheric Propagation Research. High Frequency Standards. Microwave Standards.

15. **Missile Development.** Missile Engineering. Missile Dynamics. Missile Intelligence. Missile Instrumentation. Technical Services. Combustion.

NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT | | **NBS REPORT**

1103-21-5119 August 7, 1952 1846

SOME PROPERTIES OF STRONGLY CONTINUOUS STOCHASTIC PROCESSES*

by

Eugene Lukacs

*The preparation of this paper was sponsored by the U.S. Naval Ordnance Test Station, Inyokern

NBS

SOME PROPERTIES OF STRONGLY CONTINUOUS STOCHASTIC PROCESSES

by

Eugene Lukacs
National Bureau of Standards

1. Introduction

This report is a continuation of an earlier report [1] dealing with strongly continuous processes. In the following we use the condition obtained in our previous study and restate therefore this result as theorem 1. We reformulate also this theorem so as to give it a wording which appeals more to the intuition. We derive also several new theorems on strongly continuous stochastic processes. Theorem 2 gives a characterization of the fundamental random process (Wiener process) while theorem 3 characterizes another stochastic process with independent increments.

2. Normality of the increments.

Theorem 1: Let $y(t)$ be a stochastic process and assume that

(i) $y(t)$ is a process with independent increments

(ii) $y(t)$ is strongly continuous in the interval $[a,b]$.

Then $y(b) - y(a)$ is normally distributed.

The concepts "strong continuity" and "process with independent increments" are defined in [1] where a complete proof of theorem 1 is given.

H. B. Mann proved recently [2] that a process is strongly continuous if and only if it can be realized in the space of all continuous functions $\{$Theorem 3 in reference [2]$\}$. In view of this fact we obtain the following reformulation of theorem 1.

Corollary to theorem 1. A stochastic process with independent increments which can be realized in the space of continuous functions has normally distributed increments.

A formal definition of the realizability of a process in a function space is given in [2].

3. Characterization of the fundamental random process.

We first establish a property of strongly continuous processes.

Lemma 1. If the process y(t) is strongly continuous in the interval [a,b] then it is continuous at every interior point of the interval, that means $\operatorname{plim}_{\tau \to 0} y(t+\tau) = y(t)$ if $a < t < b$.

Proof: We take for the set S the set consisting of the two points t and $t + \tau$, both interior to [a,b]. The event $\mathcal{E}(\delta, \epsilon, S)$ is then the event that $|y(t+\tau) - y(t)| \leqq \epsilon$ for $|\tau| < \delta$. The strong continuity of the y(t) process implies that for every $\epsilon > 0$, and $\eta > 0$ there exists a $\delta = \delta(\epsilon, \eta)$ such that $P(|y(t+\tau) - y(t)| \geqq \epsilon) \geqq 1 - \eta$ for $|\tau| < \delta$. By the definition of convergence in probability this means that

$$\operatorname{plim}_{\tau \to 0} [y(t+\tau) - y(t)] = 0.$$

We consider next a stochastic process y(t) and denote by $f(t, \tau)$ the variance of the increment $y(t+\tau) - y(t)$. We prove a lemma concerning the function $f(t, \tau)$.

Lemma 2. If the process y(t) satisfies the assumptions of theorem 1 then the function $f(t, \tau)$ is continuous in τ.

We have clearly

$$y(t+\tau+\lambda) - y(t) = [y(t+\tau+\lambda) - y(t+\tau)] + [y(t+\tau) - y(t)].$$

On account of the assumed independence of the increments on the right hand side of this equation we have

(1) $f(t, \tau + \lambda) - f(t, \tau) = f(t + \tau, \lambda)$.

We know from theorem 1 that the increment $y(t + \tau + \lambda) - y(t + \tau)$ is normally distributed. Lemma 1 shows that it converges in probability to zero. Its variance $f(t + \tau, \lambda)$ must therefore converge to zero as λ goes to zero. We see then from (1) that $\lim_{\lambda \to 0} [f(t, \tau + \lambda) - f(t, \tau)] = 0$ so that lemma 2 is proven.

Definition 1. A process $x(t)$ is said to be a fundamental random process (Wiener process) if

(i) it is a process with independent increments and initial value $x(0) = 0$.

(ii) the increment $x(t + \tau) - x(t)$ is normally distributed with mean zero and variance $c\tau$. (where $c > 0$).

We are now ready to characterize the fundamental random process.

Theorem 2. A stochastic process $y(t)$ is a fundamental random process if and only if the following three conditions are satisfied

(a) $y(t)$ is a strongly continuous process with independent increments and initial value $y(0) = 0$.

(b) The mean value function of the process is identically zero

(c) The variance of the increments $y(t + \tau) - y(t)$ depends only on the length τ of the interval over which the increment is taken but is independent of the location of the interval on the time axis.

Proof: The necessity of these conditions follows from well known properties of the fundamental random process (see [3])so that we have only to show that the conditions of theorem 2 are

sufficient. From (a) it is immediately seen that property (i) of definition 1 is satisfied. It is seen from theorem 1 that the increments $y(t+\tau) - y(t)$ are normally distributed while (b) shows that the mean of the distribution of the increments is zero. We have therefore only to show that the variance of the increment $y(t+\tau) - y(t)$ is $c\,\tau$. According to condition (c) the function $f(t,\tau)$ depends only on τ so that we may write $f(t,\tau) = f(\tau)$. From (1) we see that $f(\tau)$ satisfies the functional equation

(2) $f(\tau + h) = f(\tau) + f(h).$

Lemma 2 states that $f(\tau)$ is a continuous function. It is well known that the only continuous solution of (2) is $f(\tau) = c\,\tau$. Since $f(\tau)$ is by definition a variance we see finally that $c > 0$.

4. <u>Processes with factorable variance function.</u>

In this section we give an example which shows that in a strongly continuous process with independent increments the variance function $f(t,\tau)$ need not be independent of t.

<u>Theorem 3.</u> Let $y(t)$ be a stochastic process, defined for $t = 0$ and denote by $f(t,\tau)$ the variance of the increment $[y(t+\tau) - y(t)]$. Assume that

(i) $y(t)$ is a process with independent increments and is strongly continuous in every finite interval $[a,b]$ where $0 \leqq a < b$,

(ii) the mean value function of $y(t)$ is identically zero, i.e. $Ey(t) \equiv 0$,

(iii) the initial value is zero, $y(0) = 0$,

(iv) $f(t,\tau) > 0$ for $t \geqq 0$ and $\tau > 0$

(v) $f(t,\tau) = h(t)\, g(\tau)$

Then $y(t)$ is a Gaussian process with covariance function

(3) $$\sigma_{y(t_1)y(t_2)} = \sigma^2_{y(t)} = c\,\frac{e^{Kt}-1}{e^{K}-1}$$

where $t = \min(t_1, t_2)$.

By means of theorem 1 we conclude from assumptions (i) and (iii) that the process is Gaussian, assumption (ii) determines its mean value function. The purpose of assumption (iv) is to exclude some "improper" processes. To see this we assume that there exist values t_o and $\tau_o > 0$ such that $f(t_o, \tau_o) = 0$. From the independence of the increments and the additivity of the variance it follows then that $f(t_o, \tau_1) = 0$ if $\tau_1 \leqq \tau_o$ and also $f(t_o + \tau_1, \tau_2) \equiv 0$ if $\tau_1 + \tau_2 \leqq \tau_o$. The increments $y(t_o+\tau_1+\tau_2) - y(t_o + \tau_1)$ have therefore a degenerate distribution if $\tau_1 + \tau_2 \leqq \tau_o$. We conclude therefore from (ii) that $y(t) = y(t_o)$ if $t_o \leqq t \leqq t_o + \tau_o$. Assumption (iv) excludes therefore processes which do not change over a fixed interval. We proceed now to derive formula (3). From assumption (v) and from (1) we see that

(4) $$h(t)\,[g(\tau+\lambda) - g(\tau)] = h(t+\tau)\, g(\lambda)$$

Further we conclude from (iv) that $h(t) \neq 0$ for all $t \geqq 0$ while $g(\lambda) \neq 0$ for all $\lambda > 0$. We may therefore rewrite (4) as

(5) $$\frac{g(\tau+\lambda) - g(\tau)}{g(\lambda)} = \frac{h(t+\tau)}{h(t)}$$

The left hand side of this equation is independent of t but depends on τ and λ, which the right hand side of (5) is a function of τ and t but is independent of λ. This is only

possible if the quotient on either side is a function only of γ. We denote this function by $\rho(\gamma)$ and obtain

(6) $h(t+\gamma) = \rho(\gamma)\, h(t)$.

If we set $t = 0$ we have $h(\gamma) = h(0)\,\rho(\gamma)$ hence $h(t+\gamma) = h(0)\,\rho(t+\gamma)$. Substituting these expressions into (6) we obtain

$h(t+\gamma) = h(0)\,\rho(t+\gamma) = \rho(\gamma)h(t) = \rho(\gamma)\,h(0)\,\rho(t)$ or

(7) $\rho(t+\gamma) = \rho(t)\,\rho(\gamma)$.

Since $g(\gamma)$ is a continuous function of γ the same is true for $\rho(\gamma)$ so that the only solution of (7) is

(8) $\rho(t) = e^{Kt}$

and therefore

(9) $h(t) = h(0)\, e^{Kt}$

From (5) and (8) we obtain the functional equation for $g(\gamma)$

(10) $g(\gamma+\lambda) = g(\gamma) + e^{Kt}\, g(\lambda)$

In case $K = 0$, this reduces to

(10a) $g(\gamma+\lambda) = g(\gamma) + g(\lambda)$

and the process $y(t)$ is then a fundamental random process. We may therefore assume in the following that $K \neq 0$. We obtain easily from (10)

$$g(t_1+t_2 \dots +t_n) = g(t_1) + e^{Kt_1} g(t_2) + e^{K(t_1+t_2)} g(t_3) + \dots + e^{K(t_1+t_2+\dots+t_{n-1})}\, g(t_n).$$

If in particular $t_1 = t_2 = \dots = t_n = t$ we have

(11) $$g(nt) = g(t) \sum_{s=0}^{n-1^2} e^{Kst} = g(t)\, \frac{e^{Knt}-1}{e^{Kt}-1}$$

Setting here $t = \frac{1}{n}$ we obtain for integer n

$g(\frac{1}{n}) = g(1) \frac{e^{K/n}-1}{e^K -1}$. In the same manner we obtain for integers m and n

$$(12) \quad g(\frac{m}{n}) = g(1) \frac{e^{mK/n}-1}{e^K -1}$$

From the continuity of the function $g(\lambda)$ we conclude finally that for any real

$$(13) \quad g(\lambda) = g(1) \frac{e^{\lambda K}-1}{e^K -1} \quad \text{(if } K \neq 0).$$

The variance $f(t, \tau)$ of the increment $y(t+\tau) - y(t)$ is now immediately obtained from assumption (v) and from (9) and (13) and is

$$(14) \quad f(t, \tau) = c\, e^{Kt} \frac{e^{K\tau}-1}{e^K -1}$$

where we wrote for brevity $c = h(0)\, g(1) = f(0,1)$.

By assumption (iii) $y(t) = y(t) - y(0)$ so that the variance of $y(t)$ is obtained by substituting $t = 0$ and $\tau = t$ into (14) hence $\sigma^2_{y(t)} = c \frac{e^{Kt}-1}{e^K -1}$. The formula (3) for the covariance function follows then easily from the independence of the increments.

The Gaussian process defined by theorem 3 could offer an alternative to the fundamental random process, which is different from the Ornstein - Uhlenbeck process. Its main significance lies however in the fact that it shows together with theorem 2 that there exist strongly continuous processes with independent increments which are different from the fundamental random process.

[1] Eugene Lukacs, A property of strongly continuous processes. NBS report 1665, May 1952

[2] H. B. Mann, On the realization of stochastic processes by probability distributions in function space. Sankhya 11, p. 3-8,(1951)

[3] H. B. Mann, Introduction to the theory of stochastic processes depending on a continuous parameter. NBS report 1293, to be published as AMS 24.

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

NBS